Root Finding Analysis

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Written Problems

For some parts of this assignment, I used R (I typset in RStudio with RMarkdown). After reviewing some of my answers with other students', I believe R not to be as rigorous in its precision as Maple. I tried to replace these wherever I could find them after the fact.

1

(a)

$$u_0 = f(a_0) = (2.5)^3 - 25 = -9.375$$

$$v_0 = f(b_0) = (3)^3 - 25 = 2$$

$$(-9.375)(2) = u_0 v_0 < 0$$

$$c_0 = \frac{a_0 + b_0}{2} = \frac{2.5 + 3}{2} = 2.75$$

$$f(c_0) = (2.75)^3 - 25 = -4.203125 < 0 \implies a_1 = c_0$$

$$u_1 = f(a_1) = -4.203125$$

$$v_1 = 15.625$$

$$(-4.203125)(15.625) = u_1 v_1 < 0$$

$$c_1 = \frac{a_1 + b_1}{2} = \frac{2.75 + 3}{2} = \frac{5.75}{2} = 2.875$$

$$f(c_0) = 2.875^3 - 25 = -1.236328 < 0 \implies a_2 = c_1$$
$$u_1 = f(a_2) = -1.236328$$

$$v_2 = 15.625$$

$$(-1.236328)(15.625) = u_2 v_2 < 0$$

$$c_2 = \frac{a_2 + b_2}{2} = \frac{2.875 + 3}{2} = \frac{5.875}{2} = 2.9375$$

$$f(c_2) = 2.9375^3 - 25 = 0.3474121 \implies b_3 = c_2$$

Therefore our best estimate of the root is:

$$\frac{a_3 + b_3}{2} = \frac{2.875 + 2.9375}{2} = \frac{5.8125}{2} = 2.90625$$

(b)

$$f(a_0) = (2.5)^3 - 25 = -9.375$$

$$f(b_0) = (3)^3 - 25 = 2$$

$$c_0 = \frac{a_0 f(b_0) - b_0 f(a_0)}{f(b_0) - f(a_0)} = \frac{2.5(2) - 3(-9.375)}{(2) - (-9.375)} = 2.912087912$$

$$f(2.912087912) = -.30474899 < 0 \implies a_1 = c_0$$

$$f(a_1) = -.30474899$$

$$c_1 = \frac{a_1 f(b_1) - b_1 f(a_1)}{f(b_1) - f(a_1)} = \frac{2.912087912(2) - 3(-.30474899)}{(2) - (-.30474899)} = 2.923712$$

$$f(2.923712) = -0.007841255 < 0 \implies a_2 = c_1$$

$$f(a_2) = -0.007841255$$

$$c_1 = \frac{a_1 f(b_1) - b_1 f(a_1)}{f(b_1) - f(a_1)} = \frac{2.923712(2) - 3(-0.007841255)}{(2) - (-0.007841255)} = 2.92401$$

$$f(2.92401) = -0.0001984818 < 0 \implies a_3 = c_2$$

Therefore our best estimate for the root is:

$$\frac{a_3 + b_3}{2} = \frac{2.92401 + 3}{2} = \frac{5.92401}{2} = 2.962005$$

(c)

$$x_2 = x_1 - \left(\frac{x_1 - x_0}{f(x_1) - f(x_0)}\right) f(x_1) = 2.5 - \left(\frac{2.5 - 3}{f(2.5) - f(3)}\right) f(2.5) = 2.5 - \left(\frac{2.5 - 3}{-9.375 - 2}\right)(-9.375) = 2.912087912$$

$$x_3 = 2.912087912 - \left(\frac{2.912087912 - 2.5}{f(2.912087912) - f(2.5)}\right) f(2.912087912)$$

$$= 2.912087912 - \left(\frac{2.912087912 - 2.5}{-.30474899 - (-9.375)}\right)(-.30474899) = 2.925933546$$

(d)

$$f'(x) = 3x^2$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 3 - \frac{(3)^3 - 25}{3(3)^2} = 2.925926$$

$$x_2 = 2.925926 - \frac{(2.925926)^3 - 25}{3(2.925926)^2} = 2.924019$$

(e)

The absolute error at the n^{th} step is

$$\frac{b-a}{2^{n+1}}$$

So, given $a = 2.5$ and $b = 3$:

$$\frac{.5}{2^{n+1}} < 0.0001 \implies \frac{.5}{0.0001} < 2^{n+1} \implies n = log_2(5000) - 1 = 11.288$$

So 12 iterations.

(f)

$$f(x) = x^6 - 625$$

$$x_1 = 3 - \frac{(3)^6 - 625}{6(3)^5} = 2.928669410$$

$$x_2 = 2.928669410 - \frac{(2.928669410)^6 - 625}{6(2.928669410)^5} = 2.924036170$$

$$f(x) = x^9 - 15625$$

$$x_1 = 2 - \frac{(3)^9 - 15625}{9(3)^8} = 2.931277414$$

$$x_2 = 2.931277414 - \frac{(2.931277414)^9 - 15625}{9(2.931277414)^8} = 2.924089242$$

2.

6.

- In Figure 1 (you can find the Figures at the end of this document), on the interval $[0, 4]$, it is clear that bisection would converge after two iterations. The function is:

$$f(x) = \frac{|x-3|}{2} - 1$$

- In Figure 2, on the interval $[0, 8]$, bisection will skip over the zero at the first iteration, focusing instead on the second half of the domain. Since the jump in the piecewise function is at 5.1, which is not a number that could be reached perfectly by dividing intervals in two, bisection will continue to mistakenly hone in on 5.1. Here's the function:

$$f(x) = \begin{cases} -|x-1| & x \leq 5.1 \\ x & x > 5.1 \end{cases}$$

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You could stop when the relative error between estimates of iterations becomes less than the desired relative error. To have at most a relative error of ϵ, stop at iteration n such that:

$$\frac{|(a_{n-1} - b_{n-1}) - (a_n - b_n)|}{a_{n-1} - b_{n-1}} \leq \epsilon$$

3.

(a)

$$\lim_{n\to\infty} C = \lim_{n\to\infty} \left| \frac{1/(n+1)^2}{1/n^2} \right| = \lim_{n\to\infty} \left| \frac{n^2}{n^2 + 2n + 1} \right| = 1$$

Since $\lim\limits_{n\to\infty} C = 1$, a_n converges to 0 sublinearly.

(b)

$$\lim_{n\to\infty} C = \lim_{n\to\infty} \left| \frac{e^{-n-1}}{e^{-n}} \right| = e^{-1}$$

Since $\lim\limits_{n\to\infty} C = e^{-1}$, a_n converges to 0 linearly.

(c)

$$\lim_{n\to\infty} C = \lim_{n\to\infty} \left| \frac{10^{-8n-8}}{10^{-8n}} \right| = 10^{-8}$$

Since $\lim\limits_{n\to\infty} C = 10^{-8}$, a_n converges to 0 linearly.

(d)

$$\lim_{n\to\infty} C = \lim_{n\to\infty} \left| \frac{10^{-2^{n+1}}}{10^{2\cdot(-2^n)}} \right| = \lim_{n\to\infty} |10^{-(2^n \cdot 2) + 2\cdot(2^n)}| = 1$$

4.

n	x_n	$\frac{e_{n+1}}{e_n}$
1	1.864864864864865	0.7577139739549537
2	1.7559978841183803	0.7423969691856867
3	1.6705514884977428	0.7276594268101231
4	1.6053016781199145	0.7141950847586408
5	1.5568007512213087	0.7025433921137829

Clearly for some constant C on the interval $[0, 1)$:

$$\frac{e_{n+1}}{e_n} \leq C$$

The convergence theorem of Newton's Method requires that $f'(r)$ be nonzero. Since $(x^3 - 3)^3$ is a factor of the polynomial, $f'(r)$ is zero, r is a multiple zero with multiplicity 3. The quadratic convergence can be recovered with the modified Newton's Method:

n	x_n	$\frac{e_{n+1}}{e_n}$
1	1.5945945945945945	0.2731419218648604
2	1.457713794919963	0.10150790736298042
3	1.4424305293346824	0.011701784719758633
4	1.442249595196821	0.00013754170282604961

The 5th iteration actually ran into the zero derivative constraint within the code, but it is clear that the convergence was accelerated.

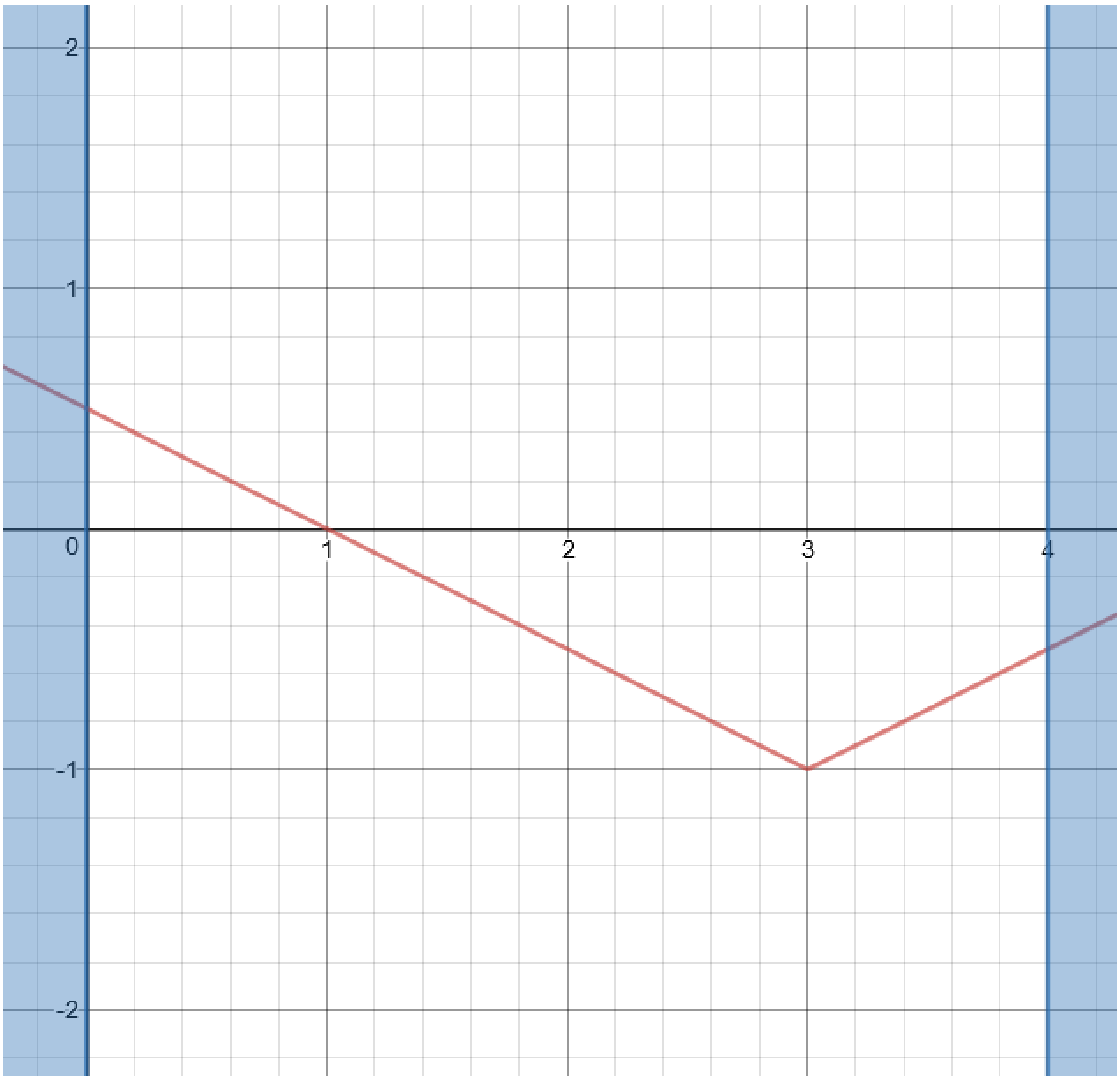

Figure 1: Bisection Converges

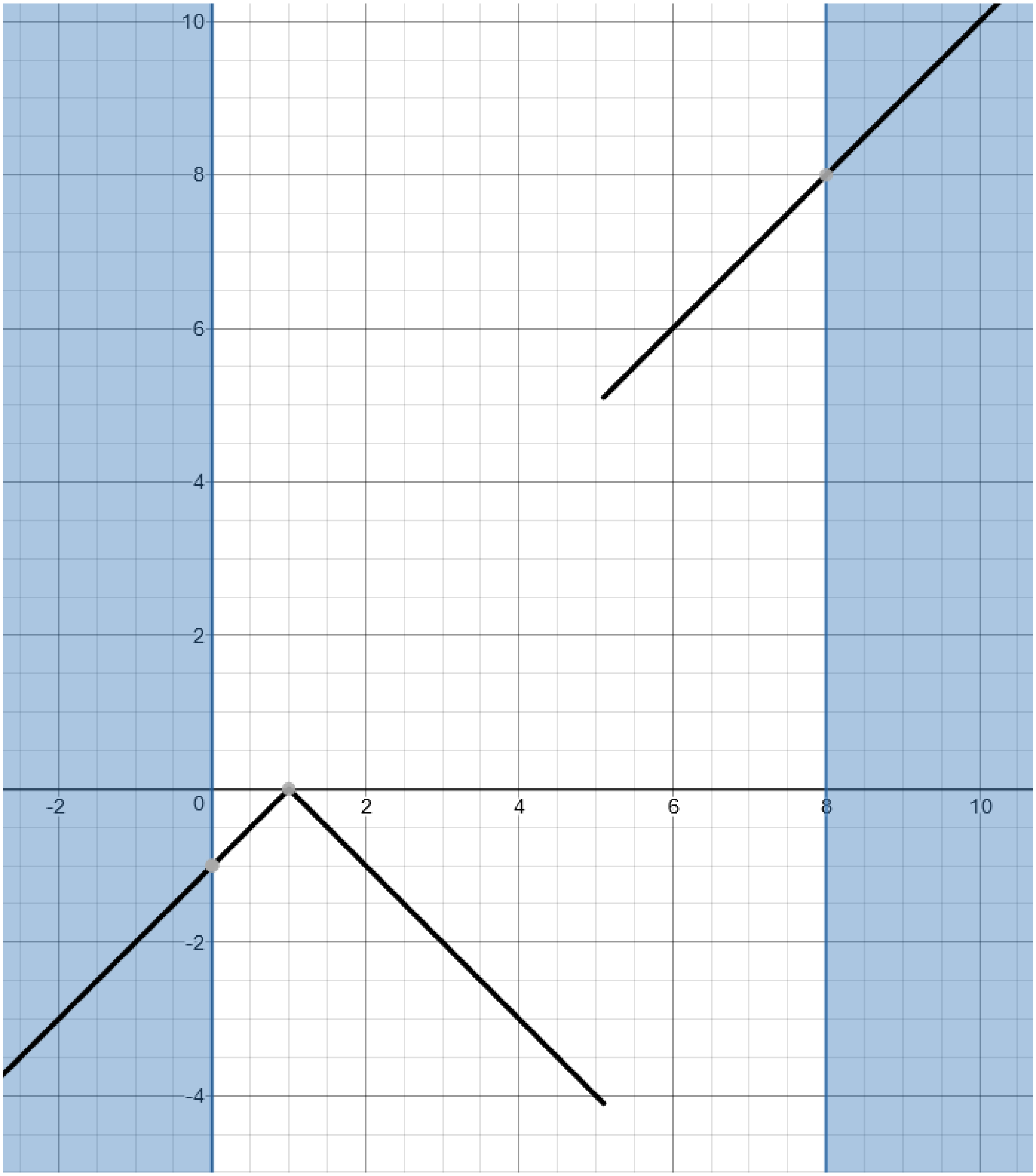

Figure 2: Bisection Diverges