Taylor Series and Error Analysis

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Text Problems

Page 54

38

$x - y = 0.0000183092$

$fl(x) = 0.53215$ and $fl(y) = 0.53213$ so $fl(x) - fl(y) = 0.00002$. Therefore the relative error in the calculation is:

$$\frac{|0.00002 - 0.0000183092|}{0.0000183092} = 0.0923470168 \approx 9\%$$

Page 63

2

Using Taylor series:

$$e^{0.01} \approx 1 + 0.01 + \frac{0.01^2}{2!} + \frac{0.01^3}{3!} + \frac{0.01^4}{4!} + \frac{0.01^5}{5!} = 1.010050167$$

$$f(10^{-2}) = 1.010050167 - 0.01 - 1 = 5.0167 \times 10^{-5}$$
$$f(10^{-2}) = [1.0101] - 0.01 - 1 = 10^{-4}$$

The absolute error between the two is 4.9833×10^{-5} and the relative error is $\approx 99\%$, which means that rounding $e^{0.01}$ to 1.0101 results in a large amount of error.

5

There can be cancellation error in the subtraction $e^x - e^{-x}$ when $|x| \leq \frac{1}{2}$. You can rewrite this using taylor series and the even terms will cancel out, leaving you with no subtraction in your approximation calculation.

$$\frac{1}{2}(e^x - e^{-x}) = \frac{1}{2}((1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + ...) - (1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - ...))$$

$$\implies \sinh(x) = \frac{1}{2}(2x + \frac{2x^3}{3!} + \frac{2x^5}{5!} + \frac{2x^7}{7!} + ...)$$

$$\implies \sinh(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + ...$$

10

There is potential loss of signifigance in the subtraction as x grows larger. This statement can be rewritten by rationalizing the numerator.

$$(\sqrt{1+x^2}-x)\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}+x}=\frac{(\sqrt{1+x^2})^2-(x)^2}{\sqrt{1+x^2}+1}=\frac{1}{\sqrt{1+x^2}+1}$$

17

You can rationalize the denominator.

$$\frac{\sin(x)}{x-\sqrt{x^2-1}}=\frac{\sin(x)(x+\sqrt{x^2-1})}{(x-\sqrt{x^2-1})(x+\sqrt{x^2-1})}=\sin(x)(x+\sqrt{x^2-1})$$

21

$$|\sin(x)-x|=(\sin(x)-x)+(\cos(x)x-1)-\frac{\sin(x)x^2}{2}-\frac{\cos(x)x^3}{6}+E_4(x)$$

$$E_4(x)=\frac{\sin(x)x^4}{24}$$

Per Maple:

$$|E_4(x)|\leq\frac{\max_{0\leq x\leq 0.1}|\sin(x)|}{24}\max_{0\leq x\leq 0.1}|x^4|=\frac{0.09983341665}{24}0.0001=4.159725694\times 10^{-7}$$

Extra Problems

1.

If x, y, z are machine numbers, then $fl(x)=x, fl(y)=y, fl(z)=z$, so

$$fl(fl(fl(x)\times fl(y))\times fl(z)=fl(fl(x\times y)\times z)=((xy)(1+\delta_{xy}))z(1+\delta_{xyz})$$

$$=xyz(1+\delta_{xy})(1+\delta_{xyz})=xyz(1+\delta_{xy}+\delta_{xyz}+\delta_{xy}\delta_{xyz})\approx xyz(1+\delta_{xy}+\delta_{xyz})$$

The approximation comes from letting the multiplication of two errors be 0. Since $\delta xy\leq$ eps and $\delta xyz\leq$ eps, the error in this calculation is $\leq 2\cdot$ eps.

If they're not machine numbers...

$$fl(fl(fl(x)\times fl(y))\times fl(z)=((x(1+\delta_y)y(1+\delta_y))(1+\delta_{xy}))z(1+\delta_z)(1+\delta_{xyz})$$

$$=xyz(1+\delta_x)(1+\delta_y)(1+\delta_z)(1+\delta_{xy})(1+\delta_{xyz})$$

Again, the approximation here comes from letting the multiplication of two errors be 0:

$$\approx xyz(1+\delta x+\delta y)(1+\delta z+\delta xy)(1+\delta xyz)$$

$$\approx xyz((1+\delta z+\delta xy+\delta x+\delta y)(1+\delta xyz)\approx xyz(1+\delta z+\delta xy+\delta x+\delta y+\delta xyz)$$

Since $\delta x, \delta y, \delta z, \delta xy, \delta xyz \leq \text{eps}$, the error in this calculation is $\leq 5\cdot\text{eps}$.

2.

a

$$y-y_0=\frac{y_1-y_0}{x_1-x_0}(x-x_0)$$

Let $y=0$.

$$(0-y_0)\frac{x_1-x_0}{y_1-y_0}=(x-x_0)\implies -\frac{(x_1-x_0)y_0}{y_1-y_0}=(x-x_0)\implies x=x_0-\frac{(x_1-x_0)y_0}{y_1-y_0}$$

b

$$x=x_0-\frac{(x_1-x_0)y_0}{y_1-y_0}=-\frac{x_0(y_1-y_0)-(x_1-x_0)y_0}{y_1-y_0}=\frac{x_0y_1-x_0y_0-x_1y_0+x_0y_0}{y_1-y_0}$$

$$=\frac{x_0y_1-x_1y_0}{y_1-y_0}$$

c

Per Maple with Digits:=3 :

$$\frac{(1.31)(4.76)-(1.93)(3.24)}{4.76-3.24}=-0.00658$$

$$1.31-\frac{(1.93-1.31)3.24}{4.76-3.24}=0.902$$

The actual x-intercept is -0.0116 (at 3 digits). The loss of signifigance from the subtractions in the second method (there are more, and the operands of the subtraction in the numerator are nearer) lead to much more relative error. The first formula is therefore more accurate.

Oddly, I changed Digits:=300 and even then the relative error was about the same. The results from these operations must simply be awful to represent on a base two machine? Desmos gave the same error, but my TI-84 gave no error at all.