CS 477: Homework #1

Due on September 13, 2016 at 2:30pm

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Problem 1

(U & G-required)[30 points] Arrange the following list of functions in ascending order of growth rate. That is, if function g(n) immediately follows function f(n) in your list.

$f_1(n) = n^{4.5}$

$f_2(n) = \sqrt{2n^2 + 1}$

$f_3(n) = n^2 + 10$

$f_4(n) = 10^n$

$f_5(n) = 100^n$

$f_6(n) = n^2 \log n$

Solution:

Formula	Complexity
$f_2(n) = \sqrt{2n^2 + 1}$	$\mathcal{O}(n)$
$f_6(n) = n^2 \log n$	$\mathcal{O}(n^2 \log n)$
$f_3(n) = n^2 + 10$	$\mathcal{O}(n^2)$
$f_1(n) = n^{4.5}$	$\mathcal{O}(n^{4.5})$
$f_4(n) = 10^n$	$\mathcal{O}(10^n)$
$f_5(n) = 100^n$	$\mathcal{O}(100^n)$

Problem 2

(U & G-required)[30 points] Using mathematical induction, show that the following relations are true for every $n \geq 1$:

a) $\displaystyle\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$

b) $\displaystyle\sum_{i=1}^{n} i(i+1) = \frac{n(n+1)(n+2)}{3}$

Solution:

Part A

Proof: *We will prove by induction that, for all* $n \in \mathbb{Z}_+$,

$$\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2 \tag{1}$$

Base Case: *When* $n = 1$, *the left side of (1) is* $(1)^3 = 1$, *and the right side is* $\left[\frac{1(1+1)}{2}\right]^2 = \left[\frac{2}{2}\right]^2 = 1$, *so both sides are equal and (1) is true for* $n = 1$.

Induction Step: *Let* $k \in \mathbb{Z}_+$ *be given and suppose (1) is true for* $n = k$. *Then*

$$\begin{aligned}
\sum_{i=1}^{k+1} i^3 &= \sum_{i=1}^{k} + (k+1)^3 \\
\left[\frac{(k+1)(k+2)}{2}\right]^2 &= \left[\frac{k(k+1)}{2}\right]^2 + (k+1)^3 \\
&= \frac{k^2(k+1)}{2^2}^2 + (k+1)^3 \\
&= \frac{k^2(k+1)}{4}^2 + \frac{4(k+1)^3}{4} \\
&= \frac{(k+1)^2(k^2+4k+4)}{4} \\
&= \frac{(k+1)^2(k+2)^2}{2^2} \\
&= \left[\frac{(k+1)(k+2)}{2}\right]^2
\end{aligned}$$

Thus, (1) holds for $n = k + 1$, *and the proof of the induction step is complete.*

Conclusion: *By the principle of induction, (1) is true for all* $n \in \mathbb{Z}_+$

Problem 2 continued on next page...

Part B

Proof: *We will prove by induction that, for all* $n \in \mathbb{Z}_+$,

$$\sum_{i=1}^{n} i(i+1) = \frac{n(n+1)(n+2)}{3} \tag{1}$$

Base Case: *When* $n = 1$, *the left side of (1) is* $(1)(1+1) = 2$, *and the right side is* $\frac{(1)(1+1)(1+2)}{3} = 2$, *so both sides are equal and (1) is true for* $n = 1$.

Induction Step: *Let* $k \in \mathbb{Z}_+$ *be given and suppose (1) is true for* $n = k$. *Then*

$$\begin{aligned}
\sum_{i=1}^{k+1} i(i+1) &= \sum_{i=1}^{k} + (k+1)(k+2) \\
\frac{(k+1)(k+2)(k+3)}{3} &= \frac{k(k+1)(k+2)}{3} + (k+1)(k+2) \\
&= \frac{k(k+1)(k+2)}{3} + \frac{3(k+1)(k+2)}{3} \\
&= \frac{(k+1)(k+2)(k+3)}{3}
\end{aligned}$$

Thus, (1) holds for $n = k + 1$, *and the proof of the induction step is complete.*

Conclusion: *By the principle of induction, (1) is true for all* $n \in \mathbb{Z}_+$

Problem 3

(U & G-required)[40 points] Indicate whether the first function of each of the following pairs has a smaller, same, or larger order of growth (to within a constant multiple) than the second function:

a) $(n-1)!$ and $n!$

b) $\log_2 n$ and $\ln n$

c) 2^{n-1} and 2^n

d) $\log_2^2 n$ and $\log_2 n^2$

Solution

a) *same*

b) *smaller*

c) *same*

d) *larger*

Problem 5

[**Extra credit - 20 points**] Using the formal definition of the asymptotic notations, prove the following statements:

a) $5n^2 + 20 \in \mathcal{O}(n^2)$

b) $n + 23 \in \mathcal{O}(n^3)$

Solution

a) $0 \leq 5n^2 + 20 \leq cn^2 \rightarrow c \geq \frac{20}{n^2} + 5$ and $n_0 \geq 1$
$\therefore c = 25$ and $n_0 = 1$ and $5n + 20 \in \mathcal{O}(n^2)$

b) $0 \leq n + 23 \leq cn^3 \rightarrow c \geq \frac{n+23}{n^3}$ and $n_0 \geq 1$
$\therefore c = 24$ and $n_0 = 1$ and $n + 23 \in \mathcal{O}(n^3)$