Dirac Notation
Complete Reference for Quantum Mechanics

1 Fundamental Concepts

Dirac Notation Definition

Bra-Ket Notation (Dirac notation) is a standard notation for describing quantum states and operations in quantum mechanics.

Three Key Components:

- **Ket**: $|\psi\rangle$ - column vector representing a quantum state
- **Bra**: $\langle\phi|$ - row vector (complex conjugate transpose)
- **Bracket**: $\langle\phi|\psi\rangle$ - inner product (probability amplitude)

Basic Definitions

$$\text{Ket:} \quad |\psi\rangle \equiv \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \tag{1}$$

$$\text{Bra:} \quad \langle\psi| \equiv (c_1^*, c_2^*, \ldots, c_n^*) \tag{2}$$

$$\text{Inner Product:} \quad \langle\phi|\psi\rangle = \sum_i \phi_i^* \psi_i \tag{3}$$

Normalization

For normalized states:

$$\langle\psi|\psi\rangle = 1$$

This ensures the total probability equals unity.

2 Mathematical Operations

2.1 Linear Combinations

Superposition Principle

Any quantum state can be written as:

$$|\psi\rangle = \sum_i c_i |i\rangle$$

where $|i\rangle$ are basis states and c_i are complex coefficients.

2.2 Operators

Operator Action

Operators $\hat{A}$ act on kets:

$$\hat{A}|\psi\rangle = |\phi\rangle$$

Expectation Values

The expectation value of operator $\hat{A}$:

$$\langle\hat{A}\rangle = \langle\psi|\hat{A}|\psi\rangle$$

Operators are represented by hats (e.g., $\hat{H}, \hat{p}, \hat{x}$)

2.3 Outer Products

Projection Operators

The outer product creates projection operators:

$$|\psi\rangle\langle\phi| = \hat{P}_{\psi,\phi}$$

Special case - projector onto state $|\psi\rangle$:

$$\hat{P}_\psi = |\psi\rangle\langle\psi|$$

3 Standard Quantum States

3.1 Spin-1/2 States

Pauli Basis States

$$|+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{4}$$

$$|+_x\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad |-_x\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{5}$$

$$|+_y\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}, \quad |-_y\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix} \tag{6}$$

3.2 Position and Momentum

Continuous Basis States

$$\text{Position:} \quad |x\rangle \text{ - eigenstate of } \hat{x} \tag{7}$$

$$\text{Momentum:} \quad |p\rangle \text{ - eigenstate of } \hat{p} \tag{8}$$

$$\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar} \tag{9}$$

Wave Functions

The wave function is the position representation:

$$\psi(x) = \langle x|\psi\rangle$$

4 Key Relations & Identities

4.1 Completeness Relations

Resolution of Identity

For discrete basis:

$$\sum_i |i\rangle\langle i| = \hat{I}$$

For continuous basis:

$$\int |x\rangle\langle x| dx = \hat{I}$$

4.2 Orthogonality

Orthonormal Basis

$$\langle i|j\rangle = \delta_{ij} \quad \text{(discrete)} \tag{10}$$

$$\langle x|x'\rangle = \delta(x - x') \quad \text{(continuous)} \tag{11}$$

Be careful with normalization factors in continuous bases!

5 Time Evolution

5.1 Schrödinger Equation

Time-Dependent Schrödinger Equation

$$i\hbar \frac{\partial}{\partial t}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle$$

where $\hat{H}$ is the Hamiltonian operator.

5.2 Time Evolution Operator

Unitary Evolution

$$|\psi(t)\rangle = \hat{U}(t, t_0)|\psi(t_0)\rangle$$

where $\hat{U}(t, t_0) = e^{-i\hat{H}(t-t_0)/\hbar}$

Properties of $\hat{U}$

- Unitary: $\hat{U}^\dagger\hat{U} = \hat{I}$
- $\hat{U}(t_0, t_0) = \hat{I}$
- $\hat{U}(t_2, t_0) = \hat{U}(t_2, t_1)\hat{U}(t_1, t_0)$

6 Quantum Measurement

6.1 Born Rule

Probability of Measurement

Probability of measuring eigenvalue a_n of operator $\hat{A}$:

$$P(a_n) = |\langle a_n|\psi\rangle|^2$$

where $|a_n\rangle$ is the corresponding eigenstate.

6.2 Post-Measurement State

State Collapse

After measuring a_n, the state becomes:

$$|\psi'\rangle = \frac{\hat{P}_n|\psi\rangle}{\sqrt{\langle\psi|\hat{P}_n|\psi\rangle}}$$

where $\hat{P}_n = |a_n\rangle\langle a_n|$

Measurement Types:

- **Projective**: Uses projection operators
- **POVM**: Positive Operator-Valued Measure
- **Weak**: Minimal disturbance measurements

7 Advanced Concepts

7.1 Tensor Products

Composite Systems

For two systems A and B:

$$|\psi_{AB}\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$$

In general:

$$|\psi_{AB}\rangle = \sum_{i,j} c_{ij} |i_A\rangle \otimes |j_B\rangle$$

7.2 Entanglement

Bell States

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \tag{12}$$

$$|\Phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \tag{13}$$

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \tag{14}$$

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \tag{15}$$

Entanglement Criterion

A state is entangled if it cannot be written as:

$$|\psi\rangle \neq |\psi_A\rangle \otimes |\psi_B\rangle$$

8 Essential Formulas

8.1 Commutation Relations

Canonical Commutators

$$[\hat{x}, \hat{p}] = i\hbar \tag{16}$$

$$[\hat{L}_i, \hat{L}_j] = i\hbar\epsilon_{ijk}\hat{L}_k \tag{17}$$

$$[\hat{a}, \hat{a}^\dagger] = 1 \quad \text{(harmonic oscillator)} \tag{18}$$

8.2 Uncertainty Relations

Heisenberg Uncertainty Principle

For any two operators $\hat{A}$ and $\hat{B}$:

$$\Delta A \cdot \Delta B \geq \frac{1}{2}|\langle[\hat{A}, \hat{B}]\rangle|$$

Most famous case:

$$\Delta x \cdot \Delta p \geq \frac{\hbar}{2}$$

The uncertainty principle is a fundamental feature of quantum mechanics, not a limitation of measurement!

9 References & Resources

Recommended Reading

Classic Textbooks:

- Griffiths - Introduction to Quantum Mechanics
- Sakurai - Modern Quantum Mechanics
- Nielsen & Chuang - Quantum Computation

Key Papers:

- Dirac - The Principles of Quantum Mechanics
- Born - Statistical Interpretation
- Bell - On the Einstein-Podolsky-Rosen Paradox

Quick Reference

Common Notation:

- $\hbar = \frac{h}{2\pi}$ - reduced Planck constant
- $\hat{H}$ - Hamiltonian (energy operator)
- $\hat{U}$ - unitary time evolution operator
- δ_{ij} - Kronecker delta
- $\delta(x)$ - Dirac delta function