AP Calculus BC | Complete Review Cheat Sheet

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How to Use This Cheat Sheet

- Find your topic in the section headers
- Look for boxed formulas and key concepts
- Review examples and common mistakes
- Use the tips for exam strategy

1 Limits & Continuity

Limit Definition

$\lim_{x\to a} f(x) = L$ means that $f(x)$ can be made arbitrarily close to L by taking x sufficiently close to a.

Key Limit Properties:

- $\lim_{x\to a}[f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$
- $\lim_{x\to a}[f(x) \cdot g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$
- $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$, if $\lim_{x\to a} g(x) \neq 0$

Special Limits

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$
$$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$$
$$\lim_{x\to 0} \frac{e^x - 1}{x} = 1$$
$$\lim_{x\to 0} (1+x)^{1/x} = e$$

Continuity

A function f is continuous at $x = a$ if:

1. $f(a)$ is defined
2. $\lim_{x\to a} f(x)$ exists
3. $\lim_{x\to a} f(x) = f(a)$

When finding limits, watch for indeterminate forms - expressions whose values cannot be determined by direct substitution. These include $\frac{0}{0}$, $\frac{\infty}{\infty}$, $0\cdot\infty$, $\infty-\infty$, 0^0, 1^∞, ∞^0. For example, $\lim_{x\to 0} \frac{\sin x}{x}$ gives $\frac{0}{0}$ but equals 1. Such forms require L'Hôpital's Rule or algebraic manipulation to evaluate.

2 Derivatives

Definition

$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

2.1 Basic Derivative Rules

Power Rule

$\frac{d}{dx}[x^n] = nx^{n-1}$

Product Rule

$\frac{d}{dx}[f(x)\cdot g(x)] = f'(x)\cdot g(x) + f(x)\cdot g'(x)$

Quotient Rule

$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)\cdot g(x) - f(x)\cdot g'(x)}{[g(x)]^2}$

Chain Rule

$\frac{d}{dx}[f(g(x))] = f'(g(x))\cdot g'(x)$

2.2 Derivative Table

Common Derivatives

$$\frac{d}{dx}[c] = 0 \quad \text{(constant)}$$
$$\frac{d}{dx}[e^x] = e^x$$
$$\frac{d}{dx}[\ln x] = \frac{1}{x}$$
$$\frac{d}{dx}[\sin x] = \cos x$$
$$\frac{d}{dx}[\cos x] = -\sin x$$
$$\frac{d}{dx}[\tan x] = \sec^2 x$$
$$\frac{d}{dx}[\sec x] = \sec x \tan x$$
$$\frac{d}{dx}[a^x] = a^x \ln a$$
$$\frac{d}{dx}[\log_a x] = \frac{1}{x\ln a}$$

2.3 Implicit Differentiation

Steps

1. Differentiate both sides with respect to x
2. When differentiating y terms, multiply by $\frac{dy}{dx}$
3. Solve for $\frac{dy}{dx}$

Example

For $x^2 + y^2 = 25$:

$$\frac{d}{dx}[x^2 + y^2] = \frac{d}{dx}[25]$$
$$2x + 2y\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = -\frac{x}{y}$$

2.4 Related Rates

Set up an equation relating the quantities, then differentiate with respect to time.

Steps for Related Rates:

- Draw a diagram if helpful
- Identify known and unknown rates
- Write equation relating variables
- Differentiate with respect to time
- Substitute known values and solve

3 Applications of Derivatives

3.1 Mean Value Theorem

MVT

If f is continuous on $[a,b]$ and differentiable on (a,b), then there exists at least one $c \in (a,b)$ such that:

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

3.2 Curve Analysis

Critical Points

Critical points occur where $f'(x) = 0$ or $f'(x)$ is undefined.

Increasing/Decreasing

- $f'(x) > 0 \Rightarrow f(x)$ is increasing
- $f'(x) < 0 \Rightarrow f(x)$ is decreasing

Concavity

- $f''(x) > 0 \Rightarrow f(x)$ is concave up
- $f''(x) < 0 \Rightarrow f(x)$ is concave down
- Inflection points occur where $f''(x) = 0$ or $f''(x)$ is undefined and $f''(x)$ changes sign

3.3 Optimization

Steps for Optimization:

- Identify the quantity to optimize
- Express as a function of one variable
- Find critical points
- Determine max/min using first or second derivative test

3.4 L'Hôpital's Rule

L'Hôpital's Rule

If $\lim_{x\to a} \frac{f(x)}{g(x)}$ gives $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then:

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

provided: (1) f, g are differentiable near a (except possibly at a), (2) $g'(x) \neq 0$ near a, and (3) the right limit exists.

You may need to apply L'Hôpital's Rule multiple times.

4 Integration

Indefinite Integral

$\int f(x)\,dx = F(x) + C$ where $F'(x) = f(x)$

Definite Integral

$\int_a^b f(x)\,dx = F(b) - F(a)$ where $F'(x) = f(x)$

4.1 Basic Integration Rules

Power Rule

$\int x^n\,dx = \frac{x}{n+1} + C, \quad n \neq -1$

Common Integrals

$$\int e^x\,dx = e^x + C$$
$$\int \frac{1}{x}\,dx = \ln|x| + C$$
$$\int \sin x\,dx = -\cos x + C$$
$$\int \cos x\,dx = \sin x + C$$
$$\int \sec^2 x\,dx = \tan x + C$$
$$\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + C$$
$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C$$

4.2 Integration Techniques

Substitution

Let $u = g(x)$, then:

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du$$

Integration by Parts

$$\int u\,dv = uv - \int v\,du$$

Choose u and dv using LIATE priority (a helpful guideline, not a strict rule):

- L: Logarithmic functions
- I: Inverse trigonometric functions
- A: Algebraic functions
- T: Trigonometric functions
- E: Exponential functions

Some texts use ILATE or ALTI – all work similarly. Choose the leftmost function for u.

Partial Fractions

For rational functions $\frac{P(x)}{Q(x)}$ where $\deg P < \deg Q$:

- Factor denominator $Q(x)$
- For each linear factor $(ax+b)$, add term $\frac{A}{ax+b}$
- For each quadratic factor (ax^2+bx+c), add term $\frac{Ax+B}{ax^2+bx+c}$
- Solve for constants

5 Applications of Integration

5.1 Area

Area Between Curves

Area between $f(x)$ and $g(x)$ from $x = a$ to $x = b$:

$$A = \int_a^b [f(x) - g(x)]\,dx$$

where $f(x) \geq g(x)$ on $[a,b]$.

Area in Polar Coordinates

$$A = \frac{1}{2}\int_\alpha^\beta [r(\theta)]^2\,d\theta$$

5.2 Volume

Disk Method

Volume of solid of revolution around x-axis:

$$V = \pi\int_a^b [f(x)]^2\,dx$$

Washer Method

Volume of solid of revolution with hole:

$$V = \pi\int_a^b [(R(x))^2 - (r(x))^2]\,dx$$

where $R(x)$ is outer radius and $r(x)$ is inner radius.

Shell Method

Volume of solid of revolution around y-axis:

$$V = 2\pi \int_a^b x \cdot h(x)\,dx$$

where x is the radius of each shell and $h(x)$ is the height.

For rotation around x-axis, use $V = 2\pi \int_a^b y \cdot h(y)\,dy$

5.3 Arc Length

Arc Length Formula

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx$$

Parametric Arc Length

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

6 Differential Equations

First-Order ODE

$\frac{dy}{dx} = f(x, y)$

6.1 Separation of Variables

Method

For equations of form $\frac{dy}{dx} = g(x)h(y)$:

1. Rewrite as $\frac{1}{h(y)}\,dy = g(x)\,dx$
2. Integrate both sides
3. Solve for y

6.2 Slope Fields

A slope field is a graphical representation of a differential equation, showing the slope at various points in the xy-plane.

6.3 Exponential Growth/Decay

Model

$$\frac{dy}{dt} = ky \quad \Rightarrow \quad y = Ce^{kt}$$

- $k > 0$: exponential growth
- $k < 0$: exponential decay

Logistic Model

$$\frac{dy}{dt} = ky\left(1 - \frac{y}{L}\right) \quad \Rightarrow \quad y = \frac{L}{1 + Ae^{-kt}}$$

where L is carrying capacity.

7 Sequences & Series

Sequence

A sequence $\{a_n\}$ is a function whose domain is the set of positive integers.

Series

A series is the sum of terms of a sequence: $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots$

7.1 Convergence Tests

Geometric Series

$\sum_{n=0}^{\infty} ar^n$ converges to $\frac{a}{1-r}$ if $|r| < 1$, diverges otherwise.

n-th Term Test

If $\lim_{n\to\infty} a_n \neq 0$, then $\sum a_n$ diverges.

Integral Test

If $f(x) \geq 0$, continuous, and decreasing for $x \geq 1$ with $f(n) = a_n$, then: $\sum_{n=1}^{\infty} a_n$ converges if and only if $\int_1^{\infty} f(x)\,dx$ converges.

Comparison Tests

- Direct: If $0 \leq a_n \leq b_n$ and $\sum b_n$ converges, then $\sum a_n$ converges.
- Limit: If $\lim_{n\to\infty} \frac{a}{b} = c > 0$, then $\sum a_n$ and $\sum b_n$ either both converge or both diverge.

Ratio Test

If $\lim_{n\to\infty} \left|\frac{a}{a}\right| = L$, then:

- If $L < 1$: series converges absolutely
- If $L > 1$ or $L = \infty$: series diverges
- If $L = 1$: test is inconclusive

Alternating Series Test

If $a_n > 0$, $\{a_n\}$ is decreasing, and $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges.

7.2 Power Series

Power Series

$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots$

Radius of Convergence

For a power series centered at a, the radius of convergence R is:

$$R = \frac{1}{\limsup_{n\to\infty} \sqrt{|c_n|}} = \lim_{n\to\infty} \left|\frac{c_n}{c_{n+1}}\right|$$

The series converges for $|x-a| < R$ and diverges for $|x-a| > R$.

7.3 Taylor Series

Taylor Series

Taylor series of $f(x)$ centered at $x = a$:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Maclaurin Series

Taylor series centered at $x = 0$:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$$

Common Maclaurin Series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots, |x| < 1$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots, |x| < 1$$

7.4 Series Manipulations

Operations

If both series converge absolutely:

- Addition: $\sum a_n \pm \sum b_n = \sum(a_n \pm b_n)$
- Multiplication: $(\sum a_n)(\sum b_n) = \sum c_n$ where $c_n = \sum_{k=0}^{n} a_k b_{n-k}$
- Term-by-term differentiation/integration of power series

8 Parametric, Polar & Vector

8.1 Parametric Equations

Definition

Parametric equations express x and y in terms of a third variable t: $x = f(t)$, $y = g(t)$.

Derivatives

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{y'(t)}{x'(t)}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{y'(t)}{x'(t)}\right)}{\frac{dx}{dt}}$$

$$= \frac{y''(t)x'(t) - y'(t)x''(t)}{[x'(t)]^3}$$

8.2 Polar Coordinates

Conversion

$$x = r\cos\theta, \quad y = r\sin\theta$$

$$r = \sqrt{x^2 + y^2}, \quad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Derivatives

$$\frac{dy}{dx} = \frac{r'(\theta)\sin\theta + r(\theta)\cos\theta}{r'(\theta)\cos\theta - r(\theta)\sin\theta}, \quad \text{where } r'(\theta) = \frac{dr}{d\theta}$$

8.3 Vectors

Vector Operations

For vectors $\vec{u} = \langle u_1, u_2, u_3\rangle$ and $\vec{v} = \langle v_1, v_2, v_3\rangle$:

- Addition: $\vec{u} + \vec{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3\rangle$
- Scalar multiplication: $c\vec{u} = \langle cu_1, cu_2, cu_3\rangle$
- Dot product: $\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + u_3v_3$
- Cross product: $\vec{u} \times \vec{v} = \langle u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1\rangle$

Vector Properties

- Magnitude: $|\vec{u}| = \sqrt{u_1^2 + u_2^2 + u_3^2}$
- Unit vector: $\hat{u} = \frac{\vec{u}}{|\vec{u}|}$
- Angle between vectors: $\cos\theta = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}$

9 AP Exam Tips

Free-Response Questions:

- Show all work clearly
- Include units when appropriate (some answers are dimensionless)
- Use correct mathematical notation
- Justify answers when asked
- Interpret results in context of the problem

Common Mistakes

- Forgetting to include $+C$ in indefinite integrals
- Errors in chain rule application
- Incorrect use of L'Hôpital's Rule
- Misidentifying convergence/divergence of series
- Not checking endpoints in optimization problems

When in doubt on a multiple-choice question, try substituting values or checking limiting cases.