An algorithm for finding factorizations of complete graphs

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Abstract

We show how to find a decomposition of the edge set of the complete graph into regular factors where the degree and edge-connectivity of each factor is given.

Let t, n, $k_1, k_2, \ldots, k_t$, $l_1, l_2, \ldots, l_t$ be nonnegative integers such that $\sum_i k_i = n-1$, if n is odd then each k_i is even, and for $1 \leq i \leq t$, $l_i \leq k_i$ and $l_i = 0$ if $k_i = 1$. We shall show how to find a factorization of K_n that contains an l_i-edge-connected k_i-factor, $1 \leq i \leq t$.

In the case where each $k_i = l_i = 2$, these factorizations are Hamiltonian decompositions, and there is an ancient and well-known method of construction. This method can be extended in a fairly obvious way to k-regular k-connected factorizations of K_{kn+1}, as shown in [1]; the only difficulty is showing that the k-factors are k-connected. Being k-connected, they are *a fortiori* k-edge-connected. The method is further extended in [3] to the case where the factors have different degrees; see also [2]. Our algorithmic method is completely different. Perhaps the most interesting case is again the classical one where each $k_i = l_i = 2$, and even in this case we believe our method is new.

Let an edge between vertices u and v of K_n be denoted uv. Let $\mathcal{F} = (F_1, \ldots, F_t)$ be a collection of graphs on $V(K_n)$ such that for $1 \le i \le t$, F_i is l_i-edge-connected and k_i-regular. The graphs may have multiple edges but not loops. It is not difficult to construct such graphs; see, for example, [1]. Let $f(uv)$ be the number of times that uv appears in the graphs of $\mathcal{F}$. Let the *discrepancy* of $\mathcal{F}$ be denoted and defined by

$$\delta(\mathcal{F}) = \sum_{uv \in E(K_n)} |f(uv) - 1|.$$

The graphs of $\mathcal{F}$ form a factorization of K_n if and only if $\delta(\mathcal{F}) = 0$. If $\delta(\mathcal{F}) > 0$, then we will find $\mathcal{G} = (G_1, \ldots, G_t)$, another collection of graphs on $V(K_n)$ such that for $1 \le i \le t$, G_i is l_i-edge-connected and k_i-regular, with $\delta(\mathcal{G}) < \delta(\mathcal{F})$. By repetition, a factorization is found.

Note that

$$\sum_{u \in V(K_n) \setminus \{v\}} f(uv) = n - 1. \tag{1}$$

Thus the existence of an edge ac_1 such that $f(ac_1) > 1$ implies and is implied by the existence of an edge bc_1 such that $f(bc_1) < 1$. So there is no loss of generality in assuming that

$$f(ac_1) > 1, \tag{2}$$

and

$$f(bc_1) < 1. \tag{3}$$

For each graph $F_i \in \mathcal{F}$, we can find k_i neighbours of a and k_i neighbours of b in F_i by counting a vertex u as a neighbour of a more than once if there is more than one edge au. We create k_i (a,b)-*pairs* in F_i by pairing off the neighbours of a with the neighbours of b in the following way. First, since F_i is l_i-edge-connected we can find l_i edge-disjoint a–b paths: let these be $au_j \cdots v_j b$, $1 \leq j \leq l_i$, and let (u_j, v_j) be an (a,b)-pair. Then for each edge ab in F_i, let (b,a) be an (a,b)-pair. Finally, pair off any remaining neighbours of a and b arbitrarily.

Assertion 1 *There exists a sequence of vertices of K_n, $c_1, c_2, \ldots c_m$, $m \geq 2$, such that*

(I) $c_j \notin \{a,b\}$, $1 \leq j \leq m$,

(II) $c_j \neq c_h$ *if* $j \neq h$,

(III) *either* $f(ac_m) = 0$ *or* $f(bc_m) > 1$ *(possibly both), and*

(IV) *for* $1 \leq j \leq m-1$, *there is a graph* $F_{i_j} \in \mathcal{F}$ *such that* (c_j, c_{j+1}) *is an* (a,b)-*pair in* F_{i_j}.

Before we prove the Assertion, we find $\mathcal{G}$. For $1 \leq j \leq m-1$, in F_{i_j} delete the edges ac_j and bc_{j+1} and replace them with the edges ac_{j+1} and bc_j (as $c_j \notin \{a,b\}$, $1 \leq j \leq m$, we are not adding any loops). We call this an (a,b)-*swap* of c_j and c_{j+1}. Note that as the F_{i_j} are not necessarily distinct we may make several changes to a single graph. For $1 \leq i \leq t$, let G_i be the same as F_i except for these alterations; this yields $\mathcal{G}$.

It is clear that for $1 \leq i \leq t$, G_i is a k_i-regular graph on $V(K_n)$. We must show that it is l_i-edge-connected. Let F_i be a graph in $\mathcal{F}$. Recall that F_i contains the paths $au_j \cdots v_j b$, $1 \leq j \leq l_i$. Therefore any graph obtained from F_i by (a,b)-swaps also contains l_i edge-disjoint a–b paths since it contains either $au_j \cdots v_j b$ or $av_j \cdots u_j b$, $1 \leq j \leq l_i$. We use induction to show that any graph obtained from F_i by (a,b)-swaps is l_i-edge-connected. We know that F_i is l_i-edge-connected. Suppose that after some number of (a,b)-swaps we have obtained a graph L_1 that is l_i-edge-connected, and then we (a,b)-swap an (a,b)-pair (u,v) to obtain a graph L_2. That is, au and bv are deleted in L_1 and replaced by av and bu to obtain L_2. If L_2 is not l_i-edge connected, then we can find a minimal separating set of edges E such that $|E| < l_i$. We show that this implies that L_1 has a separating set of edges of the same size as E, a contradiction. Let L_3 and L_4 be the two connected components of $L_2 - E$. In L_2 there are l_i edge-disjoint a–b paths so a and b must be in the same component of $L_2 - E$, say L_3. If u and v were both in L_3 as well, then we could reverse the (a,b)-swap of u and v to obtain $L_1 - E$ which would also have two components, a contradiction. If u and v were both in L_4, then av and bu must both be in E. Thus $(E \setminus \{av, bu\}) \cup \{au, bv\}$ would be a separating set of L_1, a contradiction. Finally, suppose that u is in L_3 and v is in L_4. Then $av \in E$ and $bu \in L_3$. Let $E' = (E \setminus \{av\}) \cup \{bv\}$ and $L_3' = (L_3 - \{bu\}) \cup \{au\}$. Thus $L_1 - E'$ has two connected components, L_3' and L_4. This final contradiction shows that G_i is l_i-edge-connected.

We show that $\mathcal{G}$ has a smaller discrepancy than $\mathcal{F}$. For $2 \leq j \leq m-1$, ac_j is added to $F_{i_{j-1}}$ but removed from F_{i_j} so $f(ac_j)$ is unchanged; similarly $f(bc_j)$ is unchanged. The edge ac_1 is deleted from F_{i_1} (and not added to any other graph) and bc_1 is added to F_{i_1}. Also ac_m is added to $F_{i_{m-1}}$ and bc_m is removed from $F_{i_{m-1}}$. Thus

- $f(ac_1)$ is reduced by 1,
- $f(bc_1)$ is increased by 1,
- $f(ac_m)$ is increased by 1, and
- $f(bc_m)$ is reduced by 1.

By (2) and (3), the changes in $f(ac_1)$ and $f(bc_1)$ will cause the discrepancy to be reduced by 2. By (III), the changes in $f(ac_m)$ and $f(bc_m)$ either have no effect on the discrepancy or cause it to be further reduced by 2. So the discrepancy of $\mathcal{G}$ is at least 2 less than the discrepancy of $\mathcal{F}$. It only remains to prove the Assertion.

The first term of the sequence is known already. By (2), we can find a graph $F_{i_1} \in \mathcal{F}$ that contains ac_1 and so there is an (a,b)-pair (c_1, u) in F_{i_1}. As there are no loops $u \neq b$, and since for each edge ab we formed an (a,b)-pair (b,a), $c_1 \neq b$ implies that $u \neq a$. If $u = c_1$, then (3) is contradicted. So we can let $c_2 = u$.

Suppose that we have found the first s terms of the sequence and that this sequence satisfies (I), (II) and (IV) with $m = s$. If for some $h \in \{2, \ldots, s\}$ either $f(ac_h) = 0$ or $f(bc_h) > 1$, then we choose the smallest such h and let the complete sequence be $c_1, c_2, \ldots, c_h$. Otherwise for $2 \leq j \leq s$,

$$\begin{aligned} f(ac_j) &\geq 1, \\ f(bc_j) &\leq 1. \end{aligned}$$

In fact, considering also (IV), for $2 \leq j \leq s$,

$$f(bc_j) = 1. \tag{4}$$

Now we find c_{s+1}. As $f(ac_s) \geq 1$ we can find a graph $F_{i_s} \in \mathcal{F}$ that contains ac_s and hence an (a,b)-pair (c_s, v); $v \notin \{a, b\}$ for if $ab \in F_{i_s}$ then (b,a) is an (a,b)-pair. By (4), there is exactly one graph in $\mathcal{F}$ that contains $c_j b$, $2 \leq j \leq s$, and, by (IV), c_j belongs to the (a,b)-pair (c_{j-1}, c_j); thus $c_s \neq c_{j-1}$ implies that $v \neq c_j$. By (3), no graph in $\mathcal{F}$ contains $c_1 b$ so $v \neq c_1$. So we can let $c_{s+1} = v$.

The sequence must terminate since there is a finite number of vertices and, by (1) and (2), there is at least one vertex w such that $f(aw) < 1$.

References

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[3] W. R. Johnstone, Decompositions of complete graphs, *Bull. London Math. Soc.*, **32** (2000), 141-145.