Bogoliubov Transformations in Black-Hole Evaporation

A.N.St.J.Farley and P.D.D'Eath

Department of Applied Mathematics and Theoretical Physics, Centre for Mathematical Sciences, University of Cambridge, Wilberforce Road, Cambridge CB3 0WA, United Kingdom

Abstract

In previous papers and letters on quantum amplitudes in black-hole evaporation, a boundary-value approach was developed for calculating (for example) the quantum amplitude to have a prescribed slightly non-spherical configuration of a massless scalar field ϕ on a final hypersurface Σ_F at a very late time T, given initial almost-stationary spherically-symmetric gravitational and scalar data on a space-like hypersurface Σ_I at time $t = 0$. For definiteness, we assumed that the gravitational data are also spherically symmetric on Σ_F. Such boundary data can correspond to a classical solution for the Einstein/scalar system, describing gravitational collapse from an early low-density configuration to a nearly-Schwarzschild black hole. This approach provides the quantum amplitude (not just the probability) for a transition from an initial to a final state. For a real Lorentzian time-interval T, the classical boundary-value problem refers to a set of hyperbolic equations (*modulo* gauge), and is badly posed. Instead, the boundary-value approach of the previous letters and papers requires (following Feynman) a rotation into the complex: $T \to |T|\exp(-i\theta)$, for $0 < \theta \leq \pi/2$, of the time-separation-at-infinity T. The classical boundary-value problem, for a complex solution of the coupled nonlinear classical field equations, is expected to be well-posed for $0 < \theta \leq \pi/2$. For a locally-supersymmetric Lagrangian, containing supergravity coupled to supermatter, the classical Lorentzian action $S_{\rm class}$, a functional of the boundary data (which include the complexified T), yields a quantum amplitude proportional to $\exp(iS_{\rm class})$, apart from possible loop corrections which are negligible for boundary data with frequencies below the Planck scale. Finally (still following Feynman), one computes the Lorentzian quantum amplitude by taking the limit of $\exp(iS_{\rm class})$ as $\theta \to 0_+$. In the present paper, a connection is made between the above boundary-value approach and the original approach to quantum evaporation in gravitational collapse to a black hole, *via* Bogoliubov coefficients. This connection is developed through consideration of the radial equation obeyed by the (adiabatic) non-spherical classical perturbations. When one studies the probability distribution for configurations of the final scalar field, based on our quantum amplitudes above, one finds that this distribution can also be interpreted in terms of the Wigner quasi-probability distribution for a harmonic oscillator.

1. Introduction

In previous letters and papers [1-3], we calculated the quantum amplitude for a given configuration of the massless scalar field ϕ on a space-like hypersurface Σ_F at a very late time T, measured at spatial infinity, given (for simplicity) that both the gravitational and scalar field configurations $(h_{ij},\, \phi)_I$ on an initial hypersurface Σ_I are exactly spherically symmetric, and that the gravitational data h_{ijF} on Σ_F are also exactly spherically symmetric. Here, $h_{ij} = g_{ij}$ denotes the intrinsic spatial metric $(i, j = 1, 2, 3)$, while $g_{\mu\nu}$ denotes the 4-dimensional space-time metric ($\mu, \nu = 0, 1, 2, 3$). Mathematically, this calculation involves analytic continuation of the quantum amplitude into the lower complex T-plane. In [2,3] on the 'complex approach' and the 'spin-0 amplitude' in black-hole evaporation, this was phrased in terms of a rotation of T into the complex: $T \to |T| \exp(-i\theta)$, where $0 < \theta \leq \pi/2$. Provided that the appropriate system of coupled Einstein/massless-scalar classical field equations is strongly elliptic [4], up to gauge, for $0 < \theta \leq \pi/2$, the complex *classical* boundary-value problem, corresponding to the above quantum amplitude, should be well posed. In the extreme case $\theta = \pi/2$, the boundary data have been rotated so as to provide a boundary-value problem for a real Riemannian (positive-definite) 4-metric $g_{\mu\nu}$, coupled to a real scalar field ϕ. The field equations are 'elliptic *modulo* gauge', for which one expects good behaviour of the boundary-value problem, including analyticity of the solutions.

It is assumed throughout that the original bosonic theory, such as Einstein gravity with a minimally-coupled massless scalar field, is embedded in a locally-supersymmetric theory containing supergravity with supermatter. In this case, the smallest such theory [5] contains $N = 1$ supergravity with its spin-3/2 gravitino, a *complex* scalar field and its massless spin-1/2 partner. Only for locally-supersymmetric theories does one expect that the loop terms $A_0, A_1, A_2, \ldots$ in a semi-classical expansion of the quantum amplitude $\mathrm{Amp} \sim (A_0 + \hbar A_1 + \hbar^2 A_2 + \ldots) \exp(-I_B/\hbar)$ will be finite [6-8]. Here, for our bosonic boundary data above, I_B is the classical 'Euclidean action' of the boundary-value solution of the coupled Einstein and bosonic-matter field equations. For simplicity, in our example, at present the only non-zero boundary data allowed are a real Riemannian 3-metric h_{ij} and a real scalar field ϕ. Note that, for dimensional reasons, the loop corrections will only be significant for boundary data which contain frequencies at and beyond the Planck scale.

In [3], the quantum amplitude was then evaluated semi-classically for $\theta > 0$, and finally the limit of the amplitude was taken as $\theta \to 0_+$, to obtain the Lorentzian amplitude (that is, the amplitude for T real), following Feynman's $+i\epsilon$ prescription [2]. In particular (Eq.(4.16) of [3]), we computed the imaginary part of the (necessarily) complex spin-0 classical action by taking the contribution from the poles along the real-frequency axis. The discrete set of real frequencies $\sigma_n = n\pi/|T|$, for $n = 1, 2, 3, \ldots,$ was found to dominate in late-time field configurations. So far, however, we have made no explicit mathematical connection with the usual theory of black-hole evaporation [9-12]. We rectify this omission in this paper, by considering Bogoliubov transformations between (real) massless spin-0-field modes propagating in a slowly-varying Schwarzschild-like solution, such as appears in Sec.2 of [3] for adiabatic modes. Such Vaidya-like approximate classical solutions will subsequently be treated in more detail [13].

The original derivation of black-hole radiance involved the calculation of Bogoliubov transformations between initial and final quantum states. Because of the time-dependence of the gravitational collapse, an initial basis of (linearised) positive- and negative-frequency massless-scalar modes (for example) differs from a final basis. By completeness, there exists a (Bogoliubov) transformation between the initial and final bases. When this transformation mixes positive and negative frequencies, as in the black-hole case, one interprets the result physically in terms of particle creation.

The connection between the approach used in [3] and the approach using Bogoliubov coefficients arises from the radial equation satisfied by the (adiabatic) perturbations. The real (spin-dependent) potential in the radial equation vanishes sufficiently rapidly at spatial infinity that the radial functions near infinity (for each spin s) are superpositions of complex exponentials with complex coefficients; see Eq.(3.3) of [3] for the case $s = 0$. In the approach of [3], regularity conditions for the fields at small radius imply that the radial functions are real. An early-time basis of modes, corresponding to waves incoming from past null infinity, can be related, by completeness, to a late-time basis, which is a superposition of waves outgoing at future null infinity. It is then possible to relate, on a common late-time space-like hypersurface, the Bogoliubov coefficients to the complex coefficients above, which arose in the boundary-value method, since the Bogoliubov coefficients are independent of space-like surface. The usual (Bogoliubov) approach gives a density matrix and probabilities for final configurations, but not the phase information present in a quantum amplitude.

Working instead with the boundary-value approach [3], we have, as seen above, a pure state, and can evaluate probabilities as $|\text{amplitude}|^2$. Thus, our approach exhibits a large qualitative difference, as compared to the usual approach. Correspondingly, there should be some differences between the physical predictions based on the Bogoliubov approach and on the present approach.

In Sec.2 of the present paper, after a review of the basic properties of Bogoliubov coefficients, we begin the above process of relating them, in our problem, to the complex coefficients $z_{n\ell}$ of Eq.(3.3) of [3]. Sec.3 treats particle emission rates. In Sec.4, we consider the resulting probability distribution, in terms of the steady-state Bogoliubov coefficients. Sec.5 contains a brief Conclusion. The Appendix relates this probability distribution to the Wigner quasi-probability distribution function for harmonic oscillators.

2. Bogoliubov transformations

Consider first the (nearly-)Lorentzian collapse problem. We assume the existence of a spherically-symmetrical Lorentzian-signature 'reference' or 'background' metric in the form

$$ds^2 \;=\; -\,e^{b(t,r)}dt^2 \,+\, e^{a(t,r)}dr^2 \,+\, r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \quad . \tag{2.1}$$

In this context, it is conventional to define the 'mass function' $m(t,r)$ by

$$\exp\Big(-a\big(t,r\big)\Big) \;=\; 1 - \frac{2m(t,r)}{r} \quad . \tag{2.2}$$

In the boundary-value problem outlined above, we write $(\gamma_{\mu\nu}\,,\Phi)$ for the 'background' spherically-symmetric metric and scalar field, and ∇_μ for the background covariant derivative.

Now consider a classical linearised solution $\phi^{(1)}$ of the massless scalar wave equation

$$\nabla^\mu\nabla_\mu\,\phi^{(1)} \;=\; 0 \quad . \tag{2.3}$$

Suppose first that one is given two spherically-symmetric Cauchy surfaces in the space-time [14], an initial Cauchy surface $\mathcal{S}^-$ and a final Cauchy surface $\mathcal{S}^+$. Then, subject to regularity and spatial fall-off conditions, one can expand $\phi^{(1)}$ in terms either of a basis $\{f_{\omega' \ell m}(x)\}$ of mode solutions Eq.(2.12,16) of [3] adapted to $\mathcal{S}^-$, or of a basis $\{p_{\omega\ell m}(x)\}$ adapted to $\mathcal{S}^+$:

$$\phi^{(1)}(x) \;=\; \sum_{\ell=0}^{\infty} \quad \sum_{m=-\ell}^{\ell} \int_0^\infty d\omega' \, \Big[c_{\omega'\ell m}\, f_{\omega'\ell m}(x) + \text{c.c.}\Big] \quad , \tag{2.4}$$

$$\phi^{(1)}(x) \;=\; \sum_{\ell=0}^{\infty} \quad \sum_{m=-\ell}^{\ell} \int_0^\infty d\omega \, \Big[b_{\omega\ell m}\, p_{\omega\ell m}(x) + \text{c.c.}\Big] \quad . \tag{2.5}$$

The c-number complex coefficients $\{c_{\omega'\ell m}\}$ are a set of position-independent 'Fourier amplitudes' labelling the configuration of the field on $\mathcal{S}^-$, while the $\{b_{\omega\ell m}\}$ refer correspondingly to $\mathcal{S}^+$. On $\mathcal{S}^-$, the $\{f_{\omega'\ell m}\}$ are an orthonormal, complete, family of complex solutions of the wave equation, which contain only positive frequencies ($\omega' > 0$). Here, in addition to making use of the spherical symmetry of the background 4-geometry, one adopts a simple definition of 'positive-frequency' in regions where the background gravitational and scalar fields are approximately static. Similarly, on $\mathcal{S}^+$, the $\{p_{\omega\ell m}\}$ are an analogous family of positive-frequency solutions ($\omega > 0$).

In the presence of a future event horizon, $\mathcal{H}^+$, the final Cauchy surface $\mathcal{S}^+$ (say) will typically need to cross $\mathcal{H}^+$. This is the case in the familiar treatments of black-hole evaporation [9-12], which are based on Bogoliubov transformations. Of course, given our analytic-continuation strategy, as developed in earlier papers [2,3], with the time-interval at infinity taken to be $T = |T| \exp(-i\theta)$ for $0 < \theta \leq \pi/2$, one has effectively no event horizon in the nearly-Lorentzian problem. Rather, one expects the boundary-value problem to be strongly elliptic, and so to resemble the real Riemannian boundary-value problem, with no obstruction across any 'horizon'. Thus, in our case, one can replace the usual Cauchy surface S^+, which in the centre of the space-time lies close to the Lorentzian curvature singularity, with the final surface Σ_F above, on which the final data $(h_{ij}\,\phi)_F$ are posed. One can take the intrinsic geometry of Σ_F to be nearly flat, with Σ_F diffeomorphic to $\mathbb{R}^3$. The time-separation from the initial surface Σ_I to Σ_F , as measured at spatial infinity, is $T = |T| \exp(-i\theta)$, with $|T|$ very large. A possible choice for a hyper-surface which crosses $\mathcal{H}^+$ smoothly is a constant 'time' slice in a Painlevé-Gullstrand-like coordinate system, modified slightly to account for the slow change of the black-hole mass due to emission; these coordinates are stationary and non-singular across $\mathcal{H}^+$ and have been used in a tunnelling interpretation of black-hole evaporation [15,16].

Thus, loosely speaking, we shall consider a 'Bogoliubov transformation to a smooth surface, long after the Lorentzian singularity'. In the same context, one may say that 'the singularity is simply by-passed in the analytic continuation'. That is, there should be an analytic complexified classical solution to the boundary-value problem, which only reaches a singular boundary precisely at Lorentzian signature ($\theta = 0$). One thus expects to have the possibility of circumventing Lorentzian singularities by deforming time-intervals suitably into the complex, much as one avoids singularities of functions $f(z)$ in the ordinary complex z-plane by deforming contours.

Suppose that the surfaces Σ_I and Σ_F are taken to be spherically symmetric. Then, the bases above may be chosen to have the form

$$\phi_{\omega\ell m}(x) \;=\; N(\omega)\,\frac{R_{\omega\ell}(r)}{r}\,e^{-i\omega t}\,Y_{\ell m}(\Omega) \tag{2.6}$$

in terms of the spherical harmonics $Y_{\ell m}(\Omega)$ [17], where $N(\omega)$ is a normalisation factor, and where positive frequency corresponds to $\omega > 0$. On Σ_I, one chooses the large-r behaviour of $R_{\omega\ell}(r)$ such that $\phi_{\omega\ell m}(x)$ is proportional to $e^{-i\omega v}$ at large r, where $v = t + r^*$. Thus, the $\{f_{\omega'\ell m}\}$ are purely ingoing at infinity. At Σ_F, one requires that $\phi_{\omega\ell m}(x)$ be proportional to $e^{-i\omega u}$ at large r, where $u = t - r^*$; thus, the $\{p_{\omega\ell m}\}$ are purely outgoing at infinity. We repeat the reasonable assumption for the collapse problem, that the initial spatial 3-geometry and scalar field (the 'star') can be taken to be approximately static. Typically, the initial Cauchy surface Σ_I would be taken to be in the early nearly-static region, whereas Σ_F would be at a time long after the collapse, where the background geometry would be approximately that of a Vaidya solution with slowly-varying mass [13].

The normalisation factor $N(\omega)$ is determined through use of the natural inner product [11]

$$(\phi_{\omega\ell m},\,\phi_{\omega'\ell'm'}) \;=\; -\,i\int_{\Sigma} d\sigma^{\mu}\,\left(\phi_{\omega\ell m}\,\nabla_{\mu}\phi^{*}_{\omega'\ell'm'} \;-\; \phi^{*}_{\omega'\ell'm'}\,\nabla_{\mu}\phi_{\omega\ell m}\right) \quad . \tag{2.7}$$

For a pair of classical solutions $\phi_{\omega\ell m}$, $\phi_{\omega'\ell'm'}$, this inner product is independent of the particular choice of (asymptotically-flat) space-like hypersurface Σ. The surface element $d\sigma^{\mu}$ in Eq.(2.7) is evaluated in the background space-time. The inner product (,) has the properties

$$\begin{aligned}
(\phi_{\omega\ell m}\,,\,\lambda\,\phi_{\omega'\ell'm'}) \;&=\; \lambda^{*}\,(\phi_{\omega\ell m}\,,\,\phi_{\omega'\ell'm'}) \quad , &(2.8)\\
(\lambda\,\phi_{\omega\ell m}\,,\,\phi_{\omega'\ell'm'}) \;&=\; \lambda\,(\phi_{\omega\ell m}\,,\,\phi_{\omega'\ell'm'}) \quad , &(2.9)\\
(\phi_{\omega\ell m}\,,\,\phi_{\omega'\ell'm'})^{*} \;&=\; (\phi_{\omega'\ell'm'}\,,\,\phi_{\omega\ell m}) \quad , &(2.10)
\end{aligned}$$

where λ is a complex number. For positive-frequency solutions of the massless scalar wave equation, this inner product is positive-definite [11]. By a suitable change of basis on each of Σ_I and Σ_F independently, one can normalise the $\{f_{\omega\ell m}\}$ and the $\{p_{\omega\ell m}\}$ such that

$$\begin{aligned}
(f_{\omega\ell m}\,,\,f_{\omega'\ell'm'}) \;&=\; (p_{\omega\ell m}\,,\,p_{\omega'\ell'm'}) \;=\; \delta_{\ell\ell'}\,\delta_{mm'}\,\delta(\omega\,,\omega') \quad , &(2.11)\\
(f_{\omega\ell m}\,,\,f^{*}_{\omega'\ell'm'}) \;&=\; (p_{\omega\ell m}\,,\,p^{*}_{\omega'\ell'm'}) \;=\; 0 \quad , &(2.12)\\
(f^{*}_{\omega\ell m}\,,\,f^{*}_{\omega'\ell'm'}) \;&=\; (p^{*}_{\omega\ell m}\,,\,p^{*}_{\omega'\ell'm'}) \;=\; -\,\delta_{\ell\ell'}\,\delta_{mm'}\,\delta(\omega\,,\omega') \quad . &(2.13)
\end{aligned}$$

Since the $\{f_{\omega'\ell m}\}$ form a complete orthonormal set on Σ_I, one may expand out a typical basis function (solution) $p_{\omega\ell m}$ on the surface Σ_I in the form

$$p_{\omega\ell m} \;=\; \int_0^{\infty} d\omega'\,\left(\alpha_{\omega'\omega\ell m}\,f_{\omega'\ell m} \;+\; \beta_{\omega'\omega\ell m}\,f^{*}_{\omega'\ell,-m}\right) \quad , \tag{2.14}$$

where the sets of complex numbers $\{\alpha_{\omega'\omega\ell m}\}$ and $\{\beta_{\omega'\omega\ell m}\}$ give the Bogoliubov coefficients [11]. Here, from the definition of spherical harmonics $Y_{\ell m}(\Omega)$ in [17], one has $Y^*_{\ell m} = (-1)^m\, Y_{\ell,-m}$. Of course, the (approximate) spherical symmetry of the background implies that solutions with the same ℓ and $|m|$ are connected in Eq.(2.14). Further, spherical symmetry implies that the Bogoliubov coefficients are independent of m, and we shall henceforth ignore this index. In the gravitational-collapse problem, the time-dependence of the collapse geometry gives $f_{\omega\ell m} \neq p_{\omega\ell m}$; thus, the coefficients $\beta_{\omega'\omega\ell}$ are non-zero and there is mixing between positive-and negative-frequency solutions.

The Bogoliubov coefficients may be expressed in terms of inner products as

$$\alpha_{\omega'\omega\ell} \;=\; (p_{\omega\ell m}\,,\, f_{\omega'\ell m}) \quad , \tag{2.15}$$

$$\beta_{\omega'\omega\ell} \;=\; -\,(p_{\omega\ell m}\,,\, f^*_{\omega'\ell,-m}) \quad . \tag{2.16}$$

Given these relations, one may invert Eq.(2.14) by expanding out $f_{\omega'\ell m}$ in terms of the $\{p_{\omega\ell m}\}$ and $\{p^*_{\omega\ell,-m}\}$ basis on Σ_F , as

$$f_{\omega'\ell m} \;=\; \int_0^\infty d\omega\;\left(\alpha^*_{\omega'\omega\ell}\, p_{\omega\ell m} \;-\; \beta_{\omega'\omega\ell}\, p^*_{\omega\ell,-m}\right) \quad . \tag{2.17}$$

One can similarly relate the 'Fourier amplitudes' $\{b_{\omega\ell m}\}$ and $\{c_{\omega'\ell m}\}$ above:

$$b_{\omega\ell m} \;=\; \int_0^\infty d\omega'\;\left(\alpha^*_{\omega\omega\ell}\, c_{\omega'\ell m} \;-\; \beta^*_{\omega'\omega\ell}\, c^*_{\omega'\ell,-m}\right) \quad , \tag{2.18}$$

$$c_{\omega'\ell m} \;=\; \int_0^\infty d\omega\;\left(\alpha_{\omega'\omega\ell}\, b_{\omega\ell m} \;+\; \beta^*_{\omega'\omega\ell}\, b^*_{\omega\ell,-m}\right) \quad . \tag{2.19}$$

Further, on substituting Eqs.(2.14,17) into Eqs.(2.11,13), one finds that the Bogoliubov coefficients must obey the quadratic relations

$$\int_0^\infty d\omega\;\left(\alpha_{\omega''\omega\ell}\,\alpha^*_{\omega'\omega\ell} \;-\; \beta_{\omega'\omega\ell}\,\beta^*_{\omega''\omega\ell}\right) \;=\; \delta(\omega',\,\omega'') \quad , \tag{2.20}$$

$$\int_0^\infty d\omega\;\left(\alpha^*_{\omega'\omega\ell}\,\beta_{\omega''\omega\ell} \;-\; \beta_{\omega'\omega\ell}\,\alpha^*_{\omega''\omega\ell}\right) \;=\; 0 \quad , \tag{2.21}$$

$$\int_0^\infty d\omega''\;\left(\alpha_{\omega''\omega\ell}\,\beta_{\omega''\omega'\ell} \;-\; \beta_{\omega''\omega\ell}\,\alpha_{\omega''\omega'\ell}\right) \;=\; 0 \quad . \tag{2.22}$$

These equations (2.20-22), which naturally express the property that the Bogoliubov coefficients are the matrix components for a map relating orthonormal bases on Σ_I and Σ_F , are discussed further in [11].

This formalism, involving the use of Bogoliubov coefficients as in the treatment of black-hole radiance and other consequences of quantum field theory in curved space-time [9-12], can now be connected with the alternative formalism set out in Secs.2-4 of [3]. Consider again the (nearly-) Lorentzian-signature case, where T denotes the proper-time interval between the initial and final hypersurfaces. Making the mode decomposition with respect to 'Lorentzian coordinates' (t, r, θ, ϕ):

$$\phi^{(1)}(t, r, \theta, \phi) \;=\; \frac{1}{r} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} Y_{\ell m}(\Omega)\, R_{\ell m}(t, r) \quad , \tag{2.23}$$

one arrives at the $(\ell\,, m)$ mode equation:

$$\Bigl(e^{(b-a)/2}\,\partial_r\Bigr)^2 R_{\ell m} - \;\bigl(\partial_t\bigr)^2 R_{\ell m} - \frac{1}{2}\Bigl(\partial_t\bigl(a-b\bigr)\Bigr)\bigl(\partial_t R_{\ell m}\bigr) - \;V_\ell(t, r)\, R_{\ell m} \;=\; 0 \quad . \tag{2.24}$$

Here,

$$V_\ell(t, r) \;=\; \frac{e^{b(t,r)}}{r^2}\left(\ell(\ell+1) \;+\; \frac{2m(t, r)}{r}\right) \quad , \tag{2.25}$$

where $m(t, r)$ is defined in Eq.(2.2). For adiabatic modes (that is, for frequencies k of oscillation which are rapid compared to the time rate of change of the background geometry), one has approximately-separable solutions of the mode equation (2.24), of the form

$$R_{\ell m}(t, r) \;\sim\; \exp(ikt)\;\xi_{k\ell m}(t, r) \quad , \tag{2.26}$$

where $\xi_{k\ell m}(t, r)$ varies 'slowly' with respect to t. The definition of $\xi_{k\ell m}(t, r)$ is tied down completely at spatial infinity $(r \to \infty)$, where $R_{\ell m}(t, r)$ is required to reduce to a flat space-time separated solution, for which $\xi_{k\ell m}(t, r)$ loses its t-dependence. By completeness, the general solution $\phi^{(1)}(t, r, \theta, \phi)$ of the linear wave equation (2.3) may be written as in Eq.(2.18) of [3]:

$$\phi^{(1)} \;=\; \frac{1}{r} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \int_{-\infty}^{\infty} dk\;\, a_{k\ell m}\, \xi_{k\ell m}(t, r)\, \frac{\sin(kt)}{\sin(kT)}\, Y_{\ell m}(\Omega) \quad . \tag{2.27}$$

where the $\{a_{k\ell m}\}$ are real coefficients. Here, for simplicity, we are taking, as in [2,3], boundary conditions on $\phi^{(1)}$ for which $\phi^{(1)}|_{\Sigma_I} = 0$ but $\phi^{(1)}|_{\Sigma_F} \neq 0$. Physically, if the gravitational initial data $h_{ij}|_{\Sigma I}$ were also taken to be spherically symmetric, this would correspond to scalar particle production resulting from isotropic collapse data. At very late times, near the final hypersurface Σ_F, the geometry also varies extremely slowly with respect to time, and the 'adiabatic separation functions' $\xi_{k\ell m}(t, r)$ again reduce to functions of r only, namely $\xi_{k\ell m}(r)$. These obey the mode equation [1-3]

$$e^{(b-a)/2}\,\frac{\partial}{\partial r}\left(e^{(b-a)/2}\,\frac{\partial \xi_{k\ell}}{\partial r}\right) \;+\; \bigl(k^2 - V_\ell\bigr)\,\xi_{k\ell} \;=\; 0 \quad , \tag{2.28}$$

with boundary conditions described below. The spherical symmetry of the background implies that $\xi_{k\ell m}(r)$ is independent of the quantum number m. Thus,

$$\xi_{k\ell m}(r) \;=\; \xi_{k\ell}(r) \quad . \tag{2.29}$$

Evaluating Eq.(2.27) at the final boundary Σ_F, one finds that

$$\phi^{(1)}(x)\Big|_{\Sigma_F} \;=\; \frac{1}{r}\sum_{\ell=0}^{\infty}\;\sum_{m=-\ell}^{\ell}\;\int_{-\infty}^{\infty} dk\;\, a_{k\ell m}\,\xi_{k\ell}(r)\,Y_{\ell m}(\Omega) \quad . \tag{2.30}$$

This relation can be inverted (see Sec.3 of [3]), given the normalisation of the $\{\xi_{k\ell}(r)\}$, to find the coefficients $a_{k\ell m}$ in terms of the final data $\phi^{(1)}(x)$. Indeed, the $a_{k\ell m}$ characterise the final scalar data. Since the mode functions $\{\xi_{k\ell}(r)\}$ on Σ_F are real, one can rewrite this in the form

$$\phi^{(1)}(x)\Big|_{\Sigma_F} \;=\; \frac{1}{r}\sum_{\ell=0}^{\infty}\;\sum_{m=-\ell}^{\ell}\;\int_{0}^{\infty} dk\;\,(a_{k\ell m}\,+\,a_{-k\ell m})\,\xi_{k\ell}(r)\,Y_{\ell m}(\Omega) \quad , \tag{2.31}$$

involving only positive k-values. Here, the geometry in this space-time region is expected to be approximated very accurately by a Vaidya metric [13], corresponding to a smoothed-out luminosity in the radiation, which varies only slowly with time. Such a metric can be put in the diagonal form (2.1), with [13]:

$$e^{-a} \;=\; 1\,-\,\frac{2m(t,r)}{r} \quad ; \qquad e^{b} \;=\; \left(\frac{\dot m}{f(m)}\right)^{2} e^{-a} \quad . \tag{2.32}$$

Here, $m(t,r)$ is a slowly-varying function, with $\dot m \;=\; (\partial m/\partial t)$, and the function $f(m)$ depends on the details of the radiation.

The 'left' boundary condition (3.1) of [3] reads

$$\xi_{k\ell} \;=\; 0 \tag{2.33}$$

at $r=0$. The 'right' boundary condition (3.3) of [3] requires that:

$$\xi_{k\ell}(r) \;\sim\; \Big(z_{k\ell}\;\exp(ikr_s^*)\;+\;z_{k\ell}^*\;\exp(-ikr_s^*)\Big) \quad , \tag{2.34}$$

as $r\;\rightarrow\;\infty$. A generalisation r^* of the standard Regge-Wheeler coordinate r_s^* for the Schwarzschild geometry [18] may be defined, by

$$\frac{\partial}{\partial r^*} \;=\; e^{(b-a)/2}\,\frac{\partial}{\partial r} \quad . \tag{2.35}$$

Here, for each (k,ℓ), $z_{k\ell}$ is a dimensionless complex coefficient. In our space-time, for sufficiently large r , a coordinate r_s^* may also be defined by

$$r_s^* \;=\; r\,+\,2M\,\ln\Big(\big(r/2M\big)-1\Big) \quad , \tag{2.36}$$

where M is the total ADM (Arnowitt-Deser-Misner) mass [18,19]. For very large r , r^* and r_s^* are asymptotically equal.

In the nearly-Lorentzian case, the late-time behaviour of quantum amplitudes will (as in Sections 3,4 of [3]) be dominated by the eigen-frequencies k_n defined in Eq.(4.1) of [3]:

$$k_n \;=\; \frac{n\pi}{T} \quad , \tag{2.37}$$

$(n = 1, 2, 3, \ldots\,)$. Accordingly, let us discretise the frequency integral in Eq.(2.31) above. Define real functions $\{f_{n\ell}(r)\}$, for $n = 1, 2, \ldots\,;\ \ell = 0, 1, 2, \ldots$, such that

$$\xi_{n\ell}(r) \;=\; \left(z_{n\ell}\, e^{ik_n r^*} \,+\, z^*_{n\ell}\, e^{-ik_n r^*}\right) f_{n\ell}(r) \quad , \tag{2.38}$$

where $f_{n\ell}(r) \to 1$ as $r \to \infty$, $f_{n\ell}(r) \to 0$ as $r \to 0$, and $r^* \sim r^*_s$ as $r \to \infty$.

The discrete description of Eq.(2.31) above is appropriate for the final data $\phi^{(1)}(x)|_{\Sigma_F}$, but we further need a discrete description of the space-time behaviour of $\phi^{(1)}(x)$. For this, define $\Delta k_n = \pi/T$, and then (at least in a neighbourhood of Σ_F) define complex coefficients $A_{n\ell m}$, complex numbers $\hat{z}_{n\ell}$ and further real functions $\{g_{n\ell}(r)\}$ such that

$$\phi^{(1)}(x) \;=\; \frac{1}{r} \sum_{\ell m n} \Delta k_n\, A_{n\ell m} \left(\hat{z}_{n\ell}\, e^{-ik_n(t-r^*)} + \hat{z}_{n\ell}\, e^{ik_n(t-r^*)}\right) g_{n\ell}(r)\, Y_{\ell m}(\Omega) \quad , \tag{2.39}$$

where the $\{g_{n\ell}(r)\}$ obey the same boundary conditions as the $\{f_{n\ell}(r)\}$. Eq.(2.39) agrees with the discretised version of Eq.(2.31), provided that

$$\begin{aligned} g_{n\ell}(r) \;&=\; f_{n\ell}(r) \quad , \qquad z_{n\ell} \;=\; \hat{z}_{n\ell}\, e^{-ik_n|T|} \quad , \\ A_{n\ell m} \;&=\; a_{n\ell m} + a_{-n\ell m} \quad . \end{aligned} \tag{2.40}$$

In order to make a comparison with the Bogoliubov description, consider now the discretised version of Eq.(2.5), appropriate to the final surface Σ_F , taking

$$p_{n\ell m}(x) \;=\; N(k_n)\, \frac{p_{n\ell}(r)}{r}\, e^{-ik_n(t-r^*)}\, Y_{\ell m}(\Omega) \quad . \tag{2.41}$$

Given the boundary condition of regularity at $r = 0$, Eq.(2.7) shows that the normalisation factor is

$$N(k_n) \;=\; (2\pi k_n)^{-1/2} \quad . \tag{2.42}$$

Then, comparing the discretised form of Eq.(2.31) with Eq.(2.39), we find

$$p_{n\ell}(r) \;=\; f_{n\ell}(r) \quad , \tag{2.43}$$

$$b_{n\ell m} \;=\; \frac{(a_{n\ell m} + a_{-n\ell m})\,\hat{z}_{n\ell}}{N(k_n)} \quad . \tag{2.44}$$

In calculating particle emission rates in the following Section 3, we shall need

$$|b_{n\ell m}|^2 \;=\; 2\pi\, k_n\, |z_{n\ell}|^2\, |a_{n\ell m} + a_{-n\ell m}|^2 \;=\; 2\, f_{\ell m}(k_n) \quad , \tag{2.45}$$

where we have used the fall-off property as $|k| \to \infty$:

$$|f_{\ell m}(k)| \;\sim\; |k|^{-3} \tag{2.46}$$

of Eq.(4.6) of [3]. For later use, we also record an expression for $|b_{\omega\ell m}|^2$ in terms of Bogoliubov coefficients, where Eq.(2.18) has been used. Including an implicit summation over the index m, we find

$$\begin{aligned} |b_{\omega\ell m}|^2 \;=\; & \int_0^\infty d\omega' \int_0^\infty d\omega'' \left(\alpha^*_{\omega'\omega\ell}\, \alpha_{\omega''\omega\ell} \;+\; \beta^*_{\omega''\omega\ell}\, \beta_{\omega'\omega\ell} \right) c_{\omega'\ell m}\, c^*_{\omega''\ell m} \\ & - 2 \int_0^\infty d\omega' \int_0^\infty d\omega''\; \mathrm{Re}\Big(\beta_{\omega''\omega\ell}\, \alpha^*_{\omega'\omega\ell}\, c_{\omega'\ell m}\, c_{\omega''\ell,-m} \Big) \quad . \end{aligned} \tag{2.47}$$

3. Particle emission rates

As remarked in Sec.5 of [2], the total energy of the final Einstein/scalar field configuration must equal that of the original configuration, namely M, the true ADM mass of the 'space-time'. Recall that we are here considering gravitational and massless-scalar perturbations about a given spherically-symmetric background solution, of the form $g_{\mu\nu} = \gamma_{\mu\nu} + \epsilon\, h^{(1)}_{\mu\nu} + \epsilon^2\, h^{(2)}_{\mu\nu} + \ldots\,,\quad \phi = \Phi + \epsilon\, \phi^{(1)} + \epsilon^2\, \phi^{(2)} + \ldots$. The 'second-variation' $O(\epsilon^2)$ contribution to M (that is, the leading contribution beyond the background contribution) on Σ_F is

$$E^{(2)} \;=\; \int_{\Sigma_F} d^3x\, (-\gamma)^{\frac{1}{2}}\, \mathcal{H} \quad , \tag{3.1}$$

where $\mathcal{H} \;=\; e^{-b}\, T^{(2)}_{tt}$ and where $T^{(2)}_{\mu\nu}$ is the perturbed energy-momentum tensor, given in Eq.(3.28) of [2], which is of quadratic order in the scalar-field perturbations, and, if evaluated at a late time, has no contribution from the background scalar field $\Phi(t,r)$, in view of the final boundary conditions that the gravitational data are spherically symmetric on Σ_F. From the wave equation (2.3), integrating by parts, using the boundary conditions of regularity and the asymptotic behaviour $r^2\, \phi^{(1)}\, \partial_r \phi^{(1)} \;=\; O(r^{-1})$ as $r \to \infty$, we find

$$E^{(2)} \;=\; \frac{1}{2} \int d\Omega \int_{\Sigma_F} dr\, r^2\, e^{(a-b)/2} \left[\dot{\phi}^{(1)2} \;-\; \phi^{(1)}\, \ddot{\phi}^{(1)} \;-\; \left(\frac{1}{2}\right) (\dot{a} - \dot{b})\, \phi^{(1)}\, \dot{\phi}^{(1)} \right] \quad . \tag{3.2}$$

We may neglect the terms in Eq.(3.2) with integrand proportional to $(\dot{a} - \dot{b})$, compared with the $\left[\, \dot{\phi}^{(1)2} - \phi^{(1)} \ddot{\phi}^{(1)} \,\right]$ terms, provided that the adiabatic approximation of Sec.2 of [3] holds; that is, that typical perturbative frequencies ω obey $\omega \gg \frac{1}{2} |\dot{a} - \dot{b}|$.

In the adiabatic case, $E^{(2)}$ decomposes into a 'sum' over frequencies. On substituting the representation (2.5) into Eq.(3.2) and using Eq.(2.41), we obtain

$$E^{(2)} \;=\; \sum_{\ell m} \int_0^\infty d\omega\;\, \omega\, |b_{\omega\ell m}|^2 \quad . \tag{3.3}$$

This can also be re-written in the 'harmonic-oscillator coordinates' $\{c_{\omega\ell m}\}$, on using Eq.(2.47).

One can read off the final particle-number spectrum from the total energy by writing

$$E^{(2)} \;=\; \sum_{\ell m}\int_0^\infty d\omega\;\omega\;\frac{dN_{\omega\ell m}}{d\omega} \quad . \tag{3.4}$$

On using Eq.(2.45), one obtains

$$\frac{dN_{\omega\ell m}}{d\omega} \;=\; |b_{\omega\ell m}|^2 \;=\; 2\,f_{\ell m}(\omega) \quad , \tag{3.5}$$

where $f_{\ell m}(\omega)$ is defined through Eqs.(2.41-43). Let ΔG_ω be the number of states or phase-space cells in mode ω, and let $N'_{\omega\ell m}$ be the number of (scalar) particles in these states. Then

$$\frac{N'_{\omega\ell m}}{\Delta G_\omega} \;=\; < n_{\omega\ell m} > \tag{3.6}$$

is the mean number of particles in each of the (quantum) states in mode $(\omega\ell m)$. Since the number of states emitted in the frequency interval $(\omega\,,\,\omega+d\omega)$ is

$$\Delta G_\omega \;=\; \frac{|T|}{\pi}\,d\omega \quad , \tag{3.7}$$

the number of particles in the range $(\omega\,,\,\omega+d\omega)$ is

$$dN'_{\omega\ell m} \;=\; \frac{|T|}{\pi}\;< n_{\omega\ell m} >\; d\omega \quad . \tag{3.8}$$

In the continuum limit,

$$d\dot{N}'_{\omega\ell m} \;=\; \frac{< n_{\omega\ell m} >}{\pi}\,d\omega \tag{3.9}$$

is the total emission rate in the range $(\omega\,,\,\omega+d\omega)$, where $\dot{N}'_{\omega\ell m}$ is the number of particles emitted over the duration $|T|$. (The factor π, rather than the usual factor 2π, occurs in the denominator in Eq.(3.9) due to the initial time being $t=0$ rather than $t=-|T|$.) Alternatively, Eq.(3.9) just gives the number of particles passing out through the surface of a sphere centred on the collapsing 'star' or black hole.

In the case (as here) that the particles emitted do not decay further, one identifies $< n_{\omega\ell m} >$ as [11]

$$< n_{\omega\ell m} > \;=\; |\beta_{\omega\ell}|^2 \quad , \tag{3.10}$$

where $|\beta_{\omega\ell}|^2$ is defined by

$$\int_0^\infty d\omega'\;\beta_{\omega'\omega\ell}\,\beta^*_{\omega'\omega''\ell} \;=\; |\beta_{\omega\ell}|^2\,\delta(\omega,\omega'') \quad . \tag{3.11}$$

Then the total number spectrum, added up over the black-hole life-time, is

$$\begin{aligned}\frac{dN'_\omega}{d\omega} \;&=\; \frac{1}{\pi}\sum_{s\ell m} N_{s\omega\ell m}\int_{t_I}^{t_F} dt\;|\beta_{s\omega\ell m}|^2 \\ &=\; \frac{1}{\pi}\sum_{s\ell m} N_{s\omega\ell m}\int_0^E dM\,\left(\frac{-dM}{dt}\right)^{-1}|\beta_{s\omega\ell m}|^2 \quad ,\end{aligned} \tag{3.12}$$

where $N_{s\omega\ell m}$ counts the number of states with spin s and quantum numbers $(\omega\ell m)$. In the original calculation of Bogoliubov coefficients and of probabilities for particle emission by a non-rotating black hole [10], it was found (neglecting the effects of back-reaction) that

$$|\beta_{s\omega\ell m}|^2 \;=\; \Gamma_{s\omega\ell m}(\tilde{m})\left(e^{4\pi\tilde{m}} - (-1)^{2s}\right)^{-1} \quad , \tag{3.13}$$

where $\Gamma_{s\omega\ell m}(\tilde{m})$ is the transmission probability over the centrifugal barrier [20] and $\tilde{m} = 2M\omega$ is dimensionless. This calculation, of course, referred to the case where one did not have a final surface Σ_F of topology $\mathbb{R}^3$. But, because of the very-high-frequency (adiabatic) method through which this expression for $|\beta_{s\omega\ell m}|^2$ was calculated, it should still be valid (up to tiny corrections) in our present $\mathbb{R}^3$ case.

The semi-classical expression for the rate of mass loss is [21]

$$\frac{dM}{dt} \;=\; -\,\frac{\alpha_0(M)}{M^2} \quad , \tag{3.14}$$

where $\alpha_0(M) \simeq$ constant for massless or ultra-relativistic particles, and $M = M(M_I, t)$ is the mass to which a black hole of initial mass M_I has been reduced after time t. Turning again to the total number spectrum over the black-hole life-time, given by N'_ω as in Eq.(3.12), note that, for massless particles, the Bogoliubov coefficients $|\beta_{s\omega\ell m}|^2$ depend on ω only through $\tilde{m}$ (on dimensional grounds). Hence,

$$\frac{dN'_\omega}{d\omega} \;=\; \frac{1}{8\pi\alpha_0\,\omega^3}\sum_{s\ell m}\int_0^{2M_I\omega} d\tilde{m}\;\tilde{m}^2\,|\beta_{s\omega\ell m}(\tilde{m})|^2 \quad . \tag{3.15}$$

In particular, in considering the high-frequency limit $\omega M_I \gg 1$ of this expression, we may replace the upper limit in the $\tilde{m}$-integral by infinity. From this, one finds that the time-integrated energy distribution of all the massless particles radiated by the black hole (of initial mass M_I) is

$$\frac{dN'_\omega}{d\omega} \;\sim\; c_1\,\omega^{-3} \quad , \tag{3.16}$$

for $\omega \gg (M_I)^{-1}$, where c_1 is a real number, so giving the form of the high-energy tail of the spectrum. However, most of the particles produced during the quasi-stationary regime of the radiating black hole, before its expected explosive phase with $M \to 0$, are at a temperature corresponding to the initial mass of the hole $\left(\omega \sim (M_I)^{-1}\right)$, where the energy distribution peaks.

How is the above analysis related to our theory which includes back-reaction? One might expect that the high-energy behaviour of the $\{a_{\omega\ell m}\}$ 'coordinates', which describe the perturbed scalar field $\phi^{(1)}$ on the final surface Σ_F, would be related to the emission in the final moments of evaporation. This can be seen from Eq.(3.5), taking the frequencies $k_n = \sigma_n = n\pi/|T|$, without an explicit calculation of the Bogoliubov coefficients. The high-energy behaviour of the function $f_{\ell m}(\omega)$, which is defined through Eq.(4.3) of [3], was found in Eq.(4.6) of [3] from the requirement that the contour at infinity in the integral (4.2) of [3] for the action should contribute zero. For massless scalar particles, this requirement, expressed through Eq.(3.5), reads

$$\frac{dN_{n\ell m}}{d\omega} \;\propto\; (\sigma_n)^{-3} \quad , \tag{3.17}$$

for $\sigma_n \gg (M_I)^{-1}$, in agreement with the high-energy behaviour in Eq.(3.16). Here, again, the requirement for a finite imaginary part for the classical Lorentzian action $S^{(2)}_{\rm class}$, corresponding to a non-trivial probability distribution over final configurations, is linked with the behaviour of the high-energy number spectrum; the latter is of course subject in principle to observational test.

It is well known that the thermal equilibrium between a black hole and the exterior radiation is unstable, as the specific heat in the canonical ensemble is negative [22]. The canonical ensemble breaks down for black holes since the canonical partition function diverges for all temperatures. Rather than work at fixed temperature, one must consider the micro-canonical ensemble, which is tailored to configurations of fixed energy. At the high-energy end of the emission spectrum, therefore, when the black hole approaches the Planck scale, the canonical distribution, as given by Eq.(3.13), must be replaced by the micro-canonical distribution. Naturally, the mass-loss rates for the two ensembles differ considerably. The micro-canonical decay rate modifies the small-mass behaviour of the black hole, where Eq.(3.14) breaks down. For the low-frequency quanta ($\omega \ll M$) characteristic of the majority of the evaporation process, the canonical and micro-canonical ensembles are almost equivalent, and one obtains a Planck-like number spectrum Eq.(3.13) [23].

Prior to the black hole's total evaporation, it is possible for the black hole to emit a single high-energy quantum with energy comparable to the initial black-hole mass. As $|\beta_j|^2$ is the average number of quanta present, then in the high-energy tail of the spectrum, one has $|\beta_j|^2 \sim e^{-S}$, where S is the Boltzmann entropy of the system, since there is only one state out of a total of e^S states for which all the energy is concentrated into one quantum [24]. More precisely,

$$\frac{|\beta_j|^2}{|\alpha_j|^2} \;\sim\; e^{-\Delta S_{BH}} \quad , \tag{3.18}$$

where ΔS_{BH} is the difference in black-hole entropy before and after emission. This result is expected to be valid for a general spherically-symmetric black hole.

4. Probabilistic interpretation

When $|T|$ is large but finite, one finds from Eq.(4.16) of [3], that the density function

$$P\Big[\{a_{k\ell m}\};|T|\Big] \;=\; \hat{N}\, e^{-\delta|T|M_I}\, \exp\Big(-2\,{\rm Im}\, S^{(2)}_{\rm class}\big[\{a_{k\ell m}\};|T|\big]\Big) \quad , \tag{4.1}$$

where $\hat{N}$ is a suitable normalisation factor, describes the conditional probability density over the final perturbative scalar boundary data $\phi^{(1)}|_{\Sigma_F}$, the condition being that the perturbations obey the initial conditions $\phi^{(1)}|_{\Sigma_I} = 0$ of Eq.(2.7) of [3]. The limit $\delta \rightarrow 0$

should then be taken; the only data at spatial infinity itself consist of the proper time $|T|$. This probability is, of course, the squared norm of a complex quantum amplitude, in the context of [3].

Even though the probability distribution (4.1) arises from squaring a quantum amplitude, one can still ask, given the thermal nature of black-hole evaporation in the usual description, whether the probability distribution (4.1) can be viewed in terms of the diagonal components of some non-trivial 'density matrix' in a suitable basis [9-12,32]. To evaluate more explicitly the probability distribution, consider Eq.(2.36), using also Eqs.(2.20,45,47) and Eq.(4.16) of [3]. One finds, in the continuum limit of large $|T|$, taking the $\{a_{k\ell m}\}$ as the final 'coordinate' variables, that:

$$
\begin{aligned}
P\Big[\{a_{\omega\ell m}\}\Big] &= \hat{N}\,\exp\biggl(-2\sum_{\ell m}\int_0^\infty d\omega\;|b_{\omega\ell m}|^2\biggr)\\
&= \hat{N}\,\exp\biggl\{-2\sum_{\ell m}\biggl[\int_0^\infty d\omega\;|c_{\omega\ell m}|^2\\
&+\,2\int_0^\infty d\omega'\int_0^\infty d\omega''\;c_{\omega'\ell m}\,c^*_{\omega''\ell m}\int_0^\infty d\omega\;\beta^*_{\omega''\omega\ell}\,\beta_{\omega'\omega\ell}\\
&-\,2\int_0^\infty d\omega'\int_0^\infty d\omega''\;\mathrm{Re}\biggl(c_{\omega'\ell m}\,c_{\omega''\ell,-m}\int_0^\infty d\omega\;\beta_{\omega''\omega\ell}\,\alpha^*_{\omega'\omega\ell}\biggr)\biggr]\biggr\}\quad .
\end{aligned}
\tag{4.2}
$$

The first two terms inside the square bracket are positive-definite, whereas the third is of indefinite sign. The situation becomes much clearer in the case that

$$\int_0^\infty d\omega\;\beta^*_{\omega''\omega\ell}\,\beta_{\omega'\omega\ell} \;=\; |\beta_{\omega'\ell}|^2\;\delta(\omega',\omega'')\quad , \tag{4.3}$$

$$\int_0^\infty d\omega\;\beta_{\omega''\omega\ell}\,\alpha^*_{\omega'\omega\ell} \;=\; 0\quad . \tag{4.4}$$

This holds, in particular, for the steady-state Bogoliubov coefficients in the calculation which neglects back-reaction on the metric [10]. It remains to check whether this diagonal form persists under adiabatic propagation through a slowly-varying potential. From Eq.(2.20), one has

$$|\alpha_{\omega'\ell}|^2 - |\beta_{\omega'\ell}|^2 \;=\; 1\quad . \tag{4.5}$$

Thence,

$$
\begin{aligned}
P\Big[\{a_{\omega\ell m}\}\Big] &= P\Big[\{c_{\omega\ell m}\}\Big]\\
&= \hat{N}\,\exp\biggl[-2\sum_{\ell m}\int_0^\infty d\omega\;|c_{\omega\ell m}|^2 - 4\sum_{\ell m}\int_0^\infty d\omega\;|\beta_{\omega\ell}|^2\,|c_{\omega\ell m}|^2\biggr]\\
&= \hat{N}\prod_{n\ell m}\exp\Bigl[-2\,(\Delta\omega_n)\,|c_{n\ell m}|^2 - 4\,(\Delta\omega_n)\,|\beta_{n\ell}|^2\,|c_{n\ell m}|^2\Bigr]\quad ,
\end{aligned}
\tag{4.6}
$$

where $\Delta\omega_n = \pi/|T|$. The product over $n\ell m$ tells us that the modes evolve independently; such uncorrelated modes are a consequence of the linearised theory.

A further argument for the diagonal form of the Bogoliubov coefficients is through the tunnelling interpretation of black-hole evaporation. Particles are created in pairs just outside the future horizon $\mathcal{H}^+$, with one member always falling into the singularity whilst the other escapes to infinity, or is reflected back down the hole. By constructing a basis of scalar field modes which is continuous across the future horizon, we automatically incorporate both the positive- and negative-energy particles [25]. To achieve this in the Schwarzschild picture, however, one must avoid the coordinate singularity at the future horizon by adding a small imaginary part to the mass M. Such a procedure is in keeping with Feynman's complex-time technique – which we expound in this paper – since the conjugacy between the intial mass M_I and asymptotic time T implies that moving T slightly into the lower complex plane is equivalent to adding a small positive imaginary part to M_I. The Bogoliubov coefficients are just the weights of the outgoing and ingoing particle components of the scalar field across the future horizon.

Equation (4.6) expresses the probability distribution of final field configurations in terms of amplitudes associated with the $\mathcal{S}^-$ Cauchy surface. The nature of Eq.(4.6) suggests that there is a shift in the width of the ground-state probability distribution. That is, $P[\{c_{n\ell m}\}]$ appears as an excited state relative to the ground state probability density $P[\{b_{n\ell m}\}]$. Having averaged over $\{c_j\}$ amplitudes, the stability of the $\mathcal{S}^-$ state is guaranteed by the fact that $|\alpha j|^2 + |\beta_j|^2 > 0$ for all j. However, the shift only occurs for the infra-red or low-frequency components of the field, if we assume that the Bogoliubov coefficients $|\beta_j|^2$ decay (exponentially) rapidly at high frequency and are finite for low frequency. This is just an expression of local (coordinate) covariance, in that the Σ_F and $\mathcal{S}^-$ representations are equivalent for the ultra-violet properties of the theory. An infra-red ambiguity is manifest through the occurrence of the Hawking effect with non-zero $|\beta_j|^2$. Hence, the Hawking effect is a consequence of the ambiguity in the choice of space-like evolution hypersurfaces [26].

5. Conclusion

In this paper, we have made a connection between the present boundary-value approach, as used in this study of quantum amplitudes in black-hole evaporation, and the original approach by means of Bogoliubov coefficients. This connection is established through consideration of the radial equation obeyed by the (adiabatic) perturbations; in this paper, it is the spin-0 perturbations which are studied. The discussion in Sec.4 gives the probability distribution for configurations of the perturbative scalar field, on a final hypersurface Σ_F at a very late time, by taking the squared norms of amplitudes, as a product of independent Gaussians. Since our approach gives a final pure state, whereas the traditional approach gives a non-trivial density matrix, one might expect some difference (not necessarily great) between the probabilistic predictions of our boundary-value approach and of the Bogoliubov-coefficient approach, in the case of gravitational collapse to a black hole. In the Appendix, we consider a somewhat different question, and find a possible interpretation of our probability distribution, in terms of the Wigner quasi-probability distribution for harmonic oscillators.

Further work in this area has involved the study of approximately Vaidya-like metrics [13], as giving an accurate approximation to the gravitational field in the region of space-time containing the flux of outgoing black-hole radiation. At the same time, we discuss the radiative spin-0 (scalar) and spin-2 (graviton) fields which both act as a source for the gravitational field and propagate within it. This Vaidya-like description is essential in the treatment above of the adiabatic perturbation modes. In later work, we shall give a further alternative description of the quantum states found in our boundary-value approach, namely, a description in terms of coherent and squeezed states. In [27,28] we repeat the spin-0 boundary-value quantum calculation of [3] (as used in the present paper), but for other spins, including the fermionic spin-$\frac{1}{2}$ (neutrino) case.

Appendix: Density-Matrix Interpretation and Wigner Distribution Functions

Our probability distribution $P[\{a_{\omega\ell m}\}] = P[\{c_{\omega\ell m}\}]$ above naturally arises from the pure state that we have discussed. But we now check whether or not $P[\{c_{\omega\ell m}\}]$ can be expressed in terms of the diagonal elements of some other kind of 'density-matrix distribution'. One might expect that a probability density for the final scalar field would incorporate not only the randomness in the particle-emission process but also possible choices for the boundary data. In light of the loss of phase information in Eq.(4.6) with respect to the $\{c_{\omega\ell m}\}$ amplitudes, a diagonal density-matrix-like expression for $P[\{c_{\omega\ell m}\}]$ might now be possible. Eq.(4.6) can be rewritten so as to make this clear, when one employs the generating function for the Laguerre polynomials $\{L_k(x)\}$ [29]:

$$\frac{1}{(1-s)}\exp\left[\frac{-xs}{(1-s)}\right] \;=\; \sum_{k=0}^{\infty} L_k(x)\, s^k, \qquad |s| \,<\, 1 \quad , \tag{A1}$$

where

$$L_k(x) \;=\; \sum_{n=0}^{k} \binom{k}{n} \frac{(-x)^n}{n!} \quad . \tag{A2}$$

Here, we set

$$s \;=\; \left|\frac{\beta_{n\ell}}{\alpha_{n\ell}}\right|^2 \;<\; 1 \tag{A3}$$

and

$$x \;=\; 2\,X_{n\ell m} \;=\; 4\,(\Delta\omega_n)\,|c_{n\ell m}|^2 \quad , \tag{A4}$$

a dimensionless quantity. Then, from Eqs.(4.6,A1), and writing $j = n\ell m$, we have

$$\begin{aligned} P\Big[\{c_j\}\Big] \;&=\; \hat{N}\prod_j \exp(-X_j)\,\exp\Big(-2\,|\beta_j|^2\,X_j\Big) \\ &=\; \hat{N}\prod_j \exp(-X_j)\sum_{k_j=0}^{\infty} P(k_j)\;L_{k_j}(2X_j) \\ &=\; \hat{N}\sum_{\{k_j\}} P\big(k_j\big)\prod_j \exp(-X_j)\;L_{k_j}(2X_j) \quad , \end{aligned} \tag{A5}$$

where $P(k_j)$ is defined to be

$$P(k_j) \;=\; \frac{1}{|\alpha_j|^2} \, \left| \frac{\beta_j}{\alpha_j} \right|^{2k_{n\ell m}} \quad , \tag{A6}$$

namely the probability to observe the field in the state k, such that from Eq.(2.20) one has [30]

$$\sum_{k_j\,=0}^{\infty} P(k_j) \;=\; 1 \quad . \tag{A7}$$

Alternatively, $P(k_j)$ is the probability of finding k particles outgoing at $\mathcal{I}^+$ (future null infinity), in the mode labelled by $n\ell m$. The average number of particles outgoing at $\mathcal{I}^+$ per unit frequency around ω_n per unit time is then $< k_j >= |\beta_j|^2$, independently of m. The factor $\frac{|\beta_j|^2}{|\alpha_j|^2}$ effectively controls the probability per unit time to emit particles in the frequency range around ω_n. Each radiation state is uniquely specified by the set of occupation numbers $\{k_j\}$ of the various modes $\{n\ell m\}$ and occurs with probability $P(\{k_j\})$.

Eq.(A5) describes the probability distribution $P[\{c_j\}]$ for the final scalar field, with the help of the Bose-Einstein distribution (A6), which describes the randomness in the particle emission process. Since, from Eq.(4.6), $P[\{c_j\}]$ depends only on the modulus $|c_j|$ of the complex number c_j, the sum in Eq.(A5) is diagonal. Of course, the explicit expression for the 'probability' (A5) for a given total mass M_I is clearly complicated, since, as in Eq.(3.13), the Bogoliubov coefficients $|\beta_j|^2$ depend on the transmission probability Γ_j.

Before we investigate further the probability distribution $P[\{a_j\}]$ (equivalently $P[\{c_j\}]$), we define, for fixed j, the measure d^2a_j on the complex a_j-space to be the natural measure on $\mathbb{R}^2$:

$$d^2a_j \;=\; d\big[\mathrm{Re}(a_j)\big]\; d\big[\mathrm{Im}(a_j)\big] \quad . \tag{A8}$$

Since the probability density only depends on $|a_j|$, it is further convenient to rewrite the measure in terms of polar coordinates, through $a_j = R_j\, e^{i\theta_j}$, for each j. We then choose the normalisation factor $\hat{N}$ in Eqs.(4.1,2,6,A5) such that

$$\frac{1}{\pi} \int \prod_j d^2a_j \; P\Big[\{a_j\}\Big] \;=\; 1 \quad . \tag{A9}$$

In the language of Eq.(4.6), this involves a Gaussian integral over the infinitely many variables a_j or c_j, which will require regularisation, for example by the ζ-function technique [31]. This procedure will be simpler when one works with a locally-supersymmetric theory, such as the theory of $N = 1$ supergravity with massless-scalar/spin-$\frac{1}{2}$ supermatter [2]. In that case, there should not be any divergences in the calculation of $\hat{N}$, because of cancellations between the bosonic and fermionic degrees of freedom, and such cancellations should make the detailed expression for $\hat{N}$ much simpler.

In Eq.(A5), the Laguerre polynomial $L_k(x)$ has k real, distinct roots; further, for positive argument $X_j > 0$ and excited states ($k_j > 0$), the function $e^{-X_j}\; L_{k_j}(2X_j)$ takes negative values for certain ranges of X_j . This function cannot therefore be interpreted as a probability density. Regarding the question of whether the probability distribution $P[\{a_j\}]$ might arise from the diagonal components of some kind of 'density matrix', let us consider an Hermitian position-space density matrix of the form

$$\rho(x,y) \;=\; \sum_i w_i \; \psi_i(x)\; \psi_i^*(y) \quad , \tag{A10}$$

where the $\{\psi_i(x)\}$ form a complete set and w_i is the probability to be in the state $\psi_i(x)$. If the system is in the single state defined by $\psi_i(x)$, then the probability density for observing x is $|\psi_i(x)|^2$. The full $\rho(x,y)$ does not have a probabilistic interpretation, but the diagonal contribution

$$P(x) \;=\; \rho(x,x) \;=\; \sum_i w_i \; |\psi_i(x)|^2 \tag{A11}$$

is the probability density for observing the coordinate x . For a pure state, one has

$$\rho(x,y) \;=\; \Psi(x)\Psi^*(y) \quad . \tag{A12}$$

Therefore, $P(x)$ for a pure state is the squared norm of a complex quantum amplitude.

In our case, if the field is in the k-th state, then according to Eq.(A11), $\hat{N}\, e^{-X_j}\; L_{k_j}(2X_j)$ would be the 'probability density' for the 'harmonic oscillator coordinate' X_j . Although this function takes on negative values, the Bose-Einstein distribution Eq.(A6) effectively smoothes out the oscillations in $\hat{N}e^{-X_j}L_{k_j}(2X_j)$ to give a positive probability density when we sum over all possible states $\{k_j\}$. Although this provides an interesting way of breaking down the probability distribution (A5), it does not provide us with a sensible density matrix, derived *post hoc* from the quantum amplitude $\exp\Big(iS^{(2)}_{\rm class}[\{a_{k\ell m}\};T]\Big)$, where $S^{(2)}_{\rm class}$ is given by Eq.(A10) of [3].

The density $\hat{N}e^{-X_j}L_{k_j}(2X_j)$, however, is closely related to the Wigner quasi-probability distribution for the harmonic oscillator [32]. Wigner functions are quantum-mechanical analogues of classical phase-space distributions, which are formally equivalent to more familiar Hilbert-space or density-matrix formulations of quantum mechanics. However, as the uncertainty principle forbids the existence of a probability distribution for simultaneously well-defined conjugate variables, such as coordinates and momenta, Wigner functions do not have all the properties of a classical phase-space probability density. In particular, Wigner functions must become negative in some domain of phase space.

For a system in a mixed state described by a density matrix $\rho(x,y)$, the one-dimensional Wigner function is defined as [32]

$$W(p,q) \;=\; \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dz\; e^{\frac{ipz}{\hbar}}\;\rho\Big(q-\frac{z}{2},q+\frac{z}{2}\Big). \tag{A13}$$

From Eq.(A10),

$$W(p,q) \;=\; \sum_i w_i\; W_i(p,q), \tag{A14}$$

where

$$W_i(p,q) \;=\; \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dz\; e^{\frac{ipz}{\hbar}}\;\psi_i\Big(q-\frac{z}{2}\Big)\;\psi_i^*\Big(q+\frac{z}{2}\Big). \tag{A15}$$

For a pure state,

$$W(p,q) \;= \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dz\; e^{\frac{ipz}{\hbar}}\;\Psi\Big(q-\frac{z}{2}\Big)\;\Psi^*\Big(q+\frac{z}{2}\Big). \tag{A16}$$

The $\hbar$-dependent function $W(p,q)$ is manifestly real and has the properties

$$\int_{-\infty}^{\infty} dp\;\; W(p,q) \;=\; |\Psi(q)|^2, \tag{A17}$$

$$\int_{-\infty}^{\infty} dq\;\; W(p,q) \;=\; |\tilde{\Psi}(p)|^2, \tag{A18}$$

$$\int_{-\infty}^{\infty} dp\, dq\;\; W(p,q) \;=\; 1, \tag{A19}$$

where $\tilde{\Psi}(p)$ is the Fourier transform of $\Psi(q)$. For the k-th state of the harmonic oscillator with the wave function

$$\psi_k(x) \;=\; \frac{1}{\sqrt{2^k k!}}\,\Big(\frac{\omega}{\pi\hbar}\Big)^{\frac{1}{4}}\exp\big(-\omega x^2/2\hbar\big)\;H_k\Big(x\sqrt{\omega/\hbar}\Big), \tag{A20}$$

then

$$W_k(p,q) \;=\; 2\,(-1)^k\;\exp\Big(-2H(p,q)/\hbar\omega\Big)\;L_k\bigg(\frac{4H(p,q)}{\hbar\omega}\bigg), \tag{A21}$$

where $H(p,q) = (1/2)(p^2+\omega^2 q^2)$ is the Hamiltonian and ω is the oscillator frequency. Further, using Eq.(A1), $W_k(p,q)$ is normalised according to

$$\sum_{k=0}^{\infty} W_k(p,q) \;=\; 1. \tag{A22}$$

As H increases, $W_k(p,q)$ oscillates with a decreasing amplitude and with increasing broad peaks. Introducing the complex variables $\{a_j\}$ which represent a phase-space point (p_j,q_j), with a mode index j for the infinite-dimensional phase space, where

$$a_j \;=\; \frac{1}{(2\hbar)^{1/2}}\,\Big(\omega^{1/2}q_j + i\omega^{-1/2}p_j\Big), \tag{A23}$$

one can express $W_k(p,q)$ as

$$W_k(a_j) \;=\; 2\,(-1)^k\,e^{-2|a_j|^2}L_k\Big(4|a_j|^2\Big). \tag{A24}$$

Using $L_k(0)=1$, one has

$$W_k(0) \;=\; 2\,(-1)^k, \tag{A25}$$

which is the maximum value of $W_k(a_j)$. Thus, with the replacement $2|a_j|^2 \rightarrow X_j$, the final term in Eq.(A5) can be written as

$$\hat{N}\, e^{-X_j} L_{k_j}(2X_j) \;\;=\;\; \frac{1}{4}\,\hat{N}\, W_{k_j}(X_j)\, W_{k_j}(0) \;\;=\;\; \hat{N}\,\frac{W_{k_j}(X_j)}{W_{k_j}(0)}. \qquad (A26)$$

We can interpret the product – or ratio – of two Wigner functions in the following way. Consider two statistically independent subsystems A and B. As in statistical physics, the Wigner function W describing the system $A \cup B$ is given by the product of the Wigner functions for the two subsystems

$$W(p_A, q_A, p_B, q_B) \;\;=\;\; W_A(p_A, q_A)\, W_B(p_B, q_B). \qquad (A27)$$

Thus, the state of the subsystem B does not affect the 'probabilities' of various states of the other subsystem A. Such a factorisation would not be possible if the subsystems were statistically dependent, in which case, the Wigner function of each subsystem is defined by integrating over the other system's variables:

$$W_A(p_A, q_A) \;\;=\;\; \int dq_B \int dp_B \;\; W(p_A, q_A, p_B, q_B). \qquad (A28)$$

In our case, one might interpret the system B, say, as corresponding to the weak initial state of the perturbations at early times, and A as corresponding to the final state of the fluctuations at late times. Thus, referring to Eq.(A11), the $|\psi_i(x)|^2$ term is replaced by $W_{k_j}(X_j)/W_{k_j}(0)$, which gives a Wigner function for the system A at late times.

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