ON THE INTERSECTION FORM

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Contents

1. Introduction

The intersection form of an oriented compact four-manifold is a special symmetric bilinear form on the second (co)homology group of the four-manifold. It reflects much of the topology of four-manifolds, such as the existence of a smooth structure on a topological four-manifold. In this paper, we define the intersection form, explain its significance, and discuss open questions in the area.

1.1. **Outline.** In Section 2, we define the intersection form and motivate its definition. In Section 3, we give some classification results related to the intersection form and four-manifold topology, including Donaldson's diagonalization theorem. In Section 4, we give some examples of intersection forms. In Section 5, we return to the classification problem by describing the "Geography Problem" and recent progress. In Section 6, we give a glimse into Donaldson's proof of his diagonalization theorem. We review the necessary algebraic topology in Appendix A and define a spin manifold in Appendix B.

2. The intersection form

This exposition in this section follows Donaldson–Kronheimer's book [DK90]. Let X be a closed (compact, no boundary) oriented smooth 4-manifold. An orientation of X specifies a fundamental class $[X] \in H_4(X;\mathbb{Z})$ of the manifold. See Appendix A.1 for the relevant definitions of cup product, cap product, and Poincaré duality.

Definition 2.1. The *intersection form* on the second cohomology group

$$Q_X : H^2(X;\mathbb{Z}) \times H^2(X;\mathbb{Z}) \to \mathbb{Z}$$

is given by the formula

$$Q_X(a,b) = \langle a \smile b, [X] \rangle.$$

It is clear that Q_X is a symmetric form, and moreover, Q_X is *unimodular*, a condition we define now. A bilinear form $B : V \times V \to k$ defines a map $V \to V^*$ given by $v \mapsto B(v, \cdot)$. Call B unimodular if this associated map $V \to V^*$ is an isomorphism. Then the Poincaré duality isomorphism between the groups H^2 and H_2 is the assertion that the intersection form is unimodular. By Poincaré duality and functoriality of the cup product, we have the following commutative diagram, where the top horizontal arrow is defined by the other three arrows:

$$\begin{tikzcd} H_2(X;\mathbb{Z}) \times H_2(X;\mathbb{Z}) \arrow[r] \arrow[d, "\mathrm{PD} \times \mathrm{PD}"] & H_0(X;\mathbb{Z}) \arrow[d, "\mathrm{PD}"] \\ H^2(X;\mathbb{Z}) \times H^2(X;\mathbb{Z}) \arrow[r, "\smile"] & H^4(X) \end{tikzcd}$$

So, the intersection form can be written on homology

$$Q_X : H_2(X;\mathbb{Z}) \times H_2(X;\mathbb{Z}) \to \mathbb{Z}. \tag{1}$$

Now suppose X is a compact, oriented, simply-connected four-manifold. The simply-connected assumption tells us that H_1 is zero. By the universal coefficient theorem for cohomology (see Appendix A.2),

$$H^2(X;\mathbb{Z}) \cong \mathrm{Hom}(H_2(X;\mathbb{Z}), \mathbb{Z})$$

is a free abelian group. By Poincaré duality, this implies that $H_2 = H^2$ is free as well.

Choose a basis for the free abelian group H_2. Then the intersection form is represented by a matrix with integer entries.

Definition 2.2. The *rank* of the oriented four-manifold is the rank of its associated quadratic form Q_X, which is the second Betti number $b_2 = \dim H^2(X;\mathbb{Z})$. Let b^+ and b^- are the dimensions of the maximal positive and negative subspaces of the form Q_X so that

$$b_2 = b^+ + b^-.$$

The *signature* of the oriented four-manifold is the signature of the form Q_X:

$$\tau = b^+ - b^-.$$

2.1. **Why is it called the "intersection" form?** In this subsection we clarify why the form associated to a four-manifold is called an "intersection" form. Let $\Sigma_1, \Sigma_2 \subset X$ be two oriented surfaces placed in general position. Then Σ_1 and Σ_2 meet in a finite number of points. To each intersection point p, associate a sign ± 1 according to the matching of orientations in the isomorphism of tangent bundles at p:

$$T_pX = T_p\Sigma_1 \oplus T_p\Sigma_2. \tag{2}$$

The *intersection number* $\Sigma_1.\Sigma_2$ is given by

$$\Sigma_1.\Sigma_2 = \#\{\text{intersection points } p \text{ counted with sign}\}.$$

It can be proven that the pairing passes to the homology classes $[\Sigma_1], [\Sigma_2]$ to yield the intersection form Q_X in (1), i.e. $\Sigma_1.\Sigma_2 = Q([\Sigma_1], [\Sigma_2])$.

2.2. **Connection to differential forms.** Let ω_1, ω_2 be closed 2-forms representing (de Rham) cohomology classes dual to Σ_1, Σ_2.

Proposition 2.3. *The intersection number is given by*

$$\Sigma_1.\Sigma_2 = Q(\Sigma_1, \Sigma_2) = \int_X \omega_1 \wedge \omega_2.$$

Proof sketch. Choose the forms ω_i to be supported in a small neighborhood of Σ_i for $i = 1, 2$. Near an intersection point of $\Sigma_1 \cap \Sigma_2$, we can choose local coordinates (x, y, z, w) on X such that Σ_1 is given by $x = y = 0$ and Σ_2 is given by $y = z = 0$. Then we can choose dual forms

$$\omega_1 = \psi(x, y)\, dxdy \ \text{ and } \ \omega_2 = \psi(z, w)\, dzdw,$$

where ψ is a bump function on $\mathbb{R}^2$ supported near $(0, 0)$ with total integral 1. Then the 4-form

$$\omega_1 \wedge \omega_2 = \psi(x, y)\psi(z, w)\, dxdydzdw$$

is supported near the intersection points of $\Sigma_1 \cap \Sigma_2$. For each intersection point $p \in \Sigma_1 \cap \Sigma_2$, we choose a neighborhood U_p of p containing the support of ψ. Then we can evaluate the integral

$$\int_{U_p} \omega_1 \wedge \omega_2 = \pm 1$$

depending on the orientation of the tangent bundles in (2). If we sum the integral of $\omega_1 \wedge \omega_2$ over the neighborhoods of every intersection point, this is equal to the integral of $\omega_1 \wedge \omega_2$ over X. This concludes the proof. □

3. Classification results

One central question in four-manifold theory is to what extent the intersection form determines a simply connected four-manifold? There is also the complementary question of which forms are realized by compact four-manifolds. We start with the former direction by a result deduced in 1958 by Milnor [Mil58] from Whitehead's theorem (we recall Whitehead's theorem in Appendix A.3).

Theorem 3.1. *The oriented homotopy type of a simply connected, compact, oriented four-manifold X is determined by its intersection form.*

Proof outline. Remove a small ball B^4 from X. First we compute the homology groups of $X \setminus B^4$, starting with H_1 and H_2. For this, we use the decomposition $X = A \cup B$, where $B = B^4$ and A is a neighborhood of $X \setminus B^4$. Consider the following part of the Mayer–Vietoris sequence:

$$H_i(A \cap B) \longrightarrow H_i(A) \oplus H_i(B) \longrightarrow H_i(X).$$

For $i = 1, 2$, $H_i(B) = H_i(B^4) = 0$ and $H_i(A \cap B) = H_i(S^3) = 0$. This means that $H_1(A) = H_1(X \setminus B^4) = 0$ and $H_2(A) = H_2(X \setminus B^4) = H_2(X)$. Now we compute H_4. Since X is a simply-connected compact four manifold, it has a CW structure with only one 4-cell, so that removing a B^4 from X is the same as popping the four-cell. This implies $X \setminus B^4$ deformation retracts onto a three manifold, implying $H_4(X \setminus B^4)$ vanishes. Now we compute H_3. By Poincaré duality, $H_3(X \setminus B^4)$ is isomorphic to $H^1(X \setminus B^4)$. By the Universal

Coefficient Theorem A.4, $H^1 = \mathrm{Hom}(H_1, \mathbb{Z}) = 0$ because H_0 is free so the $\mathrm{Ext}(H_0)$ term vanishes. In total,

$$H_i(X \setminus B^4) = \begin{cases} H_2(X) & \text{if } i = 2, \\ 0 & \text{if } i = 1, 3, 4. \end{cases}$$

From the fact that $\pi_0(X \setminus B^4) = \pi_1(X \setminus B^4) = 0$, by the Hurewicz theorem (Theorem A.5), we see that $\pi_2(X \setminus B^4)$ is isomorphic to $H_2(X \setminus B^4)$, so the generators of $H_2(X \setminus B^4)$ can be represented by maps $f_i : S^2 \to X \setminus B^4$. This gives us a map

$$f = \bigvee f_i : \bigvee S^2 \to X \setminus B^4.$$

Since f induces an isomorphism on all homology groups and $\bigvee S^2$ and $X \setminus B^4$ are simply-connected CW complexes, f is actually a homotopy equivalence (see Corollary A.7). Thus, up to homotopy, X is obtained by attaching a four-cell to a wedge of two-spheres:

$$X = \left(\bigvee S^2\right) \cup_h e^4$$

by an attaching map $h : S^3 \to \bigvee S^2$. This implies that the homotopy type of X is determined by the homotopy class of the attaching map h. The next and final step is to prove that homotopy classes of maps $S^3 \to \bigvee S^2$ are in one-to-one correspondence with symmetric matrices. This is done on page 16 of Donalson–Kronheimer's book [DK90]. □

The following is a result of Wall [Wal64].

Theorem 3.2 (Wall, 1964)**.** *If X and Y are simply-connected smooth oriented four-manifolds with isomorphic intersection forms, then there exists an integer $k \geq 0$ such that there is a diffeomorphism*

$$X \# k(S^2 \times S^2) \cong Y \# k(S^2 \times S^2).$$

It is not known whether such a diffeomorphism requires greater than one copy of $S^2 \times S^2$.

Now we move to the latter direction: given a symmetric unimodular quadratic form Q, which compact four-manifolds have intersection form Q?

One can first look at *topological* four-manifolds and their classification up to *homoemorphism*. In 1982, a seminal result by Freedman [Fre82] gave a complete classification for compact, simply connected topological four-manifolds. We state it below. Call an integer quadratic form Q *even* if $Q(x, x)$ is even for all x in the lattice, and call Q *odd* if it is not even.

Theorem 3.3. *For any symmetric unimodular bilinear form Q, there exists a simply connected closed topological four-manifold with intersection for Q. Moreover,*

(1) if Q is even, there exists exactly one manifold, up to homeomorphism, with intersection Q.
(2) if Q is odd, there exist two manifolds with intersection Q, at least one of which does not have a smooth structure.

We can say a bit about the *signature of a manifold*, which is the signature of its intersection form, see Definition 2.2. In 1954, Thom [Tho54] showed that the signature is a cobordism invariant. For the next theorem, a result by Rohlin [Roh52], we first state an arithmetic result which can be found in Gompf–Stipcitz' book [GS99, Lemma 1.2.20].

Lemma 3.4. *If Q is an even integer form, its signature is divisible by 8.*

We can say more about the signature if we are given an even form.

Theorem 3.5 (Rohlin's theorem, 1952)**.** *Any even form coming from a smooth simply-connected* 4*-manifold has signature divisible by* 16*.*

An equivalent version of Rohlin's theorem is as follows. If a smooth, closed four-manifold M has a spin structure (see Appendix B for details about spin manifolds), then the manifold has signature divisible by 16.

Rohlin's theorem points us towards an obstruction for the existence of a smooth structure on a four-manifold, as we explain now. Let E_8 be a certain positive definite, even form of rank 8 given by the matrix

$$E_8 = \begin{pmatrix} 2 & 0 & -1 & & & & & \\ 0 & 2 & 0 & 1 & & & & \\ -1 & 0 & 2 & -1 & & & & \\ & -1 & -1 & 2 & -1 & & & \\ & & & -1 & 2 & -1 & & \\ & & & & -1 & 2 & -1 & \\ & & & & & -1 & 2 & -1 \\ & & & & & & -1 & 2 \end{pmatrix}.$$

In 1982, Michael Freedman [Fre82] discovered a compact, simply connected manifold with intersection form E_8, suitably called the *E_8-manifold*, and proved that such a manifold is unique. The E_8-manifold can be proven to be spin, and we already know that this manifold has intersection form of signature 8. Thus, Rohlin's theorem implies that the E_8-manifold has no smooth structure on it.

In the same vein, below is a Donaldson's seminal result [Don83].

Theorem 3.6 (Donaldson's theorem, 1982)**.** *Let X be a compact smooth simply-connected oriented* 4*-manifold such that its intersection form Q is positive definite. Then Q is diagonalizable over the integers, so in some basis,*

$$Q(u_1, u_2, \ldots, u_r) = u_1^2 + u_2^2 + \cdots + u_r^2.$$

Similarly, if Q is negative definite, then Q can be diagonalized to the negative identity matrix.

The proof of this result is so intriguing that we give a glimpse of it in Section 6.

4. Examples of intersection forms

Intersection form of S^4. The four-sphere has vanishing second (co)homology group, so its intersection form is the zero form.

Intersection form of $\mathbb{CP}^2$. The complex projective plane $\mathbb{CP}^2$ has $H_2 = \mathbb{Z}$, with generator given by the complex projective line $\mathbb{CP}^1$. Displacing $\mathbb{CP}^1$ slightly, and noting that two lines in $\mathbb{CP}^2$ meet at one point, we see that the *self-intersection number* (not rigorously defined here) of $\mathbb{CP}^1$ is 1. Thus, the intersection form is represented by the 1×1 matrix (1). Note that if we consider the same manifold $\mathbb{CP}^2$ with its orientation reversed, call this manifold $\overline{\mathbb{CP}^2}$, then its intersection form is (-1).

Intersection form of $S^2 \times S^2$. In the product manifold $S^2 \times S^2$, the generators for H_2 are represented by the embedded spheres $S^2 \times \{\text{pt}\}$ and $\{\text{pt}\} \times S^2$. These spheres intersect transversely at one point in the four-manifold and each has zero self-intersection. Therefore, the intersection matrix of $S^2 \times S^2$ is

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Intersection form of $X_1 \# X_2$. If X_1 and X_2 are any compact simply-connected four-manifolds, then their connected sum $X_1 \# X_2$ is also a compact simply-connected four-manifold. Also, $H_2(X_1 \# X_2)$ is isomorphic to $H_2(X_1) \oplus H_2(X_2)$. Concatenating the bases of $H_2(X_1)$ and $H_2(X_2)$, we see that the intersection matrix of $X_1 \# X_2$ is the direct sum of the intersection matrices of X_1 and X_2.

For example, the direct sum of l copies of $\mathbb{CP}^2$ and m copies of $\overline{\mathbb{CP}^2}$ is the $(l+m) \times (l+m)$ diagonal matrix

$$l(1) \oplus m(-1).$$

5. The Geography Problem and the $\frac{11}{8}$-Conjecture

In this section, we focus on what topologists call the "Geography Problem". This section takes inspiration from Hopkins, Lin, Shi, and Xu's recent paper [HLSX22].

Question 5.1 (The Geography Problem)**.** Given a symmetric unimodular bilinear form, can it be realized as the intersection form of a closed simply connected smooth four-manifold?

Donaldson's Theorem 3.6 gives the answer when Q is definite: there is a smooth structure on the manifold if and only if its intersection form is diagonalizable.

For indefinite forms Q, there is a powerful classification result of Hasse and Minkowski:

Theorem 5.2 (Hasse–Minkowski)**.** *Let Q be an indefinite form. If Q is not even, then it must be isomorphic to a diagonal form with entries ± 1. If Q is even, it must be isomorphic to*

$$kE_8 \oplus q \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{3}$$

for some $k \in \mathbb{Z}$ and $q \in \mathbb{Z}_{>0}$. For negative k, kE_8 deenotes the direct sum of $|k|$ copies of $-E_8$.

Thus, when the indefinite bilinear form Q is not even, by the above theorem, Q can always be realized by a connected sum of $\mathbb{CP}^2$ and $\overline{\mathbb{CP}}^2$.

The remaining case is when Q an even indefinite bilinear form. The complete classification for this case still remains open. We mention relevant results in this direction. Wu's formula, tells us that the manifold X realizing Q must be spin (see Appendix B for the definition of a spin manifold). The signature of the matrix in (3) from the theorem of Hasse–Minkowski has signature $8k$, so by Rohlin's Theorem 3.5, k must be even. By reversing orientation of X, we can assume that $k \geq 0$. The following conjecture of Matsumoto [Mat82] serves as the final missing piece:

Conjecture 5.3 (The $\frac{11}{8}$-Conjecture, version 1)**.** *The form*

$$2pE_8 \oplus q \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

can be realized as the intersection form of a closed smooth spin four-manifold if and only if $q \geq 3p$.

The "if" part of the $\frac{11}{8}$-conjecture is known. If $q \geq 3p$, then the form can be realized by the connected sum of p copies of the so-called $K3$ surface and $q - 3p$ copies of $S^2 \times S^2$. The "only-if" part of the $\frac{11}{8}$-conjecture can be phrased as:

Conjecture 5.4 (The $\frac{11}{8}$-Conjecture, version 2)**.** *Any closed smooth spin four-manifold X must satisfy the inequality*

$$b_2(X) \geq \frac{11}{8}\sigma(X),$$

where $b_2(X)$ and $\sigma(X)$ are the second Betti number and signature of X, respectively.

Furuta [Fur01] proves that for $p \geq 1$, the bilinear form

$$2pE_8 \oplus q \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

is realizable by a spin manifold if and only if $q \geq 2p + 1$, now called Furuta's $\frac{10}{8}$-Theorem.

Recently, Hopkins, Lin, Shi, and Xu [HLSX22] prove a strengthening of this result: for $p \geq 2$, the bilinear form

$$2pE_8 \oplus q \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

is realizable by a spin manifold only if

$$q \geq \begin{cases} 2p+2 & p \equiv 1, 2, 5, 6 \pmod 8 \\ 2p+3 & p \equiv 3, 4, 7 \pmod 8 \\ 2p+4 & p \equiv 0 \pmod 8 \end{cases}$$

and as a corollary, obtain that any closed simply connected smooth spin 4-manifold X not homeomorphic to S^4, $S^2 \times S^2$, or $K3$ must satisfy the inequality

$$b_2(X) \geq \frac{10}{8}\sigma(X) + 4.$$

See [HLSX22, Corollary 1.13] for further details.

6. A glimpse into Donaldson's proof

In this section, we give a glimpse into Donaldson's proof of his Diagonalization Theorem 3.6. The content of his proof is gauge-theoretic, but the overarching scope can be described without this analytic technique. We follow his outline given in [Don83, Section I.2].

The idea is to associate to the given compact smooth simply connected oriented four-manifold X a space $\mathcal{M}(X)$ such that $\mathcal{M}(X)$ can be compactified to an orientable 5-manifold with boundary X and a number of point singularities. Each singularity corresponds to a pair of points $\pm\alpha$ of solutions to

$$Q(\alpha, \alpha) = 1, \quad \alpha \in H^2(X; \mathbb{Z}).$$

Let $n(Q)$ be the number of point singularities or pairs of points $\pm\alpha$ solving the above equation. For each point singularity p, a neighborhood U_p of p in $\mathcal{M}(X)$ looks like a cone with base $\mathbb{CP}^2$. If we remove the cone U_p from $\mathcal{M}(X)$ for every point singularity p and glue

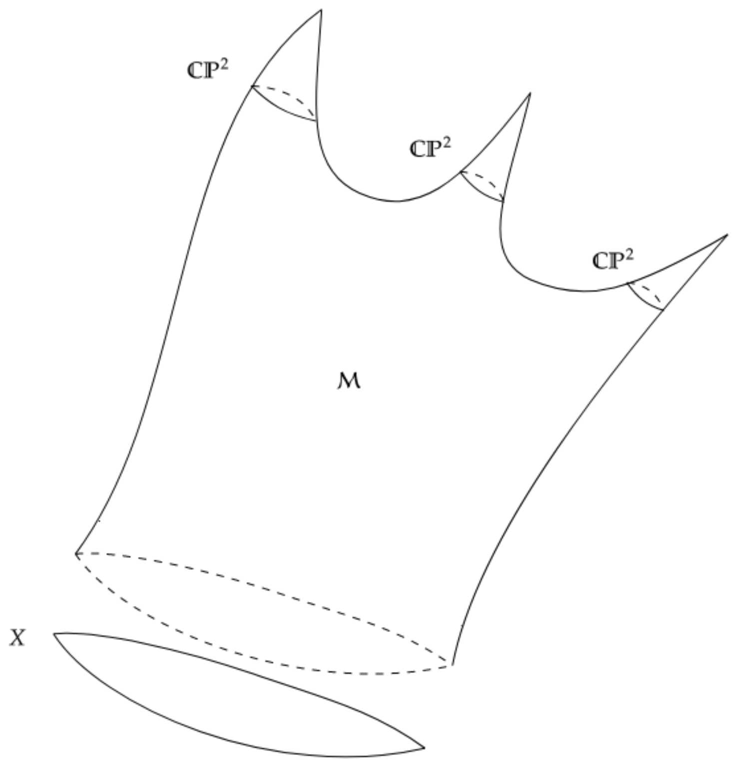

Figure 1.

in a copy of $\mathbb{CP}^2$, the resulting space is a cobordism between X and a disjoint union of $n(Q)$ copies of $\mathbb{CP}^2$. The signature of a positive definite quadratic form is equal to its rank, and a short inductive argument shows that the signature of Q is equal to $n(Q)$. In total, we have that $n(Q) = \mathrm{rk}(Q)$. Since the intersection form of a four-manifold is a cobordism invariant up to isomorphism of quadratic forms, one concludes that the intersection form of X is equal to the intersection form of the disjoint union of $\mathrm{rk}(Q)$ copies of $\mathbb{CP}^2$, which is the identity $r \times r$ matrix.

A picture of $\mathcal{M}(X)$ is shown in Figure 1 (from Wikipedia[1]).

7. Acknowledgement

I would like to thank Professor Ian Zemke for all his help and guidance throughout the writing process.

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Appendix A. Algebraic topology

A.1. **Cup product, cap product, and Poincaré duality.** This section follows Hatcher [Hat02]. Let X be a topological space and R be a commuatative ring. Let $C_n(X)$ be the free abelian group with basis the set of singular n-simplices in X, i.e. each basis element is a map $\sigma : \Delta^n \to X$ where Δ^n is an n-simplex. Define the group $C^n(X;R)$ of *singular n-cochains* with coefficients in R to be the dual group $\mathrm{Hom}(C_n(X), R)$.

Definition A.1. For cochains $\varphi \in C^k(X;R)$ and $\psi \in C^\ell(X;R)$, the *cup product* $\varphi \smile \psi \in C^{k+\ell}(X;R)$ is the cochain whose value on a singular simplex $\sigma : \Delta^{k+\ell} \to X$ is given by

$$(\varphi \smile \psi)(\sigma) = \varphi(\sigma|[v_0, \dots, v_k])\psi(\sigma|[v_k, \dots, v_{k+\ell}]).$$

Let $\delta : C^k(X;R) \to C^{k+1}(X;R)$ be the coboundary map, we can derive the following formula for $\varphi \in C^k(X;R)$ and $\psi \in C^\ell(X;R)$:

$$\delta(\varphi \smile \psi) = \delta\varphi \smile \psi + (-1)^k \varphi \smile \delta\psi.$$

See [Hat02, Lemma 3.6] in Hatcher. From this formula, we immediately see that the cup product of two cocycles is a cocycle, and the cup product of a cup product of a cocycle and a coboundary, in either order, is a coboundary. So there is an induced cup product

$$H^k(X;R) \times H^\ell(X;R) \xrightarrow{\smile} H^{k+\ell}(X;R).$$

This cup product is associative and distributive because it is on the level of cochains.

To obtain the statement of Poincaré duality, it is useful to have the notion of cap product.

Definition A.2. The *cap product* is a map on singular chains and cochains

$$\frown : C_p(X;R) \times C^q(X;R) \to C_{p-q}(X;R)$$

defined for a chain $\sigma \in C_p(X;R)$ and a cochain $\psi \in C^q(X;R)$ by

$$\sigma \frown \psi = \psi(\sigma|[v_0, \dots, v_q])\sigma|[v_q, \dots, v_p].$$

The cap product similarly descends to a map on singular homology and cohomology:

$$H_p(X;R) \times H^q(X;R) \xrightarrow{\frown} H_{p-q}(X;R).$$

For a connected, oriented, closed n-manifold M, we have the isomorphism $H_n(M;\mathbb{Z}) \cong \mathbb{Z}$. An *orientation* is a choice of generator for the free group $H_n(M;\mathbb{Z})$. Such a generator is called the *fundamental class*, denoted by $[M]$.

Theorem A.3 (Poincaré duality). *If M is a closed oriented n-manifold, for any integer $k = 0, \dots, n$, there is a canically defined isomorphism*

$$H^k(M;\mathbb{Z}) \to H_{n-k}(M;\mathbb{Z})$$

given by

$$\alpha \mapsto \alpha \frown [M].$$

A.2. **Universal coefficient theorem.** Let G be an abelian group. Below is [Hat02, Theorem 3.2].

Theorem A.4 (Universal coefficient theorem for cohomology). *If a chain complex C of free abelian groups has homology groups $H_n(C)$, then the cohomology groups $H^n(C;G)$ of the cochain complex* $\mathrm{Hom}(C_n, G)$ *are determined by the split exact sequences*

$$0 \longrightarrow \mathrm{Ext}(H_{n-1}(C), G) \longrightarrow H^n(C; G) \longrightarrow \mathrm{Hom}(H_n(C), G) \longrightarrow 0.$$

Since the singular chain groups $C_n(X)$ are free, the algebraic universal coefficient theorem takes on the following topological form:

$$0 \longrightarrow \mathrm{Ext}(H_{n-1}(X), G) \longrightarrow H^n(X; G) \longrightarrow \mathrm{Hom}(H_n(X), G) \longrightarrow 0.$$

A.3. **Hurewicz and Whitehead theorems.** Call a topological space X *n-connected* if $\pi_i(X, x_0) = 0$ for all $i \leq n$.

Theorem A.5 (Hurewicz theorem)**.** *If a space X is $(n-1)$-connected, $n \geq 2$, then the reduced homology $\widetilde{H}_i(X) = 0$ for all $i < n$ and $\pi_n(X)$ is isomorphic to $H_n(X)$.*

Theorem A.6 (Whitehead's theorem)**.** *Let X and Y be path-connected spaces. Given a continuous map $f : X \to Y$, if*

$$\pi_n(f) : \pi_n(X) \to \pi_n(Y)$$

is bijective for $n \geq 1$, then f is a homotopy equivalence.

It can be shown that combining Whitehead's theorem and the Hurewicz theorem that

Corollary A.7. *If X, Y are simply connected CW complexes, a map $f : X \to Y$ which induces isomorphisms on all integral homology groups is a homotopy equivalence.*

Appendix B. What is a spin manifold?

In this section, we give a rapid definition of a spin structure on a manifold, following Milnor–Stasheff's book [MS74] for a bit.

Let X be a fixed topological space, called the *base space*.

Definition B.1 (Vector bundle and morphisms)**.** A real vector bundle ξ over X consists of the following data:

(1) a topological space $E = E(\xi)$ called the *total space*,
(2) a continuous map $\pi : E \to X$ called the *projection map*, and
(3) for each $x \in X$, the fiber $\pi^{-1}(x)$ over x has the structure of a vector space,

such that the following *local triviality* condition is satisfied. For each $x \in X$, there exists a neighborhood $U \subset X$, an integer $n \geq 0$, and a homeomorphism

$$h : U \times \mathbb{R}^n \to \pi^{-1}(U)$$

such that for each $x \in U$, the map $\mathbb{R}^n \to \pi^{-1}(x)$ defined by $y \mapsto h(x, y)$ is a linear isomorphism.

A *morphism* from the vector bundle $\pi_1 : E_1 \to X$ to the vector bundle $\pi_2 : E_2 \to X$ is a continuous map $f : E_1 \to E_2$ such that for all $x \in X$, the map $\pi_1^{-1}(\{x\}) \to \pi_2(\{x\})$ induces by f is a linear map, i.e. the following diagram commutes:

$$\begin{tikzcd}
E_1 \arrow[rr, "f"] \arrow[dr, "\pi_1"'] & & E_2 \arrow[dl, "\pi_2"] \\
& X &
\end{tikzcd}$$

The dimension n_x of the fiber $\pi^{-1}(x)$ over x is a locally constant function of x. If n_x is the same constant n for all $x \in B$, then we call ξ a *rank n* vector bundle.

The collection of tangent spaces of a smooth manifold M forms vector bundle called the *tangent bundle* TM.

We can specify an *orientation* for a real vector bundle $\pi : E \to B$ by giving an orientation to each vector space $\pi^{-1}(x)$ and requiring that each trivialization

$$h : \pi^{-1}(U) \to U \times \mathbb{R}^n$$

is fiberwise orientation-preserving, i.e. for every $x \in U$, the maps $\mathbb{R}^n \to \pi^{-1}(x)$ given by $y \mapsto h(x, y)$ is an orientation-preserving linear map. Call the vector bundle *orientable* if there exists an orientation of the vector bundle.

Definition B.2 (Frame bundle)**.** Let $E \to X$ be a real vector bundle of rank k over a topological space X. A *frame* at $x \in X$ is an ordered basis for the fiber E_x over x, or equivalently linear isomorphism $p : \mathbb{R}^k \to E_x$. The set of all frames at x, denoted F_x, has a right action by $\mathrm{GL}(k, \mathbb{R})$ of invertible $k \times k$ matrices: the frame $p : \mathbb{R}^k \to E_x$ is sent to the frame $pg : \mathbb{R}^k \to E_x$. The *frame bundle* of E is the disjoint union of all the F_x, and can be given the structure of a vector bundle.

Definition B.3 (Principal bundle)**.** If G is any topological group, a *principal G-bundle* is a fiber bundle $\pi : P \to X$ together with a continuous right action $P \times G \to P$ such that

(1) G preserves the fibers of P, i.e. if $y \in P_x$, then $yg \in P_x$ for all $g \in G$,
(2) G acts freely and transitively on each fiber such that for each $x \in X$ and $y \in P_x$, the map $G \to P_x$ defined by $g \mapsto gx$ is a homeomorphism.

In particular, each fiber of the bundle is homeomorphic to the group G.

If $E \to X$ is a real vector bundle, it can be shown that the frame bundle of E becomes a principal bundle with structure group $\mathrm{GL}(k, \mathbb{R})$.

Similarly, if $E \to M$ is an oriented vector bundle on M, the collection of all orthonormal frames forms a frame bundle $P_{\mathrm{SO}}(E)$, which can be proven to be a principal bundle under the action of the special orthogonal group $\mathrm{SO}(n)$. We also use the fact that since $\pi_1(\mathrm{SO}(n)) = \mathbb{Z}/2\mathbb{Z}$, there exists a 2-to-1 covering space, called the *spin group* $\mathrm{Spin}(n)$. Let $\rho : \mathrm{Spin}(n) \to \mathrm{SO}(n)$ be the 2-to-1 covering map.

Definition B.4 (Spin structure)**.** A *spin structure* for $P_{\mathrm{SO}}(E)$ is a lift of $P_{\mathrm{SO}}(E)$ to a principal bundle $P_{\mathrm{Spin}(E)}$ under the action of the spin group $\mathrm{Spin}(n)$, i.e. there is a bundle map

$$\phi : P_{\mathrm{Spin}}(E) \to P_{\mathrm{SO}}(E)$$

such that

$$\phi(pg) = \phi(p)\rho(g)$$

for all $p \in P$, $g \in \mathrm{Spin}(n)$, where $\rho : \mathrm{Spin}(n) \to \mathrm{SO}(n)$ is the 2-to-1 covering map.

Finally, we arrive at the definition of a spin manifold:

Definition B.5 (Spin manifold)**.** A manifold M is a *spin manifold* if a spin structure exists on its tangent bundle TM.

Equivalently, M is spin if the $\mathrm{SO}(n)$-principal bundle $P_{\mathrm{SO}}(TM)$ of orthonormal bases of the tangent fibers of M is a $\mathbb{Z}/2\mathbb{Z}$-quotient of the principal spin bundle. Borel and Hirzebruch proved in 1958 the following theorem:

Theorem B.6. *For an oriented vector bundle $E \to M$, a spin structure on E exists if and only if the second Stiefel–Whitney class $w_2(E)$ vanishes.*

Thus some of the previous intersection form results related to spin manifolds, particularly Rohlin's Theorem 3.5, stated in terms of Stiefel–Whitney classes. However, we make no further mention of it here.