DIAGONALIZABLE OPERATORS PROBLEM

MATTHEW KENDALL

Abstract. We prove that two diagonalizable operators on a finite dimensional vector space are simultaneously diagonalizable if and only if they commute.

1. Introduction

An *operator* on a finite dimensional vector space V over a field k is a linear function $\varphi : V \to V$. An operator is *diagonalizable* if there exists a basis $\mathcal{B}$ of V consisting of eigenvectors: for every $v \in \mathcal{B}$, $\varphi(v) = \lambda v$ for some constant $\lambda \in k$. A pair of operators φ, ψ *commute* if $\varphi \circ \psi = \psi \circ \varphi$. A pair of operators φ, ψ is *simultaneously diagonalizable* if there exists a basis $\mathcal{B}$ of V consisting of eigenvectors for both φ and ψ. Notice that if φ, ψ are simultaneously diagonalizable, then they commute because they commute on the basis of eigenvectors. We will show that the previous statement holds in both directions.

Problem 1. *If φ, ψ are diagonalizable operators, then they are simultaneously diagonalizable if and only if they commute.*

In Section 2, we provide two proofs of Problem 1, one using elementary methods and one using the minimal polynomial. In Section 3, we give extensions of Problem 1.

2. Proofs

Proof of Problem 1. Let $\lambda_1, \ldots, \lambda_m$ be the distinct eigenvalues of φ. Since φ is diagonalizable, there exists a decomposition of V into eigenspaces $V_{\lambda_i} = \{v \in V : \varphi(v) = \lambda_i v\}$,

$$V = V_{\lambda_1} \oplus V_{\lambda_2} \oplus \cdots \oplus V_{\lambda_m}. \tag{1}$$

Note that for an eigenspace V_λ of φ, if we take an eigenvector $v \in V_\lambda$, then $\varphi(\psi(v)) = \psi(\varphi(v)) = \psi(\lambda v) = \lambda\psi(v)$. This means that $\psi(V_\lambda) \subset V_\lambda$, so it makes sense to restrict the operator φ to the space V_λ. If we show that $\psi|_{V_{\lambda_i}}$ is diagonalizable for every eigenspace V_{λ_i}, then take a basis $\mathcal{B}_i$ of eigenvectors for ψ in V_{λ_i}. Since the vectors in $\mathcal{B}_i$ belong to V_{λ_i}, they are eigenvectors for φ too. Then the basis $\mathcal{B} = \bigcup_i \mathcal{B}_i$ simultaneously diagonalizes φ and ψ. Therefore, it suffices to prove the following statement.

Lemma 2. *Let $W \subset V$ be a subspace of a finite dimensional vector space V. If φ is a linear operator on V such that φ is diagonalizable and $\varphi(W) \subset W$, then $\varphi|_W$ is diagonalizable.*

We provide two proofs of Lemma 2.

Elementary proof of Lemma 2. It would be nice if we can choose an basis of eigenvectors for W from the decomposition of V in (1) because the bases of $W \cap V_{\lambda_i}$ would be eigenvectors of $\varphi|_W$ and we obtain a basis for W by concatenating the bases of $W \cap V_{\lambda_i}$. It turns out that this is true. Specifically, we will show that we can decompose W into a direct sum of its factors

$$W = (W \cap V_{\lambda_1}) \oplus (W \cap V_{\lambda_2}) \oplus \cdots \oplus (W \cap V_{\lambda_m}). \tag{2}$$

Since $W \cap V_{\lambda_i} \subset W$, it suffices to show the reverse inclusion in (2). For that, we show that if we decompose $w = v_1 + v_2 + \cdots + v_m$, we have $v_i \in W \cap V_{\lambda_i}$. We induct on the number of eigenvectors we need to decompose $w \in W$. If $w = v_i$ where $v_i \in V_{\lambda_i}$, then there is nothing to show. Suppose $w = v_1 + \cdots + v_k \in W$, where $v_i \in V_{\lambda_i}$. Then

$$\varphi(w) - \lambda_1 w = (\lambda_2 - \lambda_1)v_2 + (\lambda_3 - \lambda_1)v_3 + \cdots + (\lambda_k - \lambda_1)v_k. \tag{3}$$

Since $\varphi(w) \in W$, we know $\varphi(w) - \lambda_1 w \in W$. Applying the inductive hypothesis on (3), $v_2, \ldots, v_k \in W$. Also, $v_1 = w - (v_2 + \cdots + v_k) \in W$. This completes the induction. □

Remark. When considering the decomposition of any $w = v_1 + \cdots + v_m \in W$, we can automatically show $v_i \in W$ for every $i = 1, \ldots, m$ by considering the operator $\pi_i = \prod_{j \neq i} \frac{\varphi - \lambda_j}{\lambda_j - \lambda_i}$. The operator is a polynomial in φ, so $\pi_i(W) \subset W$. Notice that $\pi_i(v_i) = v_i$ and $\pi_i(v_j) = 0$ for all $j \neq i$. So, $\pi_i(w) = \sum_{j=1}^{m} \pi_i(v_j) = v_i$, implying $v_i \in W$.

Now we provide a second proof of Lemma 2 using the minimal polynomial. The *minimal polynomial* of an operator $\varphi : V \to V$ is a polynomial $m_\varphi(x) \in k[x]$ such that m_φ is monic and it has least degree for which $m_\varphi(\varphi) = 0$. We will use the basic properties of the minimal polynomial (e.g. see [Con]).

Theorem 3. *Let $\varphi : V \to V$ be an operator on V. A polynomial $p(x) \in k[x]$ satisfies $p(\varphi) = 0$ if and only if $m_\varphi(x) \mid p(x)$.*

Theorem 4. *Let $\varphi : V \to V$ be an operator on V. Then φ is diagonalizable if and only if m_φ can be written as a product of linear factors in $k[x]$ and m_φ has distinct roots.*

Using these properties, the proof of Lemma 2 follows quickly.

Second proof of Lemma 2. Let $\phi = \varphi|_W$. From Theorem 3, $m_\varphi(\varphi) = 0$, which means the restriction of m_φ to W is 0. Then

$$m_\varphi(\phi) = m_\varphi(\varphi|_W) = m_\varphi(\varphi)|_W = 0.$$

So, $m_\phi \mid m_\varphi$. Since φ is diagonalizable, by Theorem 4, m_φ splits into a product of linear factors and has distinct roots. This means that the minimal polynomial of ϕ has no repeated factors and splits. Therefore, ϕ is diagonalizable. □

As mentioned before, the proof of Lemma 2 completes the proof of the problem. □

3. Extensions

The following is an extension of Problem 1.

Problem 5. *Let $\mathcal{F} = \{\varphi_i\}_{i \in I}$ be a collection of commuting linear operators on a finite dimensional vector space V. If each φ_i is diagonalizable on V, then the operators in $\mathcal{F}$ are simultaneously diagonalizable.*

See [Con] for a proof that follows two steps: prove the statement for a finite number of operators inductively, and then prove the general statement by finding a basis for the subspace spanned by $\{\varphi_i\}_{i \in I}$ inside $\mathrm{Hom}(V, V)$.

There is also an extension of Lemma 2 if we consider the induced mapping on the quotient space V/W, defined by $\overline{\varphi} : v + W \mapsto \varphi(v) + W$.

Lemma 6. *Let $W \subset V$ be a subspace of a finite dimensional vector space V. If φ is a linear operator on V such that φ is diagonalizable and $\varphi(W) \subset W$, then $\varphi|_W$ and $\overline{\varphi}$ are diagonalizable.*

References

[Con] Keith Conrad, *The minimal polynomial and some applications.*