Representations of S_d

Summer 2020 Reading Project

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June 25, 2020

Partitions

Definition

For a positive integer d, $p(d)$ is the number of partitions of d: the number of solutions $(\lambda_1, \ldots, \lambda_k)$, $\lambda_1 \geq \cdots \geq \lambda_k \geq 1$ to $d = \lambda_1 + \cdots + \lambda_k$ over positive integers k.

Generating function of $p(d)$

The generating function of $p(d)$ is

$$\sum_{d=0}^{\infty} p(d)t^d = (1 + t + t^2 + \cdots)(1 + t^2 + t^4 + \cdots)(1 + t^3 + t^6 + \cdots)\cdots$$
$$= \prod_{n=1}^{\infty} \left(\frac{1}{1 - t^n} \right).$$

Example

Coefficient of t^3 is 3: partition 3 as $(3), (2, 1), (1, 1, 1)$, where the multiplicity c_i of i corresponds to choosing t^{ic_i} in factor i.

The Young Diagram

To a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ is associated a **Young diagram**:

λ_1			
λ_2			
λ_3			
λ_4			

Define a **tableau** on a Young diagram to be a numbering of the boxes by the integers $1, \ldots, d$. In this section, we will use the canonical numbering shown below:

1	2	3
4	5	
6	7	
8		

The Young Diagram

Given partition λ and a tableau, define two subgroups of S_d:

$$P_\lambda = \{g \in S_d : g \text{ preserves each row}\}$$
$$Q_\lambda = \{g \in S_d : g \text{ preserves each column}\}$$

Introduce two elements a_λ and b_λ corresponding to P_λ and Q_λ in the group algebra $\mathbb{C}[S_d]$:

$$a_\lambda = \sum_{g \in P_\lambda} e_g \quad \text{and} \quad b_\lambda = \sum_{g \in Q_\lambda} \operatorname{sgn}(g) \cdot e_g .$$

- The Young tableau will be used to describe projection operators for the regular representation of S_d.

$\mathbb{C}[S_d]$ acting on $V^{\otimes d}$

If λ is a partition of d, we first look at how a_λ and b_λ act on the space $V^{\otimes d}$, where V is any vector space.

- $\sigma \in S_d$ acts on a basis of $V^{\otimes d}$ by

$$v_1 \otimes \cdots \otimes v_d \mapsto v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(d)},$$

  and extending this action to $V^{\otimes d}$ linearly.

Proposition

The image of a_λ, b_λ viewed as elements of $\mathrm{End}(V^{\otimes d})$ are isomorphic to

$$\mathrm{Im}(a_\lambda) \cong \mathrm{Sym}^{\lambda_1} V \otimes \mathrm{Sym}^{\lambda_2} V \otimes \cdots \otimes \mathrm{Sym}^{\lambda_k} V,$$
$$\mathrm{Im}(b_\lambda) \cong \Lambda^{\lambda_1} V \otimes \Lambda^{\lambda_2} V \otimes \cdots \otimes \Lambda^{\lambda_k} V.$$

$\mathbb{C}[S_d]$ acting on $V^{\otimes d}$

Example

Suppose $d = 4$, $\lambda = (2,2)$: $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array}$.

Write $v_{1234} = v_1 \otimes v_2 \otimes v_3 \otimes v_4 \in V^{\otimes 4}$. Then

$$\begin{aligned} a_\lambda v_{1234} &= (e_1 + e_{(12)} + e_{(34)} + e_{(12)(34)}) v_{1234} \\ &= v_{1234} + v_{2134} + v_{1243} + v_{2143} = (v_1 \otimes v_2 + v_2 \otimes v_1) \otimes (v_3 \otimes v_4 + v_4 \otimes v_3). \end{aligned}$$

These elements span $\operatorname{Sym}^2 V \otimes \operatorname{Sym}^2 V$. For $\operatorname{Im} b_\lambda$,

$$b_\lambda v_{1234} = (e_1 - e_{(13)} - e_{(24)} + e_{(13)(24)}) v_{1234} = v_{1234} - v_{3214} - v_{1432} + v_{3412},$$

which under the isomorphism $v_{1234} \mapsto v_{1324}$ becomes

$$v_{1324} - v_{3124} - v_{1342} + v_{3142} = (v_1 \otimes v_3 - v_3 \otimes v_1) \otimes (v_2 \otimes v_4 - v_4 \otimes v_2).$$

These elements span $\Lambda^2 V \otimes \Lambda^2 V$.

The Young Symmetrizer

Definition

The **Young symmetrizer** is $c_\lambda = a_\lambda b_\lambda \in \mathbb{C}[S_d]$.

Examples:

- When $\lambda = (d)$, b_λ fixes every element, so $c_\lambda = a_\lambda = \sum_{g \in S_d} e_g$.
- When $\lambda = (1, \ldots, 1)$, a_λ fixes every element, so $c_\lambda = b_\lambda = \sum_{g \in S_d} \operatorname{sgn}(g) e_g$.

The Big Theorem

We will focus on another action of $\mathbb{C}[S_d]$:

- $\mathbb{C}[S_d]$ acts on itself by right multiplication.

Theorem

Some scalar multiple of c_λ is idempotent, i.e., $c_\lambda^2 = n_\lambda c_\lambda$, and the image of c_λ by right multiplication on $\mathbb{C}[S_d]$ is an irreducible representation V_λ of S_d. Every irreducible reresentation of S_d can be obtained from a partition λ in this way.

- This theorem tells us that there is a direct correspondance between conjugacy classes in S_d and irreducible representations of S_d.
- To be proved in the next lecture!

Example: S_3

Recall that for a fixed irreducible representation W and a representation V, the projection map

$$\dim W \cdot \frac{1}{|G|} \sum_{g \in G} \overline{\chi_W(g)} \cdot g : V \to V$$

projects V onto the direct sum of copies of W appearing in V ([F-H] equation 2.31).

- Think of c_λ as a projection operator for the regular representation $\mathbb{C}[S_d]$ onto the reprensetation $V_\lambda = \mathbb{C}[S_d]c_\lambda$.
- $\lambda = (3)$ corresponds to the trivial representation U: $c_\lambda = \sum_{g \in S_3} e_g$, $c_\lambda^2 = 6c_\lambda$, and

$$V_{(3)} = \mathbb{C}[S_3] \cdot \sum_{g \in S_3} e_g = \mathbb{C} \cdot \sum_{g \in S_3} e_g$$

- $\lambda = (1,1,1)$ corresponds to the alternating representation U': $c_\lambda = \sum_{g \in S_3} \mathrm{sgn}(g) e_g$, $c_\lambda^2 = 6c_\lambda$, and

$$V_{(1,1,1)} = \mathbb{C}[S_3] \cdot \sum_{g \in S_3} \mathrm{sgn}(g) e_g = \mathbb{C} \cdot \sum_{g \in S_3} \mathrm{sgn}(g) e_g$$

Example: S_3

- $\lambda = (2,1)$ corresponds to the standard representation V: In the block

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array},$$

$c_\lambda = a_\lambda b_\lambda = (e_1 + e_{(12)})(e_1 - e_{(13)}) = 1 + e_{(12)} - e_{(13)} - e_{(132)}$, $c_\lambda^2 = 3c_\lambda$. For the basis $\{1, (12), (13), (23), (123), (132)\}$, the matrix of c_λ is

$$\begin{bmatrix} 1 & 1 & -1 & 0 & -1 & 0 \\ 1 & 1 & 0 & -1 & 0 & -1 \\ -1 & -1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 & 1 & -1 \\ -1 & -1 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

Columns 1 and 3 form a basis for the column space, so the basis of the image $V_\lambda = \mathbb{C}[S_3]c_\lambda$ is

$$e_1 + e_{(12)} - e_{(13)} - e_{(132)} \text{ and } -e_1 + e_{(13)} - e_{(23)} + e_{(123)}.$$

$\dim V_\lambda = 2$, and assuming V_λ is irreducible, it must be the standard representation V.

  - Note c_λ projects $\mathbb{C}[S_d]$ onto *one* copy of V, unlike the projection formula.

Notation for Frobenius Formula

Now we talk about Frobenius's formula for the charater χ_λ of V_λ.

- $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a partition of d.
- k is the number of rows in the Young Diagram of λ.
- $C_{\mathbf{i}}$ is the conjugacy class in S_d formed by i_1 1-cycles, i_2 2-cycles, ..., and i_d d-cycles.
- The character of V_λ on $C_{\mathbf{i}}$ is written $\chi_\lambda(C_{\mathbf{i}})$.
- $P_j(\mathbf{x}) = x_1^j + x_2^j + \cdots + x_k^j$ is the **power sum**.
- $\Delta(\mathbf{x}) = \prod_{1 \le i < j \le k}(x_i - x_j)$ is the **discriminant**.
- If $f(x_1, \ldots, x_k)$ is a formal power series, let $[f(\mathbf{x})]_{(l_1, \ldots, l_k)}$ be the coefficient of $x_1^{l_1} \cdots x_k^{l_k}$ in f.
- For the partition λ and an $1 \le i \le k$, let $l_i = \lambda_i + k - i$. Since $\lambda_1 \ge \cdots \ge \lambda_k$, the l_i are a strictly decreasing sequence of k nonnegative integers.

Frobenius's Formula

The Frobenius Formula

On the conjugacy class $C_{\mathbf{i}}$ of S_d, the character of V_λ on $C_{\mathbf{i}}$ is

$$\chi_\lambda(C_{\mathbf{i}}) = \left[\Delta(\mathbf{x}) \cdot \prod_{1 \le j \le d} P_j(\mathbf{x})^{i_j}\right]_{(l_1,\ldots,l_k)}.$$

An Example

The Frobenius Formula

$$\chi_\lambda(C_{\mathbf{i}}) = \left[\Delta(\mathbf{x}) \cdot \prod_{1\le j\le d} P_j(\mathbf{x})^{i_j}\right]_{(l_1,\dots,l_k)}.$$

Example

The 2×2 block corresponding to $d = 4$, $\lambda = (2,2)$, and $C_{\mathbf{i}}$ is the conjugacy class of $(12)(34)$.

- $\mathbf{i} = (i_1, i_2, i_3, i_4, i_5) = (0, 2, 0, 0, 0)$.
- $\lambda = (\lambda_1, \lambda_2) = (2,2)$.
- $k = 2$ because there are two rows in the Young tableau.
- $P_j(x) = x_1^j + x_2^j$.
- $\Delta(\mathbf{x}) = (x_1 - x_2)$.
- $(l_1, l_2) = (2+2-1, 2+2-2) = (3,2)$.

By Frobenius' formula,

$$\chi_\lambda(C_{\mathbf{i}}) = \left[(x_1 - x_2)(x_1^2 + x_2^2)^2\right]_{(3,2)} = 1 \cdot 2 = 2 = \chi_W((12)(34)).$$

This is the character of the irrep W in the character table of S_4.

Computing $\dim V_\lambda$

We compute $\dim V_\lambda$ as an application of the Frobenius formula.

Vandermonde determinant

For variables $x_1, \ldots, x_k$, the matrix

$$V = \begin{bmatrix} 1 & x_k & \cdots & x_k^{k-1} \\ \vdots & \vdots & & \vdots \\ 1 & x_1 & \cdots & x_1^{k-1} \end{bmatrix}$$

is called the **Vandermonde matrix**. We have

$$\det V = \prod_{1 \le i < j \le k} (x_i - x_j) = \Delta(\mathbf{x}).$$

Vandermonde Determinant

Proof.

Let $\det V = f(x_1, \ldots, x_k)$. f contains products of terms from each column, so all terms have degree $0 + 1 + \cdots + (k-1) = k(k-1)/2$. If $x_i = x_j$, $i \neq j$, then $\det V = 0$, so $(x_i - x_j) | f$. Write

$$f = Q \prod_{1 \leq i < j \leq k} (x_i - x_j)$$

for some polynomial Q. Degree of terms in f is $k(k-1)/2$, so Q is constant. Product of diagonal entries in $\det V$ is $1 \cdot x_{k-1} \cdot x_{k-2}^2 \cdots x_1^{k-1}$, which is also obtained from the first term of each factor in the product, so $Q = 1$. This suffices for the proof. $\square$

Computing $\dim V_\lambda$

The conjugacy class of the identity corresponds to $\mathbf{i} = (d, 0, \ldots, 0)$, so using the Frobenius formula,

$$\begin{aligned}\dim V_\lambda = \chi_\lambda(C_{\mathbf{i}}) &= [\Delta(\mathbf{x}) \cdot (x_1 + \cdots + x_k)^d]_{(l_1,\ldots,l_k)} \\ &= \left[\det V \cdot \sum_{r_1+\cdots+r_k=d} \frac{d!}{r_1! \cdots r_k!} x_1^{r_1} x_2^{r_2} \cdots x_k^{r_k}\right]_{(l_1,\ldots,l_k)}\end{aligned}$$

$$= \left[\sum_{\sigma \in S_k} (\operatorname{sgn} \sigma) x_k^{\sigma(1)-1} \cdots x_1^{\sigma(k)-1} \cdot \sum_{r_1+\cdots+r_k=d} \frac{d!}{r_1! \cdots r_k!} x_1^{r_1} x_2^{r_2} \cdots x_k^{r_k}\right]_{(l_1,\ldots,l_k)}$$

Pick $\sigma(k) - 1$ x_1's from first sum, $\sigma(k-1) - 1$ x_2's from second sum, and so on so that $r_j = l_j - \sigma(k+1-j) + 1$.

Computing dim V_λ

The coefficient of $x_1^{l_1}\cdots x_k^{l_k}$ is

$$\sum_\sigma \operatorname{sgn}\sigma \cdot \frac{d!}{(l_1-\sigma(k)+1)!\cdots(l_k-\sigma(1)+1)!}$$

for $\sigma \in S_k$ such that $l_j - \sigma(k+1-j)+1 \geq 0$. This becomes

$$= \frac{d!}{l_1!\cdots l_k!}\sum_\sigma \operatorname{sgn}(\sigma)\cdot \frac{l_1!\cdots l_k!}{(l_1-\sigma(k)+1)!\cdots(l_k-\sigma(k)+1)!}$$

$$= \frac{d!}{l_1!\cdots l_k!}\sum_\sigma \operatorname{sgn}(\sigma)\cdot \prod_{j=1}^{k} l_j(l_j-1)\cdots(l_j-\sigma(k)+2)$$

$$= \frac{d!}{l_1!\cdots l_k!}\det\begin{bmatrix} 1 & l_k & l_k(l_k-1) & l_k(l_k-1)(l_k-2) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & l_1 & l_1(l_1-1) & l_1(l_1-1)(l_1-2) & \cdots \end{bmatrix}$$

$$= \frac{d!}{l_1!\cdots l_k!}\prod_{1\leq i<j\leq k}(l_i-l_j). \qquad \text{(column reduction)}$$

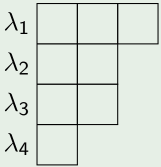
The dim V_λ Formula

Frobenius formula for dim V_λ

$$\dim V_\lambda = \frac{d!}{l_1! \cdots l_k!} \prod_{i<j} (l_i - l_j).$$

Example

For $d = 8$, $\lambda = (3, 2, 2, 1)$, $(l_1, l_2, l_3, l_4) = (6, 4, 3, 1)$,



we have

$$\dim V_\lambda = \frac{8!}{6!4!3!1!}(6-1)(6-3)(6-4)(4-1)(4-3)(3-1) = 70.$$

Hook Length Formula

The **hook length** of a box in a Young diagram is the number of squares directly below or directly to the right of the box.

6	4	1
4	2	
3	1	
1		

Hook length formula

$$\dim V_\lambda = \frac{d!}{\prod(\text{Hook lengths})}.$$

Using the same example:

$$\frac{8!}{6 \cdot 4 \cdot 4 \cdot 2 \cdot 3} = 70.$$

- To be proven from the Frobenius formula.

Exercises

Exercise 4.19

If V is the standard representation of S_d, prove the decomposition into irreducible representations:

$$\operatorname{Sym}^2 V \cong U \oplus V \oplus V_{(d-2,2)},$$
$$V \otimes V = \operatorname{Sym}^2 V \oplus \Lambda^2 V \cong U \oplus V \oplus V_{(d-2,2)} \oplus V_{(d-2,1,1)}.$$

Solution

We will do the first one. Suffices to show for any $g \in S_d$,

$$\chi_{\operatorname{Sym}^2 V}(g) = \chi_U(g) + \chi_V(g) + \chi_{V_{(d-2,2)}}(g).$$

Let g be in conjugacy class $C_{\mathbf{i}}$, $\mathbf{i} = (i_1, \ldots, i_d)$, i_j j-cycles. Use Frobenius formula for $\lambda = (d-2, 2)$, $(l_1, l_2) = (d-2+2-1, 2+2-2) = (d-1, 2)$:

$$\chi_{V_{(d-2,2)}}(C_{\mathbf{i}}) = \left[(x_1 - x_2)(x_1 + x_2)^{i_1}(x_1^2 + x_2^2)^{i_2} \cdots (x_1^d + x_2^d)^{i_d}\right]_{(d-1,2)}.$$

Exercises

Solution

To get coefficient of $(d-1,2)$ *term, we can choose the* x_2^2 *term in the first factor and second factor in* $(-1)i_1$ *ways, second factor twice in* $\binom{i_1}{2}$ *ways, or third factor in* i_2 *ways, so*

$$\chi_{V_{(d-2,2)}}(g) = -i_1 + \binom{i_1}{2} + i_2.$$

Since g *fixes* i_1 *points and* g^2 *fixes* $i_1 + 2i_2$ *points, we compute*

$$\chi_V(g) = \chi_{\mathbb{C}^d}(g) - \chi_U(g) = i_1 - 1,$$
$$\chi_V(g^2) = \chi_{\mathbb{C}^d}(g^2) - \chi_U(g^2) = i_1 + 2i_2 - 1.$$

Finally, check that

$$\chi_{\mathrm{Sym}^2 V}(g) = \frac{1}{2}(\chi_V(g)^2 + \chi_V(g^2)) = \frac{1}{2}((i_1-1)^2 + (i_1 + 2i_2 - 1))$$
$$= \chi_U(g) + \chi_V(g) + \chi_{V_{(d-2,2)}}(g).$$

Exercises

Exercise 4.13

Deduce the hook-length formula from the Frobenius Formula.

Solution

We induct on the number of columns in the Young diagram of $\lambda = (\lambda_1, \dots, \lambda_k)$*. For a Young diagram* D *with* k *rows, let* Π_D *be the product of the hook lengths in* D*. Suffices to show*

$$\Pi_D \cdot \prod_{i<j} (l_i - l_j) = l_1! \cdots l_k!. \qquad (*)$$

Note that $l_i = \lambda_i + k - i$ *is the hook length of leftmost entry in row* i*.*
Base case. 1 *column and* k *rows,* $l_i = (k+1) - i$, $\Pi_D = l_1 \cdots l_k$,

$$\Pi_D \cdot \prod_{1 \le i < j \le k} (l_i - l_j) = l_1 \cdots l_k \cdot \prod_{i=1}^{k-1} (l_i - l_{i+1}) \cdots (l_i - l_k)$$

$$= k(k-1) \cdots 1 \cdot \prod_{i=1}^{k-1} (k+1-i-1)! = l_1! l_2! \cdots l_k!.$$

Exercises

Solution

Inductive step. Assume $(*)$ *works for* $c \geq 1$ *columns and* k *rows. Consider Young diagram* D *with* $c+1$ *columns,* $\lambda_1, \ldots, \lambda_r > 1$, $\lambda_{r+1}, \ldots, \lambda_k = 1$. *Note that for* $r < j \leq k$, $l_j = \lambda_j + k - j = (k+1) - j$.

Let D' *be diagram created from columns* $2, \ldots, c+1$ *of* D. *In first column of* D', *there are* r *entries, and entry* j *has hook length* $l_j - (k-r) - 1$. *The rest is algebra:*

$$\Pi_D \cdot \prod_{1 \leq i < j \leq k} (l_i - l_j) = l_1 \cdots l_k \Pi_{D'} \cdot \prod_{i \leq r} (l_i - l_{i+1}) \cdots (l_i - l_r)(l_i - (k-r)) \cdots (l_i - 1)$$

$$\cdot \prod_{r < i \leq k} (l_i - l_{i+1}) \cdots (l_i - l_k)$$

$$= \left[\prod_{i \leq r} l_i (l_i - 1) \cdots (l_i - k + r)\right] l_{r+1}! \cdots l_k! \cdot \left[\Pi_{D'} \prod_{1 \leq i < j \leq r} ((l_i - (k-r)) - (l_j - (k-r)))\right]$$

$$= \left[\prod_{i \leq r} (l_i - 1) \cdots (l_i - k + r)\right] l_{r+1}! \cdots l_k! \cdot [(l_1 - (k-r) - 1)! \cdots (l_r - (k-r) - 1)!]$$

$$= l_1! \cdots l_r! l_{r+1}! \cdots l_k!.$$

References

- W. Fulton and J. Harris, Representation Theory: A First Course, Springer Science+Business Media, Inc., New York, 2004.