Grassmannians and Fulton-Harris §15.4

Summer 2020 Reading Project

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Objectives

Learn about:

- Grassmannians
  - As subsets of projective varieties
  - As matrices
- Exterior powers
- Representation theory of Grassmannians in Fulton-Harris §15.4

- For our field we use $K = \mathbb{C}$.

Definition

The **Grassmannian** $G(k, n)$ is the set of k-dimensional subspaces of $\mathbb{C}^n$.

Example

1. $G(1, n)$ is the set of 1-dimensional subspaces of $\mathbb{C}^n$, or $\mathbb{P}^{n-1}$.
2. $G(2, 3)$ is the set of 2-dimensional subspaces of $\mathbb{C}^3$. Correspond a plane with its normal to get an identification between $G(2, 3)$ and $G(1, 3) = \mathbb{P}^2$.

Definition

To a k-dimensional subspace $W \subset \mathbb{C}^n$ spanned by vectors $v_1, \ldots, v_k$, associate to W the multivector

$$\lambda_W = v_1 \wedge \cdots \wedge v_k \in \Lambda^k \mathbb{C}^n.$$

The map $\psi : G(k,n) \to \mathbb{P}(\Lambda^k(\mathbb{C}^n))$ such that $W \mapsto [\lambda_W]$ is the **Plücker embedding**.

The Plücker embedding is well defined:

- For another basis $v_1', \ldots, v_k'$ of W find change of basis matrix $P = \{p_{ij}\}_{1 \leq i,j \leq k}$ satisfying $v_j' = v_1 p_{1j} + v_2 p_{2j} + \cdots + v_k p_{kj}$. So

$$\lambda_W' = v_1' \wedge \cdots \wedge v_k' = \det P \cdot v_1 \wedge \cdots \wedge v_k = \det P \cdot \lambda_W. \tag{1}$$

  See (7.73) in [Gun18] for details about above equation.

The Plücker embedding is an inclusion:

- For a point in the image $[\lambda] = \psi(W) = \psi(W')$,

$$\lambda = v_1 \wedge \cdots \wedge v_k = v_1' \wedge \cdots \wedge v_k',$$

  so $0 = v_i' \wedge \lambda = v_i' \wedge v_1 \wedge \cdots \wedge v_k$. We get v_i' is a linear combination of v_i, so $W' \subset W$. Symmetric argument gives $W \subset W'$.

Plücker coordinates

Recall from Reed's lecture that the standard coordinates $Z_0, \dots, Z_n$ on $\mathbb{C}^{n+1}$ are called **homogeneous coordinates** on $\mathbb{P}^n$. Choosing a basis for V gives homogeneous coordinates for $\mathbb{P}V$.

Definition (Intrinsic)

The **Plücker coordinates** on $G(k,n)$ are the homogeneous coordinates on $\mathbb{P}(\Lambda^k\mathbb{C}^n)$ relative to the standard basis of $\Lambda^k\mathbb{C}^n$.

Note $\mathbb{P}(\Lambda^k\mathbb{C}^n) = \mathbb{P}^{\binom{n}{k}-1}$.

Example ([Gat])

1. The Plücker embedding of $G(1,n)$ maps the line

$$W = \operatorname{span}\{a_1e_1 + \cdots + a_ne_n\} \mapsto [\lambda_W] = [a_1e_1 + \cdots + a_ne_n].$$

   Coordinates of W are $[a_1, \dots, a_n] \in \mathbb{P}(\Lambda^1\mathbb{C}^n) = \mathbb{P}^{n-1}$. So $G(1,n) = \mathbb{P}^{n-1}$.

2. Consider $W = \operatorname{span}\{e_1 + e_2, e_1 + e_3\} \in G(2,3)$. Since

$$[(e_1 + e_2) \wedge (e_1 + e_3)] = [-e_1 \wedge e_2 + e_1 \wedge e_3 + e_1 \wedge e_3],$$

   W maps to the point $[-1, 1, 1] \in \mathbb{P}^2$

The Grassmannian as a projective variety

- Recall that a **projective variety** of $\mathbb{P}^n$ is the zero locus of a set of homogeneous polynomials $F_\alpha \in \mathbb{C}[Z_0, \ldots, Z_n]$.

Theorem

The Grassmannian $G(k, n)$ is a projective variety of $\mathbb{P}(\Lambda^k \mathbb{C}^n) = \mathbb{P}^{\binom{n}{k}-1}$.

To prove this, we need some facts about exterior powers.

Definition

- A multivector $\omega \in \Lambda^k V$ is **totally decomposable** or a **simple tensor** if $\omega = v_1 \wedge \cdots \wedge v_k$ for some $v_i \in V$.
- The vector $v \in V$ **divides** the multivector $\omega \in \Lambda^k V$ if $\omega = v \wedge \varphi$ for some $\varphi \in \Lambda^{k-1} V$.

Proposition

A vector $v \in V$ divides $\omega \in \Lambda^k V$ if and only if $\omega \wedge v = 0$.

Proof.

Forward direction is immediate. Suppose $\omega \wedge v = 0$ and let $v_1, \ldots, v_n$ be a basis for V such that $v_n = v$. Write

$$\omega = \sum_{i_1 < \cdots < i_k} a_{i_1,\ldots,i_k} v_{i_1} \wedge \cdots \wedge v_{i_k}.$$

Then

$$0 = \omega \wedge v = \sum_{i_1 < \cdots < i_k} a_{i_1,\ldots,i_k} v_{i_1} \wedge \cdots \wedge v_{i_k} \wedge v,$$

so for $i_k \neq n$, $a_{i_1,\ldots,i_k} = 0$. This means that every nonzero simple tensor in ω contains a v. □

Proving $G(k, n)$ is a projective variety of $\mathbb{P}(\Lambda^k\mathbb{C}^n)$

Lemma

The vector $[\omega] \in \mathbb{P}(\Lambda^k\mathbb{C}^n)$ will lie in the Grassmannian $G(k, n)$ if and only if the map

$$\begin{aligned} \varphi(\omega) &: \mathbb{C}^n \to \Lambda^{k+1}\mathbb{C}^n \\ &: v \mapsto \omega \wedge v \end{aligned}$$

has rank at most $n - k$.

Proof.

It suffices to show $[\omega]$ lies in the Grassmannian if and only if $\dim \ker \varphi(\omega) \geq k$.

($\rightarrow$) If $[\omega]$ lies in the Grassmannian, then $\omega = v_1 \wedge \cdots \wedge v_k$. So $v_i \in \ker \varphi(\omega)$.

($\leftarrow$) $v \in \ker \varphi(\omega)$ if and only if v divides ω by the previous proposition. At most k linearly independent vectors divide ω, so $\ker \varphi = \operatorname{span}\{v_1, \ldots, v_k\}$. Applying the proposition k times allows us to write $\omega = cv_1 \wedge \cdots \wedge v_k$.

□

Proving $G(k,n)$ is a projective variety of $\mathbb{P}(\Lambda^k\mathbb{C}^n)$

Theorem

The Grassmannian $G(k,n)$ is a projective variety of $\mathbb{P}(\Lambda^k\mathbb{C}^n)$.

Proof.

- For an $\omega \in \Lambda^k\mathbb{C}^n$, write its homogeneous coordinates $[a_1, \dots, a_N]$ in terms of the standard basis $e_{i_1} \wedge \cdots \wedge e_{i_k}$, $i_1 < \cdots < i_k$, for $\Lambda^k\mathbb{C}^n$.
- Write the matrix of $\varphi(\omega) \in \mathrm{Hom}(\mathbb{C}^n, \Lambda^{k+1}\mathbb{C}^n)$ in terms of the standard basis $e_1, \dots, e_n$ of $\mathbb{C}^n$ and $e_{i_1} \wedge \cdots \wedge e_{i_{k+1}}$, $i_1 < \cdots < i_{k+1}$, for $\Lambda^{k+1}\mathbb{C}^n$.
- By the lemma, $[\omega]$ is in the Grassmannian if and only if $\varphi(\omega)$ has rank at most $n-k$, which happens if and only if the $(n-k+1)\times(n-k+1)$ minors of $\varphi(\omega)$ vanish. All these equations are in terms of the coordinates a_i of ω, so the set of $[\omega]$ in the Grassmannian is precisely the solutions to the vanishing of these minors.

$\square$

Plücker coordinates in matrix form

Represent a k-plane $W \subset \mathbb{C}^n$ spanned by coordinate vectors $v_1, \ldots, v_k$ as a matrix $k \times n$ matrix M_W with rows v_i.

Definition (Matrix)

The **Plücker coordinates** of W are the determinants of the $k \times k$ submatrices of M_W for some ordering of the columns.

Example ([Gat])

The Plücker coordinates of $W = \text{span}\{e_1 + e_2, e_1 + e_3\} \in G(2,3)$ are the 2×2 minors of

$$M_W = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

For the ordering of columns $(1,2), (1,3), (2,3)$, this is the point $[-1, 1, 1] \in \mathbb{P}^2$.

- Note that performing row operations on M_W changes the $k \times k$ determinants by at most a constant factor.

Important Plücker coordinates example

Example: $G(2,4)$

If the plane $W \subset \mathbb{C}^4$ is spanned by the rows of

$$M_W = \begin{bmatrix} v_{11} & v_{12} & v_{13} & v_{14} \\ v_{21} & v_{22} & v_{23} & v_{24} \end{bmatrix}.$$

The determinants of the 2×2 submatrices are

$$\begin{aligned} W_{12} &= v_{11}v_{22} - v_{12}v_{21}, & W_{13} &= v_{11}v_{23} - v_{13}v_{21}, & W_{14} &= v_{11}v_{24} - v_{14}v_{21}, \\ W_{23} &= v_{12}v_{23} - v_{13}v_{22}, & W_{24} &= v_{12}v_{24} - v_{14}v_{22}, & W_{34} &= v_{13}v_{24} - v_{14}v_{23}. \end{aligned}$$

The W_{ij} are the Plücker coordinates of W in $\mathbb{P}(\Lambda^2\mathbb{C}^4) = \mathbb{P}^5$.
The coordinates of any $W \in G(2,4)$ satisfy one equation

$$W_{12}W_{34} - W_{13}W_{24} + W_{14}W_{23} = 0.$$

We have that $G(2,4)$ is a subvariety of $\mathbb{P}^5$ cut out by the above equation.

- $G(2,4)$ is a *quadric hypersurface* in $\mathbb{P}^5$: it is four dimensional and the degree of the polynomial in the W_{ij}'s is two.

Grassmannian as a manifold

Theorem

The dimension of $G(k, n)$ over the complex numbers is $k(n-k)$.

Proof.

- Take a k-plane $W \subset \mathbb{C}^n$ spanned by basis $v_1, \ldots, v_k$. Let M_W be the $k \times n$ matrix with v_i as row i. M_W has k linearly independent columns $i_1 < i_2 < \cdots < i_k$. Let $W_{i_1,\ldots,i_k}$ be the $k \times k$ submatrix of M_W with columns $i_1, \ldots, i_k$.
- The matrix $W_{i_1,\ldots,i_k}^{-1} M_W$ represents W. Its column i_j is $\delta_j \in \mathbb{C}^k$. For example, if $i_j = j$,

$$W_{i_1,\ldots,i_k}^{-1} M_W = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & a_{1,1} & a_{1,2} & \cdots & a_{1,n-k} \\ 0 & 1 & 0 & \cdots & 0 & a_{1,1} & a_{1,2} & \cdots & a_{2,n-k} \\ \vdots & & & & & & & & \vdots \\ 0 & 0 & 0 & \cdots & 1 & a_{k,1} & a_{k,2} & \cdots & a_{k,n-k} \end{bmatrix}.$$

- Let $U_{i_1,\ldots,i_k}$ be the set of W such that $W_{i_1,\ldots,i_k}$ is invertible. Then $U_{i_1,\ldots,i_k}$ are open subset of $G(k, n)$.
- Take charts $A^{i_1,\ldots,i_k} : U_{i_1,\ldots,i_k} \to \mathbb{C}^{k(n-k)}$ that send W to the $k \times (n-k)$ submatrix of $W_{i_1,\ldots,i_k}^{-1} M_W$ whose columns are complementary to $i_1, \ldots, i_k$. The charts are isomorphic to $\mathbb{C}^{k(n-k)}$.

□

Plücker coordinates on a chart

As we know, the Plücker coordinates of k-plane $W \subset \mathbb{C}^n$ are the $\binom{n}{k}$ determinants of size $k \times k$ in M_W.

Proposition

On the chart $A^{i_1,i_2,\dots,i_k}$, the $k \times k$ determinants of M_W become $\ell \times \ell$ determinants of $A^{i_1,\dots,i_k}(W) \in \mathbb{C}^{k(n-k)}$.

For example, for $G(3,7)$ any matrix in $U_{1,2,3}$ can be represented as

$$\begin{bmatrix} 1 & 0 & 0 & a_{1,4} & a_{1,5} & a_{1,6} & a_{1,7} \\ 0 & 1 & 0 & a_{2,4} & a_{2,5} & a_{2,6} & a_{2,7} \\ 0 & 0 & 1 & a_{3,4} & a_{3,5} & a_{3,6} & a_{3,7} \end{bmatrix}$$

so determinants can look like

$$\begin{vmatrix} 1 & 0 & a_{1,5} \\ 0 & 0 & a_{2,5} \\ 0 & 1 & a_{3,5} \end{vmatrix} = -a_{2,5}, \quad \begin{vmatrix} 0 & a_{1,5} & a_{1,7} \\ 0 & a_{2,5} & a_{2,7} \\ 1 & a_{3,5} & a_{3,7} \end{vmatrix} = a_{1,5}a_{2,7} - a_{1,7}a_{2,5}.$$

Plücker relations on a chart

Consider any $\ell \times \ell$ submatrix $A^{i_1,\dots,i_k}(W) \in \mathbb{C}^{k(n-k)}$. The $\ell \times \ell$ determinant can be expanded by rows. It is a sum of ℓ terms, each one of the form

$$\det(1 \times 1 \text{ matrix}) \cdot \det((\ell-1) \times (\ell-1) \text{ matrix}).$$

Each of these terms are products of Plücker coordinates, so we get a quadratic relation in terms of these Plücker coordinates:

$$(\text{Pl coord}) = \sum_{j=1}^{\ell} (-1)^{j-1} (\text{Pl coord})(\text{Pl coord}).$$

These quadratic relations are called the **Plücker relations**.

Example: $G(2,4)$ on the $A^{1,2}$ chart

A subspace in $U_{1,2}$ can have matrix

$$\begin{bmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \end{bmatrix},$$

so the six Plücker coordinates are $1, c, d, -a, -b, ad-bc$. The Plücker relation among these coordinates is

$$(1)(ad-bc) - (c)(-b) + (d)(-a) = 0.$$

It corresponds $W_{12}W_{34} - W_{13}W_{24} + W_{14}W_{23} = 0$ in $\mathbb{P}^5$.

It turns out that we can write Plücker relations for $G(k, n)$ globally in terms of Plücker coodinates.

Plücker coordinates for $G(2, n)$

For sequences of integers $1 \leq i < j_1 < j_2 < j_3 \leq n$, the Plücker coordinates for $G(2, n)$ are

$$W_{i,j_1} W_{j_2,j_3} - W_{i,j_2} W_{j_1,j_3} + W_{i,j_3} W_{j_1,j_2} = 0.$$

See [Pos] for proof.

- The Plücker relation for $G(2, 4)$ in coordinates $\{W_{12}, W_{13}, W_{14}, W_{23}, W_{24}, W_{34}\}$ for $\mathbb{P}(\Lambda^2\mathbb{C}^4) = \mathbb{P}^5$ is
$$W_{12}W_{34} - W_{13}W_{24} + W_{14}W_{23} = 0.$$
- The Plücker relations for $G(2, 5)$ in terms of coordinates W_{ij}, $1 \leq i < j \leq 5$, are

$$\begin{aligned}
&W_{12}W_{34} - W_{13}W_{24} + W_{14}W_{23} = 0, \qquad &&W_{12}W_{34} - W_{13}W_{24} + W_{14}W_{23} = 0,\\
&W_{12}W_{34} - W_{13}W_{24} + W_{14}W_{23} = 0, \qquad &&W_{12}W_{34} - W_{13}W_{24} + W_{14}W_{23} = 0,\\
&W_{12}W_{34} - W_{13}W_{24} + W_{14}W_{23} = 0.
\end{aligned}$$

We return to Fulton-Harris §15.4.

Proposition

All automorphisms of the Grassmannian $G(k, n)$ are induced by automorphisms of $\mathbb{C}^n$.

See (10.19) in [Har95] for full proof.

Half-Proof Sketch.

We show an automorphism of $\mathbb{C}^n$ induces an automorphism of $G(k, n)$. Represent a k-plane W in $\mathbb{C}^n$ as a $k \times n$ matrix M_W. Multiplication on the right by an $n \times n$ matrix preserves the rank of M_W. The resulting matrix represents another k-plane in $G(k, n)$ and this is a correspondence.

- It turns out there are more automorphisms for the case $n = 2k$: see (10.19) in [Har95].

- Restrict attention to $\mathrm{SL}_n \mathbb{C}$.
- Scalar multiples λI act trivially on k-planes, so we consider the action of $\mathrm{PSL}_n \mathbb{C} = \mathrm{SL}_n \mathbb{C} / \{\lambda I : \lambda^n = 1\}$.

Polynomials on the Grassmannian

Proposition

For an n-dimensional vector space V, the space of all homogeneous polynomials of degree m on $\mathbb{P}(\Lambda^k V^*)$ is the symmetric power $\mathrm{Sym}^m(\Lambda^k V)$.

- From Reed's §11.3 lecture, interpret $\mathrm{Sym}^k W$ as the space of homogeneous polynomials on $\mathbb{P}(W^*)$ by choosing a basis.
- It is easier to see how the homogeneous polynomials of degree m on $\mathbb{P}(\Lambda^k V)$ are $\mathrm{Sym}^m(\Lambda^k V^*)$, but [FH91] wishes to analyze $\mathrm{Sym}^m(\Lambda^k V)$.

Notation

Let the subspace $I(G)_m \subset \mathrm{Sym}^m(\Lambda^k \mathbb{C}^n)$ be the polynomials of degree m on $\mathbb{P}(\Lambda^k (\mathbb{C}^n)^*)$ that vanish on $G(k, n)$.

Definition

Define the **homogeneous ideal** of a Grassmannian $I(G) = \bigoplus I(G)_m$. It is the set of polynomials that vanish on the Grassmannian $G = G(k, n)$.

- It is called an ideal because if $f \in I(G)$ vanishes on the Grassmannian, so does any multiple.

$I(G)_m$ is a representation of $\mathfrak{sl}_n\mathbb{C}$

Fact [Har95]

The Plücker relations generate the ideal of the Grassmannian.

- Proof omitted.

Proposition

Each $I(G)_m$ is a representation of $\mathfrak{sl}_n\mathbb{C}$.

Proof.

- Elements $\mathrm{PSL}_n\,\mathbb{C}$ carry $G(k, n)$ to itself bijectively.
- $\mathrm{PSL}_n\,\mathbb{C}$ carries Plücker coordinates of $G(k, n)$ to linear combinations of Plücker coordinates.
- $\mathrm{PSL}_n\,\mathbb{C}$ sends polynomials of degree m to themselves by the fact.
- Note that the Lie algebra of $\mathrm{PSL}_n\,\mathbb{C}$ is $\mathfrak{sl}_n\mathbb{C}$. Differentiate $\mathrm{PSL}_n\,\mathbb{C} \to \mathrm{Aut}(I(G)_m)$ at the identity to get desired representation.

□

Exercise 15.35

For $G = G(2,4)$, deduce an isomorphism

$$I(G)_m \cong \mathrm{Sym}^{m-2}(\Lambda^2\mathbb{C}^4).$$

Proof.

- Write a polynomial $P \in \mathrm{Sym}^m(\Lambda^2\mathbb{C}^4)$ in terms of coordinates $W_{12}, W_{13}, W_{14}, W_{23}, W_{24}, W_{34}$ by choosing the standard basis $e_i \wedge e_j$, $i < j$, for $\Lambda^2\mathbb{C}^4$.
- We know G is a quadric hypersurface defined by the vanishing of the quadratic $f = W_{12}W_{34} - W_{13}W_{24} + W_{14}W_{23}$, so P vanishes on G if and only if P is divisible by f.
- This associates P with a polynomial P/f in $\mathrm{Sym}^{m-2}(\Lambda^2\mathbb{C}^4)$. It is an isomorphism.

$\square$

$I(G)_2$ for $G(2, n)$

Part of Exercise 15.34

The quadratic part of the ideal of the Grassmannian $G(2, n)$ satisfies

$$I(G)_2 \cong \Lambda^4 \mathbb{C}^n.$$

The proof is in four steps:

1. A nonzero tensor $\varphi \in \Lambda^2 \mathbb{C}^n$ can be written as $v_1 \wedge v_2$ if and only if $\varphi \wedge \varphi = 0$.

§7.2 Problem 10, [Gun18]

Show that an exterior 2-form $\omega = \sum_{1 \leq i < j \leq n} a_{ij} u_i \wedge u_j$ in an n-dimensional vector space V in terms of a basis u_i also can be written as the sum $\omega = (v_1 \wedge v_2) + (v_3 \wedge v_4) + \cdots + (v_{2r-1} \wedge v_{2r})$ for an appropriate basis $v_1, \ldots, v_n$ of V, where r is the largest integer such that the product $\omega \wedge \cdots \wedge \omega \neq 0$ of r copies of the form ω is nonzero.

Proof.

Use $V = \mathbb{C}^n$ and $r = 1$. □

2. Write the matrix for $\varphi \in \Lambda^2\mathbb{C}^n$ in skew-symmetric form in terms of the standard basis. $\varphi \wedge \varphi = 0$ if and only if the Pfaffians of the symmetric 4×4 minors vanish.

Definition

The **Pfaffian** of an $2n \times 2n$ skew-symmetric matrix $A = (a_{ij})$ is the unique polynomial pf in the a_{ij} such that $\det A = \mathrm{pf}(a_{ij})^2$.

The Pfaffian of a 4×4 skew-symmetric matrix:

$$\mathrm{pf}\begin{bmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{bmatrix} = af - be + dc.$$

Proof.

Write $\varphi = \sum_{1 \le i < j \le n} a_{ij} e_i \wedge e_j$. The coefficient of $e_{i_1} \wedge e_{i_2} \wedge e_{i_3} \wedge e_{i_4}$ for $i_1 < i_2 < i_3 < i_4$ in $\varphi \wedge \varphi$ is $a_{i_1 i_2} a_{i_3 i_4} - a_{i_1 i_3} a_{i_2 i_4} + a_{i_1 i_4} a_{i_2 i_3}$. This is the Pfaffian of the 4×4 symmetric minor of φ with columns i_1, i_2, i_3, i_4. $\square$

$I(G)_2$ for $G(2, n)$

3. The vector space generated by linear combinations of the Pfaffians is isomorphic to $\Lambda^2\mathbb{C}^n$.

Proof.

The Pfaffians are $\binom{n}{4}$ quadratic relations corresponding to a choice of four columns. They are independent. Correspond the Pfaffian $a_{i_1 i_2}a_{i_3 i_4} - a_{i_1 i_3}a_{i_2 i_4} + a_{i_1 i_4}a_{i_2 i_3}$ to the basis vector $e_{i_1} \wedge e_{i_2} \wedge e_{i_3} \wedge e_{i_4}$ in $\Lambda^4\mathbb{C}^n$. □

4. The vector space generated by the Pfaffians is the quadratic ideal $I(G)_2$ of $G(2, n)$!

So $I(G)_2$ is isomorphic to the vector space generated by the Pfaffians which is isomorphic to $\Lambda^2\mathbb{C}^n$.

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