$\mathrm{SU}(2)$ and Analysis on Compact Groups

Summer 2020 Reading Project

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Summary of real forms and compact Lie groups I

Let $\mathfrak{g}$ be a complex semisimple Lie algebra.

Definitions

- A *real form* of $\mathfrak{g}$ is a real Lie algebra $\mathfrak{g}_0$ such that $\mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C} \cong \mathfrak{g}$.
- Let $\mathfrak{g}_0$ be a real form of $\mathfrak{g}$. A Lie subalgebra $\mathfrak{h}_0 \subset \mathfrak{g}_0$ is called a *Cartan subalgebra* of $\mathfrak{g}_0$ if $\mathfrak{h}_0 \otimes_{\mathbb{R}} \mathbb{C}$ is a Cartan subalgebra of $\mathfrak{g}$.
- A *compact form* of $\mathfrak{g}$ is a real form $\mathfrak{g}_0$ such that there exists a Cartan subalgebra $\mathfrak{h}_0$ of $\mathfrak{g}_0$ whose action on $\mathfrak{g}_0$ has all purely imaginary eigenvalues.

Proposition ([FH91] §26.1)

- The compact form of $\mathfrak{g}$ is unique.
- If the real form $\mathfrak{g}_0$ is compact, every Lie group G_0 with Lie algebra $\mathfrak{g}_0$ is compact. Conversely, if we are given a real compact Lie group G_0 and a real form $\mathfrak{g}_0$ of $\mathfrak{g}$ such that G_0 has Lie algebra $\mathfrak{g}_0$, then $\mathfrak{g}_0$ is the compact form of $\mathfrak{g}$.

Notation

We'll denote the compact form of $\mathfrak{g}$ as $\mathfrak{g}_0$ from now on.

Summary of real forms and compact Lie groups II

Important tool

There is a *conjugate linear involution* $\sigma : \mathfrak{g} \to \mathfrak{g}$ ($\sigma^2 = \mathrm{id}$) associated to a fixed real form $\mathfrak{g}_0$ that takes $x \otimes z \mapsto x \otimes \overline{z}$ for $x \in \mathfrak{g}_0$ and $z \in \mathbb{C}$.

- The fixed points of σ can be written as $X \otimes 1$, $X \in \mathfrak{g}_0$. They form a Lie subalgebra that is the real form $\mathfrak{g}_0$. Conversely, given any conjugate linear involution σ, the fixed subalgebra $\mathfrak{g}^\sigma$ is a real form of $\mathfrak{g}$.
- It follows that the real forms of a Lie algebra are in one-to-one correspondance with conjugate linear forms.

Definition

Suppose G is a group of matrices that is closed under conjugate transpose: $A \in G$ implies $\overline{A}^t \in G$. The *Cartan involution* of G is the map $\Theta : A \to (\overline{A}^t)^{-1}$.

Fact ([Kna01] §1)

The fixed points of the Cartan involution on G form a maximal compact subgroup of G.

Most of the groups G we studied are closed under conjugate transpose: $\mathrm{GL}_n\,\mathbb{C}$, $\mathrm{SL}_n\,\mathbb{C}$, $\mathrm{SO}_m\,\mathbb{C}$ ($AA^t = I$ implies $\overline{A}^t\overline{A} = I$), $\mathrm{Sp}_{2n}\,\mathbb{C}$.

Examples of compact groups

$$\Theta : A \mapsto (\overline{A}^t)^{-1}, \quad A \in \mathrm{SL}_n\,\mathbb{C}.$$

- In $\mathrm{SL}_n\,\mathbb{C}$, the fixed points of Θ are A for which $A = (\overline{A}^t)^{-1}$. This implies $\overline{A}^t A = I$, or $A \in \mathrm{SU}(n)$.
- In $\mathrm{SO}_m\,\mathbb{C}$, we know $A^t A = I$, and the condition $A\overline{A}^t = I$ is equivalent to $A^t\overline{A} = I$. Combining, $A = \overline{A}$, so entries of A are real. The compact subgroup is $\mathrm{SO}(m)$, the orthogonal matrices.
- In $\mathrm{Sp}_{2n}\,\mathbb{C}$, similarly, the compact subgroup is the subgroup satisfying the unitary condition. It is denoted $\mathrm{Sp}(2n) = \mathrm{Sp}_{2n}\,\mathbb{C} \cap \mathrm{SU}(2n)$.

Representations of compact Lie groups

Let G be a Lie group with complex semisimple Lie algebra $\mathfrak{g}$. If $G_0 \subset G$ is a maximal (no other Lie groups properly nested between G_0 and G) compact Lie group:

- The irreducible representations $G_0 \to \mathrm{GL}(V)$ are the same as the irreducible representations of the complex group G (restriction).
- If G is the simply connected form, the irreducible representations are one-to-one with the irreducible representations of $\mathfrak{g}$ ([FH91] Second Principle).

Studying compact groups directly

- [FH91] gave us an algebraic perspective on representation theory: Lecture 9 was abstract algebra: ideals, radicals, semisimplicity. Lectures 10-22 were direct algebraic calculations.
- We also only worked with finite-dimensional representations!
- Representation theory is also an analytic theory. It will be the correct way when we study infinite dimensional vector spaces, and it will work well when we study *non*-compact groups such as $\mathrm{SL}_2 \mathbb{R}$.
- The analytic approach is used in physics, applied math, and in modern number theory when we look at representations of $\mathrm{GL}_n(\mathbb{Q}_p)$, where $\mathbb{Q}_p$ is a p-adic field.

Motivation and where we're going I

Let G be a compact Lie group.

1. The concept of *averaging* for *complete reducibility* of finite dimensional representations: if W is a subrepresentation of a finite dimensional representation V, then there exists a complementary G-invariant subspace $W^\perp$ of V such that $V = W \oplus W^\perp$.

$|G|$ is finite case

Recall in the case where G is finite we used a G-invariant inner product:

$$H(v, w) = \frac{1}{|G|} \sum_{g \in G} H_0(gv, gw), \; H_0 \text{ any Hermitian IP on } V.$$

If W was a subrepresentation of V, we could construct its orthogonal complement $W^\perp$ with respect to H.

Let G be a compact Lie group.

2. Finding irreducible representations of G in an analogue of the finite group regular representation.

Recall:

Definition

The *regular representation* r of a finite group G is the vector space V with basis e_g for every $g \in G$ such that $b \in G$ acts on the vectors in V by $r(b) \cdot \sum c_g e_g = \sum c_g e_{bg}$.

Proposition ([FH91] §2.2)

Any irreducible representation V of (a finite group) G appears in the regular representation $\dim V$ times.

- We will see an analogue of this corollary for compact Lie groups G.

Regular representation for compact Lie groups on $\mathscr{F}(G)$

Notation

Let $\mathscr{F}(G)$ be the complex vector space of all continuous functions $f : G \to \mathbb{C}$.

To motivate the definition of a regular representation for a compact Lie group:

Regular representation for finite groups

- Recall in the regular representation of G, multiplication by b sends $\sum_{g \in G} c_g e_g$ to $\sum_{g \in G} c_g e_{bg}$. For $a \in G$, the coefficient of the e_g term is $c_{b^{-1}g}$.
- We can also describe the regular representation by functions $f : G \to \mathbb{C}$ where $f(g) = c_g$ are some complex consants. Because of the $c_{b^{-1}g}$ coefficient, we similarly define $b \in G$ acting on a function f by

$$r(b)f(g) = f(b^{-1}g), \quad g \in G.$$

Definition

The *regular representation* r of a compact Lie group G on the complex vector space $\mathscr{F}(G)$ has the action $r(b)f = (g \mapsto f(b^{-1}g))$, $f \in \mathscr{F}(G)$, $b, g \in G$.

Invariant measure and integral I

For averaging, we will need integration, and for integration, we will need a measure.

Definition

For a set X and a set of subsets Σ_X, a *measure* is a function $\mu : \Sigma_X \to \mathbb{R}^+$ satisfying $\mu(\emptyset) = 0$ and *countable additivity*: for a countable collection of pairwise disjoint sets $\{E_k\}_{k=1}^{\infty}$ in Σ_X, $\mu(\bigcup_{k=1}^{\infty} E_k) = \sum_{k=1}^{\infty} \mu(E_k)$.

Examples:

- The *Lebesgue measure* on $\mathbb{R}$ assigns to an interval $[a, b]$ its length $b - a$.
- The *probability measure* on a set of events assigns the value 1 to the entire (sample) space.
- The *Haar measure* on a compact Lie group...

Invariant measure and integral II

The Haar measure is our key tool.

Notation warning

If G has dimension n, we will often represent measures using differential n-forms Ω. The measure associates a subset S to the integral $\int_G 1_S \Omega$.

Morally important fact ([Ser77] §4 and [BD95] §1)

There exists a unique measure dg on a compact Lie group G satisfying

1. Invariance of dg under left translation: $\int_G f(g)\,dg = \int_G f(hg)\,dg$, $f \in \mathscr{F}(G)$, $h \in G$.
2. Unit volume of G: $\int_G 1\,dg = 1$.

Definition

- A measure dg on the compact Lie group G satisfying (1) and (2) is called a *Haar measure*.
- The integral $\int f\,dg$ is called an *invariant integral*.

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2. Unit volume of G: $\int_G 1\,dg = 1$.

Note that $r(b)f$ and f yield the same values (our precursor to an invariant inner product!):

$$\int_G (r(b)f)(g)\,dg = \int_G f(b^{-1}g)\,dg = \int_G f(g)\,dg.$$

Now our goal will be to explicitly compute integrals $\int_G f(g)\,dg$ for example cases G.

Example of Haar measure I

The circle S^1

- We construct the Haar measure on the multiplicative circle group S^1. Consider the function f from $[0, 2\pi]$ onto S^1 given by $f(t) = e^{it}$. Then define the Haar measure μ by
$$\mu(S) = \frac{1}{2\pi} m(f^{-1}(S)),$$
where m is the Lebesgue measure (length) of a set in $[0, 2\pi]$. The Haar measure has total volume 1 because $\mu(S^1) = \frac{1}{2\pi} 2\pi = 1$.
- The measure is invariant because an action in S^1 corresponds to a translation in $[0, 2\pi]$, which is invariant under m.

Measure as a differential form

The circle S^1 of radius 1 is isomorphic to the additive group $\mathbb{R}/(2\pi)$. We can also define the Haar measure as a differential 1-form $\Omega = \frac{1}{2\pi} dx$, where the length of the set $S \subset \mathbb{R}/(2\pi)$ is
$$\int_{\mathbb{R}/(2\pi)} 1_S \Omega = \int_S \frac{1}{2\pi} dx.$$

Example of Haar measure II

The multiplicative group $\mathbb{R}^+$

- Define the Haar measure μ on $G = \mathbb{R}^+$ by
$$\mu(S) = \int_S \frac{dt}{t},$$
or in terms of differential forms, $\mu = dt/t$. If S is an interval $[a, b]$, then $\mu([a, b]) = \log b - \log a = \log(b/a)$.
- Invariance: let g act on S by $gS = \{gs : s \in S\}$. Then if $S = [a, b]$, $gS = [ga, gb]$, and so we get
$$\mu(gS) = \log(gb/ga) = \log(b/a) = \mu(S).$$
- $\mathbb{R}^+$ is not compact, and so we don't expect finite volume: $\int_{\mathbb{R}^+} \frac{dt}{t} = +\infty$. It turns out we can define the Haar measure without the volume 1 condition on *locally compact* groups (every point has a compact neighborhood).

Integration of differential forms and pullback

Now we want to construct invariant integrals without any lucky guesses. For that we'll need a quick refresher on integration of differential forms.

Definition ([BT82] §1.3)

For an n-form $\Omega = f\, dx_1 \wedge \cdots \wedge dx_n$ defined on a compact Lie group $G \subset \mathbb{R}^n$, we define

$$\int_G \Omega = \int_G f\, dx_1 dx_2 \cdots dx_n.$$

Differential forms are convenient gadgets because they work well with a change of coordinates.

- Informally if $\phi : M \to M$ is a differentiable change of coordinates, its *pullback* turns a k-form ω in coordinates *after* ϕ into another k-form $\phi^*(\omega)$ in coordinates *before* ϕ.

Pullback is very nice because of these properties:

$$\phi^*(c_1\omega_1 + c_2\omega_2) = c_1\phi^*(\omega_1) + c_2\phi^*(\omega_2), \qquad \phi^*(\omega \wedge \sigma) = \phi^*(\omega) \wedge \phi^*(\sigma),$$
$$d\phi^*(\omega) = \phi^*(d\omega), \qquad \phi^*(f) = f \circ \phi.$$

For example, if $\phi : (u, v) \to (x, y)$, $x = u^2$ and $y = uv$ is a change of coordinates on $\mathbb{R}^2$, then

$$\begin{aligned}\phi^*(x^2\, dx \wedge dy) &= (\phi^*(x))^2\, d\phi^*(x) \wedge d\phi^*(y) = (u^2)^2\, d(u^2) \wedge d(uv)\\ &= u^4 \cdot (2u\, du) \wedge (v\, du + u\, dv) = 2u^6\, du \wedge dv.\end{aligned}$$

Invariant measures give invariant integrals

Let $L_b : G \to G$ be left multiplication by b^{-1}. Our key fact regarding out Haar measure is

Proposition

If we have an invariant Haar measure Ω on G with respect to left multiplication, i.e. $L_b^*\Omega = \Omega$, then we have an invariant integral on G.

Proof

Since $L_b^* f(g) = f \circ L_b(g) = f(b^{-1}g)$,

$$\int_G f(b^{-1}g)\,\Omega = \int_G L_b^* f\,\Omega$$

It is a fact from analysis that $\int L_b^*(f\Omega) = \int f\Omega$, see [Gun18] §(7.13). Then $L_b^*\Omega = \Omega$ gives

$$\int_G L_b^* f\Omega = \int_G L_b^* f\, L_b^*\Omega = \int_G L_b^*(f\Omega) = \int_G f\Omega.$$

If we have a matrix representation A of the group G on $\mathbb{R}^k$ or $\mathbb{C}^k$, we can prove that $A^{-1}dA$ is a matrix of linear differential forms invariant under L_b^*:

$$L_b^*(A^{-1}\,dA) = (L_b^*(A))^{-1}\,d(L_b^*(A)) = (B^{-1}A)^{-1}\,d(B^{-1}A) = A^{-1}BB^{-1}\,dA = A^{-1}\,dA.$$

SU(2) Example

Now we will compute the Haar measure for SU(2) just like in [Ste94] §4.1.

The special unitary group SU(2)

We can write each element $A \in \mathrm{SU}(2)$ as a matrix

$$A = \begin{pmatrix} \overline{\alpha} & -\beta \\ \overline{\beta} & \alpha \end{pmatrix} \text{ such that } |\alpha|^2 + |\beta|^2 = 1.$$

We compute $A^{-1}\,dA$ to get our invariant Haar measure.

$$dA = \begin{pmatrix} d\overline{\alpha} & -d\beta \\ d\overline{\beta} & d\alpha \end{pmatrix} \text{ and } A^{-1} = A^* = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix},$$

so

$$A^{-1}dA = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \begin{pmatrix} d\overline{\alpha} & -d\beta \\ d\overline{\beta} & d\alpha \end{pmatrix} = \begin{pmatrix} \alpha d\overline{\alpha} + \beta d\overline{\beta} & -\alpha d\beta + \beta d\alpha \\ -\overline{\beta} d\overline{\alpha} + \overline{\alpha} d\overline{\beta} & \overline{\beta} d\beta + \overline{\alpha} d\alpha \end{pmatrix}.$$

Since $A^{-1}dA$ is a matrix of invariant forms, each real and imaginary matrix entry is invariant. Wedge products of invariant forms is invariant, so after a lot of computation

$$(\overline{\alpha} d\alpha + \overline{\beta} d\beta) \wedge (-\alpha d\beta + \beta d\alpha) \wedge (-\overline{\beta} d\overline{\alpha} + \overline{\alpha} d\overline{\beta}) = \frac{1}{\beta} d\alpha \wedge d\beta \wedge d\overline{\alpha}.$$

The special unitary group SU(2)

If we let $\alpha = w + iz$ and $\beta = x + iy$, we have $x^2 + y^2 + z^2 + w^2 = 1$. In terms of polar coordinates,

$$w = \cos\theta, \qquad z = \sin\theta\cos\psi,$$
$$x = \sin\theta\sin\psi\cos\phi, \qquad y = \sin\theta\sin\psi\sin\phi,$$

where $0 \leq \theta \leq \pi$, $0 \leq \psi \leq \pi$, and $0 \leq \phi \leq 2\pi$. Substitution gives

$$\frac{1}{\beta} d\alpha \wedge d\beta \wedge d\overline{\alpha} = -2\sin^2\theta \sin\psi \, d\theta \wedge d\psi \wedge d\phi$$

Scaling so that the total volume of the differential is 1, we find that our Haar measure is

$$\Omega = \frac{1}{2\pi^2} \sin^2\theta \sin\psi \, d\theta \wedge d\psi \wedge d\phi.$$

Our invariant integral is $\int_{\mathrm{SU}(2)} f\Omega$.

L_2 inner products

Given our invariant measure and integral, we can now construct an G-invariant inner product on $\mathscr{F}(G)$.

Definition

The L_2 *inner product* of two functions $f_1, f_2 \in \mathscr{F}(G)$ on a compact Lie group G is

$$(f_1, f_2)_G = \int_G f_1 \overline{f}_2 \Omega,$$

where Ω is the Haar measure of G.

The L_2 inner product is a Hermitian inner product: it maps into the complex numbers because continuous functions on a compact set are bounded, it is linear in the first slot and conjugate linear in the second slot and positive definite: note that

$$(f, f)_G = \int_G f \overline{f} \Omega > 0.$$

A function f such that $(f, f)_G < \infty$ is a *square-integrable* function. The space of all square integrable functions is $L^2(G)$, which we will encounter soon.

L_2 inner product for SU(2)

- The L_2 inner product for any two functions f_1, f_2 on SU(2) is

$$(f_1, f_2)_G = \int_G f_1 \overline{f}_2 \Omega = \frac{1}{2\pi^2} \int f_1(\theta, \psi, \phi) \overline{f_2(\theta, \psi, \phi)} \sin^2\theta \sin\psi \, d\theta \wedge d\psi \wedge d\phi,$$

- A *central* or *class* function $f \in \mathscr{F}(G)$ only depends on the conjugacy classes of G.
- Lemma: The conjugacy class of $A \in \mathrm{SU}(2)$ determines and is determined by $\operatorname{tr} A = \alpha + \overline{\alpha} = 2\cos\theta$.
  - Any element $A \in \mathrm{SU}(2)$ is diagonalizable by the Spectral Theorem (in fact any element of $\mathrm{SU}(n)$, $\mathrm{SO}(m)$, or $\mathrm{Sp}(2n)$ can be diagonalized because it is normal and a subgroup of $\mathrm{SU}(n)$).
  - Writing $A = \left(\begin{smallmatrix} \overline{\alpha} & -\beta \\ \overline{\beta} & \alpha \end{smallmatrix}\right)$, the characteristic polynomial of A is

    $$\det(\lambda I - A) = \lambda^2 - (\alpha + \overline{\alpha})\lambda + \alpha\overline{\alpha} + \beta\overline{\beta} = \lambda^2 - (\alpha + \overline{\alpha})\lambda + 1.$$

    The trace is preserved under conjugation, so A's conjugacy class determines the trace of A. Recalling that the real part of α is $\cos\theta$, we find $\operatorname{tr} A = \alpha + \overline{\alpha} = 2\cos\theta$.
  - Now we show the trace determines the conjugacy class. Fix a trace T. Diagonal matrices in SU(2) with trace T have diagonal entries with product 1 and sum T, so they have characteristic polynomial $\lambda^2 - T\lambda + 1$. Call the roots of this polynomial γ_1 and γ_2. If $\gamma_1 \neq \gamma_2$, diagonal matrices with this characteristic polynomial are either $\operatorname{diag}(\gamma_1, \gamma_2)$ or $\operatorname{diag}(\gamma_2, \gamma_1)$, which are conjugate by $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. If $\gamma_1 = \gamma_2$, then $D = \lambda I$, which has conjugacy class $\{\lambda I\}$.
    Therefore, the conjugacy class is determined by the trace.

L_2 inner product for central functions in SU(2)

- The L_2 inner product for any two functions f_1, f_2 on SU(2) is

$$(f_1, f_2)_G = \int_G f_1 \overline{f}_2 \Omega = \frac{1}{2\pi^2} \int f_1(\theta, \psi, \phi) \overline{f_2(\theta, \psi, \phi)} \sin^2 \theta \sin \psi \, d\theta \wedge d\psi \wedge d\phi,$$

- A *central* or *class* function $f \in \mathscr{F}(G)$ only depends on the conjugacy classes of G.
- Lemma: The conjugacy class of $A \in \mathrm{SU}(2)$ determines and is determined by $\operatorname{tr} A = \alpha + \overline{\alpha} = 2\cos\theta$.
- So a function f is central if and only if it depends on $\cos\theta$.
- The L_2 inner product for central functions is

$$(f_1, f_2)_G = \frac{2}{\pi} \int_0^{\pi} f_1(\theta) \overline{f_2(\theta)} \sin^2 \theta \, d\theta.$$

by integrating over $0 \leq \psi \leq \pi$ and $0 \leq \phi \leq 2\pi$.

We will build to our promised claim of finding the irreducible representations of a compact Lie group. We will need some analogues to representations of finite groups that we learned in [FH91] §1 and §2.

Definitions

- A *representation* of a compact group G on a complex vector space V is a homomorphism $\rho : G \to \mathrm{Hom}(V, V)$ such that the map $G \times V \to V$ sending $(g, v) \to \rho(g)v$ is continuous.
- V is *topologically irreducible* if the only closed invariant (under the action of G) subspaces of V are V and $\{0\}$.

- The definition holds for *topological groups*: groups that have a topology on which group multiplication and inversion are continuous.
- Since the group action is a continuous homomorphism, we need to specify a topology on V.
  - When V is finite dimensional, it can be identified with $\mathbb{C}^N$ by a choice of basis. There is a standard norm, metric, and topology on $\mathbb{C}^N$.
  - In the case $V = \mathscr{F}(G)$, we can make V into a normed space by setting $||f|| = \sqrt{(f, f)_G}$. This gives $\mathscr{F}(G)$ a topology by the same principle.
- In all cases, $\rho(G) \subset \mathrm{GL}(V)$, because $\rho(g)$ is invertible: $\rho(g)\rho(g^{-1}) = \rho(e) = I$.

Definition

The *character* of a finite-dimensional representation V is a function $\chi_V : G \to \mathbb{C}$ that takes $g \in G$ to $\operatorname{tr} \rho(g)$.

Properties of the character, just like in the finite dimensional case:

- χ_V is a central (class) function.
- There is an inner product on characters, $(\chi_1, \chi_2)_G = \int_G \chi_1(g)\overline{\chi_2(g)}\, dg$. This is the inner product in $\mathscr{F}(G)$.
- V is irreducible if and only if $(\chi_V, \chi_V) = 1$.
- If V and W are representations of G with characters χ_V and χ_W,

$$\chi_{V \oplus W} = \chi_V + \chi_W, \qquad \chi_{V \otimes W} = \chi_V \cdot \chi_W,$$
$$\chi_{V^*} = \overline{\chi}_V, \qquad \chi_V(1) = \dim V.$$

The irreducible representations lying inside $\mathscr{F}(G)$

Irreducible representations lying inside $\mathscr{F}(G)$

Let V be finite dimensional and topologically irreducible. Fix $l \in V^*$. The map

$$\begin{aligned} \phi_l &: V \to \mathscr{F}(G) \\ &: x \mapsto f_x^l = \langle \rho(\cdot)^{-1} x, l \rangle \end{aligned}$$

is a map of reprentations because it is G-linear: for any $a \in G$,

$$\begin{aligned} [\phi_l(\rho(g)x)](a) = f^l_{\rho(g)x}(a) = \langle \rho(a)^{-1}\rho(g)x, l\rangle = \langle \rho(g^{-1}a)^{-1}x, l\rangle \\ = f_x^l(g^{-1}a) = r(g)f_x^l(a) = [r(g)\phi_l(x)](a). \end{aligned}$$

Since V is topologically irreducible, note that $\ker \phi_l = \phi_l^{-1}(\{0\})$ is closed and invariant, so ϕ_l must be injective. Therefore, $\phi_l(V) \cong V$ and $\phi_l(V) \subset \mathscr{F}(G)$.

We will get to know more about the multiplicities of the irreducible representations lying inside a space similar to $\mathscr{F}(G)$ in our big theorem. We will need a few more terms.

Definition

Pick a function $f \in \phi_l(V)$. If the set of all $r(a)f$ lie in a finite dimensional subspace of $\mathscr{F}(G)$, then f is called a *representative function*.

Completing $\mathscr{F}(G)$

Let G be a compact Lie group.

- As we saw, the space $\mathscr{F}(G)$ came with a Hermitian inner product, called the L_2 inner product:
$$(f_1, f_2)_G = \int_G f_1(g)\overline{f_2(g)}\,dg.$$
This makes $\mathscr{F}(G)$ into a *pre-Hilbert* space, or inner product space. The $\mathscr{F}(G)$ is necessarily a normed space, $||f|| = \sqrt{(f,f)_G}$, and hence a metric space, $\rho(f,g) = ||f - g||$.
- However, $\mathscr{F}(G)$ is not *complete*: not every Cauchy sequence with respect to this metric converges inside $\mathscr{F}(G)$.
- When we *complete* $\mathscr{F}(G)$ we create a space that contains the limit of every Cauchy sequence and that contains $\mathscr{F}(G)$ as a dense subspace. It turns out, $\overline{\mathscr{F}(G)} = L^2(G)$!
- The regular representation on $\mathscr{F}(G)$ extends to one on $L^2(G)$, and it actually is a *unitary representation*: a representation where $(r(g)f_1, r(g)f_2)_G = (f_1, f_2)_G$ for all $g \in G$ and $f_1, f_2 \in L^2(G)$.
  - We can see that $\mathscr{F}(G)$ is a unitary representation: for two continuous functions $f_1, f_2 \in \mathscr{F}(G)$, we have that $f_1\overline{f_2}$ is continuous, so $(r(b)f_1, r(b)f_2)_G = \int_G f_1(b^{-1}g)\overline{f_2(b^{-1}g)}\,dg = \int_G (f_1\overline{f_2})(b^{-1}g)\,dg = \int_G (f_1\overline{f_2})(g)\,dg = (f_1, f_2)_G$.

The main theorem

Peter-Weyl theorem

Let G be a compact group.

1. The representative functions are dense in $L^2(G)$.
2. The space $L^2(G)$ decomposes into a Hilbert space direct sum of irreducible representations of G, each of which is finite dimensional.
   - The *Hilbert space direct sum* is a countable sum of subspaces, each of which is orthogonal to every other via our L_2 inner product.
3. Every irreducible representation is finite dimensional.
4. Each irreducible representation of G occurs in $L^2(G)$ with a multiplicity equal to its dimension.
5. Any unitary representation of G on any Hilbert space decomposes into a Hilbert space direct sum of finite-dimensional irreducible representations.
6. The irreducible characters form an orthonormal basis of the Hilbert space of square integrable *central* functions.

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