

DIAGONALIZABLE OPERATORS EXERCISE

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ABSTRACT. One of my favorite linear algebra exercises.

1. INTRODUCTION

An *operator* on a finite dimensional vector space V over a field k is a linear function $\varphi : V \rightarrow V$. An operator is *diagonalizable* if there exists a basis \mathcal{B} of V consisting of eigenvectors: for every $v \in \mathcal{B}$, $\varphi(v) = \lambda v$ for some constant $\lambda \in k$. A pair of operators φ, ψ *commute* if $\varphi \circ \psi = \psi \circ \varphi$. A pair of operators φ, ψ is *simultaneously diagonalizable* if there exists a basis \mathcal{B} of V consisting of eigenvectors for both φ and ψ . Notice that if φ, ψ are simultaneously diagonalizable, then they commute because they commute on the basis of eigenvectors. We will show that the converse is also true:

Exercise 1.1. *If φ, ψ are diagonalizable operators on a finite dimensional vector space V , then they are simultaneously diagonalizable if and only if they commute.*

Let's start the proof. Let $\lambda_1, \dots, \lambda_m$ be the distinct eigenvalues of φ . Since φ is diagonalizable, there exists a decomposition of V into eigenspaces $V_{\lambda_i} = \{v \in V : \varphi(v) = \lambda_i v\}$,

$$V = V_{\lambda_1} \oplus V_{\lambda_2} \oplus \dots \oplus V_{\lambda_m}. \quad (1)$$

Note that for an eigenspace V_{λ} of φ , if we take an eigenvector $v \in V_{\lambda}$, then $\varphi(\psi(v)) = \psi(\varphi(v)) = \psi(\lambda v) = \lambda \psi(v)$. This means that $\psi(V_{\lambda}) \subseteq V_{\lambda}$, so it makes sense to restrict the operator φ to the space V_{λ} . Once we show that $\psi|_{V_{\lambda_i}}$ is diagonalizable for every eigenspace V_{λ_i} , we may take a basis \mathcal{B}_i of eigenvectors for ψ in V_{λ_i} . Since the vectors in \mathcal{B}_i belong to V_{λ_i} , they are eigenvectors for φ too. Then the basis $\mathcal{B} = \bigcup_i \mathcal{B}_i$ simultaneously diagonalizes φ and ψ . Therefore, it suffices to prove the following statement.

Lemma 1.2. *Let W be a subspace of a finite dimensional vector space V . If φ is a linear operator on V such that φ is diagonalizable and $\varphi(W) \subseteq W$, then $\varphi|_W$ is diagonalizable.*

We provide two proofs of Lemma 1.2. One argument is elementary, in the sense that it uses no further definitions than the ones given above, but there is a clever induction argument. The other uses the minimal polynomial. Then we give some extensions of Exercise 1.1 and Lemma 1.2.

2. ELEMENTARY PROOF OF LEMMA 1.2

As above, let $\lambda_1, \dots, \lambda_m$ be the distinct eigenvalues of φ , and decompose V into the eigenspaces of φ as in equation (1). It would be nice if we can choose an basis of eigenvectors for W from the decomposition of V in (1) because the bases of $W \cap V_{\lambda_i}$ would be eigenvectors of $\varphi|_W$ and we obtain a basis for W by concatenating the bases of

$W \cap V_{\lambda_i}$. It turns out that this is true. Specifically, we will show that we can decompose W into a direct sum of its factors

$$W = (W \cap V_{\lambda_1}) \oplus (W \cap V_{\lambda_2}) \oplus \cdots \oplus (W \cap V_{\lambda_m}). \quad (2)$$

Since $W \cap V_{\lambda_i} \subseteq W$, it suffices to show the inclusion $W \subseteq \bigoplus_i W \cap V_{\lambda_i}$. For that, we show that we can decompose $w = v_1 + v_2 + \cdots + v_m$ for eigenvectors $v_i \in W \cap V_{\lambda_i}$. We prove the following statement by induction on m : if m is minimal such that $w = v_1 + \cdots + v_m$ with $v_i \neq 0$, then $v_i \in W \cap V_{\lambda_i}$. Note that since φ is diagonalizable, any vector can be decomposed into a sum of eigenvectors, so such an m must exist.

If $w = v_i$ where $v_i \in V_{\lambda_i}$, then $w \in W$ implies $v_i \in W \cap V_{\lambda_i}$. Suppose $w = v_1 + \cdots + v_k \in W$, where $v_i \in V_{\lambda_i}$ and are nonzero. Then

$$\varphi(w) - \lambda_1 w = (\lambda_2 - \lambda_1)v_2 + (\lambda_3 - \lambda_1)v_3 + \cdots + (\lambda_k - \lambda_1)v_k. \quad (3)$$

Since $\varphi(w) \in W$, we know that $\varphi(w) - \lambda_1 w \in W$. Thus the right side is a vector in W which is a sum of $k - 1$ eigenvectors the inductive hypothesis on (3), $v_2, \dots, v_k \in W$. Also, $v_1 = w - (v_2 + \cdots + v_k) \in W$. This completes the induction.

Remark. When considering the decomposition of any $w = v_1 + \cdots + v_m \in W$, we can automatically show $v_i \in W$ for every $i = 1, \dots, m$ by considering the operator

$$\pi_i = \prod_{j \neq i} \frac{\varphi - \lambda_j}{\lambda_j - \lambda_i},$$

where product in this case means composition of operators. The operator is a polynomial in φ , so $\pi_i(W) \subseteq W$. Notice that $\pi_i(v_i) = v_i$ and $\pi_i(v_j) = 0$ for all $j \neq i$. So, $\pi_i(w) = \sum_{j=1}^m \pi_i(v_j) = v_i$, implying $v_i \in W$.

3. PROOF OF LEMMA 1.2 VIA THE MINIMAL POLYNOMIAL

The *minimal polynomial* of an operator $\varphi : V \rightarrow V$ is a polynomial $m_\varphi(x) \in k[x]$ such that m_φ is monic and it has least degree for which $m_\varphi(\varphi) = 0$. We will use the basic properties of the minimal polynomial, see for example Keith Conrad's notes [1].

Theorem 3.1. *Let $\varphi : V \rightarrow V$ be an operator on V . A polynomial $p(x) \in k[x]$ satisfies $p(\varphi) = 0$ if and only if $m_\varphi(x) \mid p(x)$.*

Theorem 3.2. *Let $\varphi : V \rightarrow V$ be an operator on V . Then φ is diagonalizable if and only if m_φ can be written as a product of linear factors in $k[x]$ and m_φ has distinct roots.*

Using these properties, we can give a shorter proof of Lemma 1.2.

Proof. Let $\phi = \varphi|_W$. From Theorem 3.1, $m_\varphi(\varphi) = 0$, which means the restriction of m_φ to W is 0. Then

$$m_\varphi(\phi) = m_\varphi(\varphi|_W) = m_\varphi(\varphi)|_W = 0.$$

So, $m_\phi \mid m_\varphi$. Since φ is diagonalizable, by Theorem 3.2, m_φ splits into a product of linear factors and has distinct roots. This means that the minimal polynomial of ϕ has no repeated factors and splits. Therefore, ϕ is diagonalizable. \square

4. EXTENSIONS

The following is an extension of Exercise 1.1.

Exercise 4.1. *Let $\mathcal{F} = \{\varphi_i\}_{i \in I}$ be a collection of commuting linear operators on a finite dimensional vector space V . If each φ_i is diagonalizable on V , then the operators in \mathcal{F} are simultaneously diagonalizable.*

See [1] for a proof that follows two steps: prove the statement for a finite number of operators inductively, and then prove the general statement by finding a basis for the subspace spanned by $\{\varphi_i\}_{i \in I}$ inside $\text{Hom}(V, V)$.

There is also an extension of Lemma 1.2 if we consider the induced mapping on the quotient space V/W , defined by $\bar{\varphi} : v + W \mapsto \varphi(v) + W$.

Lemma 4.2. *Let W be a subspace of a finite dimensional vector space V . If φ is a linear operator on V such that φ is diagonalizable and $\varphi(W) \subseteq W$, then $\varphi|_W$ and $\bar{\varphi}$ are diagonalizable.*

REFERENCES

- [1] Keith Conrad. The minimal polynomial and some applications. Online notes.