# The surgery exact triangle in Heegaard Floer homology

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# Declaration

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# Introduction

Heegaard Floer homology is a package of invariants for closed and oriented three-manifolds, originally defined by Peter Ozsváth and Zoltán Szabó [30, 31]. The surgery exact triangle is a powerful computational tool in Heegaard Floer homology. Given a knot $K$ in a closed three-manifold $Y$, the surgery exact triangle relates the Heegaard Floer homology of $Y$ with the Heegaard Floer homologies of a pair of surgered manifolds obtained from framings of $K$. Some of its immediate applications are examples many three-manifolds with the simplest possible Heegaard Floer invariants, called $L$-space. The maps in the surgery exact triangle can be interpreted in terms of cobordisms between three-manifolds, which can be used to define an invariant of closed, smooth four-manifolds [33]. The properties of the maps inside the surgery exact triangle can also be used [20, 34] to prove the Dehn surgery of the unknot, stating that if $K$ is a knot with the property that some surgery on $K$ is a lens space, then $K$ is the unknot. The goal of this thesis is to give the construction of Heegaard Floer homology, as well as the proof and some immediate applications of the surgery exact triangle.

The construction of Heegaard Floer homology starts with a connected closed oriented three-manifold $Y$. Any such manifold has a decomposition $Y = U_0 \cup_\Sigma U_1$, where $U_0$ and $U_1$ are handlebodies joined along their common boundary $\Sigma$, a closed genus $g$ surface. The invariants are defined by studying the $g$-fold symmetric product of $\Sigma$, the set of unordered $g$-tuples of points in $\Sigma$. The symmetric product is an even-dimensional manifold that can be endowed with a symplectic structure, i.e. has a closed and non-degenerate 2-form $\omega$, and the manifold contains a pair of Lagrangian submanifolds, which are half-dimensional submanifolds on which $\omega$ vanishes. Heegaard Floer homology is a variant of a homology theory defined for symplectic manifold and a pair of Lagrangian submanifolds constructed by Floer [12]. The generators correspond to intersection points of the Lagrangian submanifolds, and the boundary maps count pseudo-holomorphic disks with appropriate boundary conditions. This construction, known as Lagrangian Floer homology, has its roots in Morse homology [42], an illuminating way to understand the homology of a smooth manifold in terms of a a generic choice of a smooth function and a Riemannian metric on the manifold.

In Chapter 1, we construct Morse homology and Lagrangian Floer homology. In Chapter 2, we construct Heegaard Floer homology and set up the background necessary to show the surgery exact triangle. In Chapter 3, we move to the proof and immediate applications of the surgery exact triangle. This thesis draws from Ozsváth–Stipsicz–Szabó's book in preparation *Heegaard Floer homology* [29]. My intention is to make this thesis readable for anybody who is familiar with algebraic topology up to Poincaré duality and differential geometry up to geodesics and connections.

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# Chapter 1

# Morse theory to Floer homology

This chapter goes through two stories relevant to Heegaard Floer homology. The first is Morse theory, allowing one to analyze the topology of a finite-dimensional smooth manifold by studying differentiable functions on it. In Section 1.1, we define Morse functions and illustrate how they induce a handle decomposition of a manifold. We then define the Morse complex, which recovers the singular homology of a manifold from an analysis of gradient flowlines of a Morse function. The second story is Lagrangian Floer homology, developed by Floer [12] to answer a conjecture of Arnold. Arnold [1, page 419] conjectured that the number of fixed points of a Hamiltonian diffeomorphism of a symplectic manifold is at least the number of critical points of a Morse function on the manifold. In Section 1.2, we study a few basic properties symplectic manifolds and in Section 1.3 we outline Floer's approach to Arnold's conjecture.

## 1.1 Morse Theory

Let $M$ be a smooth manifold, and let $f : M \to \mathbb{R}$ be a smooth function on $M$. Recall that the point $p \in M$ is a critical point if the differential $df$ vanishes at $p$. The critical point $p$ is *non-degenerate* if in some local chart $x_1, \ldots, x_n$ of $p$, the Hessian matrix $(\frac{\partial^2 f}{\partial x_i \partial x_j}(p))$ is invertible.

**Definition 1.1.1.** A smooth function $f : M \to \mathbb{R}$ is *Morse* if each critical point of $f$ is non-degenerate. If $p$ is a non-degenerate critical point of $f$, its *index* $\lambda(p)$ is the maximal dimension of a subspace of $T_pM$ for which the Hessian is negative definite.

Many Morse functions exist on a given manifold. For example, if one embeds the manifold into Euclidean space $\mathbb{R}^n$, then for almost all $q \in \mathbb{R}^n$, the function $p \mapsto \|p - q\|^2$ is Morse, see [24, Theorem 6.6].

Particularly useful to us is the fact that a Morse function gives rise to a handlebody decomposition of $M$, which is a convenient way to study smooth manifolds. We now define the notion of a handle and a handlebody decomposition.

**Definition 1.1.2.** Suppose that $X$ is a smooth $n$-manifold with boundary. For an integer $k \in \{0, \ldots, n\}$, an *$n$-dimensional $k$-handle* $h$ is a copy of $D^k \times D^{n-k}$ that is attached to the boundary of $X$ along $(\partial D^k) \times D^{n-k}$ by a smooth embedding $\varphi : (\partial D^k) \times D^{n-k} \to \partial X$. The manifold after handle attachment $X_h = X \cup_\varphi h$ is a smooth $n$-manifold after rounding out the corners.

**Definition 1.1.3.** Let $X$ be a compact $n$-dimensional manifold with boundary decomposing as a disjoint union $\partial X = \partial_+ X \cup \partial_- X$ of two compact submanifolds. Suppose $X$ is oriented so that $\partial X = \partial_+ X \cup -\partial_- X$ in the boundary orientation, where $-\partial_- X$ is the manifold $\partial_- X$ with the reversed orientation. A *handle decomposition of* $X$ relative to $\partial_- X$ is a diffeomorphism of $X$ with a manifold obtained from $[0, 1] \times \partial_- X$ by attaching handles to $\{1\} \times \partial_- X$. If $\partial_- X = \emptyset$, then $X$ equipped with a handle decomposition is called a *handlebody.*

Let $f$ be a Morse function on a smooth closed $n$-manifold $X$. Let $X_t$ denote $f^{-1}((-\infty, t])$, so that $X_t$ is empty for large negative $t$ and $X_t = X$ for large positive $t$. Then $X$ inherits a handlebody decomposition from $f$, as follows compare [24, 29]. Let $t_1, t_2$ be a pair of regular values of $f$ with $t_1 < t_2$. If $f^{-1}([t_1, t_2])$ contains no critical point, then $X_{t_1}$ is diffeomorphic to $X_{t_2}$. If $f^{-1}([t_1, t_2])$ contains $m$ critical points of index $k$ all of which have the same value, then $X_{t_2}$ is constructed from $X_{t_1}$ by attaching $m$ disjoint smooth $n$-dimensional $k$-handles.

In fact, we can arrange for handles to be added in increasing order of index, by the following theorem. Call a Morse function $f$ is called *self-indexing* if for any critical point $p$, $f(p)$ is the index of $p$. A proof of the below theorem can be found in Smale [38], also see Milnor's book on the $h$-cobordism theorem [25].

**Theorem 1.1.4.** *Let $M$ be a compact smooth $n$-manifold with connected boundary. Then there exists a self-indexing Morse function on $M$. If $M$ is also closed, then the self-indexing Morse function can be chosen to have a unique local minimum and a unique local maximum.*

It is worth mentioning that Morse theory has many other applications to the topology of manifolds:

1. Morse originally applied his theory to geodesics to show that there are infinitely any pair of non-conjugate points on the $n$-sphere $S^n$ can be connected by infinitely many geodesics. See [24, Chapter 17] for a proof.

2. Morse theory can be used to prove the Lefschetz hyperplane theorem on the topology of complex projective varieties. Let $V \subseteq \mathbb{C}\mathrm{P}^n$ be an complex projective variety of complex dimension $k$. Let $P$ be a hyperplane in $\mathbb{C}\mathrm{P}^n$ which contains the singular points of $V$. Then the inclusion map $V \cap P \to V$ induces isomorphisms $H_i(V \cap P; \mathbb{Z}) \to H_i(V; \mathbb{Z})$ for $i < k-1$. Moreover, the induced homomorphism $H_{k-1}(V \cap P; \mathbb{Z}) \to H_{k-1}(V; \mathbb{Z})$ is onto. See [24, Chapter 7] for a proof.

3. Bott [6] applied Morse theory to the topology of Lie groups to prove a periodicity theorem for classical groups.

### 1.1.1 The Morse complex

Let $M$ be a finite dimensional manifolds equipped with a Morse function $f : M \to \mathbb{R}$ and a Riemannian metric $g$. For a pair of critical points $\mathbf{x}, \mathbf{y}$ of $f$, a *gradient flowline* from $\mathbf{x}$ to $\mathbf{y}$ is a path $\gamma : \mathbb{R} \to M$ such that $\lim_{t \to -\infty} \gamma(t) = \mathbf{x}$ and $\lim_{t \to +\infty} \gamma(t) = \mathbf{y}$ and $\gamma$ satisfies the gradient flow equation

$$\frac{d\gamma}{dt}(t) = (-\nabla_g f)_{\gamma(t)}.$$

One can collect all the gradient flowlines from $\mathbf{x}$ to $\mathbf{y}$ into a space $\mathcal{M}(\mathbf{x}, \mathbf{y})$. There is a natural $\mathbb{R}$-action on $\mathcal{M}(\mathbf{x}, \mathbf{y})$ by time translation: if $\gamma$ is a gradient flowline from $\mathbf{x}$ to $\mathbf{y}$ and $s$ is a real number, then the path $t \mapsto \gamma(t+s)$ is also a gradient flowline. Let $\widehat{\mathcal{M}}(\mathbf{x}, \mathbf{y})$ be the quotient of $\mathcal{M}(\mathbf{x}, \mathbf{y})$ by this $\mathbb{R}$-action. The beautiful fact is that under sufficiently generic conditions of the Riemannian metric $g$, these sets $\widehat{\mathcal{M}}(\mathbf{x}, \mathbf{y})$ turn out to be smooth finite-dimensional manifolds, as we will see below.

**Definition 1.1.5.** Let $\varphi^s : M \to M$ for $s \in \mathbb{R}$ denote the flow of the gradient of $f$. Define the *ascending manifold* at a critical point $p$ of $f$ to be

$$A(p) = \{x \in M \mid \lim_{x \to +\infty} \varphi^s(x) = p\}$$

and the *descending manifold* at $p$ to be

$$D(p) = \{x \in M \mid \lim_{x \to -\infty} \varphi^s(x) = p\}.$$

Call the pair $(f, g)$ *Morse–Smale* for all pairs of critical points $p, q$ of $f$, the ascending manifold $A(p)$ and the descending manifold $D(p)$ intersect transversely.

A theorem of Smale [38] states that given a Morse function $f$ on a closed manifold $M$, there exists a Riemannian metric $g$ on $M$ such that the pair $(f, g)$ is Morse–Smale. Below is the aforementioned theorem about the space of gradient flowlines, compare [2, Section 2.2] and [29, Theorem 5.1.2].

**Theorem 1.1.6.** *Let $M$ be a finite-dimensional closed manifold and let $f : M \to \mathbb{R}$ be a Morse function on $M$. Choose a Riemannian metric $g$ on $M$ such that the pair $(f, g)$ is Morse–Smale. Fix a pair of distinct critical points $\mathbf{x}$ and $\mathbf{y}$ of $f$ and consider the space $\widehat{\mathcal{M}}(\mathbf{x}, \mathbf{y})$ of gradient flowlines from $\mathbf{x}$ to $\mathbf{y}$ modulo time translation. Then $\widehat{\mathcal{M}}(\mathbf{x}, \mathbf{y})$ is a smooth manifold of dimension $\lambda(\mathbf{x}) - \lambda(\mathbf{y}) - 1$. In particular, if $\lambda(\mathbf{x}) = \lambda(\mathbf{y})$, the space $\widehat{\mathcal{M}}(\mathbf{x}, \mathbf{y})$ is empty.*

Now we proceed to define the Morse complex.

**Definition 1.1.7.** Let $M$ be a smooth closed manifold, let $f$ be a Morse function on $M$, and let $g$ be a Riemannian metric on $M$ such that the pair $(f, g)$ is Morse–Smale. The *Morse chain complex* $\mathrm{CM}(M, f, g)$ is the vector space over $\mathbb{Z}/2\mathbb{Z}$ generated by the critical points of $f$ and equipped with an endomorphism $\partial$ defined on the generators by

$$\partial \mathbf{x} = \sum_{\mathbf{y} \in \mathrm{Crit}(f) | \lambda(\mathbf{x}) - \lambda(\mathbf{y}) = 1} \#\widehat{\mathcal{M}}(\mathbf{x}, \mathbf{y}) \cdot \mathbf{y}.$$

Here, $\#\widehat{\mathcal{M}}(\mathbf{x}, \mathbf{y})$ is the parity of the number of elements in $\widehat{\mathcal{M}}(\mathbf{x}, \mathbf{y})$.

Having arrived at the definition, there are a few components to resolve.

1. First of all, one needs to show that the definition of $\partial \mathbf{x}$ is a finite sum. The moduli spaces $\widehat{\mathcal{M}}(\mathbf{x}, \mathbf{y})$ are zero-dimensional manifolds by Theorem 1.1.6, and they are in fact compact, by an argument that we sketch below. One can show that any sequence of gradient flowlines from $\mathbf{x}$ to $\mathbf{y}$ has a convergent subsequence in the $C^{\infty,\mathrm{loc}}$-topology to a *broken flowline* from $\mathbf{x}$ to $\mathbf{y}$, which is informally a collection of gradient flowlines starting at $\mathbf{x}$ and ending at $\mathbf{y}$. Note that the sum of the index differences of the gradient flowlines from $\mathbf{x}$ to $\mathbf{y}$ is still one, so if there is more than one gradient flowline, then the index difference of one component is nonpositive. However, there is no class of flowlines in with nonpositive index difference by Theorem 1.1.6, so the subsequence converges to an unbroken flowline.

2. The endomorphism $\partial$ satisfies $\partial^2 = 0$, making $\mathrm{CM}(M, f, g)$ into a chain complex with differential $\partial$. In this case, to show $\partial^2 \mathbf{x} = 0$, we examinine the space of gradient flowlines from $\mathbf{x}$ to another critical point $\mathbf{z}$ with $\lambda(\mathbf{x}) - \lambda(\mathbf{z}) = 2$. The boundary of this space $\widehat{\mathcal{M}}(\mathbf{x}, \mathbf{z})$ corresponds to the terms appearing in the coefficient of $\mathbf{z}$. The key insight here is that by a gluing argument, $\widehat{\mathcal{M}}(\mathbf{x}, \mathbf{z})$ is a compact one-dimensional manifold, so its boundary is a zero-manifold contains an even number of boundary components.

3. The Morse complex $\mathrm{CM}(M, f, g)$ is independent of the choice of Morse function $f$ and Riemannian metric $g$ on $M$, and is in fact isomorphic to the singular homology

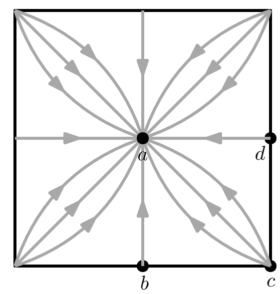

Figure 1.1: Gradient flow on the torus.

$H_*(M; \mathbb{Z}/2\mathbb{Z})$. Relying on the formulation of singular homology adapted to CW-complexes, one can directly prove that $\mathrm{CM}(M, f, g)$ is isomorphic to the singular homology of $M$ with $\mathbb{Z}/2\mathbb{Z}$ coefficients. Alternatively, one can prove independence of the metric by connecting any pair of metrics $g_0, g_1$ by a path of metrics $\{g_t\}_{t\in[0,1]}$ and finding maps between $\mathrm{CM}(M, f, g_0)$ and $\mathrm{CM}(M, f, g_1)$ which count certain time-dependent gradient flowlines. To show independence of the Morse function, one similarly interpolates between a pair of Morse functions. This latter approach generalizes better to showing the invariance of Lagrangian Floer homology, a construction outlined later in this chapter.

We conclude this section with the following example.

**Example 1.1.8.** Let the ambient manifold be $T^2$, thought of as a quotient of $[0, 1] \times [0, 1]$, and consider the function $f : T^2 \to \mathbb{R}$ given by

$$f(x, y) = \cos 2\pi x + \sin 2\pi x.$$

This function is Morse, and has a single critical point $a$ of index zero, two critical points $b, c$ of index 1, and one critical point $d$ of index two. Giving the torus the flat metric, the gradient flowlines are as drawn in Figure 1.1. We conclude that

$$\partial a = 0, \qquad \partial b = \partial c = 2a = 0, \qquad \partial d = 0,$$

so the Morse homology groups $\mathrm{CM}_0 = \mathbb{Z}/2\mathbb{Z}$, $\mathrm{CM}_1 = (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})$, and $\mathrm{CM}_2 = \mathbb{Z}/2\mathbb{Z}$.

These are the singular homology groups of $T^2$, as expected.

## 1.2 Symplectic geometry

This section starts with the properties of symplectic vector spaces, which will be the structure of the tangent spaces of a symplectic manifold. On the way, we study the Lagrangian Grassmannian and the Maslov cycle.

### 1.2.1 Symplectic vector spaces

A vector space $V$ over $\mathbb{R}$ equipped with a non-degenerate anti-symmetric bilinear form $\omega : V \times V \to \mathbb{R}$ is called a *symplectic vector space.* Any such vector space has even dimension. For example, $\mathbb{R}^{2n}$ equipped with the basis $\{e_i, f_i\}_{i=1}^n$ with the *standard symplectic form* $\Omega_0$ determined by $\Omega_0(e_i, e_j) = \Omega_0(f_i, f_j) = 0$ and $\Omega_0(e_i, f_j) = \delta_{ij}$ is a symplectic vector space. For any symplectic vector space $V$ of dimension $2n$, there is a choice of basis $\{u_i, v_i\}_{i=1}^n$ for which $\omega(u_i, u_j) = \omega(v_i, v_j) = 0$ and $\omega(u_i, v_j) = \delta_{ij}$. Equivalently, there is a *symplectomorphism* $\varphi : (V^{2n}, \omega) \to (\mathbb{R}^{2n}, \Omega_0)$, which is a linear isomorphism for which $\varphi^*\Omega_0 = \omega$. See [7, Chapter 1].

One can enhance a symplectic vector space to have two other compatible structures: a complex structure and a positive-definite bilinear form, defined below. Suppose that $V$ is an even-dimensional vector space. A *complex structure* on $V$ is a linear endomorphism $J$ so that $J \circ J = -\mathrm{Id}_V$. A non-degenerate anti-symmetric bilinear form $\omega$ and a complex structure $J$ on $V$ are *compatible* if $\omega(Ju, Jv) = \omega(u, v)$ for all $u, v \in V$ and the symmetric form $g(u, v) = \omega(u, Jv)$ is positive definite, meaning that $\omega(v, Jv) > 0$ for all nonzero $v \in V$; say that $(J, \omega, g)$ form a *compatible triple.* The standard symplectic vector space $(\mathbb{R}^{2n}, \Omega_0)$ also has a *standard complex structure* $J_0$ determined by $J_0 e_i = f_i$ and $J_0 f_i = -e_i$ in the basis $\{e_i, f_i\}_{i=1}^n$ of $\mathbb{R}^{2n}$. Let $g_0$ be the induced form defined by $g_0(u, v) = \Omega_0(u, J_0 v)$. By pulling back $g_0$ and $J_0$ under the symplectomorphism $\varphi$, the symplectic structure $\omega$ can be enhanced to a compatible triple.

Call a linear subspace $\Lambda$ of a symplectic vector space $(V^{2n}, \omega)$ *Lagrangian* if $\dim \Lambda = n$

and $\omega|_\Lambda = 0$. The *Lagrangian Grassmannian* $LGr(V, \omega)$ is the space of Lagrangian subspaces of $(V^{2n}, \omega)$.

**Proposition 1.2.1.** *Let $(V, \omega)$ be a symplectic vector space of dimension $2n$. The Lagrangian Grassmannian $LGr(V, \omega)$ is is a compact smooth manifold diffeomorphic to $U(n)/O(n)$.*

*Proof.* Fix a complex structure $J$ on $V$ compatible with $\omega$, and let $g(u, v) = \omega(u, Jv)$ be the induced positive definite symmetric bilinear form. One can then define a hermitian form $\langle \cdot, \cdot \rangle$ on $V$ via

$$\langle v, w \rangle = g(v, w) + i\omega(v, w)$$

for $v, w \in V$. By the Gram–Schmidt process, any $n$-dimensional subspace $\Lambda$ in $V^{2n}$ can be given an orthonormal basis $e_1, \ldots, e_n$ with respect to $g$. Then $\Lambda$ is Lagrangian if and only if $\omega(e_i, e_j) = 0$ for all $i, j \in \{1, \ldots, n\}$, i.e. if $e_1, \ldots, e_n$ is an orthonormal basis for $V$ with respect to the hermitian form $\langle \cdot, \cdot \rangle$. Such bases are parametrized by elements of $U(n)$, and two orthonormal bases represent the same subspace if and only if they can be transformed onto one another by an element of $O(n)$. This gives the desired diffeomorphism. $\square$

**Definition 1.2.2.** Let $(V, \omega)$ be a symplectic vector space, and let $\Lambda_0$ be a Lagrangian subspace of $V$. Define the *Maslov cycle relative to* $\Lambda_0$ to be the space

$$\Sigma(\Lambda_0) = \{L \in LGr(V, \omega) \mid L \cap \Lambda_0 \neq \emptyset\}.$$

The Maslov cycle decomposes as

$$\Sigma(\Lambda_0) = \bigcup_{k=1}^{n} \Sigma_k(\Lambda_0)$$

where $\Sigma_k(\Lambda_0) = \{L \in LGr(V, \omega) \mid \dim(L \cap \Lambda) = k\}$. Now, choose a symplectomorphism between $(V^{2n}, \omega)$ and $(\mathbb{R}^{2n}, \Omega_0)$ such that $\Lambda_0$ in $V^{2n}$ maps to $\mathbb{R}^n = \operatorname{span}\{e_1, \ldots, e_n\}$ in $\mathbb{R}^{2n} = \operatorname{span}\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$. This choice specifies an identification of $LGr(V, \omega)$ with $U(n)/O(n)$ under which $\Lambda_0$ corresponds to the identity coset and $\Sigma_k(\Lambda_0)$ corresponds to the set $\Sigma_k = \{A \in U(n) \mid \dim(A(\mathbb{R}^n) \cap \mathbb{R}^n) = k\}$.

### 1.2.2 Symplectic manifolds

Let $M$ be a $2n$-dimensional manifold. A *symplectic form* on $M$ is a smooth 2-form $\omega \in \Omega^2(M;\mathbb{R})$ that is closed, i.e. $d\omega = 0$, and is non-degenerate, i.e. the $n$-fold wedge product of $\omega$ vanishes nowhere. A *symplectic manifold* is a pair $(M,\omega)$, where $\omega$ is a symplectic form on $M$. A diffeomorphism $\varphi : (M,\omega) \to (M',\omega')$ is a *symplectomorphism* if $\varphi^*\omega' = \omega$, where $\varphi^*\omega'$ is a 2-form on $M$ defined on a pair of tangent vectors $u, v \in T_pM$ by $(\varphi^*\omega')_p(u,v) = \omega'_{\varphi(p)}(d\varphi_p(u), d\varphi_p(v))$.

Products of symplectic manifolds are symplectic, by the following proposition.

**Proposition 1.2.3.** *If $(M,\omega), (N,\nu)$ are symplectic manifolds of the same dimension, and let $\pi_1 : M \times N \to M$ and $\pi_2 : M \times N \to N$ be projection maps. Then $M \times N$ is a symplectic manifold with symplectic form $\pi_1^*\omega + \pi_2^*\nu$.*

*Proof.* Since

$$d(\pi_1^*\omega + \pi_2^*\nu) = \pi_1^*(d\omega) + \pi_2^*(d\nu) = 0 + 0 = 0,$$

the form $\pi_1^*\omega + \pi_2^*\nu$ is closed. Moreover,

$$(\pi_1^*\omega + \pi_2^*\nu)^{2n} = \binom{2n}{n}(\pi_1^*\omega)^n \wedge (\pi_2^*\nu)^n$$

is nonzero as $\omega$ and $\nu$ are volume forms. Thus $\pi_1^*\omega + \pi_2^*\nu$ is nondegenerate. $\square$

An *almost complex structure* $J$ on a smooth manifold is a bundle automorphism $J : TX \to TX$ such that $J \circ J = -\operatorname{Id}_{TX}$. For a symplectic manifold $(M,\omega)$, an almost complex structure $J$ is *compatible* with the symplectic form $\omega$ if at each $p \in M$ and $v, w \in T_pM$, we have that $\omega(v,w) = \omega(Jv, Jw)$, and for any nonzero $v \in T_pM$, $\omega(v, Jv)$ is positive. A weaker condition than compatibility is tameness. An almost-complex structure $J$ on a symplectic manifold $(M,\omega)$ is *$\omega$-tame* if for any $p \in M$ and for any nonzero $v \in T_pM$, $\omega(v, Jv) > 0$. A symplectic manifold always admits $\omega$-tame almost-complex structures; in fact, the space of $\omega$-tame almost-complex structures on $(M,\omega)$ is contractible, see [29, Theorem 4.3.19] or [23, Theorem 4.1.1]. We will need the notion of $\omega$-tameness later in this section.

We will also work with complex manifolds and Kähler manifolds. A *complex manifold* $X$ is a smooth manifold equipped with local charts modeled on $\mathbb{C}^n$, whose transition functions are holomorphic. The local charts gives the underlying smooth manifold a *complex structure*. A complex structure on $X$ induces an almost complex structure on $X$ by multiplication by $i$ on each tangent space. A metric $g$ on a complex manifold $(X, J)$ is a *Kähler metric* if it is compatible with $J$, in the sense that $g(u, v) = g(Ju, Jv)$ for any $u, v \in T_pX$ and $p \in X$, and the 2-form $\omega(u, v) = g(Ju, v)$ is a symplectic form. The symplectic form $\omega$ is called a *Kähler form* and the manifold $X$ with the data $(\omega, g, J)$ is called a *Kähler manifold*.

## 1.3 Lagrangian Floer homology

Lagrangian Floer homology can be viewed as an obstruction to making a pair of Lagrangian submanifolds $L_0$ and $L_1$ in a symplectic manifold disjoint. The generators of the complex are the intersection points $L_0 \cap L_1$. The differential informally counts disks between pairs of intersection points satisfying certain boundary conditions. We define these notions presently.

Let $M$ be a symplectic manifold and let $L_0, L_1$ be a pair of Lagrangian submanifolds in $M$ which intersect transversally. Given two points $\mathbf{x}, \mathbf{y} \in L_0 \cap L_1$, a *Whitney strip from* $\mathbf{x}$ *to* $\mathbf{y}$ is a continuous map $u : [0, 1] \to M$ such that $u(\mathbb{R} \times \{0\}) \subset L_0$, $u(\mathbb{R} \times \{1\}) \subset L_1$, satisfying the asymptotics $\lim_{t\to-\infty} u(t, s) = \mathbf{x}$ and $\lim_{t\to+\infty} u(t, s) = \mathbf{y}$.

Sometimes it is useful to reformulate the above notion in terms of disks. Let $\mathbb{D}$ be the closed unit disk in $\mathbb{C}$. A *Whitney disk from* $\mathbf{x}$ *to* $\mathbf{y}$ is a continuous map $u : \mathbb{D} \to M$ such that $u(z) \in L_0$ if $|z| = 1$ and $\mathrm{Re}(z) > 0$; $u(z) \in L_1$ if $|z| = 1$ and $\mathrm{Re}(z) < 0$; $u(-i) = \mathbf{x}$; and $u(i) = \mathbf{y}$. Using a conformal diffeomorphism $\mathbb{R} \times [0, 1]$ with $\mathbb{D} \setminus \{\pm i\}$ sending $\mathbb{R} \times \{0\}$ to the semicircle in the boundary of $\mathbb{D}$ with $\mathrm{Re}(z) > 0$, we see that each pseudo-holomorphic strip gives rise to a Whitney disk and vice-versa.

Fix two Whitney disks from $\mathbf{x}$ to $\mathbf{y}$. A *homotopy* from $u_0$ to $u_1$ is a map $u : \mathbb{D}\times[0, 1] \to M$ such that $u(z, t) \in L_0$ if $|z| = 1$ and $\mathrm{Re}(z) > 0$; $u(z, t) \in L_1$ if $|z| = 1$ and $\mathrm{Re}(z) < 0$; $u(-i, t) = \mathbf{x}$; $u(i, t) = \mathbf{y}$; $u(z, 0) = u_0(z)$; and $u(z, 1) = u_1(z)$. The set of homotopy classes of Whitney disks from $\mathbf{x}$ to $\mathbf{y}$ is denoted $W(\mathbf{x}, \mathbf{y})$.

Given a Whitney disk $u_1$ from $\mathbf{x}$ to $\mathbf{y}$ and a Whitney disk from $\mathbf{y}$ to $\mathbf{z}$, there is a natural composition $u_1 * u_2$, which is a Whitney disk from $\mathbf{x}$ to $\mathbf{z}$. Let $q : \mathbb{D} \to \mathbb{D} \vee \mathbb{D}$ be the composition of a quotient map given by collapsing the real interval to a point and a homeomorphism between this quotient space and $\frac{\mathbb{D} \sqcup \mathbb{D}}{(i \in \mathbb{D}_1) \sim (-i \in \mathbb{D}_2)}$. Define the map $u_1 * u_2$ to be the composite $(u_1 \vee u_2) \circ q : \mathbb{D} \to M$.

### 1.3.1 Maslov index

The Maslov index plays the role of the index difference in Morse homology. In this section, we consider two ways to define the Maslov index.

First, we define the Maslov index of a Whitney strip $u : \mathbb{R} \times [0,1] \to M^{2n}$. Note that $u^*(TM)$ is a bundle of symplectic vector spaces of $\mathbb{R} \times [0,1]$ with Lagrangian subbundles $(u|_{\mathbb{R}\times\{0\}})^*(TL_0)$ and $(u|_{\mathbb{R}\times\{1\}})^*(TL_1)$ over $\mathbb{R} \times \{0\}$ and $\mathbb{R} \times \{1\}$ respectively. In particular, for a fixed $t \in \mathbb{R}$, the vector space $V_t = u^*(TM)_{(t,0)}$ is a symplectic vector space, with a Lagrangian subspace $\Lambda_0^t = (u|_{\mathbb{R}\times\{0\}})^*(TL_0)_{(t,0)}$. Parallel transport across $\{t\} \times [0,1]$ identifies the symplectic vector space $u^*(TM)_{(t,1)}$ with the symplectic vector space $u^*(TM)_{(t,0)}$. Under the parallel transport, the Lagrangian subspace $(u|_{\mathbb{R}\times\{1\}})^*(TL_1)_{(t,1)}$ is moved to a new Lagrangian $\Lambda_1^t$ inside $V_t$. Under the identification $LGr(V_t) \cong U(n)/O(n)$ from Proposition 1.2.1 that identifies $\Lambda_0^t$ with the identity coset, $\Lambda_1^t$ corresponds to some coset $\Lambda_t \in U(n)/O(n)$. Recall that $U(n)/O(n)$ contains the Maslov cycle

$$\Sigma = \Sigma(\mathbb{R}^n) = \{A \in U(n) \mid \dim(A(\mathbb{R}^n) \cap \mathbb{R}^n) > 0\} \subset U(n)/O(n).$$

**Definition 1.3.1.** The *Maslov index* of a Whitney disk $u$ from $\mathbf{x}$ to $\mathbf{y}$ is the intersection number of $\{\Lambda_t\}_{t\in\mathbb{R}}$ with the Maslov cycle $\Sigma$.

First note that since $L_0$ and $L_1$ are tranverse, $\lim_{t\to\pm\infty} \Lambda_t \notin \Sigma$, so the Maslov index is well-defined.

**Example 1.3.2.** In the two pictures in Figure 1.2, the ambient manifold is $\mathbb{C}$, and there are two curves $L_0, L_1$ contained in the plane, intersecting at two points $x$ and $y$. As $L_0$ and $L_1$ are one-dimensional, they are Lagrangian. The cycle $\Sigma$ consists of all Lagrangians which

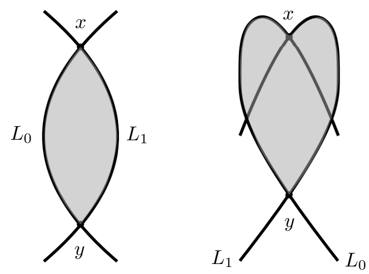


Figure 1.2: Two examples of a pair of Lagrangians $L_0$ and $L_1$ intersecting at two points $x$ and $y$.

have a nontrivial intersection with the real axis. In our case, this is just the real axis, so $\Sigma = \{[\mathbb{R}]\}$. The element $\Lambda_t$ in $U(n)/O(n)$ is equal to the coset of $\mathbb{R}$ only when the tangent spaces to the Lagrangians are parallel. Thus, the Maslov index is the number of parallel pairs of tangent subspaces, as we do parallel transport over all $t \in \mathbb{R}$. In the left figure this means that $\mu = 1$ and in the right it means that $\mu = 2$.

**Proposition 1.3.3.** *Homotopic Whitney disks have the same Maslov index, so we can think of $\mu$ as a function on $W(\mathbf{x}, \mathbf{y})$. If $\phi \in W(\mathbf{x}, \mathbf{y})$ and $\psi \in W(\mathbf{y}, \mathbf{z})$, then $\mu(\phi * \psi) = \mu(\phi) + \mu(\psi)$.*

*Proof.* A homotopy of Whitney disks moves the curve $\Lambda_t$ by homotopies whose endpoints do not intersect the Maslov cycle $\Sigma$. The intersection number of $\Lambda_t$ with $\Sigma$ is invariant under such isotopies. For the second part, note that composition of Whitney disks corresponds to a concatenation of paths. Since the intersection number is additive under such concatenations, the proposition holds. $\square$

Later, we will need to define the Maslov index for disks bounding more than two Lagrangians. In the above definition, we were comparing Lagrangian subspaces in the tangent spaces of a pair of Lagrangian submanifolds. Once there are more than two Lagrangian submanifolds, this will be difficult. Instead, it will be easier to make a loop of Lagrangians by traversing the entire boundary of the disk, and extracting data from that. To make an honest loop, we will need to find a canonical path of Lagrangians at each corner. For this, the notion of a *preferred path* of Lagrangian subspaces is helpful. If $(V, \omega)$ is a symplectic vector space, and $\Lambda_0, \Lambda_1$ are a pair of Lagrangians in $V$ meeting transversely, call a smooth path $\Lambda : [0,1] \to LGr(V,\omega)$ from $\Lambda_0$ to $\Lambda_1$ *preferred* if:

- $\Lambda'(0)$ is a positive definite bilinear form over $\Lambda_0$. Here, we implicitly are using the fact that the tangent space of $LGr(V,\omega)$ at $\Lambda$ is the vector space of symmetric bilinear forms on $\Lambda$. See [29, Lemma 4.5.10] for a proof.

- $\Lambda_t$ meets $\Sigma(\Lambda_0)$ transversely for all $t \in (0,1]$.

- The intersection number of the path $\{\Lambda(t)\}_{t\in(0,1]}$ with $\Sigma(\Lambda_0)$ is zero.

For a pair of Lagrangian subspaces $\Lambda_0, \Lambda_1$ meeting transversely, there exists a preferred path connecting them, and any two preferred paths are homotopic relative to their endpoints. See [29, Lemma 6.5.1] for a proof.

Now we proceed to the second definition of the Maslov index by first finding an appropriate loop of Lagrangians. As before, let $M$ be a symplectic manifold with a pair of Lagrangian submanifolds $L_0, L_1$ inside $M$, and let $u : \mathbb{D} \to M$ be a Whitney disk from $\mathbf{x}$ to $\mathbf{y}$ in $L_0 \cap L_1$. Choose a preferred path $\Lambda_{\mathbf{x}} : [0,1] \to LGr(T_{\mathbf{x}}M)$ from $T_{\mathbf{x}}L_0$ to $T_{\mathbf{x}}L_1$, and choose another preferred path $\Lambda_{\mathbf{y}} : [0,1] \to LGr(T_{\mathbf{y}}M)$ from $T_{\mathbf{y}}L_0$ to $T_{\mathbf{y}}L_1$. Precomposing $u : \mathbb{D} \to M$ with a quotient map $[0,1]\times[0,1] \to \mathbb{D}$ collapsing $[0,1]\times\{0\}$ to $-i$ and $[0,1]\times\{1\}$ to $i$, we get a continuous map $\overline{u} : [0,1]\times[0,1] \to M$. Then $\overline{u}^*(TM)$ is a bundle of symplectic vector spaces over $[0,1]\times[0,1]$. Then there is a Lagrangian subbundle $\mathfrak{L}$ of $\overline{u}^*(TM)$ over the boundary of $[0,1]\times[0,1]$ given by

$$\mathfrak{L}_{(s,0)} = \overline{u}^* T_{\overline{u}(s,0)} L_0, \qquad \mathfrak{L}_{(s,1)} = \overline{u}^* T_{\overline{u}(s,1)} L_1,$$
$$\mathfrak{L}_{(0,t)} = \Lambda_{\mathbf{x}}(t), \qquad \mathfrak{L}_{(1,t)} = \Lambda_{\mathbf{y}}(t).$$

Using a symplectic trivialization of $\overline{u}^*TM$ over $[0,1]\times[0,1]$, we can view $\mathfrak{L}$ as a loop of Lagrangians $\ell_u$ in a fixed vector space. Then define the Maslov index of $u$ to be the intersection number of $\ell_u$ and the Maslov cycle of the fixed vector space. By identifying an apprpriate Lagrangian Grassmannian with $U(n)/O(n)$, it follows that two definitions of Maslov index coincide. See [29, Proposition 6.5.3] for the details.

Equipped with the notion of a preferred path, we now proceed to define the Maslov index for disks bounding more than two Lagrangian submanifolds. As usual, fix a symplectic manifold $M$. Call an ordered $m$-tuple of Lagrangian submanifolds $(L_1, \ldots, L_m)$ of $M$ a *transversal chain* if $L_i$ intersects $L_j$ transversally for any distinct indices $i, j \in \{1, \ldots, m\}$, and the triple intersections $L_i \cap L_j \cap L_k$ are empty for any three distinct indices $i, j, k \in \{1, \ldots, m\}$. Now consider the unit closed disk $\mathbb{D} \subset \mathbb{C}$ with $m$ distinct points $v_1, \ldots, v_m$ marked on its boundary in the listed order. Let $\overline{v_i v_{i+1}}$ denote the arc on $\partial D$ between $v_i$ and $v_{i+1}$.

Suppose we have a transversal chain of Lagrangian submanifolds $(L_1, \ldots, L_m)$ in $M$. Let $\mathbf{x}_{1,2}, \mathbf{x}_{2,3}, \ldots, \mathbf{x}_{m,m+1}$ be points such that $\mathbf{x}_{i,i+1} \in L_i \cap L_{i+1}$ for $i \in \{1, \ldots, m\}$, where indices are taken in $\mathbb{Z}/m\mathbb{Z}$. Call a smooth map $u : \mathbb{D} \setminus \{v_1, \ldots, v_m\} \to M$ a *Whitney $m$-gon* if it extends continuously to $\mathbb{D}$ and is such that

$$\lim_{z \to v_i} u(z) = \mathbf{x}_{i,i+1}, \qquad\qquad u(\overline{v_i v_{i+1}}) \subset L_i.$$

The Maslov index of a Whitney $m$-gon can then be defined as follows. For each $\mathbf{x} \in \{\mathbf{x}_{1,2}, \mathbf{x}_{2,3}, \ldots, \mathbf{x}_{m,m+1}\}$, choose a preferred path $\gamma_{\mathbf{x}} = \{\Lambda_{\mathbf{x}}(t)\}_{t \in [0,1]}$ in $LGr(T_{\mathbf{x}} M)$ from $T_{\mathbf{x}} L_i$ to $T_{\mathbf{x}} L_{i+1}$. Using the Whitney $m$-gon $u$, we can define a closed loop of Lagrangian subspaces of $u^*_{\partial \mathbb{D}}(TM)$ by considering the pullbacks of paths in $u^*(TL_1 \cup \cdots \cup TL_m)$ and closing it up by attaching $\gamma_{\mathbf{x}_{i,i+1}}$ at the corners. Using a symplectic trivialization of $u^*(TM)$ over $\mathbb{D}$, we can view this as a closed loop $\ell_u$ of the Lagrangian Grassmannian in a fixed symplectic vector space. Then define the Maslov index of $u$ to be the intersection of $\ell_u$ and the Maslov cycle of the fixed vector space.

### 1.3.2 Pseudoholomorphic disks

In Lagrangian Floer homology, the analogue of a gradient flowline is the notion of a pseudo-holomorphic strip representing a given homotopy class of Whitney disks. Following Morse homology, we collect all pseudo-holomorphic strips into a given space. There are many beautiful properties of this space; for example, its dimension is governed by the Maslov index of the Whitney strip. Although this space is not compact, in a suitable topology, it can be compactified to a space that can be studied. This is the content of the Gromov compactness theorem, discussed later in this section.

Let $\{J^s\}_{s\in[0,1]}$ be a one-parameter family of $\omega$-tame almost-complex strctures on a symplectic manifold $(M,\omega)$. A $\{J^s\}$*-pseudo-holomorphic strip* is a Whitney strip $u : \mathbb{R}\times[0,1] \to M^{2n}$ which is smooth and satisfies

$$\frac{\partial u}{\partial t} + J^s \frac{\partial u}{\partial s} = 0,$$

i.e., at each point $(t,s) \in \mathbb{R}\times[0,1]$,

$$\frac{\partial u}{\partial t} + J^s_{u(t,s)} \frac{\partial u}{\partial s} = 0.$$

For a fixed homotopy class of Whitney strips $\phi \in W(\mathbf{x},\mathbf{y})$, denote $\mathcal{M}_{\{J^s\}}(\phi)$ to be the set of pseudo-holomorphic representatives of $\phi$.

There is an $\mathbb{R}$-action on $\mathbb{R}\times[0,1]$ given by $(t,s) \mapsto (t+\tau,s)$ for any $\tau \in \mathbb{R}$. If $u : \mathbb{R}\times[0,1] \to M$ is pseudo-holomorphic, then so is the map $(t,s) \mapsto u(t+\tau,s)$. Thus $\mathbb{R}$ acts on the moduli space of pseudo-holomorphic strips $\mathcal{M}_{\{J^s\}}(\phi)$. Denote the resulting quotient by $\widehat{\mathcal{M}}_{\{J^s\}}(\phi)$.

The relevance of the Maslov index in the study of pseudo-holomorphic curves is given by the following theorem. See [13, Theorem 5.1] for a proof. See also [12].

**Theorem 1.3.4.** *Let $\phi \in W(\mathbf{x},\mathbf{y})$ be a non-constant homotopy class with Maslov index $\mu(\phi) \leq 2$. If $\{J^s\}_{s\in[0,1]}$ is a suitably generic path of $\omega$-tame almost-complex structures, then the space $\widehat{\mathcal{M}}_{\{J^s\}}(\phi)$ is a smooth manifold of dimension $\mu(\phi)-1$. In particular, if $\phi$ is a nonconstant homotopy class with nonpositive Maslov index, then the space $\widehat{\mathcal{M}}_{\{J^s\}}(\phi)$ is empty.*

Now we proceed to informally state the Gromov compactness theorem, an essential tool in the study of pseudo-holomorphic disks and curves. The *energy* of a pseudo-holomorphic strip $u$ is defined to be

$$E(u) = \int_{\mathbb{R}\times[0,1]} \frac{1}{2}\left(\left\|\frac{\partial u}{\partial t}\right\|^2 + \left\|\frac{\partial u}{\partial s}\right\|^2\right) dtds,$$

where $\left\|\frac{\partial u}{\partial t}\right\|^2$ is defined to be $g_s(\frac{\partial u}{\partial t}, \frac{\partial u}{\partial t})$ for the metric $g_s$ on $TM$ associated to $\omega$ and almost complex structure $J^s$. The quantity $\left\|\frac{\partial u}{\partial s}\right\|^2$ is similarly defined. The Gromov compactness theorem in the case of pseudoholomorphic disks says the following, compare [29, 17, 14, 22]. Assume that $M$ is a compact symplectic manifold. Then any sequence of $\{J^s\}$-holomorphic strips from $\mathbf{x}$ to $\mathbf{y}$ such that there is a uniform energy bound has a subsequence which converges in a suitable sense to a map out of a Riemann surface with double points. This means, informally, that the limiting object in the domain may contain holomorphic strips, i.e. objects of the form $[u_0] * \cdots * [u_n]$ where $u_i$ are nonconstant $\{J^s\}$-holomorphic strips; $J$-holomorphic spheres that bubble off from some interior point in the sequence of strips; and $J$-holomorphic disks with boundary contained entirely inside $L_0$ or $L_1$, which can be thought of as a disk bubbling off the $L_0$ or $L_1$ side (a map $u : (\Sigma, j) \to (M, J)$ from a Riemann surface to a manifold with an almost complex structure is *$J$-holomorphic* if the derivative of $u$ is complex-linear, that is, $J \circ du = du \circ j$). Moreover, if the limiting object contains a sphere or disk that bubbles off, it needs to nonconstant.

### 1.3.3 Lagrangian Floer homology

Let $(M, \omega)$ be a compact symplectic manifold, and let $L$ be a compact Lagrangian submanifold of $M$. From a smooth function $H : M \times [0, 1] \to \mathbb{R}$ called a *time-dependent Hamiltonian*, one may obtain a family of vector fields $X_t$ determined by the equation

$$\omega(\cdot, X_t) = dH_t,$$

where $H_t = H(\cdot, t)$. Integrating these vector fields over $t \in [0, 1]$ yields a diffeomorphism $\psi$ of $M$, called the *Hamiltonian diffeomorphism $\psi$ generated by $H$*. For a topological disk $u : \mathbb{D} \to M$, define its area to be $\int_{\mathbb{D}} u^*\omega$.

**Theorem 1.3.5** (Floer [12]). *Suppose that the area of any topological disc in $M$ with boundary in $L$ vanishes. Assume moreover that $\psi(L)$ and $L$ intersect transversely. Then the number of intersection points of $L$ and $\psi(L)$ satisfies*

$$|\psi(L) \cap L| \geq \sum_i \dim H_i(L; \mathbb{Z}/2\mathbb{Z}).$$

Floer was motivated to prove the above theorem due to Arnold's conjecture on the behavior of Hamiltonian diffeomorphisms. Arnold [1, page 419] conjectured that the number of fixed points of a Hamiltonian diffeomorphism is at least the rank of the singular homology of $M$ has at least as many fixed points as a Morse function on this manifold has critical points. It follows from the Morse inequalities [24, Chapter 5] that the number of critical points of a Morse function is always at least the rank of $H_*(M; \mathbb{Z}/2\mathbb{Z})$. Arnold's conjecture can be viewed as a special case of the above theorem, where the ambient symplectic manifold is $(M \times M, p_1^*(\omega) - p_2^*(\omega))$ and $p_i$ is the projection of $M \times M$ onto the $i$-th factor for $i = 1, 2$.

Floer's approach is to associate to the pair of Lagrangians $(L_0, L_1) = (L, \psi(L))$ a chain complex $\mathrm{CF}(L_0, L_1)$, freely generated by the intersection points of $L_0$ and $L_1$ with a endomorphism $\partial : \mathrm{CF}(L_0, L_1) \to \mathrm{CF}(L_0, L_1)$ such that: $\partial^2 = 0$ so that if $L_1$ and $L_1'$ are Hamiltonian isotopic, so that *Lagrangian Floer homology* $\mathrm{HF}(L_0, L_1)$ is well-defined; if $L_1$ and $L_1'$ are Hamiltonian isotopic, then $\mathrm{HF}(L_0, L_1) \cong \mathrm{HF}(L_0, L_1')$; and if $L_1$ is Hamiltonian isotopic to $L_0$, in the sense that $L_1 = \phi(L_0)$ for some Hamiltonian diffeomorphism $\phi$ of $M$, then $\mathrm{HF}(L_0, L_1) \cong H_*(L_0; \mathbb{Z}/2\mathbb{Z})$. Then Theorem 1.3.5 follows immediately because the rank of $\mathrm{HF}(L, \psi(L)) \cong H^*(L; \mathbb{Z}/2\mathbb{Z})$ is bounded by the rank of the Floer complex $\mathrm{CF}(L, \psi(L))$, which is equal to $|\psi(L) \cap L|$.

As we will see, to make the differential a finite sum, it is useful to enlarge the ring over which Lagrangian Floer homology is defined. Define the *Novikov field* $N_{\mathbb{Z}/2\mathbb{Z}}$ over $\mathbb{Z}/2\mathbb{Z}$ to be the collection of formal sums $x_A = \sum_{a \in A} x_a T^a$, where $A$ is a discrete set, $x_a \in \mathbb{Z}/2\mathbb{Z}$, and $T$ is a formal variable. Given two discrete subsets $A, B \subset \mathbb{R}$, their Minkowski sum is $A + B = \{a + b \text{ for some } a \in A \text{ and } b \in B\}$, where the elements are counted with multiplicity in $\mathbb{Z}/2\mathbb{Z}$. Now define $x_A \cdot x_B = x_{A+B}$.

**Definition 1.3.6.** Let $M$ be a compact symplectic manifold with a pair of compact Lagrangian submanifolds $L_0, L_1$ in $M$. Fix a suitably generic one-parameter family $\{J^s\}_{s\in[0,1]}$ of $\omega$-tame almost-complex structures on $M$. Let $\mathrm{CF}(L_0, L_1)$ be the vector space over $N_{\mathbb{Z}/2\mathbb{Z}}$ generated by the intersection points $L_0 \cap L_1$ with an endomorphism $\partial$ given by

$$\partial \mathbf{x} = \sum_{\mathbf{y}\in L_0\cap L_1} \sum_{\{\phi\in W(\mathbf{x},\mathbf{y})|\mu(\phi)=1\}} T^{\omega(\phi)} \# \widehat{\mathcal{M}}_{\{J^s\}}(\phi) \cdot \mathbf{y}.$$

Here, $\#\widehat{\mathcal{M}}_{\{J^s\}}(\phi)$ is the parity of the number of elements in $\widehat{\mathcal{M}}_{\{J^s\}}(\phi)$, and $\omega(\phi)$ is the integral of $\omega$ on any pseudo-holomorphic representative of $\phi$. Just like in the case of the Morse complex, there are a few components to resolve:

1. The definition of $\partial\mathbf{x}$ is a finite sum. By Theorem 1.3.4, the space $\widehat{\mathcal{M}}_{\{J^s\}}(\phi)$ is a zero-manifold for Whitney disks $\phi$ of Maslov index one. By Gromov compactness, the set
$$\bigcup_{\{\phi\in W(\mathbf{x},\mathbf{y})|\mu(\phi)=1,\omega(\phi)=a\}} \widehat{\mathcal{M}}_{\{J^s\}}(\phi)$$
is a compact zero-manifold, assuming that we exclude any disks or spheres bubbling off. Under the hypotheses of Floer's Theorem 1.3.5, this holds.

2. The endomorphism $\partial$ satisfies $\partial^2 = 0$, making $\mathrm{CF}(L_0, L_1)$ into a chain complex with differential $\partial$. To show this, one analyzes moduli spaces $\widehat{\mathcal{M}}_{\{J^s\}}(\phi)$ for which $\mu(\phi) = 2$. The boundary of this moduli space under a suitable compactification is a one-manifold with boundary.

3. The complex $\mathrm{CF}(L_0, L_1)$ is independent of the choice of path of almost complex structure $\{J^s\}_{s\in[0,1]}$. To show this, one connects a pair of paths of almost complex structures with a path of paths, and defines appropriate maps counting time-dependent pseudo-holomorphic Whitney disks connecting a pair of intersection points of $L_0$ and $L_1$.

4. The complex $\mathrm{CF}(L_0, L_1)$ is invariant under Hamiltonian isotopy, in the sense that if $L_1$ and $L_1'$ are Hamiltonian isotopic, then $\mathrm{HF}(L_0, L_1) \cong \mathrm{HF}(L_0, L_1')$. To show this,

one defines appropriate maps counting pseudo-holomorphic strips that incorporate the Hamiltonian diffeomorphism.

Floer's construction [12] was extended to more general settings, see for example Oh's extension to monotone Lagrangians [28].

# Chapter 2

# Heegaard Floer homology

This chapter defines Heegaard Floer homology. In Section 2.1, we study Heegaard diagrams, which provide the data needed to decompose a closed oriented three-manifold into a union of two handlebodies glued along an orientable surface. We also define $\mathrm{Spin}^c$ structures on three-manifolds and study their relation to Heegaard diagrams. In Section 2.2, we study the symmetric product of a Riemann surface, the key object used to define Heegaard Floer invariants, and in particular provide the symmetric product with a complex structure. In Section 2.3, we provide the symmetric product with a Kähler struceture, following the work of Perutz [35] relying on a lemma of Varouchas [41]. With this structure in place, we define Heegaard Floer homology for closed oriented three-manifolds in Section 2.4, starting with rational homology spheres. In Section 2.5, we study counts of pseudo-holomorphic polygons needed for the following section on the surgery exact triangle.

## 2.1 Heegaard diagrams

Let $\Sigma$ be a closed surface of genus $g$. A *complete set of attaching circles* is a collection of disjoint embedded simple closed curves $\boldsymbol{\alpha} = \{\alpha_1, \ldots, \alpha_g\}$, whose homology classes are linearly independent in $H_1(\Sigma; \mathbb{Z}/2\mathbb{Z})$. A *Heegaard diagram* is a triple $(\mathcal{H}, \boldsymbol{\alpha}, \boldsymbol{\beta})$, where $\Sigma$ is a closed, oriented, surface of genus $g$, and $\boldsymbol{\alpha} = \{\alpha_1, \ldots, \alpha_g\}$ and $\boldsymbol{\beta} = \{\beta_1, \ldots, \beta_g\}$ are complete sets of attaching cirlces in $\Sigma$. We assume that the diagram is generic, in the sense that each $\alpha_i$ intersects each $\beta_j$ transversally.

A Heegaard diagram $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ can be used to build a closed oriented three-manifold. One first builds a handlebody $U_\alpha$ from the complete attaching circles $\boldsymbol{\alpha}$ in $\Sigma$ as follows. Attach $g$ three-dimensional 2-handles on $[0,1] \times \Sigma$ along $\{0\} \times \alpha_i$ for $i = 1, \ldots, g$. The boundary of the resulting manifold is $\{1\} \times \Sigma$, and the other one is a two-sphere. Let $U_\alpha$ be the result of attaching a three-dimensional 3-handle along the two-sphere. Similarly create the handlebody $U_\beta$. Now glue the boundary of $U_\alpha$, which is identified with $\Sigma$, to the boundary of $-U_\beta$, which is identified with $-\Sigma$. Here, $-X$ is the manifold $X$ with the opposite orientation. The result of this construction is a closed oriented three-manifold.

**Example 2.1.1.** The *standard genus one diagram for* $S^3$ is the Heegaard diagram consisting of the genus one surface equipped with a pair of curves $\alpha$ and $\beta$ that inetrsect at a single point.

One can compute the homology groups from the Heegaard diagram according to the following proposition.

**Proposition 2.1.2.** *For a closed curve $\gamma$ in $\Sigma$, let $[\gamma]$ denote the homology class of $\gamma$ in $H_1(\Sigma;\mathbb{Z})$. There is an isomorphism*

$$H_1(Y;\mathbb{Z}) \cong \frac{H_1(\Sigma;\mathbb{Z})}{\mathrm{Span}([\alpha_1], \ldots, [\alpha_g], [\beta_1], \ldots, [\beta_g])}$$

*Moreover, if $\phi : \mathbb{Z}^g \oplus \mathbb{Z}^g \to H_1(\Sigma;\mathbb{Z})$ is the map defined by*

$$(\{m_i\}_{i=1}^g, \{n_i\}_{i=1}^g) \mapsto \sum_{i=1}^g m_i[\alpha_i] + n_i[\beta_i],$$

*then* $\ker(\phi) \cong H_2(Y;\mathbb{Z})$.

*Proof.* In the following lemma, we always work with homology over $\mathbb{Z}$-coefficients, so we drop the coefficient ring from the notation. The long exact sequence of the pair $(Y, \Sigma)$ is

$$\begin{tikzcd}
\cdots \arrow[r] & H_2(\Sigma) \arrow[r] & H_2(Y) \arrow[r] & H_2(Y,\Sigma) \arrow[dll, "\delta", rounded corners] \\
& H_1(\Sigma) \arrow[r] & H_1(Y) \arrow[r] & H_1(Y,\Sigma) \arrow[r] & \cdots
\end{tikzcd}$$

By a Mayer–Vietoris argument for relative homology, we see that

$$H_i(Y,\Sigma) \cong H_i(U_\alpha,\Sigma) \oplus H_i(U_\beta,\Sigma)$$

for $i = 1, 2$. By Poincaré duality for a handlebody $U$ with $\partial U = \Sigma$, we have that $H_1(U,\Sigma) \cong H^2(U) = 0$ and $H_2(U,\Sigma) \cong H^1(U) \cong \mathbb{Z}^g$. Thus the bottom right term in the long exact sequence of the pair $(Y,\Sigma)$ is zero. Now note that by the long exact sequence of the pair $(U,\Sigma)$, the image of the boundary map $\partial_* : H_2(U,\Sigma) \to H_1(\Sigma)$ is the span of the homology classes of the attaching circles. So, $\delta$ can be identified with the map $\phi$ and thus

$$H_1(Y) \cong \frac{H_1(\Sigma)}{\operatorname{im}\delta} \cong \frac{H_1(\Sigma)}{\operatorname{im}\phi} \cong \frac{H_1(\Sigma;\mathbb{Z})}{\operatorname{Span}([\alpha_1],\ldots,[\alpha_g],[\beta_1],\ldots,[\beta_g])}.$$

For the second part, note that the map induced by inclusion $H_2(\Sigma) \to H_2(Y)$ is trivial because the Heegaard surface bounds a handlebody. Thus,

$$H_2(Y) \cong \ker\delta \cong \ker(\phi),$$

concluding the proof. $\square$

The generators of the Heegaard Floer complex are the Heegaard states, which we define now. Let $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ be a generic Heegaard diagram, and fix an ordering $\boldsymbol{\alpha} = \{\alpha_1, \ldots, \alpha_g\}$ and $\boldsymbol{\beta} = \{\beta_1, \ldots, \beta_g\}$ of the attaching circles. A Heegaard state is a $g$-tuple of points $\mathbf{x} = \{x_1, \ldots, x_g\}$ in $\Sigma$ such that there is a permutation $\sigma \in \mathfrak{S}_g$ for which $x_i \in \alpha_i \cap \beta_{\sigma(i)}$.

Heegaard states have an interpretation in terms of Morse theory. Start with a self-indexing Morse function $f : Y \to \mathbb{R}$ with a unique maximum and minimum, and $g$ index-1 critical points. Fix a metric $h$ on $Y$. A *simultaneous trajectory* is a set of $g$ gradient flowlines connecting all of the index-1 critical points to all of the index-2 critical points. For a generic metric $h$ with respect to $f$, the associated Heegaard diagram is generic, and the simultaneous

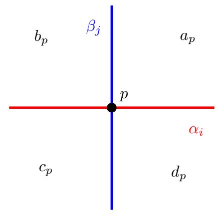


Figure 2.1: Convention for multiplcities at a region.

trajectories are in bijection with the Heegaard states.

Now we consider domains connecting Heegaard states. The curves $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$ divide $\Sigma$ into a collection of open path-connected components $\Sigma \backslash (\alpha_1 \cup \cdots \cup \alpha_g \cup \beta_1 \cup \cdots \cup \beta_g) = \bigcup \mathrm{int} D_k^\circ$. The closures $D_k = \overline{D^\circ}_k$ are the *elementary domains* of the Heegaard diagram. A *domain* is a formal linear combination of elementary domains

$$D = \sum_k m_k D_k,$$

where $m_k \in \mathbb{Z}$. The *multiplicity* of $D$ at a point $p \in \Sigma \setminus (\boldsymbol{\alpha} \cup \boldsymbol{\beta})$ is the coefficient of the elementary domain containing $p$ in the expression for $D$.

Let $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ be a generic Heegaard diagram, and fix a domain $D$. At each intersection point $p$ of $\alpha_i$ and $\beta_j$, we have at most four neighboring regions which meet at the corner $p$. Label the multiplicities of $D$ by $a_p, b_p, c_p, d_p$ according to the conventions in Figure 2.1. Call $D$ *cornerless* if at each intersection point $p \in \alpha_i \cap \beta_j$ for $i, j \in \{1, \ldots, g\}$,

$$a_p + c_p = b_p + d_p.$$

Fix a point $w \in \Sigma \setminus (\boldsymbol{\alpha} \cup \boldsymbol{\beta})$. A *periodic domain* is a cornerless domain whose multiplicity at $w$ is zero.

**Example 2.1.3.** In the standard genus 1 Heegaard diagram for $S^3$, all local multiplcities at the single intersection point are the same, so the group of cornerless domains is isomorphic to $\mathbb{Z}$ and the subgroup of periodic domains is the zero domain.

**Proposition 2.1.4.** *The group of periodic domains is isomorphic to $\mathbb{Z} \oplus H_2(Y; \mathbb{Z})$.*

*Proof.* Any domain corresponds to a 2-chain in $\Sigma$, and the cornerless domains are those for which the boundary of the corresponding 2-chain is in the span of the attaching circles. Thus the boundary $\partial D$ of a cornerless domain $D$ is in the kernel of $\phi$ because the corresponding 1-chain is a boundary, where $\phi$ is defined in Proposition 2.1.2. Now, the map $\partial : \{\text{cornerless domains}\} \to \ker(\phi)$ is onto because every element in $\ker(\phi)$ bounds a cornerless domain by definition. Moreover, $\ker \partial$ is generated by the homology class $[\Sigma]$ of the Heegaard surface $\Sigma$, thought of as a cornerless domain where all local multiplicities are one. Thus we have the short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \{\text{cornerless domains}\} \longrightarrow \ker(\phi) \longrightarrow 0.$$

This sequence splits because all groups are free. Therefore, the group of cornerless domains is isomorphic to $\mathbb{Z} \oplus \ker(\phi)$, which is isomorphic to $\mathbb{Z} \oplus H_2(Y;\mathbb{Z})$ by Proposition 2.1.2. $\square$

**Proposition 2.1.5.** *For any closed, connected, oriented three-manifold $Y$, there exists a Heegaard diagram representing $Y$.*

*Proof.* By Theorem 1.1.4, find a self indexing function $f$ with a unique minimum and maximum on $Y$. From Section 1.1, we also know that $f$ induces a handle decomposition of $Y$. Let $Y_1$ be the union of the 0-handle and all of the 1-handles in this decomposition.

The Heegaard surface $\Sigma$ is the oriented boundary of $Y_1$. The belt circles of the 1-handles are the $\alpha$-circles, and the attaching circles of the 2-handles are the $\beta$-circles. All curves are regarded as circles in $\Sigma = \partial Y_1$. If $f$ has $g$ index-1 critical points, then $Y_1$ is a genus $g$ handlebody, $\Sigma$ is a surface of genus $g$, and $\boldsymbol{\alpha}$ consists of $g$ simple closed curves. Now consider the function $3 - f$ on $Y$. It is also self-indexing Morse function, with the same critical points as $f$. The union of the 0-handle and the 1-handles induced by $3 - f$ give the complement $Y \setminus Y_1$, and since the genus of the boudnary determines the number of 1-handles in the handlebody, we also have $g$ $\boldsymbol{\beta}$-circles. The attaching circles of the index-2 critical points of $f$, which are the $\boldsymbol{\beta}$-circles, are the belt circles of the index-1 critical points of $3-f$. Thus, the resulting Heegaard diagram presents $Y$. $\square$

We can also form the sum of two Heegaard diagrams:

**Definition 2.1.6.** Let $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ and $\mathcal{H}' = (\Sigma', \boldsymbol{\alpha'}, \boldsymbol{\beta'})$ be a pair of Heegaard diagrams, and choose points $w \in \Sigma \setminus (\boldsymbol{\alpha} \cup \boldsymbol{\beta})$ and $w' \in \Sigma \setminus (\boldsymbol{\alpha}' \cup \boldsymbol{\beta}')$. Define the *connected sum of $\mathcal{H}$ and $\mathcal{H}'$*, denoted $\mathcal{H} \# \mathcal{H}'$, to be the Heegaard dyagram whose underlying Heegaard surface is the connected sum of $\Sigma$ and $\Sigma'$ at $w$ and $w'$, and whose $\alpha$-circles are $\boldsymbol{\alpha} \cup \boldsymbol{\alpha}'$ and whose $\beta$-circles are $\boldsymbol{\beta} \cup \boldsymbol{\beta}'$.

### 2.1.1 Spin$^c$ structures

Heegaard Floer homology, a topological invariant of a three-manifold, splits according to Spin$^c$-structures. In this subsection, we define Spin$^c$ structures on three-manifolds in the classical way and then give a more concrete geometric definition. This part is not strictly necessary for understanding the definition of Heegaard Floer homology and is only used to find L-space manifolds at the end of the thesis.

Recall that $SO(3)$ can be thought of as the quotient $U(2)/U(1)$, where $U(1)$ lies in $U(2)$ as the diagonal subgroup, making the projection $U(2) \to SO(3)$ a principal circle bundle over $SO(3)$. Now start with a closed oriented three-manifold $Y$. Endow $Y$ with a Riemannian metric and consider the associated principal $SO(3)$-bundle of oriented orthonormal frames. A Spin$^c$*-structure* on $Y$ is a lift of the orthonormal frame bundle to a principal $U(2)$-bundle. More precisely, a Spin$^c$-structure on $Y$ is an isomorphism class of a pair consisting of a principal $U(2)$-bundle $F \to Y$ and an isomorphism of the principal $SO(3)$-bundle $F/U(1) \to Y$ with the orthonormal frame bundle of $Y$.

Spin$^c$ structures on three-manifolds have a concrete geometric formulation defined by Turaev [39, 40]. Let $Y$ be a closed oriented three-manifold. Then the tangent bundle of $Y$ is trivial [27, Chapter 12], so in particular $Y$ admits a nowhere vanishing vector field. By rescaling, we can find a unit vector field over $Y$. Call two unit vector fields $v_1, v_2$ on $Y$ homologous if they are homotopic through unit vector fields on the complement of a point in $Y$. The homotopy class of a unit vector field $v$, denoted $[v]$, is called an *Euler structure* on $Y$. In [40, Lemma 1.4], Turaev establishes a bijection between Euler structures on $Y$ and Spin$^c$ structures on $Y$.

Spin$^c$-structures can be interpreted in terms of a pointed Heegaard diagram. Given two

Heegaard states $\mathbf{x}, \mathbf{y}$, there is an obstruction $\epsilon(\mathbf{x}, \mathbf{y})$ to the existence of a domain connecting $\mathbf{x}$ to $\mathbf{y}$, defined as follows. Pick paths $\xi_i \subset \alpha_i$ for $i = 1, \ldots g$ from $\mathbf{x} \cap \alpha_i$ to $\mathbf{y} \cap \alpha_i$, and also pick paths $\eta_i \subset \beta_i$ for $i = 1, \ldots, g$ from $\mathbf{x} \cap \beta_i$ to $\mathbf{y} \cap \beta_i$. The one-chain $\sum_{i=1}^{g}(\xi_i - \eta_i)$ is a cycle, whose homology class in $H_1(\Sigma; \mathbb{Z})$ does depend on the choices of paths $\xi_i$ and $\eta_i$. However, the corresponding element $\epsilon(\mathbf{x}, \mathbf{y})$ in the quotient

$$H_1(\Sigma; \mathbb{Z})/\mathrm{Span}(\{[\alpha_i], [\beta_i]\}_{i=1}^{g}) \cong H_1(Y; \mathbb{Z}),$$

is independent the these choices of $\xi_i$ and $\eta_i$, where we are using the isomorphism from Proposition 2.1.2. In [29, Proposition 2.4.7], it is shown that for a generic Heegaard diagram $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ and pair of Heegaard states $\mathbf{x}, \mathbf{y}$, the set of domains $D(\mathbf{x}, \mathbf{y})$ is nonempty if and only if the obstruction $\epsilon(\mathbf{x}, \mathbf{y})$ is zero.

**Definition 2.1.7.** Two Heegaard states $\mathbf{x}$ and $\mathbf{y}$ are *equivalent* if $\epsilon(\mathbf{x}, \mathbf{y})$ is zero.

Now we demonstrate that in a pointed Heegaard diagram, there is a map

$$s_w : \{\text{Heegaard states}\} \to \mathrm{Spin}^c(Y)$$

depending on the choice of basepoint $w$. Let $Y$ be a closed, connected, oriented three-manifold. Fix a self-indexing Morse function $f$ on $Y$ and a Riemannian metric $h$ on $Y$, inducing a Heegaard diagram $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ by Proposition 2.1.5. Fix a Heegaard state $\mathbf{x}$ and consider the simultaneous trajectories corresponding to $\mathbf{x}$, together with the gradient flowline passing through $w$ connecting the minimum and maximum of $f$. Outside a small tubular neighborhood of these $g + 1$ gradient flow lines, the gradient of $f$ is a nowhere vanishing vector field. Since the index of the gradient of $f$ as a vector field at an index-$i$ critical point is $(-1)^i$ and the indices of two critical points in each ball have opposite parity, the gradient of $f$ can be extended to a nonvanishing vector field over the $g + 1$ balls, see [26, Chapter 6]. In this manner, the Heegaard state, together with a choice of a basepoint, determines a nonzero vector field on $Y$, well-defined up to a homotopy away from finitely many balls. This object is an Euler structure on $Y$, which can be viewed as a $\mathrm{Spin}^c$ structure.

Thus we have a map from {Heegaard states} $\to \mathrm{Spin}^c(Y)$. In fact, for a pair of Heegaard states, $s_w(\mathbf{x}) = s_w(\mathbf{y})$ if and only if $\epsilon(\mathbf{x}, \mathbf{y}) = 0$, see [29, Lemma 2.6.1]. Thus we actually have a bijection

$$s_w : \{\text{equivalence classes of Heegaard states}\} \to \mathrm{Spin}^c(Y) \tag{2.1}$$

## 2.2 Symmetric product

In this section and the following section, we study a key object in the construction of Heegaard Floer homology, the symmetric product of a Riemann surface,.

Let $X$ be a topological space. Form the $m$-fold cartesian product $X^{\times m}$ of $X$, and quotient by the natural action of the symmetric group $\mathfrak{S}_m$ on $m$ letters, acting by permuting the factors in the Cartesian product. In this manner, we form a new topological space, the *$m$-fold symmetric product* $\mathrm{Sym}^m(X)$ of $X$. By construction, there is a quotient map $X^{\times m} \to \mathrm{Sym}^m(X)$. If $f : X \to Y$ is a continuous map, then there is a naturally induced continuous map $\mathrm{Sym}^m(f) : \mathrm{Sym}^m(X) \to \mathrm{Sym}^m(Y)$ sending the point $\{x_1, \ldots, x_m\}$ to $\{f(x_1), \ldots, f(x_m)\}$. Let $\Delta \subset \mathrm{Sym}^m(X)$ be the *diagonal* in the symmetric product, consisting of those points where at least two of the coordinates coincide. Away from $\Delta$, the symmetric group $\mathfrak{S}_m$ acts freely, so if $X$ is an $n$-dimensional manifold, then $\mathrm{Sym}^m(X) \setminus \Delta$ is an $mn$-dimensional manifold.

There is a special homeomorphism $\mathrm{Sym}^m(\mathbb{C}) \cong \mathbb{C}^m$. Given an unordered $m$-tuple of complex numbers $\{z_1, \ldots, z_n\}$, by the Fundamental Theorem of Algebra, there is a unique monic polynomial $p$ whose roots are $\{z_1, \ldots, z_n\}$. Identify the unordered $m$-tuple $\{z_1, \ldots, z_n\}$ with the ordered $m$-tuple consisting of the coefficients of $p$.

Now we show that given a complex structure on an orientable surface, we can find an appropriate complex structure on the symmetric product of the orientable surface.

**Proposition 2.2.1.** *Let $\Sigma$ be an oriented surface. Then a complex structure $j$ on $\Sigma$ induces a complex structure on $\mathrm{Sym}^m(\Sigma)$, uniquely characterized by the property that if $\Sigma^{\times m}$ is equipped with the product complex structure $j^{\times m}$ induced from $j$, then the quotient map*

$\pi : \Sigma^{\times m} \to \mathrm{Sym}^m(\Sigma)$ *is holomorphic.*

*Proof.* First we prove the following lemma:

**Lemma 2.2.2.** *Let* $D_1, D_2 \subset \mathbb{C}$ *be two bounded open sets and* $\phi : D_1 \to D_2$ *a biholomorphic map. Then* $\mathrm{Sym}^m(\phi) : \mathrm{Sym}^m(D_1) \to \mathrm{Sym}^m(D_2)$ *is a biholomorphic map between subsets of* $\mathrm{Sym}^m(\mathbb{C}) \cong \mathbb{C}^m$.

*Proof.* Let $\pi : \mathbb{C}^m \to \mathrm{Sym}^m(\mathbb{C})$ be the natural quotient map and $\theta : \mathrm{Sym}^m(\mathbb{C}) \to \mathbb{C}^m$ be the homeomorphism from the Fundamental Theorem of Algebra. To show that $\mathrm{Sym}^m(\phi)$ is a biholomorphism, it is enough to show that $\theta \circ \mathrm{Sym}^m(\phi) \circ \theta^{-1}$ defined on the open set $\theta(\mathrm{Sym}^m(D_1)) \subset \mathbb{C}^m$ is holomorphic, as its inverse is $\theta \circ \mathrm{Sym}^m(\phi^{-1}) \circ \theta^{-1}$.

The diagonal $\Delta \subset \mathrm{Sym}^m(\Sigma)$ is mapped under $\theta$ to the *discriminant locus* $\Delta_0 \subset \mathbb{C}^n$, which is the set of points $(a_1, \ldots, a_m)$ for which $p(z) = z^m + a_1 z^{m-1} + \cdots + a_m$ has a repeated root. We will later use the fact that $\Delta_0$ is the zero set of a polynomial: if $p$ has roots $\{x_1, \ldots, x_n\}$, then the discriminant locus is the zero set of $\prod_{1 \le i < j \le n}(x_i - x_j)$ which is a symmetric polynomial of the roots, so the product can be written as a polynomial in terms of the coefficients $a_1, \ldots, a_m$ of $p(z)$.

In any case, by the inverse function theorem and the fact that $\pi$ is a covering map away from $\pi^{-1}(\Delta)$, we have that

$$(\theta \circ \pi)|_{\mathbb{C}^m \setminus \pi^{-1}(\Delta)} : \mathbb{C}^m \setminus \pi^{-1}(\Delta) \to \mathbb{C}^m \setminus \Delta_0$$

is a covering map that is holomorphic. Thus, since $\phi^{\times m}$ is holomorphic, the restriction of $\theta \circ \mathrm{Sym}^m(\phi) \circ \theta^{-1}$ to $(\mathbb{C}^m \setminus \Delta_0) \cap \theta(\mathrm{Sym}^m(D_1))$ can be written as a composition of locally defined holomorphic maps. Since $\theta \circ \mathrm{Sym}^m(\phi) \circ \theta^{-1}$ is continuous and $\Delta_0 \subset \mathbb{C}^m$ is the zero set of the discriminant in the coefficients $a_1, \ldots, a_n$, it follows from the Riemann extension theorem from several complex variables, see for example [16, page 9], that $\theta \circ \mathrm{Sym}^m(\phi) \circ \theta^{-1}$ is holomorphic on $\theta(\mathrm{Sym}^m(D_1))$. $\square$

Now we construct a complex structure on $\mathrm{Sym}^m(\Sigma)$ in terms of the complex structure $j$ on $\Sigma$. The complex structure on $\Sigma$ provides us with an atlas $\{\phi : U_i \to \Sigma\}_i$ with $U_i \subset \mathbb{C}$

open for which the transition maps $\phi_j^{-1} \circ \phi_i : \phi_i^{-1} \circ \phi_j(U_j) \to U_j$ are holomorphic. The space $\mathrm{Sym}^m(\Sigma)$ is covered by charts

$$(\mathrm{Sym}^{d_1}(U_{n_1}) \times \cdots \times \mathrm{Sym}^{d_k}(U_{n_k}), \mathrm{Sym}^{d_1}(\phi_1) \times \cdots \times \mathrm{Sym}^{d_k}(\phi_k))$$

indexed by partitions $(d_1, \ldots, d_k)$ of $m$ and $k$-tuples of charts $\{\phi_{n_i} : U_{n_i} \to \Sigma\}$ whose images are disjoint. By Lemma 2.2.2, these maps are holomorphic.

Now we show that the complex structure on $\mathrm{Sym}^m(\Sigma)$ is uniquely characterized by the property that $\pi : \Sigma^{\times m} \to \mathrm{Sym}^m(\Sigma)$ is holomorphic. Suppose that $\mathrm{Sym}^m(\Sigma)$ is equipped with two complex structures $J_1, J_2$ for which the quotient map $\pi$ is holomorphic. Then the identity map $\mathrm{Id} : (\mathrm{Sym}^m(\Sigma), J_1) \to (\mathrm{Sym}^m(\Sigma), J_2)$ is continuous and holomorphic away from the diagonal $\Delta$. By the Riemann extension theorem, the identity map extends to a holomorphic map accross $\Delta$, which implies that $J_1 = J_2$. □

Denote the complex structure on $\mathrm{Sym}^m(\Sigma)$ by $\mathrm{Sym}^m(j)$. Our next goal is to find a symplectic structure on $\mathrm{Sym}^m(\Sigma)$ which is compatible with $\mathrm{Sym}^m(j)$, where compatibility is defined in Section 1.2.

Fix a symplectic form $\nu$ on $\Sigma$ compatible with $j$. Then there exists a naturally induced symplectic form $\nu^\times$ over $\Sigma^{\times m}$ by Proposition 1.2.3, defined by

$$\nu^\times = p_1^*(\nu) + \cdots + p_m^*(\nu)$$

where $p_i : \Sigma^{\times m} \to \Sigma$ is projection onto the $i$-th factor. Note moreover that this form is $\mathfrak{S}_m$-invariant.

Over $\mathrm{Sym}^m(\Sigma) \setminus \Delta$, the $\mathfrak{S}_m$-invariant form $\nu^\times$ on $\Sigma^{\times m}$ determines a symplectic form $\nu^\times/\mathfrak{S}_m$. Note that $\nu^\times$ is *not* the pull-back of a symplectic form over $\mathrm{Sym}^m(\Sigma)$, as the pull-back of any 2-form by $\pi$ is degenerate at the critical points of $\pi$, which in our case is the preimage of the diagonal $\pi^{-1}(\Delta)$. However, since $\nu^\times$ is invariant under the action of $\mathfrak{S}_m$, its underlying cohomology class is the pullback of some cohomology class over $\mathrm{Sym}^m(\Sigma)$. We can in fact do better: we can find a Kähler form on $\mathrm{Sym}^m(\Sigma)$ which represents the cohomology class of $\nu^\times$ and coincides with $\nu^\times$ outside an open neighborhood of the diagonal $\Delta$.

**Theorem 2.2.3** (Perutz [35])**.** *Let $(j, \nu)$ be a Kähler structure on an oriented two-manifold $\Sigma$, and let $\nu^\times$ be the induced product Kähler form pver $\Sigma^{\times m}$. Given an open set $U$ containing the diagonal $\Delta \subset \mathrm{Sym}^m(\Sigma)$, there is a Kähler form $\omega$ on $\mathrm{Sym}^m(\Sigma)$ equipped with its induced complex structure $\mathrm{Sym}^m(j)$ from Proposition 2.2.1, such that*

$$\pi^*(\omega) - \nu^\times = d\eta,$$

*where $\eta$ is a 1-form over $\Sigma^{\times m}$ with support inside $\pi^{-1}(U)$.*

We will give a detailed outline of the proof of Theorem 2.2.3 in Section 2.3. Perutz's result in Heegaard Floer theory applies as follows. Let $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w)$ be a pointed Heegaard diagram, and let $\mathbb{T}_\alpha = \alpha_1 \times \cdots \times \alpha_g$ and $\mathbb{T}_\beta = \beta_1 \times \cdots \times \beta_g$ be two tori inside $\mathrm{Sym}^m(\Sigma)$. Fix a complex structure $j$ and a positive area form $\nu$ on $\Sigma$. There is a quotient map $\pi : \Sigma^{\times m} \to \mathrm{Sym}^g(\Sigma)$ and a symplectic two-form $\nu^\times = \sum_{i=1}^m p_i^*(\nu)$ on $\Sigma^{\times g}$, where $p_i : \Sigma^{\times g} \to \Sigma$ is projection onto the $i$-th factor. The preimage of $\mathbb{T}_\alpha$ under $\pi$ is a union of $g!$ tori $\bigcup_{\sigma \in \mathfrak{S}_g} \alpha_{\sigma(1)} \times \cdots \times \alpha_{\sigma(g)}$, and similarly, $\pi^{-1}(\mathbb{T}_\beta) = \bigcup_{\sigma \in \mathfrak{S}_g} \beta_{\sigma(1)} \times \cdots \times \beta_{\sigma(g)}$. Note moreover that $\pi^{-1}(\mathbb{T}_\alpha)$ and $\pi^{-1}(\mathbb{T}_\beta)$ are Lagrangian with respect to $\nu^\times$. Indeed, if $q = (q_1, \ldots, q_g)$ is a point in $\pi^{-1}(\mathbb{T}_\alpha)$, say $q_i \in \alpha_{\sigma(i)}$ for some $\sigma \in \mathfrak{S}_g$, and pick tangent vectors $X, Y \in T_q(\pi^{-1}(\mathbb{T}_\alpha))$. Since $T_q(\pi^{-1}(\mathbb{T}_\alpha)) = T_q(\alpha_{\sigma(1)} \times \cdots \alpha_{\sigma(g)}) = T_{q_1}\alpha_{\sigma(1)} \times \cdots \times T_{q_g}\alpha_{\sigma(g)}$, we can write $X = (X_1, \ldots, X_g)$ and $Y = (Y_1, \ldots, Y_g)$ such that $X_i, Y_i \in T_{q_i}\alpha_{\sigma(i)}$. Then

$$\nu_q^\times(X, Y) = \sum_{i=1}^g (p_i^*(\nu))_q(X, Y) = \sum_{i=1}^g \nu_{q_i}(p_i^* X, p_i^* Y) = \sum_{i=1}^g \nu_{q_i}(X_i, Y_i) = 0,$$

where the last step follows from the fact that $\nu$ is Lagrangian on the one-manifold $\alpha_i$ for all $i = 1, \ldots, g$.

A final important fact is that $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ are disjoint from the diagonal $\Delta$, so we can choose an open neighborhood $U$ of the diagonal $\Delta$ which is disjoint from $\mathbb{T}_\alpha \cup \mathbb{T}_\beta$. For a given area form $\nu$ over $\Sigma$, Perutz's Theorem 2.2.3 gives us a symplectic form $\omega$. In summary, we have the following corollary:

**Corollary 2.2.4** (Peurtz [35])**.** *Let $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w)$ be a pointed Heegaard diagram, $j$ be a complex structure over $\Sigma$. Then there exists a Kähler form $\omega$ on the complex manifold $(\mathrm{Sym}^g(\Sigma), \mathrm{Sym}^g(j))$ for which the tori $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ are Lagrangian submanifolds. Moreover, for any positive area form $\nu$ over $\Sigma$, we can choose $\omega$ so that the cohomology classes $[\pi^*(\omega)]$ and $[\nu^\times]$ coincide when thought of as elements of $H^2(\Sigma^{\times g}, \pi^{-1}(\mathbb{T}_\alpha) \cup \pi^{-1}(\mathbb{T}_\beta); \mathbb{R})$.*

## 2.3 Symplectic structure on the symmetric product

Fix an open domain $\Omega \subseteq \mathbb{R}^2$. Recall that a smooth function $\phi : \Omega \to \mathbb{R}$ is *harmonic* if $\Delta\phi = \frac{\partial^2 \phi}{\partial x_1^2} + \frac{\partial^2 \phi}{\partial x_2^2} = 0$. A continuous function $\psi : \Omega \to \mathbb{R}$ is *subharmonic* if for any $x \in \Omega$ there exists an $r > 0$ with $\overline{B_r(x)} \subseteq \Omega$ such that for any harmonic function $h : \overline{B_r(x)} \to \mathbb{R}$ with $\psi \leq h$ on the boundary $\partial B_r(x)$, we have that $\psi \leq h$ on $\overline{B_r(x)}$. The above definitions extend to functions on domains in $\mathbb{C}^n$ as follows.

**Definition 2.3.1.** Fix an open domain $\Omega \subseteq \mathbb{C}^n$.

- A smooth function $\psi : \Omega \to \mathbb{R}$ is *pluriharmonic* if the restriction of $\phi$ to any complex line is harmonic.
- A continuous function $\psi : \Omega \to \mathbb{R}$ is *plurisubharmonic* if the restrictions of $\psi$ to any complex line are subharmonic.
- A continuous function $\psi : \Omega \to \mathbb{R}$ is *strictly plurisubharmonic* if for every $x \in \Omega$ there exists a neighborhood $U$ and a constant $c > 0$ such that the function $y \mapsto \psi(y) - c|y|^2$ is plurisubharmonic on $U$.

Composing a pluriharmonic (resp. plurisubharmonic) function with a holomorphic function gives a pluriharmonic (resp. plurisubharmonic) function, so the above notions extend to functions on complex manifolds. Suppose that $(X, J)$ is a complex manifold. For a smooth function $f : X \to \mathbb{R}$, define the 1-form $d^{\mathbb{C}} f$ by $df \circ J$. Using this notion, one can show that a smooth function $f : X \to \mathbb{R}$ is pluriharmonic if $dd^{\mathbb{C}} f = 0$, see for example [8, Chapter

1]. If the symmetric form $g_f(v, w) = -dd^{\mathbb{C}}(v, Jw)$ is positive-definite, then $f$ is strictly plurisubharmonic, and $-dd^{\mathbb{C}}$ is a Kähler form with $g_f$ a compatible Kähler metric.

**Definition 2.3.2.** For a complex manifold $X$, the set $\{(U_i, \varphi_i)\}_{i\in I}$ is a *Kähler cocycle* if $\{U_i\}_{i\in I}$ provides an open cover of $X$, and $\varphi_i : U_i \to \mathbb{R}$ are continuous functions such that $\varphi_i$ is strictly plurisubharmonic on $U_i$ for all $i \in I$, and $\varphi_i - \varphi_j$ is pluriharmonic on $U_i \cap U_j$ for all $i, j \in I$. The Kähler cocycle is *smooth* if all $\varphi_i$ are smooth functions.

If we have a smooth Kähler cocycle $\{(U_i, \varphi_i)\}_{i\in I}$ on $X$, the cocycle $-dd^{\mathbb{C}}\varphi_i$ patches together (by the second property) to a global 2-form on $X$, which (by the first property) is a $J$-compatible symplectic, and therefore Kähler, form. Note that the definition of a Kähler cocycle only requires the functions $\varphi_i$ to be continuous, so a complex manifold having Kähler cocycle is a weaker condition than having a Kähler form.

We use the following lemma to go between Kähler cocycles and Kähler forms, see for example [19, Lemma 3.A.22].

**Lemma 2.3.3** ($dd^{\mathbb{C}}$-lemma)**.** *Let $X$ be a compact Kähler manifold and let $\alpha$ be a $k$-form $d^{\mathbb{C}}$-exact and $d$-closed. Then there exists a $(k-2)$-form $\beta$ such that $\alpha = dd^{\mathbb{C}}\beta$.*

Perutz's Theorem 2.2.3 rests on the following lemma due to Varouchas [41].

**Lemma 2.3.4.** *Let $U, V, W, \Omega \subseteq \mathbb{C}^n$ be bounded open sets such that*

$$U \subseteq \overline{U} \subseteq V \subseteq \overline{V} \subseteq W \ \ \text{and} \ \ \Omega \subseteq W.$$

*Let $\phi : W \to \mathbb{R}$ be a continuous, structly plurisubharmonic function which is smooth on $\Omega$. Then there exists $\psi : W \to \mathbb{R}$ such that*

- *$\psi$ is continuous and strictly plurisubharmonic,*
- *$\psi|_{W\setminus\overline{V}} = \phi|_{W\setminus\overline{V}}$,*
- *$\psi$ is smooth on $\Omega \cup U$.*

*Proof.* Fix the following data:

- A nonnegative real valued smooth function $\alpha : \mathbb{C}^n \to \mathbb{R}_{\geq 0}$ with support in the unit ball such that $\int_{\mathbb{C}^n} \alpha = 1$. For $\rho > 0$, let $\alpha_\rho : \mathbb{C}^n \to \mathbb{R}_{\geq 0}$ be defined by $\alpha_\rho(z) = \rho^{-1}\alpha(\frac{z}{\rho})$, so that $\operatorname{supp}(\alpha_\rho) \subseteq B(0, \rho)$, where $B(0, \rho)$ is the ball of radius $\rho$ centered at $0 \in \mathbb{C}^n$, and also such that $\int_{\mathbb{C}^n} \alpha_\rho = 1$.
- By Urysohn's lemma, we can choose a smooth function $\eta : W \to [-1, 1]$ such that $\eta = 1$ on $U$ and $\eta = -1$ on $W \setminus V$.
- Let $\xi : \mathbb{R}^2 \to \mathbb{R}$ be a smooth even function with support in $(-1, 1)^2$ such that $\int_{\mathbb{R}^2} \xi = 1$.

Let $W_\rho \subseteq W$ be the set of points $x \in W$ such that the distance of $x$ to the complement $\mathbb{C}^n \setminus W$ is greater than $\rho$. Then the convolution

$$(\phi * \alpha_\rho)(x) = \int_{\mathbb{C}^n} \phi(y)\alpha_\rho(x - y)\, dy$$

is defined on $W_\rho$; indeed, for $x \in W_\rho$ and for $y \in \mathbb{C}^n \setminus W$, the distance between $x$ and $y$ is greater than $\rho$, so $\alpha_\rho(x - y) = 0$. Moreover, the convolution of a smooth and continuous function is smooth, so $\phi * \alpha_\rho$ is smooth on $W_\rho$.

Since $\phi$ is strictly plurisubharmonic, there exists an open subset $W'$ of $W$ and a constant $c > 0$ such that on $W'$, we have that $\phi = \phi_1 + \phi_2$, where $\phi_2$ is $c|x|^2$ and $\phi_1$ is plurisubharmonic. Since $\phi_2$ is strictly subharmonic and being plurisubharmonic is an open condition (see for example [18, Theorem K.2]), there exists a constant $t > 0$ such that $\phi_2 + t\eta$ is plurisubharmonic.

We claim that $\phi * \alpha_\rho$ converges uniformly to $\phi$ as $\rho \to 0$. Fix $\varepsilon > 0$. We have that for $x \in W_\rho$,

$$(\phi * \alpha_\rho)(x) - \phi(x) = \int_{\mathbb{C}^n} (\phi(y) - \phi(x))\alpha_\rho(x - y)\, dy. \tag{2.2}$$

Note that since $\phi$ is continuous, it is unifomly continuous on the compact set $\overline{V}$. Thus choose $\delta > 0$ such that $|\phi(y) - \phi(x)| \leq \varepsilon$ whenever $|y - x| < \delta$. Then (2.2) tells us that for any $x \in W_\rho$ and $\rho < \delta$, so that $\alpha_\rho = 0$ outside of $B(0, \delta)$,

$$|(\phi * \alpha_\rho)(x) - \phi(x)| = \left| \int_{B(0,\delta)} (\phi(y) - \phi(x))\alpha_\rho(x - y)\, dy \right| \leq \int_{B(0,\delta)} \varepsilon\alpha_\rho(x - y) = \varepsilon.$$

Let $\rho > 0$ be chosen such that $\overline{V} \subseteq W_\rho$ so that $\phi * \alpha_\rho$ makes sense in $\overline{V}$ and also so that

$$|\phi * \alpha_\rho - \phi| < \frac{t}{2}$$

on $\overline{V}$.

We now show that $\phi * \alpha_\rho + t\eta$ is strictly plurisubharmonic on a neighborhood of $\overline{V}$. From our splitting $\phi = \phi_1 + \phi_2$ from above, we can write

$$\phi * \alpha_\rho = \phi_1 * \alpha_\rho + \phi_2 * \alpha_\rho.$$

Since $\phi_1$ is plurisubharmonic, and since convolution preserves this property, we have that $\phi_1 * \alpha_\rho$ is plurisubharmonic. It remains to show that $\phi_2 * \alpha_\rho + t\eta$ is strictly plurisubharmonic. For this, note that for $0 < t < \frac{c}{\sup |dd^{\mathbb{C}}\eta|}$,

$$dd^{\mathbb{C}}(\phi_2 * \alpha_\rho + t\eta) = (dd^{\mathbb{C}}\phi_2) * \alpha_\rho + tdd^{\mathbb{C}}\eta = c + tdd^{\mathbb{C}}\eta > 0.$$

For $\delta \leq \frac{t}{4}$, let $M_\delta : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$M_\delta(x, y) = \int_{\mathbb{R}^2} \max\{x - \delta t_1, y - \delta t_2\}\xi(t_1, t_2)\, dt_1 dt_2.$$

**Lemma 2.3.5.** *If $|x - y| \geq 2\delta$, we have that $M_\delta = \max\{x, y\}$.*

*Proof.* If $\xi(x, y) \neq 0$, then $t_1, t_2 \in (-1, 1)$. Suppose that $x - y \geq 2\delta$. Then $x - t_1\delta > y - t_2\delta$, so

$$M_\delta(x, y) = \int \max\{x - \delta t_1, y - \delta t_2\}\xi = \int (x - \delta t_1)\xi = \int x\xi = x,$$

where we used the fact that $\xi$ is even in the second to last step. Similarly, for $y - x \geq 2\delta$, we have that $M_\delta(x, y) = y$. □

Consider the function

$$\psi_\delta(z) = M_\delta(\phi(z), (\phi * \alpha_\rho + t\eta)(z)).$$

**Lemma 2.3.6.** *$\psi_\delta$ is strictly plurisubharmonic and equal to $\phi$ in a neighborhood of $\partial V$.*

*Proof.* Indeed if $f, g$ are strictly plurisubharmonic, so are $f - \delta t_1$ and $g - \delta t_2$ and $\max\{f, g\}$ (see for example [18, Theorem K.5]). As we have shown that $\phi * \alpha_\rho + t\eta$ is strictly plurisubharmonic and $\phi$ is strictly plurisubharmonic by assumption, we have that so is $\psi_\delta$. Since on $\overline{V}$ we have that $\phi * \alpha_\rho - \phi < \frac{t}{2}$, it follows that on a sufficiently small neighborhood of $\partial V$,

$$\phi * \alpha_\rho + t\eta < \phi - \frac{t}{2} \leq \phi - 2\delta.$$

By Lemma 2.3.5, the claim follows. $\square$

By the behavior of $\psi_\delta$ near $\partial V$, we can define

$$\psi(x) = \begin{cases} \psi_\delta(x) & \text{if } x \in V, \\ \phi(x) & \text{if } x \in W \setminus V. \end{cases}$$

With this in place, we claim that $\psi$ is our desired function. First note that $\psi$ is continuous and equal to $\phi$ on $W \setminus V$ by construction. Since $\psi_\delta$ is strictly plurisubharmonic by Lemma 2.3.6 and $\phi$ is strictly plurisubharmonic by assumption, we have that $\psi$ is strictly plurisubharmonic. It remains to show that $\psi$ is smooth on $U \cup \Omega$. Since $\psi = \phi$ on $W \setminus V$, and $\phi$ is smooth on $\Omega$, it remains to show that $\phi$ is smooth on $V \cap (U \cup \Omega)$. On $U$, we have that $\eta = 1$, so using the fact that $|\phi * \alpha_\rho - \phi| < \frac{t}{2}$ on $V$, we have that

$$\phi * \alpha_\rho + t\eta > \phi + \frac{t}{2} \geq \phi + 2\delta.$$

Thus by Lemma 2.3.5, the function $\psi_\delta = \psi$ is equal to $\phi * \alpha_\rho + \eta$, which is smooth on $U$. On $V \cap \Omega$, we have that $\phi$ is smooth there, $\phi * \alpha_\rho + t\eta$ is smooth there, and $M_\delta$ is smooth on $\mathbb{R}^2$. Thus $\psi$ is a composition of smooth functions on $V \cap \Omega$, establishing smoothness of $\psi$ on $V \cap \Omega$. Therefore, $\psi$ is smooth on $U \cup \Omega$, concluding the proof. $\square$

The following proposition allows us to exchange a continuous Kähler cocycle with a smooth one.

**Proposition 2.3.7.** *Let $\{(U_i, \varphi_i)\}_{i\in I}$ be a continuous Kähler cocycle on a complex manifold $X$. Let $X_1, X_2$ be open subsets of $X$ such that $X = X_1 \cup X_2$ and $\varphi_i|_{U_i\cap X_1}$ are smooth. Then there exists a continuous function $\chi : X \to \mathbb{R}$ with support $\mathrm{supp}(\chi) \subseteq X_2$ and a refinement $V_j \subseteq U_{i(j)}$ for $j \in J$ such that*

$$\{(V_j, (\varphi_{i(j)} + \chi)|_{V_j})\}_{j\in J}$$

*is a smooth Kähler cocycle on $X$.*

*Proof.* First refine the open cover $\{U_i\}$ such that every element of the cover is either in $X_1$ or in $X_2$, and assume that the original cover is countable; if it is finite, repeat the last set infinitely many times. After performing another refinement, we may assume that the cover is locally finite, meaning that every point is contained in finitely many distinct open subsets of the cover. Let $\{V_{i,1}\}_{i\in\mathbb{N}}$ and $\{V_{j,2}\}_{j\in\mathbb{N}}$ denote the open sets in $X_1$ and $X_2$ respectively; the Kähler cocycles corresponding to this refinement is $\{(V_{i,1}, \varphi_{i,1})\}_{i\in\mathbb{N}} \cup \{(V_{j,2}, \varphi_{j,2})\}_{j\in\mathbb{N}}$.

Find open sets $V''_{j,2}$ and $V'_{j,2}$ for every $j \in \mathbb{N}$ such that

$$V''_{j,2} \subseteq \overline{V''_{j,2}} \subseteq V'_{j,2} \subseteq \overline{V'_{j,2}} \subseteq V_{j,2}$$

and such that $\{V''_{j,2}\}_{j\in\mathbb{N}}$ covers $X_2 \setminus X_1$.

We now construct Kähler cocycles $\{(V_{i,k}, \psi^n_{i,k})\}_{i\in\mathbb{N},k=1,2}$, indexed by a natural number $n$, such that $\psi_{i,k}$ is smooth on the set

$$V_{i,k} \cap (X_1 \cup \bigcup_{j=1}^{n-1} V''_{j,2}).$$

Let $\psi^1_{i,k} = \varphi_{i,k}$. To define $\psi^n_{i,k}$ for $n \geq 2$, we define appropriate functions $\chi_n : X \to \mathbb{R}$ and then let

$$\psi^n_{i,k} = \psi^{n-1}_{i,k} + \chi_n.$$

Define $\chi_n$ as follows. If $\psi^{n-1}_{n-1,2}$ is already smooth, let $\chi_n = 0$. Otherwise, apply Varouchas'

Lemma 2.3.4 to the sets

$$(U, V, W, \Omega) = \left( V''_{n-1,2}, V'_{n-1,2}, V_{n-1,2}, V_{n-1,2} \cap \left( X_1 \cup \bigcup_{j=1}^{n-2} V''_{j,2} \right) \right)$$

with the strictly plurisubharmonic function $\psi^{n-1}_{n-1,2}$. First note that for $n = 2$, the function $\psi^{n-1}_{n-1,2} = \varphi_{n-1,2}$ is strictly plurisubharmonic by assumption, and for all $n \geq 2$, the function $\psi^{n-1}_{n-1,2}$ is smooth on $\Omega$. Thus, Varouchas' lemma gives us a new function $\psi^{n}_{n-1,2}$ which is smooth on $V_{n-1,2} \cap (X_1 \cup \bigcup_{j=1}^{n-1} V''_{j,2})$. Now define

$$\chi_n = \psi^{n}_{n-1,2} - \psi^{n-1}_{n-1,2}$$

on $V_{i,k}$ and $\chi_n$ extend to zero on the complement of $V_{i,k}$. Now as above, define $\psi^n_{i,k} = \psi^{n-1}_{i,k} + \chi_n$.

Now we show that $\{(V_{i,k}, \psi^n_{i,k})\}$ is a Kähler cocycle. For $x \notin V_{n-1,2}$, we have that $\psi^n_{i,k}(x) = \psi_{i,k}$, which was already smooth on $V_{i,k} \cap (X_1 \cup \bigcup_{j=1}^{n-1} V''_{j,2})$. For $x \in V_{n-1,2}$, using the definition of $\chi_n$, we have that

$$\psi^n_{i,k} = (\psi^n_{i,k} - \psi^n_{n-1,2}) + \psi^n_{n-1,2} = (\psi^1_{i,k} - \psi^1_{n-1,2}) + \psi^n_{n-1,2} = (\varphi_{i,k} - \varphi_{n-1,2}) + \psi^n_{n-1,2}.$$

Note that the first summand on the right is pluriharmonic by the definition of Kähler cocycle, and the second summand is strictly plurisubharmonic, as we have already seen from Varouchas' Lemma. Thus $\psi^n_{i,k}$ is strictly plurisubharmonic. Moreover, the first and second summands on the right are smooth on $V_{n-1,2} \cap (X_1 \cup \bigcup_{j=1}^{n-1} V''_{j,2})$. Thus, $\{(V_{i,k}, \psi^n_{i,k})\}$ is a Kähler cocycle.

Now consider the function $\chi : X \to \mathbb{R}$ given by

$$\chi(x) = \sum_{n=1}^{\infty} \chi_n(x).$$

This definition makes sense because by construction, $\mathrm{supp}(\chi_n) \subseteq V''_{n-1,2}$, and the original cover $\{V_{i,k}\}$ is locally finite. Finally, define

$$\zeta_{i,k} = \psi^1_{i,k} + \chi + \varphi_{i,k} + \chi.$$

We show that $\{(V_{i,k}, \zeta_{i,k})\}$ is a smooth Kähler cocycle. On $X \setminus \bigcup_{j=1}^{\infty} V''_{j,2} \subseteq X_1$, the cocycle $\varphi_{i,k}$ was already smooth. Now we prove smoothness of $\zeta_{i,k}$ on $V''_{j,2}$ for a given positive integer $j$. Let $x \in V''_{j,2}$, and let $n$ be the largest integer for which $\chi_n(x)$ is nonzero. By making $n$ even larger, we may assume that $V''_{j,2} \subset X_1 \cup \bigcup_{i=1}^{n-1} V''_{i,2}$. Then $\zeta_{i,k}(x) = \psi^n_{i,k}(x)$, which is smooth on the domain $X_1 \cup \bigcup_{i=1}^{n-1} V''_{i,2}$. This concludes the proof. □

*Proof outline of Theorem 2.2.3.* For $x \in \Sigma^{\times m}$, find a neighborhood $U_x$ biholomorphic to the disk $D^{2m}$. Let $\phi_x : U_x \to \mathbb{R}$ be a solution to

$$-dd^{\mathbb{C}}\phi_x = \nu^{\times}|_{U_x}.$$

Indeed, we can find such a $\phi_x$ by the $dd^{\mathbb{C}}$-lemma 2.3.3 as follows. Using the contractibility of $U_i$, we have that $\omega$ is $d$-exact by the Poincaré Lemma. In particular $\omega$ is $d$-closed, and since $\omega$ is $J$-invariant, we know that $\omega$ is $d^{\mathbb{C}}$-exact.

For $x \in \mathrm{Sym}^m(\Sigma)$ choose an open neighborhood $V_{x'}$ such that

$$\pi^{-1}(V_x) \subset \bigcup_{x \in \pi^{-1}(x)} U_x.$$

By compactness of $\Sigma^{\times m}$ we can choose a finite subcover $\{V_{x'_1}, \dots, V_{x'_k}\}$ of $\{V_{x'} : x' \in \mathrm{Sym}^m(\Sigma)\}$. Then consider a finite subcover $\{W_1, \dots, W_K\}$ of

$$\{W_x = U_x \cap \pi^{-1}(V_{x'_i}) : \pi(x) = x'_i\}$$

covering $\Sigma^{\times m}$. Restricting $\phi_x$ to $W_x$, we have a smooth Kähler cocycle $\{(W_j, \phi_j)\}_{j=1}^K$ over

$\Sigma^{\times m}$. For a point $y \in W_x$, consider the function $\varphi_x : W_x \to \mathbb{R}$ given by

$$\varphi_x(y) = \frac{1}{m!} \sum_{\sigma \in \mathfrak{S}_m} \phi_{\sigma(x)}(\sigma(y)),$$

where the action of $\sigma \in \mathfrak{S}_m$ on a point $x = (x_1, \ldots, x_m)$ in $\Sigma^{\times m}$ is the point $(x_{\sigma(1)}, \ldots, x_{\sigma(m)})$. Since $\{(W_j, \phi_j)\}$ is a smooth Kähler cocycle, it follows that $\{(W_j, \varphi_j)\}$ is also a smooth Kähler cocycle on $\Sigma^{\times m}$. Moreover, it follows from the definition that the cocycle $\{(W_x, \varphi_x)\}$ is $\mathfrak{S}_m$-equivariant, in the sense that $\varphi_x(y) = \varphi_{\sigma(x)}(\sigma(y))$ for any $\sigma \in \mathfrak{S}_m$.

Let $\phi'_i : V_{x'_i} \to \mathbb{R}$ be the push-forward of $\{(W_j, \varphi_j)\}$ by $\pi$, in the sense that for $x' \in V_{x'_i}$,

$$\phi'_i(x') = \frac{1}{m} \sum_{x \in \pi^{-1}(x')} \varphi_i(x),$$

where the points in $\pi^{-1}(x)$ are counted with multiplicity. Since the $\varphi_i$ are continuous, so is $\phi'_i$. However, even though $\varphi_i$ is smooth, it may not be the case that $\phi'_i$ are smooth. See Example 2.3.8 for an example of a smooth function whose pushforward under a branched covering is not smooth. By the $\mathfrak{S}_m$-equivariance of $\varphi_j$, we have that the pull-back of $\{(V_{x'_i}, \phi'_i)\}$ is the cocycle $\{(W_i, \varphi_i)\}$.

Perhaps unsurprisingly, $\{(V_{x_i}, \phi'_i)\}$ is a Kähler cocycle. For the details we refer to [29, Theorem 7.6.1] and [15].

Now we replace $\{(V_{x_i}, \phi'_i)\}$ by a smooth Kähler cocycle. Recall the open neighborhood $U$ of the diagonal $\Delta$ given in the statement. Let $U'$ be an open subset of $U$ such that $\overline{U'} \subseteq U$, and let $X_1 = \mathrm{Sym}^m(\Sigma) \setminus \overline{U'}$ and $X_2 = U$. Note that $\varphi'_i$ is in fact smooth when restricted to $V_{x_i} \cap X_1$, as $X_1$ is an open subset disjoint from diagonal. So we can apply Proposition 2.3.7 to obtain a continuous functiopn $\chi : \mathrm{Sym}^m(\Sigma) \to \mathbb{R}$ such that $\mathrm{supp}(\chi) \subseteq U$ and $\{\phi'_i + \chi\}$ is a smooth Kähler cocycle. We also get a corresponding symplectic form $\omega = -dd^{\mathbb{C}}(\phi'_i + \chi)$. Since the pull-back $\pi^*(\omega)$ can be represented by the pullback of the Kähler cocycle $\{\phi'_i + \chi\}$, and the pull-back of $\{\phi'_i + \chi\}$ is a smooth Kähler cocycle defining $\nu^{\times}$, we have that the pull-back $\widetilde{\chi}$ of $\chi$ is a smooth function such that

$$\pi^*(\omega) - \nu^\times = -dd^{\mathbb{C}}\widetilde{\chi}.$$

Letting $\eta$ be the one-form $-d^{\mathbb{C}}\widetilde{\chi}$ over $\Sigma^{\times m}$, we have that $\mathrm{supp}(\chi) \subseteq \pi^{-1}(U)$. This concludes the proof. $\square$

In the proof of Perutz's lemma, we had to be careful because the push-forward of a smooth Kähler cocycle is not necessarily smooth, as the following example illustrates.

**Example 2.3.8.** For an example where the push-forward of a smooth function is not smooth, consider the double branched cover map $\pi : \mathbb{D} \to \mathbb{D}$ on the unit disc $\mathbb{D} \subseteq \mathbb{C}$, given by $\pi(z) = z^2$. Let $f : \mathbb{D} \to \mathbb{R}$ be given by $f(z) = (\mathrm{Re}(z))^2$. Then the pushforward $g : \mathbb{D} \to \mathbb{R}$ of $f$ by $\pi$, defined by $g(z') = \frac{1}{2}\sum_{z\in\pi^{-1}(z')} f(z)$ is continuous, but not smooth. Indeed, the restriction of $g$ to the real axis is identically zero for negative values and the identity map for nonnegative values.

## 2.4 Definition of Heegaard Floer homology

The Heegaard Floer complex is obtained by modifying constructions in Lagrangian Floer homology to the symplectic manifold $(\mathrm{Sym}^g(\Sigma), \omega)$. To describe these modifications, fix a basepoint $w \in \Sigma$, giving us a submanifold $\{w\} \times \mathrm{Sym}^{g-1}(\Sigma) \subset \mathrm{Sym}^g(\Sigma)$. This thesis defines the simplest version of Heegaard Floer homology, which informally corresponds to counting pseudo-holomorphic disks $u$ in $\mathrm{Sym}^g(\Sigma)$ whose algebraic intersection number with $\{w\} \times \mathrm{Sym}^{g-1}(\Sigma)$ is zero. This variant is the hat version $\widehat{\mathrm{HF}}$ of Heegaard Floer homology. There are further variants of Heegaard Floer homology if we choose to count pseudo-holomorphic disks intersecting the divisor.

We begin with the notion of the local multiplicity of a Whitney disk at a basepoint. Fix a point $w \in \Sigma \setminus (\boldsymbol{\alpha} \cup \boldsymbol{\beta})$. Given a Whitney disk $u$ conecting $\mathbf{x}$ and $\mathbf{y}$, the *local multiplicity of $u$ at $w$*, denoted $n_w(u)$, is the algebraic intersection number of $u$ with the submanifold $\{w\}\times\mathrm{Sym}^{g-1}(\Sigma) \subset \mathrm{Sym}^g(\Sigma)$. The local multiplicity of $u$ at $w$ depends only on the homotopy class of $u$, inducing a function

$$n_w : W(\mathbf{x}, \mathbf{y}) \to \mathbb{Z}.$$

We will also need to choose special kinds of paths of almost-complex structures that ensure that we have nonnegative local multiplcities of Whitney disks. Call a path of almost-complex structure $\{J^s\}_{s\in[0,1]}$ adapted to the basepoint $w$, or *$w$-adapted*, if for any $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ and $\phi \in W(\mathbf{x}, \mathbf{y})$ with local multiplicity $n_w(\phi) < 0$, the space $\mathcal{M}_{\{J^s\}}(\phi)$ of pseudo-holomorphic representatives of $\phi$ is empty. Call a path of $\omega$-tame almost-complex structures $\{J^s\}_{s\in[0,1]}$ on $\mathrm{Sym}^g(\Sigma)$ *regular* if for any pair of Heegaard states $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ and any Whitney disk $\phi \in W(\mathbf{x}, \mathbf{y})$, the moduli space $\mathcal{M}_{\{J^s\}}(\psi)$ is a smooth manifold of dimension $\mu(\psi)$. If $\mu(\psi) = 0$, then $\mathbf{x} = \mathbf{y}$ and $\mathcal{M}_{\{J^s\}}(\psi)$ contains only the constant flowline. In [29, Theorem 9.2.1], it is shown that there exist regular paths of $\omega$-tame almost-complex structures on $\mathrm{Sym}^g(\Sigma)$ that are $w$-adapted.

### 2.4.1 For rational homology spheres

First we define $\widehat{\mathrm{HF}}$ for rational homology spheres, i.e. those closed oriented three-manifolds $Y$ for which $H_*(Y;\mathbb{Q}) = H_*(S^3;\mathbb{Q})$, or equivalently, those $Y$ for which $b_1(Y) = 0$.

Start with a closed, oriented three-manifold $Y$ with $b_1(Y) = 0$. Fix the following extra data:

- A pointed Heegaard diagram $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w)$ representing $Y$,
- A complex structure $j$ over $\Sigma$,
- An area form $\nu$ compatible with $j$,
- A symplectic structure on $\mathrm{Sym}^g(\Sigma)$ chosen with respect to $\nu$, as in Corollary 2.2.4,
- A regular path of $\omega$-tame almost-complex structures $\{J^s\}$ that is $w$-adapted.

Let $\widehat{\mathrm{CF}}(\mathcal{H}, j, \{J^s\}_{s\in[0,1]})$ be the vector space over $\mathbb{Z}/2\mathbb{Z}$ generated by the intersection points

$\mathbb{T}_\alpha \cap \mathbb{T}_\beta$ equipped with the differential

$$\widehat{\partial}\mathbf{x} = \sum_{\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta} \sum_{\left\{\phi\in W(\mathbf{x},\mathbf{y}) \,\middle|\, \substack{n_w(\phi)=0\\ \mu(\phi)=1}\right\}} \#\widehat{\mathcal{M}}_{\{J^s\}}(\phi)\cdot \mathbf{y}.$$

Note that for any $\mathbf{x}, \mathbf{y}$ with nonempty $W(\mathbf{x}, \mathbf{y})$, there is a single $\phi \in W(\mathbf{x}, \mathbf{y})$ with $\mu(\phi) = 1$. Thus, $\widehat{\partial}\mathbf{x}$ makes sense. Moreover, $\widehat{\partial}^2 = 0$, making $\widehat{\mathrm{CF}}(\mathcal{H}, j, \{J^s\}_{s\in[0,1]})$ into a chain complex with differential $\widehat{\partial}$, see [29, Proposition 9.2.5]. The proof, not shown here, is an application of Gromov compactness to the space of pseudo-holomorphic disks from $\mathbf{x}$ to $\mathbf{y}$ with local multiplicity zero and Maslov index two.

### 2.4.2 For positive first Betti number

To define Heegaard Floer homology for closed connected oriented three-manifolds $Y$, one uses special kinds of Heegaard diagrams:

**Definition 2.4.1.** A pointed Heegaard diagram $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w)$ is *admissible* if every nonzero periodic domain has positive and negative local multiplicities.

The admissibility of the Heegaard diagram is used in the following sense. For an admissible pointed Heegaard diagram $\mathcal{H}$ representing a three-manifold, there exists a path $\{J^s\}_{s\in[0,1]}$ of almost-complex structures on $\mathrm{Sym}^g(\Sigma)$ such that for any $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ and an integer $k$, the moduli space

$$\bigcup_{\{\phi\in W(\mathbf{x},\mathbf{y})|\mu(\phi)=1, n_w(\phi)=k\}} \widehat{\mathcal{M}}_{\{J^s\}}(\phi)$$

is compact. See [29, Proposition 9.4.5].

**Definition 2.4.2.** Let $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w)$ be an admissible pointed Heegaard diagram. Let $\widehat{\mathrm{CF}}(\mathcal{H})$ be the vector space over $\mathbb{Z}/2\mathbb{Z}$ generated by the Heegaard states $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$, together with the endomorphism

$$\widehat{\partial}\mathbf{x} = \sum_{\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta} \sum_{\left\{\phi\in W(\mathbf{x},\mathbf{y}) \,\middle|\, \substack{n_w(\phi)=0\\ \mu(\phi)=1}\right\}} \#\widehat{\mathcal{M}}_{\{J^s\}}(\phi)\cdot\mathbf{y}.$$

for $\mathbf{x}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta$.

The admissiblity of the Heegaard diagram and the lemma above are used to show that $\widehat{\mathrm{CF}}(\mathcal{H})$ is indeed a chain complex. See [29, Section 9.4].

### 2.4.3 Decomposition into $\mathrm{Spin}^c$ structures

As mentioned before, Heegaard Floer homology decomposes along $\mathrm{Spin}^c$-structures. Recall that there is an obstruction $\epsilon(\mathbf{x},\mathbf{y})$ to finding a domain connecting Heegaard states $\mathbf{x}$ and $\mathbf{y}$. Also recall from (2.1) that the choice of a basepoint $w\in\Sigma$ gives a injective map

$$s_w : \{\text{equivalence classes of Heegaard states}\} \to \mathrm{Spin}^c(Y).$$

Given $\mathfrak{s}\in\mathrm{Spin}^c(Y)$, let $\widehat{\mathrm{CF}}(\mathcal{H},\mathfrak{s})$ be the vector space generated by $\mathbf{x}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta$ for which $s_w(\mathbf{x})=\mathfrak{s}$. Then we have a splitting of vecor spaces

$$\widehat{\mathrm{CF}}(\mathcal{H}) = \bigoplus_{\mathfrak{s}\in\mathrm{Spin}^c(Y)} \widehat{\mathrm{CF}}(\mathcal{H},\mathfrak{s}).$$

Moreover, if $\mathbf{x}\in\widehat{\mathrm{CF}}(\mathcal{H},\mathfrak{s})$, then $\widehat{\partial}\mathbf{x}$ consists of terms all of which are equivalent to $\mathbf{x}$, so $\widehat{\partial}\mathbf{x}$ also belongs to $\widehat{\mathrm{CF}}(\mathcal{H},\mathfrak{s})$. Equivalently, $\widehat{\mathrm{CF}}(\mathcal{H},\mathfrak{s})$ is a subcomplex of $\widehat{\mathrm{CF}}(\mathcal{H})$, so the above centered equation induces a splitting

$$\widehat{\mathrm{HF}}(\mathcal{H}) = \bigoplus_{\mathfrak{s}\in\mathrm{Spin}^c(Y)} \widehat{\mathrm{HF}}(\mathcal{H},\mathfrak{s}).$$

The key invariance statement is the following. See [29, Theorem 13.1.1] and [31].

**Theorem 2.4.3.** *Let $Y$ be a closed oriented three-manifold with a $\mathrm{Spin}^c$-structure $\mathfrak{s}$ over $Y$.*

*Then the Heegaard Floer homology* $\widehat{\mathrm{HF}}(Y)$ *is an invariant of the underlying three-manifold* $Y$ *and its* $\mathrm{Spin}^c$ *structure* $\mathfrak{s}$.

To prove such a result, one first shows that the homology groups of the Heegaard Floer complex are independent of the analytical choices, which follows the route in Lagrangian Floer homology outlined in Section 1.3. Then one shows invariance from the chosen Heegaard diagram by showing invariance under three operations by which any two Heegaard diagrams representing the same three-manifold can be connected.

## 2.5 Counts of pseudo-holomorphic polygons

In this section, study disks bounding more than two Lagrangian tori.

First, we adapt the notions of admissibility and almost-complex structures to triples of Lagrangian tori. A *Heegaard triple* $\mathcal{H}_{\alpha,\beta,\gamma} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$ is a surface of genus $g$ equipped with three complete sets of attaching circles $\boldsymbol{\alpha}, \boldsymbol{\beta}$, and $\boldsymbol{\gamma}$. We will assume that the pointed Heegaard triple is generic, in the sense that any two simple closed curves in $\boldsymbol{\alpha} \cup \boldsymbol{\beta} \cup \boldsymbol{\gamma}$ intersect transversally.

Fix a genus $g$ pointed Heegaard triple $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, w)$. Let $\Delta$ be the standard 2-simplex with edges labeled $e_\alpha, e_\beta, e_\gamma$ in clockwise order. A *Whitney triangle* is a continuous map $u : \Delta \to \mathrm{Sym}^g(\Sigma)$ such that $u(e_\alpha) \subset \mathbb{T}_\alpha, u(e_\beta) \subset \mathbb{T}_\beta$, and $u(e_\gamma) \subset \mathbb{T}_\gamma$. Fix three Heegaard states $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, $\mathbf{y} \in \mathbb{T}_\beta \cap \mathbb{T}_\gamma$, and $\mathbf{z} \in \mathbb{T}_\alpha \cap \mathbb{T}_\gamma$. If the three vertices, taken in clockwise order, map to $\mathbf{x}, \mathbf{y}$, and $\mathbf{z}$ respectively, then $u$ is a *Whitney triangle connecting* $\mathbf{x}, \mathbf{y}$*, and* $\mathbf{z}$. Two Whitney triangles are homotopic if they can be connected by a continuous one-parameter family of Whitney triangles. Denote $W(\mathbf{x}, \mathbf{y}, \mathbf{z})$ to be the space of homotopy classes of Whitney triangles connecting $\mathbf{x}, \mathbf{y}, \mathbf{z}$.

Given a point $w \in \Sigma \setminus (\boldsymbol{\alpha} \cup \boldsymbol{\beta} \cup \boldsymbol{\gamma})$ and a Whitney triangle $u$, one can similarly define the *local multiplicity of* $u$ *at* $w$ to be the algebraic intersection number of the image $u(\Delta)$ with the submanifold $\{w\} \times \mathrm{Sym}^{g-1}(\Sigma) \subset \mathrm{Sym}^g(\Sigma)$. Again this only depends on the homotopy class of $u$, giving us a well-defined function $n_w : W(\mathbf{x}, \mathbf{y}, \mathbf{z}) \to \mathbb{Z}$.

### 2.5.1 Triangles of almost complex structures

Let $\Delta \subset \mathbb{C}$ be the triangle with vertices $i$ and $\pm\frac{\sqrt{3}}{2} - \frac{1}{2}i$, labeled $v_0, v_1, v_2$ in some order. Let $\mathcal{J}(\mathrm{Sym}^g(\Sigma))$ denote the space of almost-complex structures over $\mathrm{Sym}^g(\Sigma)$. A triangle of almost-complex structures is a continuous map $J^{\triangle} : \Delta \setminus \{v_0, v_1, v_2\} \to \mathcal{J}(\mathrm{Sym}^g(\Sigma))$. A Whitney triangle $u : \Delta \to \mathrm{Sym}^g(\Sigma)$ is $J^{\triangle}$-*pseudo-holomorphic* if

$$(d_p u) \circ (j_p) = (J_p^{\triangle}) \circ (d_p u)$$

for every $p \in \Delta \backslash \{v_0, v_1, v_2\}$, where $j$ is the complex structures that $\Delta$ gets from $\mathbb{C}$. The space of $J^{\triangle}$-pseudo-holomorphic triangles representing a given homotopy class $\psi \in W(\mathbf{x}, \mathbf{y}, \mathbf{z})$ is denoted $\mathcal{M}(\psi)$.

Note that $\mathcal{M}(\psi)$ has no non-trivial complex automorphisms, so there is no natural $\mathbb{R}$-action to quotient by as there was in the case of pseudo-holomorphic disks.

Call a triangle of almost-structures *regular* if, informally, the restrictions of the map to neighborhoods of the corners of the triangle are regular paths of almost-complex structures, and for any $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, $\mathbf{y} \in \mathbb{T}_\beta \cap \mathbb{T}_\gamma$, and $\mathbf{z} \in \mathbb{T}_\alpha \cap \mathbb{T}_\gamma$, and Whitney triangle $\psi \in W(\mathbf{x}, \mathbf{y}, \mathbf{z})$, the moduli space $\mathcal{M}_{\{J^{\triangle}\}}(\psi)$ is a smooth manifold of dimension $\mu(\psi)$. In particular, when $\mu(\psi)$ is negative, $\mathcal{M}_{\{J^{\triangle}\}}(\psi)$ is empty. Call a triangle of almost-structures *adapted* if for any choice of basepoint $w$, there is no pseudo-holomorphic triangle $u$ for which $n_w(u) < 0$. The existence of a triangle of $\omega$-tame almost-complex structures on $\mathrm{Sym}^g(\Sigma)$ which is adapted and regular is proven in [29, Theorem 12.2.6].

### 2.5.2 Counting pseudo-holomorphic triangles

Fix a triangle of $\omega$-tame almost-complex structures on $\mathrm{Sym}^g(\Sigma)$ which is adapted and regular. Given an admissible pointed Heegaard triple $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, w)$, we may define a linear map

$$\widehat{m}_{\alpha,\beta,\gamma} : \widehat{\mathrm{CF}}(\mathcal{H}_{\alpha,\beta}) \otimes \widehat{\mathrm{CF}}(\mathcal{H}_{\beta,\gamma}) \to \widehat{\mathrm{CF}}(\mathcal{H}_{\alpha,\gamma})$$

given by

$$\widehat{m}_{\alpha,\beta,\gamma}(\mathbf{x} \otimes \mathbf{y}) = \sum_{\mathbf{z}\in\mathbb{T}_\alpha\cap\mathbb{T}_\gamma} \sum_{\left\{\psi\in W(\mathbf{x},\mathbf{y},\mathbf{z})\middle|\substack{n_w(\psi)=0\\ \mu(\psi)=0}\right\}} \#\mathcal{M}_{\{J^\triangle\}}(\psi)\cdot\mathbf{z}. \tag{2.3}$$

Note that we may compute the Maslov index of a Whitney triangle by the Maslov index formula for polygons shown in Section 1.3.

**Theorem 2.5.1.** *For an admissible Heegaard triple, the map*

$$\widehat{m}_{\alpha,\beta,\gamma} : \widehat{\mathrm{CF}}(\mathcal{H}_{\alpha,\beta}) \otimes \widehat{\mathrm{CF}}(\mathcal{H}_{\beta,\gamma}) \to \widehat{\mathrm{CF}}(\mathcal{H}_{\alpha,\gamma})$$

*well defined and is a chain map.*

*Proof.* To see that the sum $\widehat{m}_{\alpha,\beta,\gamma}(\mathbf{x}\otimes\mathbf{y})$ is finite, we consider the set

$$\bigcup_{\left\{\substack{\mathbf{x}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta\\ \mathbf{y}\in\mathbb{T}_\beta\cap\mathbb{T}_\gamma\\ \mathbf{z}\in\mathbb{T}_\alpha\cap\mathbb{T}_\gamma}\right\}} \bigcup_{\left\{\psi\in W(\mathbf{x},\mathbf{y},\mathbf{z})\,\middle|\,\substack{\mu(\psi)=0\\ n_w(\psi)=0}\right\}} \mathcal{M}_{\{J^\triangle\}}(\psi).$$

It is a zero-manifold by our choice of triangle of almost-complex structures. We show that it is compact. First, there are uniform bounds on the energy of an element in the above space because $n_w(u)$ is fixed and hence bounded, see [29, Proposition 12.2.7]. Thus, given a sequence in the space, we may take the Gromov limit. Let $(v, u_1, \ldots, u_n)$ be the components of the Gromov limit, where $v$ is the limiting bigon and $u_1, \ldots, u_n$ are limiting spheres or bigons. By additivity of the Maslov index, Proposition 1.3.3, we have that $\mu(v) + \sum_{i=1}^n \mu(u_i) = 0$. By regularity of the triangle of almost complex structures, the Maslov index of $v$ cannot be negative. Thus $\mu(v) = 0$. Again by regularity we have that all bigons and sphere have Maslov index zero, which implies that they are constant. Thus, the bigons and spheres are excluded in the Gromov limit. Thus, the limit consists of one pseudo-holomorphic triangle $v$, that the sum $\widehat{m}_{\alpha,\beta,\gamma}(\mathbf{x}\otimes\mathbf{y})$ is finite.

Now to show that $\widehat{m}_{\alpha,\beta,\gamma}(\mathbf{x}\otimes\mathbf{y})$, we consider the set

$$\bigcup_{\left\{\substack{\mathbf{x}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta \\ \mathbf{y}\in\mathbb{T}_\beta\cap\mathbb{T}_\gamma \\ \mathbf{z}\in\mathbb{T}_\alpha\cap\mathbb{T}_\gamma}\right\}} \bigcup_{\left\{\psi\in W(\mathbf{x},\mathbf{y},\mathbf{z}) \,\middle|\, \substack{\mu(\psi)=1 \\ n_w(\psi)=0}\right\}} \mathcal{M}_{\{J^\triangle\}}(\psi).$$

The above set is a one-manifold by our choice of triangle of almost-complex structures. The same argument as the one in the previous paragraph shows that the above set must be compact. Examining the ends of this space, we get the relation showing that $\widehat{m}_{\alpha,\beta,\gamma}$ is a chain map. $\square$

### 2.5.3 Counts of pseudo-holomorphic polygons

Define a *pointed Heegaard $n$-tuple*, for a positive integer $n\geq 2$, to be the data $(\Sigma,\boldsymbol{\alpha}^1,\ldots,\boldsymbol{\alpha}^n,w)$, where $\Sigma$ is an oriented surface of genus $g$, $\{\boldsymbol{\alpha}^i\}$ are each complete sets of $g$-tuples of attaching circles for all $i\in\{1,\ldots,n\}$, and $w$ is disjoint from all the $\alpha$-curves. We may define similar notions of periodic domains, leading to the notion of admissibility. Given an admissible pointed Heegaard $n$-tuple, there is a map

$$\widehat{m}_{n-1}:\widehat{\mathrm{CF}}(\mathcal{H}_{\alpha^1,\alpha^2})\otimes\widehat{\mathrm{CF}}(\mathcal{H}_{\alpha^2,\alpha^3})\otimes\cdots\otimes\widehat{\mathrm{CF}}(\mathcal{H}_{\alpha^{n-1},\alpha^n})\to\widehat{\mathrm{CF}}(\mathcal{H}_{\alpha^1,\alpha^n})$$

defined by counting pseudo-holomorphic $n$-gons with local local multiplicity zero at $w$ and Maslov index $3-n$. In our case, $\widehat{m}_1:\widehat{\mathrm{CF}}(\mathcal{H}_{\alpha^1,\alpha^2})\to\widehat{\mathrm{CF}}(\mathcal{H}_{\alpha^1,\alpha^2})$ is the differential, and $\widehat{m}_2:\widehat{\mathrm{CF}}(\mathcal{H}_{\alpha^1,\alpha^2})\otimes\widehat{\mathrm{CF}}(\mathcal{H}_{\alpha^2,\alpha^3})\to\widehat{\mathrm{CF}}(\mathcal{H}_{\alpha^1,\alpha^3})$ is the triangle counting map defined in equation (2.3). These maps satisfy the following relations.

**Theorem 2.5.2** ($\mathcal{A}_\infty$ relations)**.** *Let $n$ be a positive integer, and let $(\Sigma,\boldsymbol{\alpha}^1,\ldots,\boldsymbol{\alpha}^{n+1},w)$ be an admissible pointed Heegaard diagram. Then for any sequence of intersection points $\mathbf{x}_1,\ldots,\mathbf{x}_n$ such that $\mathbf{x}_i\in\mathbb{T}_{\alpha^i}\cap\mathbb{T}_{\alpha^{i+1}}$ for $i=1,\ldots,n$,*

$$\sum_{i+j=n+1}\sum_{\ell=1}^{n-j+1}\widehat{m}_i(\mathbf{x}_1,\ldots,\mathbf{x}_{\ell-1},\widehat{m}_j(\mathbf{x}_\ell,\ldots,\mathbf{x}_{\ell+j-1}),\mathbf{x}_{\ell+j},\ldots,\mathbf{x}_n)=0.$$

For example, when $n = 1$, this forces $i = j = \ell = 1$, which gives us the relation

$$\widehat{m}_1(\widehat{m}_1(\mathbf{x}_1)) = 0,$$

which is the fact that $\widehat{m}_1$ is a differential. When $n = 2$, we get the relation

$$\widehat{m}_1(\widehat{m}_2(\mathbf{x}_1, \mathbf{x}_2)) + \widehat{m}_2(\widehat{m}_1(\mathbf{x}_1), \mathbf{x}_2) + \widehat{m}_2(\mathbf{x}_1, \widehat{m}_1(\mathbf{x}_2)) = 0,$$

the statement that $\widehat{m}_2$ is a chain map. For $n = 3$, we get the relation

$$\begin{aligned} &\widehat{m}_1(\widehat{m}_3(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)) + \widehat{m}_2(\widehat{m}_2(\mathbf{x}_1, \mathbf{x}_2), \mathbf{x}_3) + \widehat{m}_2(\mathbf{x}_1, \widehat{m}_2(\mathbf{x}_2, \mathbf{x}_3)) \\ &+ \widehat{m}_3(\widehat{m}_1(\mathbf{x}_1), \mathbf{x}_2, \mathbf{x}_3) + \widehat{m}_3(\mathbf{x}_1, \widehat{m}_1(\mathbf{x}_2), \mathbf{x}_3) + \widehat{m}_3(\mathbf{x}_1, \mathbf{x}_2, \widehat{m}_1(\mathbf{x}_3)) = 0. \end{aligned}$$

Over $\mathbb{Z}/2\mathbb{Z}$ coefficients, we can rearrange the above to

$$\widehat{m}_2(\widehat{m}_2(\mathbf{x}_1, \mathbf{x}_2), \mathbf{x}_3) + \widehat{m}_2(\mathbf{x}_1, \widehat{m}_2(\mathbf{x}_2, \mathbf{x}_3)) = \widehat{\partial}\widehat{m}_3(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) + \widehat{m}_3\widehat{\partial}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3),$$

where the rightmost term contains the usual differential on the product complex $\widehat{\mathrm{CF}}(\mathcal{H}_{\alpha^1\alpha^2}) \otimes \widehat{\mathrm{CF}}(\mathcal{H}_{\alpha^2\alpha^3}) \otimes \widehat{\mathrm{CF}}(\mathcal{H}_{\alpha^3\alpha^4})$ From the above rearrangement, the third $\mathcal{A}_\infty$ relation says that $\widehat{m}_3$ is a chain homotopy between the maps $\widehat{m}_2 \otimes \mathrm{Id}_{\widehat{\mathrm{CF}}(\mathcal{H}_{\alpha^2\alpha^3})}$ and $\mathrm{Id}_{\widehat{\mathrm{CF}}(\mathcal{H}_{\alpha^1\alpha^2})} \otimes \widehat{m}_2$. Said differently, the associativity law holds up to homotopy, which in particular implies that the associativity level holds on homology. This fact will be useful when proving the surgery exact triangle.

*Remark* 2.5.3. In the context of Lagrangian Floer homology, there are similar operations. Roughly, given a symplectic manifold $M$ and $k+1$ Lagrangian submanifolds $L_0, \ldots, L_k$ in $M$, one can define an operation

$$\mu_k : \mathrm{CF}(L_0, L_1) \otimes \mathrm{CF}(L_1, L_2) \otimes \cdots \otimes \mathrm{CF}(L_0, L_k)$$

which on a point $p_1 \otimes p_2 \otimes \cdots \otimes p_k$ sums over all points $q \in L_0 \cap L_k$ a count of perturbed

$J$-holomorphic disks with $k+1$ punctures mapping in a clockwise order to $p_1, \dots, p_k, q$ up to reparametrization of Maslov index $2-k$. The operations $\mu^k$ satisfy the $\mathcal{A}_\infty$ relations. To get a count, one needs a correct transversality statement, so extra assumptions need to be made on the Lagrangian submanifolds and the ambient symplectic manifold. See [37, Chapter 9]. One can obtain a lot of information about the intersections of Lagrangian submanifolds by putting together all of this data into what is known as a *Fukaya cateogry.* The objects are Lagrangian submanifolds $L$ inside $M$, and the morphism between two objects $L, L'$ is a suitably defined Lagrangian Floer complex $\mathrm{CF}(L, L')$. This data does not form an honest category, because composition of morphsims is not associative, but only associative up to homotopy.

Stepping back, in practice, it is very difficult to classify all Lagrangian submanifolds of a given symplectic manifold. In contrast, one can understand the Fukaya category in terms of a small subset of generating objects, provided we understand the higher operations among these generators. This has repercussions in Heegaard Floer homology. For example, Auroux [3, 4] proved a generation criterion for the a certain Fukaya category of $\mathrm{Sym}^g(\Sigma)$. The generators have a simple description: consider a collection of $2g$ disjoint simple arcs $\gamma_1, \dots, \gamma_g$ in $\Sigma$ such that $\Sigma \setminus (\gamma_1 \cup \cdots \cup \gamma_{2g})$ is homeomorphic to a disk, then the Lagrangian submanifolds $\prod_{i \in S} \gamma_i$ for $S$ a $g$-element subset of $\{1, \dots, 2g\}$ are the generators. See Auroux's introductory article [5] and Seidel's book [37].

# Chapter 3

# The surgery exact triangle

This chapter proves and illustrates some immediate consequences of the surgery exact triangle, a key computational tool in Heegaard Floer homology. In Section 3.1, we review Dehn surgery on three-manifolds. In Section 3.2, we give the statement of the surgery exact triangle and give some examples. In Section 3.3, we prove an algebraic lemma needed for the proof of the surgery exact triangle. In Section 3.4, we show a small isotopy of an attaching circle in a Heegaard diagram does not affect the Heegaard Floer homology. In Section 3.5, we study the behavior of certain pseudo-holomorphic triangle and rectangle counting maps. In Section 3.6, we piece together the lemmas in the previous sections to prove the surgery exact triangle. Finally, in Section 3.7, we give some applications to finding manifolds.

## 3.1 Dehn Surgery

Let $M$ be a three-manifold with torus boundary and fix a homologically non-trivial closed embedded curve $\gamma$. A *Dehn filling* of $M$ is the operation of gluing a solid torus $D \times S^1$ to $M$ via a homeomorphism $\varphi : \partial D \times S^1 \to \partial M$ so that $\partial D^2 \times \{p\}$ glues onto $\gamma$.

Suppose that we have a knot $K$ in a three-manifold $Y$. Then the complement of an open tubular neighborhood of $K$ in $Y$ is a three-manifold $M$ with torus boundary. A framing on $K$ can be thought of as a normal vector field to $K$. The normal push-off of $K$ gives a closed curve in the torus $\partial M$, which is uniquely determined by the framing up to isotopy. By Dehn filling the manifold $M = Y \setminus \mathrm{nd}(K)$ along $\gamma$, we obtain the manifold $Y_\gamma(K)$.

Now suppose that $K$ is oriented. Let $\lambda \subset \partial M$ be a push-off of $K$ specified by the framing, and let $\mu \subset \partial M$ be the boundary of a normal disk to $K$, oriented such that the intersection number $\mu \cdot \lambda$ is $-1$. When $p$ and $q$ are relatively prime integers, the homology class $p\mu + q\lambda$ can be represented by an embedded closed curve $\gamma_{p,q}$ on $\partial M$, and the Dehn filling of $M$ along this curve is called the $p/q$ *Dehn surgery* along the framed knot, and denoted $Y_{p/q}(K, \phi)$. Note that the notation is justified because $\gamma_{-p,-q}$ is the curve $\gamma_{p,q}$ with reversed orientation, so that $Y_{p/q}(K, \phi) = Y_{-p/-q}(K, \phi)$. In the case when $q = 0$, we have that $Y_{1/0}(K, \phi) = Y_\infty = Y$.

**Example 3.1.1.** Let $U$ denote the unknot in $S^3$. We show that $S^3_1(U) = S^3$. Let $A = S^3 \setminus \mathrm{nd}(U)$ be the remaining solid torus, and we use a map $\phi : S^1 \times S^1 \to \partial A$ mapping the meridian $m = S^1 \times \mathrm{pt}$ to $\mu + \lambda$. Then $S^3_1(U) = A \cup_\phi D^2 \times S^1$. There is a diffeomorphism $g$ of $A$ which fixes $\mu$ and sends the meridian $\mu$ to $\mu - \lambda$; we are performing a Dehn twist on the solid torus $A$. Then note that the new attaching map $g \circ \phi : S^1 \times S^1 \to \partial A$ sends the meridian $m$ to $(\mu - \lambda) + \lambda = \mu$. Then extending $g$ gives us a well-defined map $A \cup_\phi (D^2 \times S^1) \to A \cup_{g\circ\phi} (D^2 \times S^1)$, which is a diffeomorphism because $g$ is a diffeomorphism. By construction, $g \circ \varphi$ is the standard gluing map sending meridian to meridian and longitude to longitude, so $A \cup_{g\circ\phi} (D^2 \times S^1) = S^3$. By applying $q$ Dehn twists, one can show that $S^3_{1/q}(U) = S^3$.

## 3.2 Statement and examples

Three simple closed curves $b_1, b_2, b_3$ form a *triad* if any pair of curves meet transversely in a single point. An orientation on the torus specifies a preferred cyclic ordering on the three curves, specified by the fact that the oriented intersection number $\#(b_i \cap b_{i+1})$ is $-1$ for $i \in \mathbb{Z}/3\mathbb{Z}$. We can draw a symmetric picture of the torus as a hexagon with opposite sides identified.

**Definition 3.2.1** (Triad of three-manifolds)**.** Let $M$ be an oriented three-manifold with torus boundary. Fix a triad of curves $b_1, b_2, b_3$ in $\partial M$ with the induced cyclic ordering. The three three-manifolds $Y_1, Y_2, Y_3$, where $Y_i$ is obtained from $M$ by Dehn filling $b_i$, are said to form a *triad* of three-manifolds.

Heegaard Floer homology transforms a triad of three manifolds into an exact triangle of groups. This algebraic notion is defined as follows:

**Definition 3.2.2.** Fix three Abelian groups $A^1, A^2$, and $A^3$ and three homomorphisms $f^i : A^i \to A^{i+1}$ for all $i \in \mathbb{Z}/3\mathbb{Z}$. This data forms an *exact triangle* if $\ker f^i = \operatorname{im} f^{i-1}$ for all $i \in \mathbb{Z}/3\mathbb{Z}$. An exact triangle is depicted by the diagram

$$\begin{tikzcd} A^1 \arrow[rr, "f^1"] & & A^2 \arrow[dl, "f^2"] \\ & A^3 \arrow[ul, "f^3"] & \end{tikzcd}$$

The surgery exact triangle in Heegaard Floer homology can then be stated as follows:

**Theorem 3.2.3** (Surgery exact triangle)**.** *Let $Y_1, Y_2$, and $Y_3$ be a triad of three-manifolds. Then there is an exact triangle connecting $\widehat{\mathrm{HF}}(Y_1), \widehat{\mathrm{HF}}(Y_2)$, and $\widehat{\mathrm{HF}}(Y_3)$. The maps in this triangle preserve the relative $\mathbb{Z}/2\mathbb{Z}$ grading.*

*Remark* 3.2.4. The surgery exact triangle can be generalized to surgery on links as follows, see [32]. Suppose we have a framed link $L$ in a three-manifold $Y$ with $n$ components and we know the Heegaard Floer homology groups of the $2^n$ three-manifolds obtained by performing 0 or 1 surgery on each component of $L$. When $n = 1$, we have the surgery exact triangle. In general, there is a spectral sequence whose $E_2$ term consists of the direct sum of $\widehat{\mathrm{HF}}$ of all of the $2^n$ three-manifolds, and whose $E^\infty$ term calculates $\widehat{\mathrm{HF}}(Y)$. In the proof, one counts pseudo-holomorphic $m$-gons and uses the $\mathcal{A}_\infty$ relation.

*Remark* 3.2.5. Donaldson's theory of instantons in four-manifolds [9] was adapted by Floer [11] to create an invariant of certain three-manifolds. Floer observed that this theory also contains an exact triangle [10]. Other instances of exact triangles have appeared in several other variants of Floer homology, such as Seidel's long exact sequence for Lagrangian Floer homology [36], and another exact triangle [20] which holds for Seiberg–Witten monopole Floer homology defined by Kronheimer and Mrowka [21].

Now we proceed to show some examples of triads of three-manifolds.

**Example 3.2.6.** If $K \subset S^3$ is a knot, then the triple $(S^3, S^3_n(K), S^3_{n+1}(K))$ is a triad. More generally, let $Y$ be a closed three-manifold and a knot $K$ in $Y$. Consider the three-manifolds $Y$, $Y_\lambda(K)$ obtained by $\lambda$-framed surgery of $K$, and $Y_{\lambda+\mu}(K)$ obtained by $(\lambda + \mu)$-framed surgery on $K$, where $\mu$ is a meridian of $K$. Then we claim that $(Y, Y_\lambda(K), Y_{\lambda+\mu}(K))$ is a triad of three-manifolds. Indeed, $M = Y - \mathrm{nd}(K)$ has torus boundary. Note moreover that $Y$ can be obtained by Dehn filling $M$ along $\mu$. Since $\mu, \lambda, \lambda + \mu$ form a triad of curves, it follows that $(Y, Y_\lambda(K), Y_{\lambda+\mu}(K))$ is a triad of three-manifolds.

**Example 3.2.7.** Let $L$ be a link in $S^3$ and $\Sigma(L)$ be its branched double cover. Fix a diagram for $L$ and pick a crossing in that diagram. Let $L_1$ and $L_2$ be two links obtained by resolving that crossing. Then we claim that $\Sigma(L), \Sigma(L_1)$, and $\Sigma(L_2)$ form a triad of three-manifolds.

Let $B$ be a small ball centered around the crossing. Then the ball intersects each diagram in two disjoint arcs. Note that the branched double cover of a three-ball branched at two disjoint arcs is a solid torus. This can be seen by rotating the torus by $180°$ about a central axis. Thus, the branched double cover of $L, L_1, L_2$ excluding a small neighborhood of the crossing is a a three-manifold with torus boundary. To find the curves at which we Dehn fill to obtain the branched double covers, we take the curve connecting the midpoints of the two arcs, and push it to the boundary. Since the three curves have algebraic intersection number one with each other, these three curves form a triad.

## 3.3 Revisiting the long exact sequence on homology

In this section, we work with chain complexes over the field $\mathbb{Z}/2\mathbb{Z}$. If $(C, \partial)$ is a chain complex over $\mathbb{Z}/2\mathbb{Z}$, given a cycle $x \in C$, i.e. an element such that $\partial x = 0$, let $[x]$ be the induced homology class in $H(C)$. If $f : C \to C'$ is a chain map, let $H(f) : H(C) \to H(C')$ be the induced map on homology defined by $H(f)([x]) = [f(x)]$.

First we state a simpler version of the statement that we wish prove.

**Lemma 3.3.1.** *Let $(C^1, \partial^1), (C^2, \partial^2)$, and $(C^3, \partial^3)$ be three chain complexes, let $f^i : C^i \to C^{i+1}$ be three chain maps indexed by $i \in \mathbb{Z}/3\mathbb{Z}$, and let $h^i : C^i \to C^{i+2}$ be three null-homotopies of $f^{i+1} \circ f^i$, i.e. the maps $h^i$ satisfy the equation*

$$\partial^{i+2} \circ h^i + h^i \circ \partial^i = f^{i+1} \circ f^i$$

*for $i \in \mathbb{Z}/3\mathbb{Z}$. Consider the maps*

$$\phi^i = h^{i+1} \circ f^i + f^{i+2} \circ h^i$$

*on $C^i$. Thus we have the following configuration:*

$$\begin{tikzcd}
C^1 \arrow[loop left, "\phi^1"] \arrow[rr, bend left, "f^1"] \arrow[dr, bend left, "h^1"] & & C^2 \arrow[loop right, "\phi^2"] \arrow[ll, bend left, "h^2"] \arrow[dl, bend left, "f^2"] \\
& C^3 \arrow[loop below, "\phi^3"] \arrow[ul, bend left, "f^3"] \arrow[ur, bend left, "h^3"] &
\end{tikzcd}$$

*Then the maps $\phi^i$ are chain maps for $i \in \mathbb{Z}/3\mathbb{Z}$. Moreover, if $\phi^i$ is chain homotopic to the identity map on $C^i$, then there is an exact triangle:*

$$\begin{tikzcd}
H(C^1) \arrow[rr, "H(f^1)"] & & H(C^2) \arrow[dl, "H(f^2)"] \\
& H(C^3) \arrow[ul, "H(f^3)"] &
\end{tikzcd}$$

See [29, Lemma 18.4.1] for a proof of this lemma.

*Remark* 3.3.2. The above lemma can be used to show that a short exact sequence of chain complexes of projective modules induces an exact triangle on homology. See [29, Corollary 18.4.3] for a proof.

For our purposes, it will be convenient to prove an enhanced version of this lemma. Recall that a map $\phi : C \to C'$ is a *quasi-isomorphism* if the induced map on homology is an isomorphism. In our elaboration of Lemma 3.3.1, we instead work with a sequence of chain complexes in which every third term has isomorphic homologies, and we will also assume that the given chain maps $\phi^i$ are only quasi-isomorphisms. This form will better comply with the proof of the surgery exact triangle.

**Lemma 3.3.3.** *Let $\{(C^i, \partial^i)\}_{i\in\mathbb{Z}}$ be a sequence of chain complexes, $\{f^i : C^i \to C^{i+1}\}_{i\in\mathbb{Z}}$ a sequence of chain maps, and $\{h^i : C^i \to C^{i+2}\}_{i\in\mathbb{Z}}$ a sequence of null-homotopies of $f^{i+1} \circ f^i$, i.e. such that for all $i \in \mathbb{Z}$,*

$$f^{i+1} \circ f^i = \partial^{i+2} \circ h^i + h^i \circ \partial^i.$$

*Then $\phi^i = h^{i+1} \circ f^i + f^{i+2} \circ h^i : C^i \to C^{i+3}$ is a chain map. Suppose further that $\phi^i : C^i \to C^{i+3}$ is a quasi-isomorphism, i.e. the induced map $H(\phi^i) : H(C^i) \to H(C^{i+3})$ is an isomorphism. Then there is a long exact sequence*

$$\cdots \longrightarrow H(C^i) \xrightarrow{H(f^i)} H(C^{i+1}) \xrightarrow{H(f^{i+1})} H(C^{i+2}) \longrightarrow \cdots$$

*which can be rolled up into an exact triangle*

$$\begin{tikzcd} H(C^i) \arrow[rr, "H(f^i)"] & & H(C^{i+1}) \arrow[dl, "H(f^{i+1})"] \\ & H(C^{i+2}) \arrow[ul, "H(\phi^i)^{-1} \circ H(f^{i+2})"] & \end{tikzcd} \tag{3.1}$$

*Proof.* First we show that $\phi^i$ is a chain map. We have that

$$\begin{aligned}
\partial^{i+3} \circ \phi^i + \phi^i \circ \partial^i &= \partial^{i+3} \circ (h^{i+1} \circ f^i + f^{i+2} \circ h^i) + (h^{i+1} \circ f^i + f^{i+2} \circ h^i) \circ \partial^i \\
&= \partial^{i+3} \circ h^{i+1} \circ f^i + (\partial^{i+3} \circ f^{i+2}) \circ h^i + h^{i+1} \circ (f^i \circ \partial^i) + f^{i+2} \circ h^i \circ \partial^i \\
&= \partial^{i+3} \circ h^{i+1} \circ f^i + (f^{i+2} \circ \partial^{i+2}) \circ h^i + h^{i+1} \circ (\partial^{i+1} \circ f^i) + f^{i+2} \circ h^i \circ \partial^i \\
&= (\partial^{i+3} \circ h^{i+1} + h^{i+1} \circ \partial^{i+1}) \circ f^i + f^{i+2} \circ (\partial^{i+2} \circ h^i + h^i \circ \partial^i) \\
&= (f^{i+2} \circ f^{i+1}) \circ f^i + f^{i+2} \circ (f^{i+1} \circ f^i) \\
&= 0,
\end{aligned}$$

where we used the fact that we are working over $\mathbb{Z}/2\mathbb{Z}$ coefficients, that $f^i, f^{i+2}$ are chain maps in the third line, and that $h^i, h^{i+1}$ are null-homotopies in the fifth line.

From the fact that $h^i$ are null-homotopies of $f^{i+1} \circ f^i$, we have that

$$H(f^{i+1}) \circ H(f^i) = H(f^{i+1} \circ f^i) = 0,$$

which implies that $\operatorname{im} H(f^i) \subseteq \ker H(f^{i+1})$.

To show the long exact sequence, it remains to show that $\ker H(f^{i+1}) \subseteq \operatorname{im} H(f^i)$. First we show that $f^{i+3} \circ \phi^i$ is chain-homotopic to $\phi^{i+1} \circ f^i$. Since

$$\begin{aligned}
f^{i+3} \circ \phi^i + \phi^{i+1} \circ f^i &= f^{i+3} \circ (h^{i+1} \circ f^i + f^{i+2} \circ h^i) + (h^{i+2} \circ f^{i+1} + f^{i+3} \circ h^{i+1}) \circ f^i \\
&= f^{i+3} \circ f^{i+2} \circ h^i + h^{i+2} \circ f^{i+1} \circ f^i \\
&= (\partial^{i+4} \circ h^{i+2} + h^{i+2} \circ \partial^{i+2}) \circ h^i + h^{i+2} \circ (\partial^{i+2} \circ h^i + h^i \circ \partial^i) \\
&= \partial^{i+4} \circ h^{i+2} \circ h^i + h^{i+2} \circ h^i \circ \partial^i,
\end{aligned}$$

we have that $h^{i+2} \circ h^i$ is a chain homotopy between $f^{i+3} \circ \phi^i$ and $\phi^{i+1} \circ f^i$.

Fix a homology class $[b]$ in the kernel of $H(f^{i+1})$. Our goal is to find a cycle $a \in C^i$ such that $[f^i(a)] = [b]$.

Since $\phi^{i-2}$ is an isomorphism, there exists a cycle $b' \in C^{i-2}$ such that $[\phi^{i-2}(b')] = [b]$.

Now using the fact that $f^{i+1} \circ \phi^{i+2}$ is chain homotopic to $\phi^{i-1} \circ f^{i-2}$, we have that

$$0 = H(f^{i+1}) \circ H(\phi^{i+2})[b'] = H(\phi^{i-1}) \circ H(f^{i-2})[b'].$$

Since $\phi^{i+1}$ is a quasi-isomorphism, in particular that $H(\phi^{i+1})$ is injective, we have that $H(f^{i-2})[b'] = 0$, so there exists some $c \in C^{i-1}$ such that

$$f^{i-2}(b') = \partial^{i-1}(c).$$

Let $a \in C^i$ be defined by

$$a = h^{i-2}(b') + f^{i-1}(c).$$

Then we have that $a$ is a cycle since

$$\begin{aligned}
\partial^i(a) &= \partial^i \circ h^{i-2}(b') + \partial^i \circ f^{i-1}(c) \\
&= (f^{i-1} \circ f^{i-2}(b') + h^{i-2} \circ \partial^{i-2}(b')) + f^{i-1} \circ \partial^{i-1}(c) \\
&= f^{i-1} \circ f^{i-2}(b') + f^{i-1} \circ f^{i-2}(b') \\
&= 0,
\end{aligned}$$

where we used the fact that $f^{i-1}$ is a chain map in the second line, and the fact that $f^{i-2}(b') = \partial^{i-1}(c)$ and $b'$ is a cycle in the third line. Moreover, we have that

$$\begin{aligned}
[f^i(a)] &= [f^i \circ h^{i-2}(b') + f^i \circ f^{i-1}(c)] \\
&= [\phi^{i-1}(b') + h^{i-1} \circ f^{i-2}(b') + \partial^{i+1} \circ h^i(c) + h^{i-1} \circ \partial^{i-1}(c)] \\
&= [\phi^{i-1}(b') + h^{i-1} \circ \partial^{i-1}(c) + \partial^{i+1} \circ h^i(c) + h^{i-1} \circ \partial^{i-1}(c)] \\
&= [b] + [\partial^{i+1} \circ h^i(c)] \\
&= [b].
\end{aligned}$$

This shows that $\ker H(f^{i+1}) \subseteq \operatorname{im} H(f^i)$, concluding the proof of the long exact sequence.

To prove that the long exact sequence rolls up into a exact triangle, it remains to show that

$$\operatorname{im} H(f^{i+1}) = \ker H(\phi^i)^{-1} \circ H(f^{i+2}) \tag{3.2}$$

$$\ker H(f^i) = \operatorname{im} H(\phi)^{-1} \circ H(f^{i+2}) \tag{3.3}$$

Since $\phi^i$ is a quasi-isomorphism, we have that $\operatorname{im} H(f^{i+1}) = \ker H(f^i) = \ker H(\phi^i) \circ H(f^i)$, which shows (3.2). Recall that we showed that $f^{i+3} \circ \phi^i$ is chain homotopic to $\phi^{i+1} \circ f^i$. So, we have that $f^{i+3} \circ \phi^i$ is chain homotopic to $\phi^{i+1} \circ f^i$, which implies that $H(f^{i-1}) = H(\phi)^{-1} \circ H(f^{i+2}) \circ H(\phi^{i-1})$. Using this and the fact that $H(\phi^{i-1})$ is an isomorphism, we have that

$$\ker H(f^i) = \operatorname{im} H(f^{i-1}) = \operatorname{im} H(\phi^i)^{-1} \circ H(f^{i+2}) \circ H(\phi^{i-1}) = \operatorname{im} H(\phi^i)^{-1} \circ H(f^{i+2}).$$

This proves (3.3), which concludes the proof. $\square$

Now we explain how to add gradings to Lemma 3.3.3. A chain complex $(C, \partial)$ is *graded* if $C = C_0 \oplus C_1$, where $C_0$ and $C_1$ are subcomplexes of $C$. Call a map $f : C \to C'$ *graded of degree $d$* if $f$ maps $C_i$ to $C_{i+d}$ for all $i \in \mathbb{Z}/2\mathbb{Z}$. For a graded chain map $f : C \to C'$, let $\operatorname{gr}(f)$ denote its degree.

**Lemma 3.3.4.** *Let $C^i, f^i, h^i, \phi^i$ be defined as in Lemma 3.3.3. Assume further that the chain complexes $C^i$ are $\mathbb{Z}/2\mathbb{Z}$ graded and $f^i, h^i$ are graded such that*

$$\operatorname{gr}(f^i) + \operatorname{gr}(f^{i+1}) + \operatorname{gr}(f^{i+2}) = 1, \tag{3.4}$$

$$\operatorname{gr}(h^i) = \operatorname{gr}(f^i) + \operatorname{gr}(f^{i+1}) + 1, \tag{3.5}$$

*for all $i \in \mathbb{Z}/3\mathbb{Z}$. Then the maps in the exact triangle* (3.1) *are $\mathbb{Z}/2\mathbb{Z}$ graded.*

*Proof.* It remains to show that the maps $\phi^i = h^{i+1} \circ f^i + f^{i+2} \circ h^i$ are graded. This follows from the fact that composition of graded maps is graded and the fact that $h^{i+1} \circ f^i$ and $f^{i+2} \circ h^i$ both have grading 0. $\square$

## 3.4 Clean approximations

Let $\boldsymbol{\beta} = \{\beta_1, \ldots, \beta_g\}$ be a collection of $g$ homologically linearly independent circles in a genus $g$ surface $\Sigma$. Let $\beta_i'$ be an isotopic translate of $\beta_i$ such that $\beta_i$ and $\beta_i'$ intersect transversally in two points. The circle $\beta_i$ is called a *clean approximation of* $\beta_i$. Let $\boldsymbol{\beta'} = \{\beta_1', \ldots, \beta_g'\}$. Choosing a basepoint $w$ away from the circles in $\boldsymbol{\beta}$ and $\boldsymbol{\beta'}$, form the Heegaard diagram $(\Sigma, \boldsymbol{\beta}, \boldsymbol{\beta'}, w)$.

**Lemma 3.4.1.** *In the Heegaard diagram $(\Sigma, \boldsymbol{\beta}, \boldsymbol{\beta'}, w)$ defined above, there is a unique Heegaard state $\mathbf{t}_{\beta\beta'}$ with maximal Maslov grading.*

*Proof.* Set up the Heegaard diagram so that the $\alpha$-circles lie to the left of the $\beta$-circles. Then the Heegaard state that lies at the top of the diagram differs has Maslov grading higher than the grading of any other Heegaard state. □

**Lemma 3.4.2.** *Let $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w)$ be a pointed Heegaard diagram. If $\boldsymbol{\beta}'$ is a sufficiently close approximation to $\boldsymbol{\beta}$ and $\mathbf{t}_{\beta\beta'}$ is the top dimensional homology class in $\widehat{\mathrm{HF}}(\mathcal{H}_{\beta,\beta'})$, then the map $f : \widehat{\mathrm{CF}}(\mathcal{H}_{\alpha,\beta}) \to \widehat{\mathrm{CF}}(\mathcal{H}_{\alpha,\beta'})$ given by*

$$\mathbf{x} \mapsto \widehat{m}_{\alpha,\beta,\beta'}(\mathbf{x} \otimes \mathbf{t}_{\beta\beta'})$$

*is a quasi-isomorphism.*

To prove the lemma, we need a couple of preliminary definitions. Let $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ be a Heegaard diagram. Cutting $\Sigma$ along $\boldsymbol{\alpha} \cup \boldsymbol{\beta}$ gives us a decomposition of $\Sigma$ into closed elementary domains $\mathcal{D}_1, \ldots, \mathcal{D}_m$ so that $\Sigma \setminus (\boldsymbol{\alpha} \cup \boldsymbol{\beta}) = \bigcup_{k=1}^m \operatorname{int} D_k$. Fix points $z_k \in \operatorname{int} D_k$ and a pair of intersection points $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$. Define the *shadow* of a Whitney disk $\phi \in W(\mathbf{x}, \mathbf{y})$ to be the domain

$$\mathcal{S}(\phi) = \sum_k n_{z_k}(\phi) \cdot \mathcal{D}_k.$$

Note that for any two points $z, z'$ in $\Sigma \setminus (\boldsymbol{\alpha} \cup \boldsymbol{\beta})$, we have that $n_z(\phi) = n_{z'}(\phi)$, so the shadow of $\phi$ is well-defined. Given a Whitney disk, we can also define the *area* $\mathcal{A}(\phi)$ of its shadow to be

$$\mathcal{A}(\phi) = \int_{\mathcal{S}(\phi)} \nu = \sum_k n_{z_k}(\phi) \int_{\mathcal{D}_k} \nu,$$

where $\nu$ is the area form on $\Sigma$ chosen in the construction of the symplectic form on $\mathrm{Sym}^g(\Sigma)$, see Corollary 2.2.4.

Returning to the problem, define the *area filtration* on $\widehat{\mathrm{CF}}(\mathcal{H}_{\alpha,\beta'})$ as follows. Fix $\mathbf{x}_0 \in \mathbb{T}_\alpha \cap \mathbb{T}_{\beta'}$, and let $\mathcal{F} : \mathbb{T}_\alpha \cap \mathbb{T}_\beta \to \mathbb{R}$ be defined by

$$\mathcal{F}(\mathbf{x}) = \mathcal{A}(\phi) - n_w(\phi),$$

where $\phi \in D(\mathbf{x}_0, \mathbf{x})$ is any homotopy class connecting $\mathbf{x}$ and $\mathbf{y}$, and $\mathcal{A}(\phi)$ is the area of the shadow of $\phi$. We now show that $\mathcal{F}(\mathbf{x})$ is well-defined:

**Lemma 3.4.3.** *If $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w)$ is an admissible pointed Heegaard diagram, then for any $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, the quantity*

$$\mathcal{A}(\phi) - n_w(\phi)$$

*is independent of the choice of $\phi \in W(\mathbf{x}, \mathbf{y})$.*

*Proof sketch.* First, by Lemma 2.1.4, $D(\mathbf{x}, \mathbf{y})$ is an affine space over the space of cornerless domains, which can be written as $\mathbb{Z} \oplus$ (periodic domains). By rescaling, one can choose a volume form $\nu$ on the Heegaard surface such that $\int_\Sigma \nu = 1$, and by the admissibility of the Heegaard diagram, the volume form can moreover be chosen such that for any periodic domain $\mathcal{D}$, $\int_{\mathcal{D}} \nu = 0$. Then adding $[\Sigma]$ to a given domain $\mathcal{D} \in D(\mathbf{x}, \mathbf{y})$ increases $\mathcal{A}$ and $n_w$ by one, and adding a periodic domain leave $\mathcal{A}$ and $n_w$ unchanged. $\square$

**Proposition 3.4.4.** *Let $V$ be a finite-dimensional vector space and $f, g, h : V \to V$ be linear maps such that $f = g + h$, $f$ is an isomorphism, $h$ is niloptent (i.e. $h^n = 0$ for some positive integer $n$), and $g$ and $h$ commute. Then $g$ is an isomorphism.*

*Proof.* Let $n$ be a positive integer such that $h^n = 0$. As $g$ is map from $V$ to itself, it suffices to show that $g$ is injective. Let $v \in V$ be such that $g(v) = 0$. Then

$$f^n(v) = (g+h)^n(v) = \sum_{k=0}^{n} h^k g^{n-k}(v) = h^n(v) = 0,$$

where we used the fact that $g$ and $h$ commute in the third equality. Since $f$ is an isomorphism, so is $f^n$. Therefore, $v = 0$, so $g$ is injective. $\square$

*Proof of Lemma 3.4.2.* First we show that $f$ is a chain map. Writing $\widehat{m}$ as shorthand for $\widehat{m}_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\beta'}}$, we have that by the associativity up to homotopy (Theorem 2.5.2) and the fact that $\widehat{\partial}\mathbf{t}_{\beta\beta'} = 0$,

$$\begin{aligned}(\widehat{\partial} f + f\widehat{\partial})(\mathbf{x}) &= \widehat{\partial}\widehat{m}(\mathbf{x}, \mathbf{t}_{\beta\beta'}) + \widehat{m}(\widehat{\partial}\mathbf{x}, \mathbf{t}_{\beta\beta'}) \\ &= \widehat{\partial}\widehat{m}(\mathbf{x}, \mathbf{t}_{\beta\beta'}) + \widehat{m}(\widehat{\partial}\mathbf{x}, \mathbf{t}_{\beta\beta'}) + \widehat{m}(\mathbf{x}, \widehat{\partial}\mathbf{t}_{\beta\beta'}) \\ &= 0.\end{aligned}$$

Now perform a perturbation on $\boldsymbol{\beta'}$ such that the signed area between $\beta_i$ and $\beta_i'$ is zero for $i = 1, \ldots, n$. If $\boldsymbol{\beta'}$ is sufficiently close to $\boldsymbol{\beta'}$, then for all $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, there exists a corresponding closest point $\iota(\mathbf{x}) \in \mathbb{T}_\alpha \cap \mathbb{T}_{\beta'}$. This gives us a map

$$\iota : \widehat{\mathrm{CF}}(\mathcal{H}_{\alpha,\beta}) \to \widehat{\mathrm{CF}}(\mathcal{H}_{\alpha,\beta'})$$

which is a group isomorphism.

By the Riemann mapping theorem, for all $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, there exists a smallest triangle $\psi^{\mathbf{x}} \in D(\mathbf{x}, \mathbf{t}_{\beta\beta'}, \iota(\mathbf{x}))$ which admits a unique pseudo-holomorphic representative. Now take the circles in $\boldsymbol{\beta'}$ to be so close to $\boldsymbol{\beta}$ that the area of $\psi^{\mathbf{x}}$ is smaller than the areas of any homotopy classes of disks $\phi \in D(\mathbf{x}, \mathbf{y})$ for any $\mathbf{x}, \mathbf{y}$ either in $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$ or in $\mathbb{T}_\alpha \cap \mathbb{T}_{\beta'}$.

We claim that for each $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, $f(\mathbf{x}) - \iota(\mathbf{x})$ can be written as a linear combination of $\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_{\beta'}$ such that $\mathcal{F}(\iota(\mathbf{x})) < \mathcal{F}(\mathbf{y})$ with respect to the area filtration $\mathcal{F}$ on $\widehat{\mathrm{CF}}(\mathcal{H}_{\alpha,\beta'})$. Let $\psi$ be a disk in $D(\mathbf{x}, \mathbf{t}_{\beta\beta'}, \iota(\mathbf{x}))$. Since $\psi$ has a pseudo-holomorphic representative, $\psi$ is a positive domain. Moreover, note that

$$\psi = \psi^{\mathbf{x}} * \phi$$

for some $\phi \in D(\iota(\mathbf{x}), \mathbf{y})$. Now we claim that $\phi$ is a positive domain. Let $D$ be an elementary domain appearing witha nonzero coefficient in the shadow of $\phi$. Since $\psi$ and $\phi$ agree away from the region between $\beta_i$ and $\beta_i'$ for $i = 1, \dots, n$, and the area of $D$ is greater than the area of the region between $\boldsymbol{\beta}$ and $\boldsymbol{\beta}'$, we have that $D$ appears with a positive coefficient in $\psi$. Thus $\mathcal{A}(\psi) > \mathcal{A}(\psi^{\mathbf{x}})$, from which it follows that $\mathcal{F}(\iota(\mathbf{x})) < \mathcal{F}(\mathbf{y})$.

Writing

$$f = \iota + (f - \iota),$$

we can think of $f - \iota$ as a nilpotent linear map on a finite-dimensional vector space isomorphic to $\widehat{\mathrm{CF}}(\mathcal{H}_{\alpha,\beta'})$, because the area filtration always increases. Since $\iota$ is a isomorphism of vector spaces, by Proposition 3.4.4 so is $f$. □

## 3.5 Heegaard triads

Let $\mathcal{E}_{\alpha,\beta} = (E, \{\alpha\}, \{\beta\}, w)$ be a pointed genus one Heegaard diagram such that $\alpha$ is a homologically essential simple closed curve, and $\beta$ is a curve that is isotopic to $\alpha$ and intersecting $\alpha$ transversely in two points. Given an admissible pointed Heegaard diagram $\mathcal{H}_{\gamma,\delta} = (\Sigma, \boldsymbol{\gamma}, \boldsymbol{\delta}, w)$, the *near-stabilization* is the pointed Heegaard diagram $\mathcal{H}_{\gamma,\delta} \# \mathcal{E}_{\alpha,\beta}$, see Definition 2.1.6 for the definition of a connected sum of Heegaard diagrams.

Let $\beta^{(1)}, \beta^{(2)}, \beta^{(3)}$ be a triad of curves in the torus $E$, as in Figure 3.1. Choose a basepoint $w$ in the hexagon away from the attaching curves; call the collected data $(E, \beta^{(1)}, \beta^{(2)}, \beta^{(3)}, w)$ a *genus one Heegaard triad*. The *genus $g$ Heegaard triad* is the pointed Heegaard triple $(\Sigma, \boldsymbol{\beta}^{(1)}, \boldsymbol{\beta}^{(2)}, \boldsymbol{\beta}^{(3)}, w)$ obtained by taking the $(g-1)$-fold near stabilization of the genus one Heegaard triad $(E, \beta^{(1)}, \beta^{(2)}, \beta^{(3)}, w)$.

In the proof of the exact triangle, we will be counting pseudo-holomorphic rectangles whose edges are the $\boldsymbol{\beta}$-curves. To appropriately count these rectangles, we have to perturb

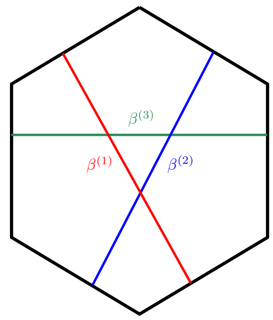


Figure 3.1: A triad of curves $\beta^{(1)}, \beta^{(2)}, \beta^{(3)}$ in the torus.

the set of $\boldsymbol{\beta}$ curves off itself. For this reason, We extend the triad of attaching circles $\boldsymbol{\beta}^{(1)}, \boldsymbol{\beta}^{(2)}, \boldsymbol{\beta}^{(3)}$ to an infinite sequence of $g$-tuples of attaching circles $\{\boldsymbol{\beta}^{(i)}\}_{i=1}^{\infty}$ such that if $i \equiv j \pmod 3$ then $\boldsymbol{\beta}^{(i)}$ is an exact Hamiltonian translate of $\boldsymbol{\beta}^{(j)}$.

**Lemma 3.5.1.** *$(\Sigma, \boldsymbol{\beta}^{(i)}, \boldsymbol{\beta}^{(i+1)}, w)$ is an admissible Heegaard diagram for $\#^{g-1}(S^2 \times S^1)$.*

*Proof.* For $g = 1$, the diagram $(\Sigma, \boldsymbol{\beta}^{(i)}, \boldsymbol{\beta}^{(i+1)}, w)$ is the standard Heegaard diagram for $S^3$, hence admissible. For $g > 1$, the periodic domains of $(\Sigma, \boldsymbol{\beta}^{(i)}, \boldsymbol{\beta}^{(i+1)}, w)$ are generated by the domains $D_i - D_i'$, shown in the picture. Therefore, the diagram is admissible. $\square$

The Heegaard diagram $(\Sigma, \boldsymbol{\beta}^{(i)}, \boldsymbol{\beta}^{(i+1)}, w)$ has the property that $\mathbb{T}_{\boldsymbol{\beta}^{(i)}} \cap \mathbb{T}_{\boldsymbol{\beta}^{(i+1)}}$ consists of $2^{g-1}$ points. Moreover, the differential of $\widehat{\mathrm{CF}}(\mathcal{H}_{\beta^{(i)},\beta^{(i+1)}})$ is trivial, so each point represents a distinct generator in $\widehat{\mathrm{HF}}(\mathcal{H}_{\beta^{(i)},\beta^{(i+1)}})$. By Lemma 3.4.1, in the diagram $\mathcal{H}_{\boldsymbol{\beta},\boldsymbol{\beta}'}$, we can find an intersection point $\mathbf{t}_i \in \mathbb{T}_{\beta^{(i)}} \cap \mathbb{T}_{\beta^{(i+1)}}$ which represents a top dimensional homology class.

**Lemma 3.5.2.** *With the above choices, for every positive integer $i$,*

$$\widehat{m}(\mathbf{t}_i, \mathbf{t}_{i+1}) = 0.$$

*Proof.* First we consider the case of a genus $g = 1$ Heegaard triad. Then $(\Sigma, \boldsymbol{\beta}^{(i)}, \boldsymbol{\beta}^{(i+1)}, w)$ is a genus one Heegaard diagram for $S^3$ with one Heegaard state.

The curves $\beta^{(i)}, \beta^{(i+1)}, \beta^{(i+2)}$ divide the torus into a hexagon that contains the basepoint $w$ and the two triangles. Since both triangles have vanishing local multiplicity at $w$ and

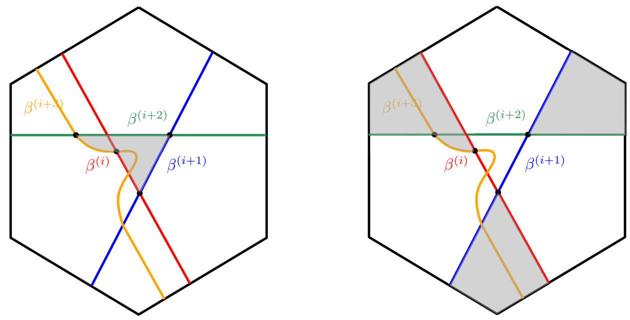


Figure 3.2: Two shadows of rectangles with local multiplicity zero at $w$.

have Maslov index 0, both contribute a term of $\mathbf{t}_{i+2}$ to $\widehat{m}(\mathbf{t}_i, \mathbf{t}_{i+1})$. Thus the contributions of these two triangles cancel.

For the $g > 1$ case, note that there is a unique positive domain in the nearby triple containing the top-dimensional generators of $\mathcal{H}_{\alpha,\beta}$ and $\mathcal{H}_{\beta,\gamma}$. So, there are exactly two positive domains connecting $\mathbf{t}_i, \mathbf{t}_{i+1}, \mathbf{y}$: the disjoint union of one of the two triangles from the hexagon with $g-1$ triangles from the nearby triples. By the Riemann mapping theorem, there is a unique pseudo-holomorphic triangle with this shadow. Therefore, these triangles cancel in the count. $\square$

Since $\boldsymbol{\beta}^{(i+3)}$ is an exact Hamiltonian translate of $\boldsymbol{\beta}^{(i)}$, we have that $\mathbb{T}_{\beta^{(i)}} \cap \mathbb{T}_{\beta^{(i+3)}}$ consists of $2^g$ intersection points, each of which represents a distinct generator in $\widehat{\mathrm{HF}}(\mathcal{H}_{\beta^{(i)},\beta^{(i+3)}}) = \widehat{\mathrm{HF}}(\#^g(S^1 \times S^2))$. Again by Lemma 3.4.1, we can an intersection point $\mathbf{t}'_i \in \mathbb{T}_{\beta^{(i)}} \cap \mathbb{T}_{\beta^{(i+3)}}$ that represents the top-dimensional homology class.

**Lemma 3.5.3.** *With the above choices, for every positive integer $i$,*

$$\widehat{m}(\mathbf{t}_i, \mathbf{t}_{i+1}, \mathbf{t}_{i+2}) = \mathbf{t}'_i.$$

*Proof.* First we prove the lemma in genus $g = 1$. The quadruple $(E, \beta^{(i)}, \beta^{(i+1)}), \beta^{(i+2)}, \beta^{(i+3)})$ has exactly two shadows of rectangles with $n_w = 0$ and corners at $\mathbf{t}_i, \mathbf{t}_{i+1}, \mathbf{t}_{i+2}, \mathbf{t}'_i$, pictured in Figure 3.2. The left embedded rectangle is a nonnegative domain, and hence determines an embedded quadrilateral in the torus. By the Riemann mapping theorem, this embedded quadrilateral has a unique holomorphic representative. The right quadrilateral has an ob-

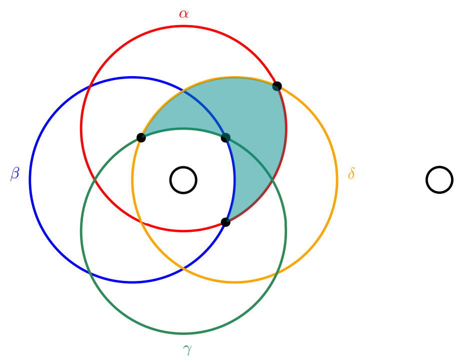


Figure 3.3: A nearby quadruple.

tuse angle, and since holomorphic maps are conformal, it does not determine a holomorphic representative.

For the $g > 1$ case, note that there is a unique positive domain in the *nearby quadruple*, a torus with four closed curves shown in Figure 3.3, containing the top-dimensional generators of $\mathcal{H}_{\alpha,\beta}$, $\mathcal{H}_{\beta,\gamma}$, and $\mathcal{H}_{\gamma,\delta}$. So, there is a unique domain connecting $\mathbf{t}_i, \mathbf{t}_{i+1}, \mathbf{t}_{i+2}$ with another vertex: the disjoint union of the embedded rectangle with the $g-1$ embedded rectangles from the nearby quadruple. By the Riemann mapping theorem, there is a unique pseudo-holomorphic rectangle with this shadow. $\square$

## 3.6 Proof of the surgery exact triangle

Let $M$ be a smooth 3-manifold with torus boundary, and let $(Y_1, Y_2, Y_3)$ be a triad of manifolds obtained by filling a triad of curves $b_1, b_2, b_3 \subset \partial M$. From Theorem 1.1.4 there exists a self-indexing Morse function $f : M \to \mathbb{R}$ with one index-0 critical point, $g$ index-1 critical points, $g-1$ index-2 critical points, and no index 3 critical points. Such a Morse function induces a handle decomposition of $M$ into one 0-handle, $g$ 1-handles, and $g-1$ 2-handles. Let $\alpha_1, \ldots, \alpha_g$ be belt circles on each of the $g$ one-handles, and let $\beta_1, \ldots, \beta_{g-1}$ be the attaching circles of the $g-1$ 2-handles. If $Y_1$ is the union of the 0-handle and all the 1-handles, then the curves $\alpha_1, \ldots, \alpha_g$ and $\beta_1, \ldots, \beta_{g-1}$ are curves on the boundary $\Sigma = \partial Y_1$. From the definition of a triad then $(\Sigma, \{\alpha_1, \ldots, \alpha_g\}, \{b_i, \beta_1, \ldots, \beta_{g-1}\}, w)$, where

$w \notin \Sigma \setminus \bigcup \alpha_i \setminus \bigcup \beta_j \setminus b_i$, is a Heegaard diagram for $Y_i$.

For brevity let

$$\boldsymbol{\alpha} = \{\alpha_1, \ldots, \alpha_g\} \text{ and } \boldsymbol{\beta}^{(i)} = \{b_i, \beta_1, \ldots, \beta_{g-1}\} \text{ for } i = 1, 2, 3.$$

As before, we extend the triad of attaching circles $\boldsymbol{\beta}^{(1)}, \boldsymbol{\beta}^{(2)}, \boldsymbol{\beta}^{(3)}$ to an infinite sequence of $g$-tuples of attaching circles $\{\boldsymbol{\beta}^{(i)}\}_{i=1}^{\infty}$ such that if $i \equiv j \pmod 3$ then $\boldsymbol{\beta}^{(i)}$ is an exact Hamiltonian translate of $\boldsymbol{\beta}^{(j)}$.

Define chain complexes

$$(C^i, \partial^i) = \widehat{\mathrm{CF}}(\mathcal{H}_{\alpha,\beta^{(i)}})$$

for every positive integer $i$. Let $\widehat{f}^i : \widehat{\mathrm{CF}}(\mathcal{H}_{\alpha,\beta^{(i)}}) \to \widehat{\mathrm{CF}}(\mathcal{H}_{\alpha,\beta^{(i+1)}})$ be given by

$$\widehat{f}^i(\mathbf{x}) = \widehat{m}_{\alpha,\beta^{(i)},\beta^{(i+1)}}(\mathbf{x}, \mathbf{t}_i).$$

Let $\widehat{h}^i : \widehat{\mathrm{CF}}(\mathcal{H}_{\alpha,\beta^{(i)}}) \to \widehat{\mathrm{CF}}(\mathcal{H}_{\alpha,\beta^{(i+2)}})$ be given by

$$\widehat{h}^{(i)}(\mathbf{x}) = \widehat{m}_{\alpha,\beta^{(i)},\beta^{(i+1)},\beta^{(i+2)}}(\mathbf{x}, \mathbf{t}_i, \mathbf{t}_{i+1}).$$

**Lemma 3.6.1.** *The following holds:*

1. *$\widehat{f}^i$ is a chain map, i.e.*
$$\widehat{\partial}^{i+1} \circ \widehat{f}^i = \widehat{f}^i \circ \widehat{\partial}^i.$$

2. *$\widehat{h}^i$ is a nullhomotopy operator for $\widehat{f}^{i+1} \circ \widehat{f}^i$, i.e.*
$$\widehat{f}^{i+1} \circ \widehat{f}^i = \widehat{\partial}^{i+2} \circ \widehat{h}^i + \widehat{h}^i \circ \widehat{\partial}^i.$$

3. *The map $\widehat{\phi}^i = \widehat{f}^{i+2} \circ \widehat{h}^i + \widehat{h}^{i+1} \circ \widehat{f}^i$ is a chain map. Moreover, $\widehat{\phi}^i$ is a quasi-isomorphism.*

*Proof.* For brevity, we will sometimes denote the multiplication maps from the $\mathcal{A}_\infty$ relation by $\hat{m}$ when it is clear which elements we are multiplying.

1. Since triangle counting map

$$\widehat{m} = \widehat{m}_{\boldsymbol{\alpha},\boldsymbol{\beta}^{(i)},\boldsymbol{\beta}^{(i+1)}} : \widehat{\mathrm{CF}}(\mathcal{H}_{\alpha,\beta^{(i)}}) \otimes \widehat{\mathrm{CF}}(\mathcal{H}_{\beta^{(i)},\beta^{(i+1)}}) \to \widehat{\mathrm{CF}}(\mathcal{H}_{\alpha,\beta^{(i+1)}})$$

   is a chain map (Theorem 2.5.1), and $\mathbf{t}_i$ is a cycle, we have that

$$\widehat{\partial}^{i+1}(\widehat{f}^i(\mathbf{x})) = \widehat{\partial}^{i+1}(\widehat{m}(\mathbf{x},\mathbf{t}_i)) = \widehat{m}(\widehat{\partial}^{i+1}\mathbf{x},\mathbf{t}_i) = \widehat{f}^i(\widehat{\partial}^i(\mathbf{x})).$$

2. Taking into account that $\partial\mathbf{t}_i = 0$, $\widehat{m}(\mathbf{t}_i,\mathbf{t}_{i+1}) = 0$ (Lemma 3.5.2), the associativity law 2.5.2 simplifies to

$$\widehat{m}(\widehat{\partial}\mathbf{x},\mathbf{t}_i,\mathbf{t}_{i+1}) + \widehat{\partial}\widehat{m}(\mathbf{x},\mathbf{t}_i,\mathbf{t}_{i+1}) + \widehat{m}(\widehat{m}(\mathbf{x},\mathbf{t}_i),\mathbf{t}_{i+1}) = 0.$$

   Therefore,

$$\widehat{f}^{i+1}(\widehat{f}^i(\mathbf{x})) = \widehat{\partial}^{i+2}(\widehat{h}^i(\mathbf{x})) + \widehat{h}^i(\widehat{\partial}^i(\mathbf{x})).$$

3. By the first part of Lemma 3.3.3, $\widehat{\phi}^i$ is a chain map. Now we show that $\widehat{\phi}^i$ is a quasi-isomorphism. Taking in account that $\partial\mathbf{t}_i = 0$, $\widehat{m}(\mathbf{t}_i,\mathbf{t}_{i+1}) = 0$ (Lemma 3.5.2), and $\widehat{m}(\mathbf{t}_i,\mathbf{t}_{i+1},\mathbf{t}_{i+2}) = \mathbf{t}_i'$ (Lemma 3.5.3), the $\mathcal{A}_\infty$ relation on four inputs (Theorem 2.5.2) simplifies to

$$\begin{aligned}
&\widehat{\partial}\widehat{m}(\mathbf{x},\mathbf{t}_i,\mathbf{t}_{i+1},\mathbf{t}_{i+2}) + \widehat{m}(\widehat{\partial}\mathbf{x},\mathbf{t}_i,\mathbf{t}_{i+1},\mathbf{t}_{i+2})\\
&= \widehat{m}(\widehat{m}(\mathbf{x},\mathbf{t}_i),\mathbf{t}_{i+1},\mathbf{t}_{i+2}) + \widehat{m}(\widehat{m}(\mathbf{x},\mathbf{t}_i,\mathbf{t}_{i+1}),\mathbf{t}_{i+2}) + \widehat{m}(\mathbf{x},\widehat{m}(\mathbf{t}_i,\mathbf{t}_{i+1},\mathbf{t}_{i+2}))\\
&= \widehat{h}^{i+1}(\widehat{f}^i(\mathbf{x})) + \widehat{f}^{i+2}(\widehat{h}^i(\mathbf{x})) + \widehat{m}(\mathbf{x},\mathbf{t}_i')
\end{aligned}$$

   From Lemma 3.4.2, $\mathbf{x} \mapsto \widehat{m}(\mathbf{x},\mathbf{t}_i')$ is a quasi-isomorphism, so we have a chain homotopy between $\widehat{f}^{i+2} \circ \widehat{h}^i + \widehat{h}^{i+1} \circ \widehat{f}^i = \widehat{\phi}^i$ and a quasi-isomorphism. Therefore, $\widehat{\phi}^i$ is a quasi-isomorphism.

□

*Proof of Theorem 3.2.3.* From Lemma 3.6.1, $(C^i, \partial^i) = (\widehat{\mathrm{CF}}(\mathcal{H}_{\alpha,\beta^{(i)}}), \widehat{\partial}^i)$ and the maps $\widehat{f}^i$ and $\widehat{h}^i$, satisfy the hypotheses of Lemma 3.3.3. Thus by Lemma 3.3.3 , there is an exact triangle connecting the triad $(\widehat{\mathrm{CF}}(\mathcal{H}_{\alpha,\beta^{(1)}}), \widehat{\mathrm{CF}}(\mathcal{H}_{\alpha,\beta^{(2)}}), \widehat{\mathrm{CF}}(\mathcal{H}_{\alpha,\beta^{(3)}})) = (Y_1, Y_2, Y_3)$. $\square$

## 3.7 Applications to finding $L$-space manifolds

Given a three-manifold $Y$, let $|H_1(Y;\mathbb{Z})|$ be the integer defined as follows. If the number of elements $n$ in $H_1(Y;\mathbb{Z})$ is finite, then $|H_1(Y;\mathbb{Z})| = n$; otherwise, $|H_1(Y;\mathbb{Z})| = 0$. In [34, Lemma 1.6] it is shown that

$$\chi(\widehat{\mathrm{HF}}(Y)) = \pm|H_1(Y;\mathbb{Z})|$$

by analyzing a CW decomposition of $Y$ obtained from a Heegaard diagram. In fact, if $\mathfrak{t} \in \mathrm{Spin}^c(Y)$, then

$$\chi(\widehat{\mathrm{HF}}(Y,\mathfrak{t})) = \begin{cases} \pm 1 & \text{if } H_1(Y;\mathbb{Z}) \text{ is finite,} \\ 0 & \text{otherwise.} \end{cases}$$

In particular, if $Y$ is a rational homology sphere, then $H_1(Y;\mathbb{Z})$ is finite, so

$$\dim \widehat{\mathrm{HF}}(Y) \geq |H_1(Y;\mathbb{Z})|.$$

**Definition 3.7.1.** A rational homology sphere $Y$ is *$L$-space* if $\dim \widehat{\mathrm{HF}}(Y) = |H_1(Y;\mathbb{Z})|$.

**Example 3.7.2.** The lens spaces $L(p,q)$, where $p$ and $q$ are coprime integers, are $L$-space. One way to define a lens space is by creating a Heegaard diagram on a genus one surface with one horizontal $\alpha$ curve horizontal and one $\beta$-curve with slope $p/q$. Then there are $p$ intersection points, which we can label $x_1, \ldots, x_p$ from left to right. See Figure 3.4 for an example for $L(5,2)$.

We show that for any two intersection points $x, y$, the obstruction $\epsilon(x,y)$ defined in Section 2.1.1 is nonzero. Let $\gamma$ be the homology class representing a meridian of the torus, where we assume that and $\delta$ be the homology class representing the longitude of the torus. Also assume that $\alpha$ is homologous to the meridian. Choosing the standard segments from

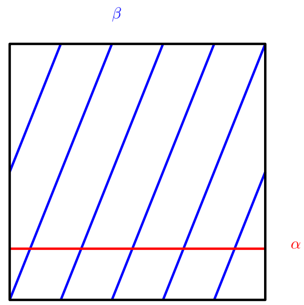


Figure 3.4: A Heegaard diagram for $L(5,2)$.

$x_i$ to $x_j$ on the $\alpha$ and $\beta$ curves, we see that $\epsilon(\mathbf{x},\mathbf{y}) = n\delta$ for some positive integer $n \in \{1,\ldots,p-1\}$.

Thus, any two Heegaard states are not equivalent, so $\dim \widehat{\mathrm{HF}}(L(p,q),\mathfrak{s}) = 1$ for all $\mathfrak{s} \in \mathrm{Spin}^c(L(p,q))$. This implies that $L(p,q)$ is $L$-space.

**Lemma 3.7.3.** *Suppose $(Y_1,Y_2,Y_3)$ is a triad of rational homology spheres such that*

$$|H_1(Y_3)| = |H_1(Y_1)| + |H_1(Y_2)|$$

*and $Y_1$ and $Y_2$ are $L$-spaces. Then $Y_3$ is an $L$-space.*

*Proof.* We have that

$$\dim \widehat{\mathrm{HF}}(Y_3) \leq \dim \widehat{\mathrm{HF}}(Y_1) + \dim \widehat{\mathrm{HF}}(Y_2) = |H_1(Y_1)| + |H_1(Y_2)| = |H_1(Y_3)|,$$

where the inequality is from the exactness of the surgery exact triangle of the triad $(Y_1,Y_2,Y_3)$, the first equality is by the fact that $Y_1$ and $Y_2$ are $L$-space, and the second equality is by assumption. Since $\dim \widehat{\mathrm{HF}}(Y_3) \geq |H_1(Y_3)|$ holds for any rational homology sphere, this concludes the proof. □

**Corollary 3.7.4.** *Let $K$ be knot in $S^3$ such that $S^3_n(K)$ is an $L$-space for some $n > 0$. Then $S^3_m(K)$ is an $L$-space for all $m \geq n$.*

*Proof.* From Example 3.2.6, $(S^3, S^3_n(K), S^3_{n+1}(K))$ form a triad of three-manifolds. Moreover, since $|H_1(S^3_{n+1}(K))| = n+1 = |H_1(S^3_n(K))| + |H_1(S^3)|$, applying Lemma 3.7.3 gives the result by induction. ☐

One can show [29, Chapter 18] that for the $(p, q)$ torus knot $T_{p,q}$, the three-manifold $S^3_{pq-1}(T_{p,q})$ is an , so by the above coroallary, $S^3_n(T_{p,q})$ is an for any $n \geq pq - 1$.

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