ON MORSE–SMALE AND LAGRANGIAN FLOER HOMOLOGY

MATTHEW KENDALL

Contents

1. Introduction

Morse theory allows one to analyze the topology of a finite dimensional smooth manifold by studying differentiable functions on the manifold. Morse homology is a particularly illuminating way to understand the homology of smooth manifolds, and can be proven to be isomorphic to the singular homology the manifold. The theory involves a generic choice of smooth function and Riemannian metric on the manifold.

The underlying complex in the construction, here called the *Morse–Smale complex*, has somewhat of an opaque history.[1] There were early suggestions of this complex by Thom [12] and Smale [11], and an essentially equivalent complex was described in Milnor's book on the h-cobordism theorem [9], but not in the language of gradient flowlines. Witten's paper "Supersymmetry and Morse Theory" [13] had a big effect on the interpretation of the Morse complex, and allowed for a generalization to an infinite-dimensional analogue of Morse theory in symplectic manifolds.

In the 1980's, Floer was working on the Arnold conjecture [1, p. 419], which was concerned with the relationship between Morse theory and fixed points of certain diffeomorphisms of a symplectic manifold. He developed [4] what is now known as *Lagrangian Floer homology*, a type of Morse theory for Lagrangian intersections, to solve a case of the conjecture for a large class of symplectic manifolds. There are obstacles, one being that that the language of gradient flowlines does not extend via the classical approach to Morse homology because the index of critical points he was led to analyze might be infinite. However, the index difference turned out to be finite, which was exploited by Floer when constructing the Lagrangian Floer complex.

The goal of this paper is to communicate the stories of Morse homology and Lagrangian Floer homology side by side and draw on their similarities and differences, mostly following the book in preparation *Heegaard Floer Homology* by Ozsváth, Szabó, and Stipsicz [10]. Lagrangian Floer homology is an important part of the construction of Heegaard Floer homology, where the ambient symplectic manifold is a symmetric product of a Riemann surface and the Lagrangian submanifolds are tori. Part of the reason I wanted to write this paper is to collect and consolidate the main ideas behind the Morse–Smale and Lagrangian Floer complexes.

1.1. **Acknowledgement.** This paper was done for my junior independent work. I would like to thank my advisor Peter Ozsváth for his constant guidance and support throughout the project.

1.2. **Delcaration.** This paper represents my own work within university regulations.

2. Morse theory

The goal of this section is to give a rapid account of the Morse theory needed for the Morse–Smale complex. In Section 2.1, we define Morse functions and their index. In Section 2.2, we emphasize the handlebody decomposition and Morse inequalities that will be useful context for the construction of the Morse–Smale complex.

2.1. **Morse functions.** Let M be a smooth manifold, and let $f : M \to \mathbb{R}$ be a smooth function on M. The point $p \in M$ is called a *critical point* if df vanishes at p. The critical point p is called *non-degenerate* if in some local coordinate chart $x_1, \ldots, x_n$ of p, the Hessian matrix $(\frac{\partial^2 f}{\partial x_i \partial x_j}(p))$ is invertible.

The following coordinate-free description of the Hessian will be useful, at least formally when generalizing the Hessian to infinite dimensional manifolds. Let M be a Riemannian manifold with a metric g, let $f : M \to \mathbb{R}$ be a Morse function, and let $p \in M$ be a critical point. The Hessian of f at p is a map

$$\mathrm{Hess}_f : T_pM \to T_pM$$

[1]This history section is taken in part from Michael Hutchings' "Lecture notes on Morse homology".

characterized by the equation

$$g(\mathrm{Hess}_f(u), v) = u(\tilde{v}f), \tag{1}$$

where $u, v \in T_pM$, and $\tilde{v}$ is a vector field extending v to a neighborhood of p. Also, $\tilde{v}f$ is the directional derivative of f in the direction specified by v. The right hand side can be shown not to depend on the extension $\tilde{v}$ of v.

Definition 2.1. A smooth function $f : M \to \mathbb{R}$ is called a *Morse function* if each critical point of f is nondegenerate.

Definition 2.2. If p is a non-degenerate critical point of f, its *index* $\lambda(p)$ is the maximal dimension of a subspace of T_pX such that the Hessian is negative definite.

2.2. **Important results.** First of all, Morse functions exist on any compact smooth manifold M, compare [8, Corollary 6.8]. To get a grip on Morse functions, the *Morse lemma* is helpful, which we state now, also see [8, I.2]. Given an n-dimensional manifold M, a Morse function $f : M \to \mathbb{R}$ and an index k critical point $p \in M$, there exists a coordinate chart of f around p on which f has the form

$$f(p) - x_1^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_n^2.$$

From the Morse lemma, it follows that the cricial points of a Morse function are isolated.

One can extract topological information from Morse functions. Given a Morse function $f : M \to \mathbb{R}$, let M_t denote $f^{-1}(-\infty, t]$. Compare [8, I.3] and [10, Theorem 1.1.6].

Theorem 2.3. *Let $f : M \to \mathbb{R}$ be a Morse function, and assume that $t_1, t_2 \in \mathbb{R}$ are regular values with $t_1 < t_2$. If $f^{-1}[t_1, t_2]$ contains no critical points, then M_{t_1} is diffeomorphic to M_{t_2}. If $f^{-1}[t_1, t_2]$ contains exactly one crtical point p of index k, then M_{t_2} can be constructed from M_{t_1} by attaching a smooth n-dimensional k-handle.*

In particular, the Morse function f gives M the structure of a handlebody and hence presents M as a CW complex.

One can also prove the *Morse inequalities*: for a Morse function $f : M \to \mathbb{R}$, if c_i is the number of critical points of index i and $b_i(M)$ is the rank of $H_i(M; \mathbb{F})$ where $\mathbb{F}$ is any field, then

$$c_i - c_{i-1} + c_{i-2} - \cdots + (-1)^i c_0 \geq b_i(M) - b_{i-1}(M) + b_{i-2}(M) + \cdots + (-1)^i b_0(M)$$

for all $i = 0, \dots, n$. See for example [8, I.5]. In particular, we inductively get that $c_i \geq b_i(M)$ for $i = 0, \dots, n$. Summing over all i, we get the following topological lower bound on the number of critical points of a Morse function:

$$\#\,\mathrm{Crit}(f) \geq \sum_{i=0}^{n} b_i(M). \tag{2}$$

3. The Morse–Smale complex

The goal of this section is to introduce the Morse–Smale complex and give the main idea behind why the Morse–Smale complex is a topological invariant. In Section 3.1, we define the Smale condition, which will be a useful hypothesis for many of the later technical results. In Section 3.2, we introduce gradient flowlines and their moduli spaces. In Section 3.3, we define the Morse–Smale complex. In Section 3.4 and Section 3.5, we introduce the compactness and gluing techniques needed to show that the definition of the Morse–Smale complex is actually a complex. In Section 3.6, we go over the main ideas behind showing that the Morse–Smale complex is a topological invariant, which in our case means that it does not depend on the auxillary choice of Morse function and Riemannian metric.

3.1. **The Smale condition.** Let M be a smooth manifold, f be a Morse function on M and g is a Riemannian metric on M. Let $\varphi^s : M \to M$ for $s \in \mathbb{R}$ denote the flow of the gradient of f. Define the *ascending manifold* at a critical point p of f to be

$$A(p) = \{x \in M \mid \lim_{x\to+\infty} \varphi^s(x) = p\}$$

and the *descending manifold* at p to be

$$D(p) = \{x \in M \mid \lim_{x\to-\infty} \varphi^s(x) = p\}.$$

For example, consider the sphere S^2 centered at the origin and standard height function $f : S^2 \to \mathbb{R}$. If p is the top point $(0, 0, 1)$ and q is the bottom point $(0, 0, -1)$, then

$$\begin{aligned} A(p) &= S^2 - \{q\}, & D(p) &= \{q\}, \\ A(q) &= \{q\}, & D(q) &= S^2 - \{p\}. \end{aligned}$$

If p is a critical point of index k, then $A(p)$ can be shown to be diffeomorphic to a disk of dimension $n - k$ and $D(p)$ can be shown to be diffeomorphic to a disk of dimension k, see [2, p. 28].

Call the pair (f, g) *Morse-Smale* for all pairs of critical points p, q of f, the ascending manifold $A(p)$ and the descending manifold $D(p)$ intersect transversely.[2] Alternatively, sometimes it is said that the gradient of f satisfies the *Smale condition*.

Suppose now that our manifold M is closed (this condition is not necessarily needed, but it will simplify our life). Then a theorem of Smale [11] states that given a Morse function f on M, there exists a Riemannian metric g on M such that (f, g) is Morse–Smale.

3.2. **Gradient flowlines and their moduli spaces.** Let M be a finite dimensional manifold equipped with a Morse function $f : M \to \mathbb{R}$. Also equip M with a Riemannian metric g.

Definition 3.1. Fix $\mathbf{x}, \mathbf{y} \in \mathrm{Crit}(f)$. A *gradient flowline* from $\mathbf{x}$ to $\mathbf{y}$ is a path $\gamma : \mathbb{R} \to M$ such that $\lim_{t\to-\infty} \gamma(t) = \mathbf{x}$ and $\lim_{t\to+\infty} \gamma(t) = \mathbf{y}$ and γ satisfies the gradient flow equation

$$\frac{d\gamma}{dt}(t) = (-\nabla_g f)_{\gamma(t)}.$$

We can collect all gradient lines from $\mathbf{x}$ to $\mathbf{y}$ into a *moduli space* $\mathcal{M}(\mathbf{x}, \mathbf{y})$. There is a natural $\mathbb{R}$-action on $\mathcal{M}(\mathbf{x}, \mathbf{y})$: if γ is a gradient flowline from $\mathbf{x}$ to $\mathbf{y}$ and s is a real number, then the path $\tau_s(\gamma) : \mathbb{R} \to M$ given by $t \mapsto \gamma(t + s)$ is also a gradient flowline. This $\mathbb{R}$-action on $\mathcal{M}(\mathbf{x}, \mathbf{y})$ is called *time translation*. Let $\widehat{\mathcal{M}}(\mathbf{x}, \mathbf{y})$ be the quotient of $\mathcal{M}(\mathbf{x}, \mathbf{y})$ by the time translation action, i.e.

$$\widehat{\mathcal{M}}(\mathbf{x}, \mathbf{y}) = \mathcal{M}(\mathbf{x}, \mathbf{y})/\mathbb{R}.$$

The moduli spaces $\widehat{\mathcal{M}}(\mathbf{x}, \mathbf{y})$ turn out to be smooth finite dimensional manifolds. Compare [2, p. 39] and [10, Theorem 6.1.2].

[2]Let P be a manifold and let M and N be two submanifolds of P. Say that M and N are *transverse* if for any point u in the intersection of M and N, $T_uP = T_uM + T_uN$.

Theorem 3.2. *Let M be a finite dimensional closed manifold and let $f : M \to \mathbb{R}$ be a Morse function on M. Choose a Riemannian metric g such that (f, g) satisfies the Smale condition. Fix two critical points $\mathbf{x}, \mathbf{y}$ of f such that $\mathbf{x} \neq \mathbf{y}$, and consider the space $\widehat{\mathcal{M}}(\mathbf{x}, \mathbf{y})$ of gradient flowlines from $\mathbf{x}$ to $\mathbf{y}$ modulo time translation. Then $\widehat{\mathcal{M}}(\mathbf{x}, \mathbf{y})$ is a smooth manifold of dimension*

$$\dim \widehat{\mathcal{M}}(\mathbf{x}, \mathbf{y}) = \lambda(\mathbf{x}) - \lambda(\mathbf{y}) - 1.$$

In particular, if $\lambda(\mathbf{x}) = \lambda(\mathbf{y})$, the moduli space $\widehat{\mathcal{M}}(\mathbf{x}, \mathbf{y})$ is empty.

Example 3.3. Consider $S^n \subseteq \mathbb{R}^{n+1}$ with the round metric, and consider the height function $f : S^n \to \mathbb{R}$. It can be seen that f is a Morse function, and the critical points of f are the north and south poles of S^n, call them $\mathbf{x}$ and $\mathbf{y}$ respectively. Then every gradient flowline is a great half-circle connecting $\mathbf{x}$ and $\mathbf{y}$, so $\widehat{\mathcal{M}}(\mathbf{x}, \mathbf{y})$ is diffeomorphic to the equator $\{x_{n+1} = 0\}$, which is an $(n-1)$-dimensional sphere S^{n-1}.

3.3. **Definition of the Morse–Smale complex.** This section is devoted to defining the Morse–Smale chain complex, and then outlining the next steps in this section to make sense of the definition. Let M be a smooth manifold, let f be a Morse function on M, let and g be a Riemannian metric on M such that the pair (f, g) is Morse–Smale.

Definition 3.4. The *Morse–Smale chain complex* $\mathrm{CM}(M, f, g)$ is the vector space over $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$ generated by the critical points of f and equipped with the endomorphism ∂ defined by

$$\partial \mathbf{x} = \sum_{\{\mathbf{y} \in \mathrm{Crit}(f) \mid \lambda(\mathbf{x}) - \lambda(\mathbf{y}) = 1\}} \# \widehat{\mathcal{M}}(\mathbf{x}, \mathbf{y}) \cdot \mathbf{y}. \tag{3}$$

Here, $\#\widehat{\mathcal{M}}(\mathbf{x}, \mathbf{y})$ is the parity of the number of elements in $\widehat{\mathcal{M}}(\mathbf{x}, \mathbf{y})$.

Having arrived at the definition, there are a number of components to resolve:

(1) The definition of $\partial \mathbf{x}$ given in (3) is a finite sum. From Theorem 3.2, the moduli spaces $\widehat{\mathcal{M}}(\mathbf{x}, \mathbf{y})$ are zero dimensional manifolds. So, to make sense of the sum in (3), we need $\widehat{\mathcal{M}}(\mathbf{x}, \mathbf{y})$ to be compact. This is what we discuss in Section 3.4.
(2) The endomorphism ∂ satisfies $\partial^2 = 0$, making $\mathrm{CM}(M, f, g)$ into a chain complex with differential ∂. In this case, we will see that we are analyzing moduli spaces $\widehat{\mathcal{M}}(\mathbf{x}, \mathbf{z})$ with $\lambda(\mathbf{x}) - \lambda(\mathbf{z}) = 2$, and their boundary will be the terms in the sum $\partial^2 \mathbf{x}$. To properly understand the boundary, we need a gluing tool. This is outlined in Section 3.5.
(3) The Morse–Smale complex $\mathrm{CM}(M, f, g)$ is independent of the choice of Morse function f and Riemannian metric g on M. The main ideas of this proof will be outlined in Section 3.6.

Remark 3.5. One could also directly prove that $\mathrm{CM}(M, f, g)$ is isomorphic to the cellular chain complex with $\mathbb{Z}/2\mathbb{Z}$ coefficients provided by the CW decomposition of M from the Morse function, see Theorem 2.3. However, the perspective that generalizes to the case of Lagrangian Floer homology by directly analyzing the moduli spaces of gradient flowlines. This is the perspective we take in Section 3.6.

Example 3.6. Let our manifold be the torus T^2, thought of as a quotient of $[0, 1] \times [0, 1]$, and consider the function $f : T^2 \to \mathbb{R}$ given by

$$f(x, y) = \cos 2\pi x + \sin 2\pi x.$$

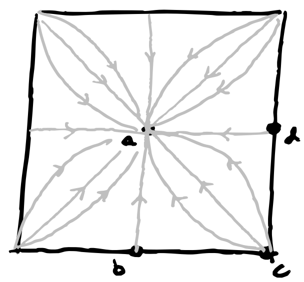

FIGURE 1. An example of gradient flow on the torus.

This function is Morse, and it has a single critical point a of index zero, two critical points b, c of index 1, and one critical point d of index 2. Giving the torus the flat metric, we see that the gradient flowlines are as drawn in Figure 1. We conclude that

$$\partial a = 0, \qquad \partial b = \partial c = 2a = 0, \qquad \partial d = 0,$$

so the Morse–Smale homology groups are equal to the complex groups $H_0 = \mathbb{Z}/2\mathbb{Z}$, $H_1 = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, $H_2 = \mathbb{Z}/2\mathbb{Z}$. These are the singular homology groups of T^2.

Remark 3.7. Morse Theory can be generalized to consider functions whose critical sets are submanifolds, called Morse–Bott theory. In this version of the story, a Morse–Bott function is a smooth function on a manifold whose critical set is a closed submanifold and whose Hessian is nondegenerate in the normal direction of every point. A Morse function is a special case where the critical points are zero dimensional, so the Hessian is critical in every direction, or equivalently, has no kernel. Bott [3] used Morse–Bott theory in his original proof of the Bott periodicity theorem.

3.4. **Compactness.** The goal of this section is to give the necessary tools needed to show that the endomorphism ∂ in the definition of $\mathrm{CM}(M, f, g)$ is finite. Let $\mathbf{x}$ and $\mathbf{y}$ be critical points of a Morse function $f : M \to \mathbb{R}$. A *broken flowline* from $\mathbf{x}$ to $\mathbf{y}$ is a sequence of distinct critical points $\mathbf{x}_1, \ldots, \mathbf{x}_{n+1}$ with $\mathbf{x}_1 = \mathbf{x}$ and $\mathbf{x}_{n+1} = \mathbf{y}$, and a collection of gradient flowlines $\alpha_1, \ldots, \alpha_n$, where $\alpha_i \in \widehat{\mathcal{M}}(\mathbf{x}_i, \mathbf{x}_{i+1})$. A sequence of flowlines $\{\gamma_m\}_{m=1}^{\infty}$ in $\widehat{\mathcal{M}}(\mathbf{x}, \mathbf{y})$ *converges to a broken flowline* $(\alpha_1, \ldots, \alpha_n)$ if for each j, we can find representatives $\gamma_i^j \in \mathcal{M}(\mathbf{x}, \mathbf{y})$ for $\gamma_i \in \widehat{\mathcal{M}}(\mathbf{x}, \mathbf{y})$ such that $\{\gamma_i^j\}_{i=1}^{\infty}$ converges to α_j in the $C^{\infty,\mathrm{loc}}$ topology. Broken flowlines give a compactification of the space of flowlines in the following sense. Compare [10, Theorem 6.2.3].

Theorem 3.8 (Compactness theorem)**.** *Let $\mathbf{x}$ and $\mathbf{y}$ be critical points of f with $\mathbf{x} \neq \mathbf{y}$. Then any sequence of gradient flowlines from $\mathbf{x}$ to $\mathbf{y}$ has a $C^{\infty,\mathrm{loc}}$-convergent subsequence to a broken flowline from $\mathbf{x}$ to $\mathbf{y}$.*

Proposition 3.9. *For a genereic enough choice of metric g, the zero-dimensional moduli spaces $\widehat{\mathcal{M}}(\mathbf{x}, \mathbf{y})$ with $\lambda(\mathbf{x}) - \lambda(\mathbf{y}) = 1$ are compact in the C^{∞} topology.*

Proof. By Theorem 3.8, any sequence of gradient flowlines from $\mathbf{x}$ to $\mathbf{y}$ has a convergent subsequence to a broken flowline $(\alpha_1, \ldots, \alpha_n)$, where $\alpha_i \in \widehat{\mathcal{M}}(\mathbf{x}_i, \mathbf{x}_{i+1})$ for some distinct critical points $\mathbf{x}_1, \ldots, \mathbf{x}_{n+1}$ such that $\mathbf{x}_1 = \mathbf{x}$ and $\mathbf{x}_{n+1} = \mathbf{y}$. The key observation is that any broken flowline connecting $\mathbf{x}$ and $\mathbf{y}$ such that $\lambda(\mathbf{x}) - \lambda(\mathbf{y}) = 1$ is in fact unbroken.

Suppose otherwise that there is more than one unbroken flowline, meaning $n > 1$. Since $\lambda(\mathbf{x}) - \lambda(\mathbf{y}) = 1$, we have that $1 = \sum_{i=1}^{n} \lambda(\mathbf{x}_i) - \lambda(\mathbf{x}_{i+1})$, which implies that $\lambda(\mathbf{x}_i) - \lambda(\mathbf{x}_{i+1}) \leq 0$ for some i. However, there is no class of flowlines in the space $\widehat{\mathcal{M}}(\mathbf{x}_i, \mathbf{x}_{i+1})$ by Theorem 3.2. □

The above proposition implies that the moduli spaces $\widehat{\mathcal{M}}(\mathbf{x}, \mathbf{y})$ do indeed have finitely many elements, so the right hand side of equation (3) is a finite sum.

3.5. **Gluing.** In this section we outline the proof of the operator ∂ in $\mathrm{CM}(M, f, g)$ satisfying $\partial^2 = 0$, making $\mathrm{CM}(M, f, g)$ into a chain complex.

For a critical point $\mathbf{x} \in \mathrm{Crit}(f)$, applying the differential twice gives the sum

$$\begin{aligned}
\partial^2 \mathbf{x} &= \sum_{\{\mathbf{y} \in \mathrm{Crit}(f) | \lambda(\mathbf{x}) - \lambda(\mathbf{y}) = 1\}} \#\widehat{\mathcal{M}}(\mathbf{x}, \mathbf{y}) \cdot \partial \mathbf{y} \\
&= \sum_{\{\mathbf{y} \in \mathrm{Crit}(f) | \lambda(\mathbf{x}) - \lambda(\mathbf{y}) = 1\}} \#\widehat{\mathcal{M}}(\mathbf{x}, \mathbf{y}) \sum_{\{\mathbf{z} \in \mathrm{Crit}(f) | \lambda(\mathbf{y}) - \lambda(\mathbf{z}) = 1\}} \#\widehat{\mathcal{M}}(\mathbf{y}, \mathbf{z}) \cdot \mathbf{z} \\
&= \sum_{\{\mathbf{y}, \mathbf{z} \in \mathrm{Crit}(f) | \lambda(\mathbf{x}) - \lambda(\mathbf{y}) = 1, \lambda(\mathbf{y}) - \lambda(\mathbf{z}) = 1\}} \#\widehat{\mathcal{M}}(\mathbf{x}, \mathbf{y}) \#\widehat{\mathcal{M}}(\mathbf{y}, \mathbf{z}) \cdot \mathbf{z}.
\end{aligned}$$

So we need to show that for any $\mathbf{z} \in \mathrm{Crit}(f)$ such that $\lambda(\mathbf{x}) - \lambda(\mathbf{z}) = 2$,

$$\#\widehat{\mathcal{M}}(\mathbf{x}, \mathbf{y}) \#\widehat{\mathcal{M}}(\mathbf{y}, \mathbf{z}) \equiv 0 \pmod 2. \tag{4}$$

To prove this, we need to consider the compactification of $\widehat{\mathcal{M}}(\mathbf{x}, \mathbf{z})$ for $\lambda(\mathbf{x}) - \lambda(\mathbf{z}) = 2$ by broken trajectories. The compactification is proven to be a compact one dimensional manifold, whose boundary is precisely the space of broken flowlines

$$\bigcup_{\{\mathbf{y} \in \mathrm{Crit}(f) | \lambda(\mathbf{x}) - \lambda(\mathbf{y}) = 1\}} \widehat{\mathcal{M}}(\mathbf{x}, \mathbf{y}) \times \widehat{\mathcal{M}}(\mathbf{y}, \mathbf{z}). \tag{5}$$

The boundary of this compact 1-manifold consists of an even number of points, showing that (4) holds.

The main ingredient for this compactification result is the following *gluing* theorem. Compare [10, Theorem 6.2.6].

Theorem 3.10 (Gluing theorem)**.** *Suppose $\mathbf{x}, \mathbf{y}, \mathbf{z}$ are critical points of f such that $\lambda(\mathbf{x}) = \lambda(\mathbf{y}) + 1 = \lambda(\mathbf{z}) + 2$. Then for any broken flowline $(\alpha_1, \alpha_2) \in \widehat{\mathcal{M}}(\mathbf{x}, \mathbf{y}) \times \widehat{\mathcal{M}}(\mathbf{y}, \mathbf{z})$, there exists a real number ρ_0 and a smooth map*

$$f : [\rho_0, \infty) \to \widehat{\mathcal{M}}(\mathbf{x}, \mathbf{z})$$

such that $f(\rho)$ converges to (α_1, α_2) as $\rho \to \infty$. Moreover, if the sequence $\{\gamma_n\}$ in $\widehat{\mathcal{M}}(\mathbf{x}, \mathbf{z})$ converges to (α_1, α_2), then for large enough n,

$$\gamma_n \in f[\rho_0, \infty).$$

The result is called a gluing result because to show that such a map $f : [\rho_0, \infty) \to \widehat{\mathcal{M}}(\mathbf{x}, \mathbf{z})$ exists, we approximate the broken flowline (α_1, α_2) by a glued flowline depending on the parameter ρ.

This theorem shows that the boundary of the compactification of $\widehat{\mathcal{M}}(\mathbf{x}, \mathbf{z})$ for $\lambda(\mathbf{x}) - \lambda(\mathbf{z}) = 2$ is indeed the union in (5): the compactification Theorem 3.8 shows that any convergent sequence of flowlines coverges to a broken flowline, and Theorem 3.10 shows that given a broken flowline $(\alpha_1, \alpha_2) \in \widehat{\mathcal{M}}(\mathbf{x}, \mathbf{y}) \times \widehat{\mathcal{M}}(\mathbf{y}, \mathbf{z})$, the function f gives a sequence of unbroken flowlines converging to (α_1, α_2).

Thus, with the gluing result in hand, this implies that $(\mathrm{CM}(M, f, g), \partial)$ is a chain complex.

3.6. **Independence and invariance.** In this section we give insight into the proof of why the Morse–Smale complex $\mathrm{CM}(M, f, g)$ only depends on the manifold M, i.e. does not depend on the choice of Morse function $f : M \to \mathbb{R}$ and the choice of Riemannian metric g on M.

First we start with the independence of the Morse–Smale complex from the choice of Riemannian metric g, while fixing f. Consider two metrics g_0, g_1 such that the gradient flow equation is Morse–Smale for both (See Section 3.3). We consider "time-dependent gradient flowlines" between two critical points $\mathbf{x}$ and $\mathbf{y}$ of f, which are maps $\gamma : \mathbb{R} \to M$ such that $\lim_{t\to-\infty} = \mathbf{x}, \lim_{t\to+\infty} \gamma(t) = \mathbf{y}$ and γ satisfies the *time-dependent gradient flow* condition

$$\frac{d\gamma}{dt} = -\nabla_{g_{\psi(t)}} f_{\gamma(t)}$$

for some smooth monotone function $\psi : \mathbb{R} \to [0, 1]$ such that $\psi(t) = 0$ for $t \leq 0$ and $\psi(t) = 1$ for $t \geq 1$. Let $\mathcal{M}_{\{g_t\}}(\mathbf{x}, \mathbf{y})$ of time-dependent flowlines from $\mathbf{x}$ to $\mathbf{y}$. Note that the time-dependent gradient flow condition is no longer invariant under time translation.

Theorem 3.2 can be adapted to the time-dependent case. Compare [10, Theorem 6.3.1].

Theorem 3.11. *Let M^n be a closed n-dimensional manifold, and let g_0 and g_1 be two Riemannian metrics on M. For a sufficiently generic path of Riemannian metrics $\{g_t\}$ from g_0 to g_1, and for two critical points $\mathbf{x}$ and $\mathbf{y}$ of f such that $\mathbf{x} \neq \mathbf{y}$, the moduli space $\mathcal{M}_{\{g_t\}}(\mathbf{x}, \mathbf{y})$ is a smooth manifold with dimension*

$$\dim \mathcal{M}_{\{g_t\}}(\mathbf{x}, \mathbf{y}) = \lambda(\mathbf{x}) - \lambda(\mathbf{y}).$$

From now on, omit M from the notation of the Morse–Smale complex, so for example, $\mathrm{CM}(f, g_0)$ is the Morse–Smale complex of M with Morse function f and Riemannian metric g_0 on M. To show that $\mathrm{CM}(f, g_0)$ and $\mathrm{CM}(f, g_1)$ have the same homology groups, we define what are known as *continuation maps*

$$\Psi_{\{g_t\}} : \mathrm{CM}(f, g_0) \to \mathrm{CM}(f, g_1)$$

and

$$\Psi_{\{g_{1-t}\}} : \mathrm{CM}(f, g_1) \to \mathrm{CM}(f, g_0).$$

These maps are designed such that such that $\Psi_{\{g_t\}}, \Psi_{\{g_{1-t}\}}$ are chain maps, and $\Psi_{\{g_t\}} \circ \Psi_{\{g_{1-t}\}}$ and $\Psi_{\{g_{1-t}\}} \circ \Psi_{\{g_t\}}$ are chain homotopic to the (respective) identity maps.

First we need to discuss the broken line compactification of $\mathcal{M}_{\{g_t\}}(\mathbf{x}, \mathbf{y})$. Given the following information:

- sequences of critical points $\mathbf{x}_1, \ldots, \mathbf{x}_n$ with $\mathbf{x}_1 = \mathbf{x}$ and $\mathbf{y}_1, \ldots, \mathbf{y}_m$ with $\mathbf{y}_m = \mathbf{y}$;
- gradient flows $\alpha_i \in \widehat{\mathcal{M}}_{g_0}(\mathbf{x}_i, \mathbf{x}_{i+1})$ for $i = 1, \ldots, n-1$ and gradient flows $\beta_i \in \widehat{\mathcal{M}}_{g_1}(\mathbf{y}_i, \mathbf{y}_{i+1})$ for $i = 1, \ldots, m-1$;
- a time-dependent gradient flowline $\gamma \in \mathcal{M}_{\{g_t\}}(\mathbf{x}_n, \mathbf{y}_1)$,

there is an analogue of the compactification Theorem 3.8 (compare [10, pp. 121]):

Theorem 3.12. *Let $\mathbf{x}$ and $\mathbf{y}$ be critical points of f with $\mathbf{x} \neq \mathbf{y}$. Any sequence of time-dependent gradient flowlines from $\mathbf{x}$ to $\mathbf{y}$ has a $C^{\infty,\mathrm{loc}}$-convergent subsequence to a broken flowline $(\alpha_1, \ldots, \alpha_n, \gamma, \beta_1, \ldots, \beta_m)$.*

Adapting the argument from the proof of Proposition 3.9, we get that the space $\mathcal{M}_{\{g_t\}}(\mathbf{x},\mathbf{y})$ is a compact, zero-dimensional manifold.

Now we can define the continuation map $\Psi_{\{g_t\}} : \mathrm{CM}(f,g_0) \to \mathrm{CM}(f,g_1)$ by

$$\Phi_{\{g_t\}}(\mathbf{x}) = \sum_{\{\mathbf{y}\in \mathrm{Crit}(f) \mid \lambda(\mathbf{x})=\lambda(\mathbf{y})\}} \#\mathcal{M}_{\{g_t\}}(\mathbf{x},\mathbf{y})\cdot \mathbf{y},$$

which is a finite sum because $\#\mathcal{M}_{\{g_t\}}(\mathbf{x},\mathbf{y})$ is finite. Analogously define $\Phi_{\{g_{1-t}\}} : \mathrm{CM}(f,g_1) \to \mathrm{CM}(f,g_0)$ by

$$\Phi_{\{g_{1-t}\}}(\mathbf{x}) = \sum_{\{\mathbf{y}\in \mathrm{Crit}(f) \mid \lambda(\mathbf{x})=\lambda(\mathbf{y})\}} \#\mathcal{M}_{\{g_{1-t}\}}(\mathbf{x},\mathbf{y})\cdot \mathbf{y}.$$

The idea of the proof that $\Phi_{\{g_t\}}$ is a chain map is as follows. We consider the ends of the moduli space $\mathcal{M}_{\{g_t\}}(\mathbf{x},\mathbf{y})$ with $\lambda(\mathbf{y}) = \lambda(\mathbf{x})-1$. In can be proven that by adding the broken flowlines, the compactification of $\mathcal{M}_{\{g_t\}}(\mathbf{x},\mathbf{y})$ is a compact 1-manifold which has boundary

$$\bigcup_{\{\mathbf{x}'\in \mathrm{Crit}(f)\mid\lambda(\mathbf{x}')=\lambda(\mathbf{y})\}} \widehat{\mathcal{M}}_{g_0}(\mathbf{x},\mathbf{x}')\times \mathcal{M}_{\{g_t\}}(\mathbf{x}',\mathbf{y}) \cup \bigcup_{\{\mathbf{y}'\in \mathrm{Crit}(f)\mid\lambda(\mathbf{y}')=\lambda(\mathbf{x})\}} \mathcal{M}_{\{g_t\}}(\mathbf{x},\mathbf{y}')\times \widehat{\mathcal{M}}_{g_1}(\mathbf{y}',\mathbf{y}).$$

There is a compactness result showing that all of the ends of $\mathcal{M}_{\{g_t\}}(\mathbf{x},\mathbf{y})$ are contained in the above union, and there is a gluing result that shows that any sequence of time dependent gradient flowlines in $\mathcal{M}_{\{g_t\}}(\mathbf{x},\mathbf{y})$ converges to a broken flowline in the above union.

The ends of $\mathcal{M}_{\{g_t\}}(\mathbf{x},\mathbf{y})$ are precisely the coefficient of $\mathbf{y}$ in

$$\Phi_{\{g_t\}}(\partial_{g_0}(\mathbf{x})) + \partial_{g_1}(\Phi_{\{g_t\}}(\mathbf{x})),$$

where ∂_{g_i} is the boundary map in $\mathrm{CM}(f,g_i)$. Thus because a compact 1-manifold has an even number of boundary components and we are working modulo 2,

$$\Phi_{\{g_t\}}\circ\partial_{g_0} - \partial_{g_1}\circ\Phi_{\{g_t\}} = 0,$$

meaning that $\Phi_{\{g_t\}}$ is a chain map.

It remains to show that we can construct a homotopy operator $H : \mathrm{CM}_*(f,g_0) \to \mathrm{CM}_{*+1}(f,g_0)$ such that

$$\partial\circ H + H\circ\partial = \mathrm{id} + \Phi_{\{g_{1-t}\}}\circ\Phi_{\{g_t\}}. \tag{6}$$

The homotopy operator H will count time-gradient flowlines in a two-dimensional sense: let $\{g_{r,t}\}_{r\in[0,\infty),t\in\mathbb{R}}$ be a two-parameter family of metrics such that the following conditions hold:

- $g_{0,t} = g_0$,
- for $t > 1$, $g_{r,t} = g_{\psi(r+t)}$,
- for $t < -1$, $g_{r,t} = g_{\psi(1-r-t)}$,

where $\psi : \mathbb{R} \to [0,1]$ is the same smooth monotone function such that $\psi(t) = 0$ for $t \le 0$ and $\psi(t) = 1$ for $t \ge 1$. Notice that for large r, the family $g_{r,t}$ looks like the path $\{g_{1-t}\}$, followed by the constant path g_0 for a long time, and then the path $\{g_t\}$.

Consider the moduli space $\mathcal{M}_{\{g_{r,t}\}}(\mathbf{x},\mathbf{y})$ of pairs of $r \in [0,\infty)$ and a path $\gamma : \mathbb{R} \to M$ satisfying the usual asymptotics $\lim_{t\to-\infty}\gamma(t) = \mathbf{x}$, $\lim_{t\to+\infty}\gamma(t) = \mathbf{y}$ and the time-dependent gradient flow equation

$$\frac{d\gamma}{dt} = -\nabla_{g_{r,t}} f_{\gamma(t)}.$$

There is an analogue of Theorem 3.11 which prove that the moduli space $\mathcal{M}_{\{g_{r,t}\}}(\mathbf{x},\mathbf{y})$ of such pairs (r,γ) is a manifold of dimension $\lambda(\mathbf{x}) - \lambda(\mathbf{y}) + 1$ (intuitively, there is an extra dimension from the r parameter).

The broken line compactification of $\mathcal{M}_{g_{r,t}}(\mathbf{x},\mathbf{y})$ is described as follows. Let (r_i,γ_i) be a sequence in $\mathcal{M}_{g_{r,t}}(\mathbf{x},\mathbf{y})$. Such a sequence (r_i,γ_i) has a convergent subsequence in the compactified space. This subsequence could converge in $\mathcal{M}_{\{g_{r,t}\}}(\mathbf{x},\mathbf{y})$, or otherwise, projecting to the r coordinate, there could be one of the following behaviors:

(1) $r_i \to \rho$ for some real number $\rho \in (0,\infty)$. In this case, γ_1 converges to a broken time-dependent flowline with respect to the path $\{g_{\rho,t}\}_{t\in[-\rho,\rho]}$.
(2) $r_i \to 0$. In this case, $\{\gamma_i\}$ has a subsequence that converges to a (possibly broken) gradient flowline for the g_0 metric.
(3) $r_i \to \infty$. In this case, $\{\gamma_i\}$ converges to a juxtaposition of two (possibly broken) time-dependent flowlines, starting with the $\{g_t\}$ family and then for the $\{g_{1-t}\}$ family.

Compactness and transversality arguments show that for $\mathbf{x},\mathbf{y}$ such that $\lambda(\mathbf{y}) = \lambda(\mathbf{x})+1$, the moduli space $\mathcal{M}_{\{g_{r,t}\}}(\mathbf{x},\mathbf{y})$ is a compact zero-dimensional manifold, i.e. a finite number of points.

Define the homotopy operator $H : \mathrm{CM}_*(f,g_0) \to \mathrm{CM}_{*+1}(f,g_0)$ by the equation

$$H(\mathbf{x}) = \sum_{\{\mathbf{y}\in\mathrm{Crit}(f)\mid\lambda(\mathbf{y})=\lambda(\mathbf{x})+1\}} \#\mathcal{M}_{\{g_{r,t}\}}(\mathbf{x},\mathbf{y})\cdot\mathbf{y}.$$

To prove equation (6), we consider moduli spaces $\mathcal{M}_{\{g_{r,t}\}}(\mathbf{x},\mathbf{y})$ where $\lambda(\mathbf{y}) = \lambda(\mathbf{x})$. A transversality argument shows that the ends of $\mathcal{M}_{\{g_{r,t}\}}(\mathbf{x},\mathbf{y})$ appearing in cases (2) and (3) are unbroken, and so are part of the union

$$\widehat{\mathcal{M}}_{g_0}(\mathbf{x},\mathbf{y}) \cup \bigcup_{\{\mathbf{z}\in\mathrm{Crit}(f)\mid\lambda(\mathbf{x})=\lambda(\mathbf{z})\}} \mathcal{M}_{\{g_t\}}(\mathbf{x},\mathbf{z})\times\mathcal{M}_{\{g_{1-t}\}}(\mathbf{z},\mathbf{y}), \tag{7}$$

and the ends appearing in case (1) belong to the union

$$\bigcup_{\{\mathbf{x}'\in\mathrm{Crit}(f)\mid\lambda(\mathbf{x}')=\lambda(\mathbf{x})-1\}} \widehat{\mathcal{M}}_{g_0}(\mathbf{x},\mathbf{x}')\times\mathcal{M}_{\{g_{r,t}\}}(\mathbf{x}',\mathbf{y}) \cup \bigcup_{\{\mathbf{y}'\in\mathrm{Crit}(f)\mid\lambda(\mathbf{y}')=\lambda(\mathbf{y})+1\}} \mathcal{M}_{\{g_{r,t}\}}(\mathbf{x},\mathbf{y}')\times\widehat{\mathcal{M}}_{g_0}(\mathbf{y}',\mathbf{y}). \tag{8}$$

and the union of (7) and (8) is precisely the ends of $\mathcal{M}_{\{g_{r,t}\}}(\mathbf{x},\mathbf{y})$ by a gluing argument. The number of ends of $\mathcal{M}_{\{g_{r,t}\}}(\mathbf{x},\mathbf{y})$ is even because it is a compact one-dimensional manifold, and the set of such ends is precisely what is counted in $\partial\circ H + H\circ\partial + \mathrm{id} + \Phi_{\{g_{1-t}\}}\circ\Phi_{\{g_t\}}$. Therefore, (6) holds, implying that $\Phi_{g_t} : \mathrm{CM}(f,g_0)\to\mathrm{CM}(f,g_1)$ is indeed a chain homotopy equivalence, i.e. there is no dependence on the metric.

To show that the homology of $\mathrm{CM}(M,f,g)$ is independent of the choice of Morse function function, we similarly interpolate between two choices of Morse functions, count time-dependent trajectories (except this time the dependence is in the Morse function), collect them into moduli spaces, define continuation maps, and so forth.

4. Symplectic geometry

The goal of this section is to give the symplectic geometry background necessary for the construction of Lagrangian Floer homology. We end with the Arnold conjecture, Floer's motivation for the development of Lagrangian Floer homology. In Section 4.1, we define symplectic manifolds and Lagrangian submanifolds. In Section 4.2, we introduce almost-complex structures. In Section 4.3, we motivate and state a few versions the Arnold conjecture.

4.1. **Symplectic manifolds and Lagrangian submanifolds.** We follow [10, Chapter 5]. Let M be a $2n$-dimensional smooth manifold. A *symplectic form* on M is a smooth 2-form $\omega \in \Omega^2(M;\mathbb{R})$ which is closed ($d\omega = 0$), and is non-degenerate, meaning that the n-fold wedge product of ω in $\Omega^{2n}(M)$ vanishes nowhere. A *symplectic manifold* is a pair (M, ω), where ω is a symplectic form on M. The top exterior power of ω can be viewed as a volume form on M, with respected to a preferred orientation of M. We assume that M is oriented with this preferred orientation.

The space $\mathbb{R}^{2n}$ is a symplectic manifold with the *standard symplecitc form* ω_{st} on $\mathbb{R}^{2n}$ given by

$$\omega_{st} = \sum_{i=1}^{n} dx_i \wedge dy_i.$$

This form is closed and non-degenerate. Another important example of a symplectic manifold is the cotangent bundle T^*L of a (real) n-dimensional manifold L. Let $\pi : T^*L \to L$ be the projection map. For fixed $\eta \in T^*L$, take the differential of π map at η to get the map $T\pi_\eta : T_\eta(T^*L) \to T_{\pi(\eta)}L$. Then the composition

$$T_\eta(T^*L) \xrightarrow{T\pi_\eta} T_{\pi(\eta)}L \xrightarrow{\eta} \mathbb{R}$$

sending $v \in T_\eta(T^*L)$ to $\eta(T\pi_\eta(v))$ induces a 1-form $\lambda \in \Omega^1(T^*M)$, where $M = T^*L$, called the *Liouville form.* The claim is that $(T^*L, -d\lambda)$ is a symplectic manifold. The form $-d\lambda$ is closed because it is exact. The form $-d\lambda$ is non-degenerate by the following local computation. On local coordinate chart with coordinates $x_1, \ldots, x_n$, we have that there is an induced local coordinate system on $T^*\mathbb{R}^n$ given by the map

$$\mathbb{R}^n \times \mathbb{R}^n \to T^*\mathbb{R}^n$$
$$(x_1, \ldots, x_n, y_1, \ldots, y_n) \mapsto \left(x_1, \ldots, x_n, \sum_{i=1}^{n} y_i dx_i \right).$$

With respect to this trivialization, $\lambda = \sum y_i dx_i$, which implies that $-d\lambda$ is equal to the standard symplectic form $\sum dx_i \wedge dy_i$ on $\mathbb{R}^n$. In particular, $-d\lambda$ is nondegenerate.

Let (M^{2n}, ω) be a symplectic manifold, and let $L^n \subseteq M^{2n}$ be an n-dimensional submanifold. Say that L is *Lagrangian* if the restriction of ω to L vanishes identically. For example, for any smooth manifold L, the zero-section of the cotangent bundle T^*L, equipped with the symplectic structure $-d\lambda$ from the previous paragraph, is a Lagrangian submanifold.

4.2. **Almost-complex structures.** An *almost complex structure* J on a smooth manifold M is a bundle automorphism $J : TM \to TM$ such that $J \circ J = -\operatorname{id}_{TM}$. If (M, ω) is a symplectic manifold, an almost-complex structure J is ω*-tame* if $\omega_p(v, J_p v) > 0$ for each $p \in M$ and nonzero $p \in T_pM$. The following is an important theorem about the existence and space of ω-tame almost complex structures. Compare [10, Theorem 5.3.12].

Theorem 4.1. *A symplectic manifold (M, ω) always admits ω-tame almost-complex structures and the space of ω-tame almost-complex structures is contractible.*

Given (M, ω) and an ω-tame J, there is an associated Riemannian metric g on M given by the formula

$$g_p(v, w) = \frac{1}{2}(\omega_p(v, J_p w) + \omega(w, J_p v)), \tag{9}$$

for each $p \in M$ and $v, w \in T_pM$.

4.3. **Hamiltonian diffeomorphisms and the Arnold conjecture.** Fix a symplectic manifold (M, ω). Let $H : [0,1] \times M \to \mathbb{R}$ be any smooth time dependent function, and for fixed t, let $H_t : M \to \mathbb{R}$ be given by $H_t(x) = H(t, x)$. Choose the time-dependent vector field X_t given by the equation

$$\omega(X_t, \cdot) = dH_t(\cdot),$$

where $\cdot$ can be replaced with any smooth vector field. Then X_t is called a *Hamiltonian vector field*. Consider the ordinary differential equation for paths $z : [0,1] \to M$,

$$\frac{dz}{dt} = X_t(z).$$

Its solutions define a smooth arc $t \mapsto \Phi_t$ for $t \in [0,1]$ of smooth diffeomorphisms starting at $\Phi_0 = \mathrm{id}_M$. The time-1 map Φ_1 is called a *Hamiltonian diffeomorphism*. Call a Hamiltonian diffeomorphism $\varphi : M \to M$ *nondegenerate* if its graph intersects the diagonal of $M \times M$ transversely.

Arnold [1, p. 419] conjectured that the number of fixed points of a nondegenerate Hamiltonian diffeomorphism is at least the minimum number of critical points of a Morse function on M. (This happens to be is true if the Hamiltonian diffeomorphism is generated by a Hamiltonian function which is small enough in C^2-norm.) Compare with [7, Conjecture 1.2.4].

Conjecture 4.2 (The Arnold conjecture)**.** *Let (M, ω) be a compact symplectic manifold. For a nondegenerate Hamiltonian diffeomorphism ϕ,*

$$\#\{\textit{fixed points of } \phi\} \geq \min_f \# \operatorname{Crit}(f),$$

where the minimum goes over all Morse functions $f : M \to \mathbb{R}$.

In view of the inequality

$$\# \operatorname{Crit}(f) \geq \sum_{i=0}^{n} b_i(M)$$

obtained from the Morse inequalities (2), we have the following conjecture, also known as the Arnold conjecture.

Conjecture 4.3 (The Arnold conjecture)**.** *Fix a field $\mathbb{F}$. Let $\phi : M \to M$ be a nondegenerate Hamiltonian diffeomorphism of M. Then the number of fixed points of ϕ is at least $\dim_{\mathbb{F}} H_*(M; \mathbb{F})$.*

The above conjecture can be obtained as a special case of the following conjecture, where the ambient symplectic manifold is $(M \times M, p_1^*(\omega) - p_2^*(\omega))$, and p_i is projection onto the i-th factor for $i = 1, 2$. Compare [10, Conjecture 7.1.5]

Conjecture 4.4 (The Arnold conjecture)**.** *Fix a field $\mathbb{F}$. If L is a closed Lagrangian submanifold in a closed symplectic manifold (M, ω) and $\phi : M^{2n} \to M^{2n}$ is a Hamiltonian such that L and $\phi(L)$ intersect transversely, then the number of intersection points of L and $\phi(L)$, denoted $|L \cap \phi(L)|$, satisfies*

$$|L \cap \phi(L)| \geq \dim_{\mathbb{F}} H_*(L; \mathbb{F}).$$

5. Lagrangian Floer homology

The goal of this section is to introduce the construction of Lagrangian Floer homology and how Floer [4] uses it to important case Arnold's conjecture. Here is a nutshell of the story. The Lagrangian Floer complex takes in a symplectic manifold M and two Lagrangian submanifolds L_0 and L_1. It is generated by the intersection points of L_0 and L_1. To understand its differential, we turn to the *action functional*, which acts on the space of paths from L_0 to L_1, and whose critical points are the constant paths at the intersection points of L_0 and L_1. The idea is to apply a variant of Morse theory on the action functional, where the critical points of the functional are the intersection points of two Lagrangians. Connecting two intersection points by a gradient flow equation gives the notion of a *pseudo-holomorphic strip*. However, when computing the index of these critical points, we run into the issue that the index might be infinite. This issue is resolved with a relative quantity, the *Maslov index*. In any case, the differential roughly counts the number of pseudo-holomorphic strips between pairs of intersection points.

With this in place, here is an outline for this section. In Section 5.1, we explain the bigger idea behind the Lagrangian Floer complex, specifically how it helps prove the Arnold conjecture. In Section 5.2, we introduce the action functional and then proceed via the story of Morse Theory by computing its critical points, its Hessian, and so forth. In Section 5.3, we introuce Whitney disks, which resemble pseudo-holomorphic strips but with no analytical constraints. In Section 5.4, we discuss the Lagrangian Grassmannian, the construction of the Maslov index, and give an example computation of the Maslov index. In Section 5.5, we introduce pseudo-holomorphic strips and their moduli spaces. In Section 5.6, we define the Lagrangian Floer complex. In Section 5.7 and Section 5.8, we introduce the compactness and gluing techniques needed to show that the Lagrangian Floer complex is a complex. In Section 5.9, we go over the main ideas behind showing that the Lagrangian Floer complex is independent of auxillary chpoices and is also invariant under Hamiltonian isotopies, a property baked into the definition to prove the Arnold conjecture.

Most of this section closely follows [10, Chapter 7].

5.1. **The bigger idea.** Floer [5] proved Arnold's Conjecture 4.4 for the case of symplectic manifolds (M, ω) such that $\pi_2(M) = 0$ and $\pi_2(M, L) = 0$. Assume that we have fixed a symplectic manifold (M, ω) with these properties. Floer's approach was to define what is now known as the *Lagrangian Floer complex* $\mathrm{CF}(L_0, L_1)$, over $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$, is associated to a pair of transversally intersecting, compact, oriented Lagrangian submanifolds L_0 and L_1. This complex has the following properties:

(1) The complex $\mathrm{CF}(L_0, L_1)$ is generated by the intersection points of L_0 and L_1.
(2) The homology groups $HF(L_0, L_1)$ only depend on the symplectic manifold and its two Lagrangian submanifolds.
(3) The homology groups $HF(L_0, L_1)$ are invariant under Hamiltonian isotopy, in the following sense: if $\phi_0, \phi_1 : M \to M$ are Hamiltonian diffeomorphisms and $\phi_0(L_0)$ and $\phi_1(L_1)$ intersect transversally, then
$$HF(L_0, L_1) \cong HF(\phi_0(L_0), \phi_1(L_1)).$$
This property allows us to extend the homology groups $HF(L_0, L_1)$ to cases where L_0 and L_1 don't intersect transversally: let L_1' be a Hamiltonian translate of L_1 which intersects L_0 transversally, and define $HF(L_0, L_1)$ to be $HF(L_0, L_1')$.

(4) The homology groups $HF(L, L)$ are isomorphic to the singular homology of L with $\mathbb{Z}/2\mathbb{Z}$ coefficients.
(5) The homology groups $HF(L_0, L_1)$ are graded by $\mathbb{Z}/2\mathbb{Z}$: each intersection point $\mathbf{x}$ of L_0 and L_1 has a local intersection number $i(\mathbf{x}) \in \{\pm 1\}$. The $\mathbb{Z}/2\mathbb{Z}$-grading $\mathrm{gr}(\mathbf{x})$ is defined by

$$(-1)^{\mathrm{gr}(\mathbf{x})} = i(\mathbf{x}).$$

First, notice that the existence of a theory satisfying (1),(2),(3),(4) implies Arnold's conjecture, which we argue as follows. Notice that $HF(L \cap \phi(L))$ is $HF(L, L)$ by property (3) and $HF(L, L)$ is the singular homology of L by property (4). Since the number of generators of a complex is at least the $\mathbb{Z}/2\mathbb{Z}$-dimension of its homology,

$$|L \cap \phi(L)| \geq \dim_{\mathbb{Z}/2\mathbb{Z}} H_*(L; \mathbb{Z}/2\mathbb{Z}).$$

Also in view of (1),(2),(3), Lagrangian Floer homology can be viewed as an obstruction to making L_0 disjoint from L_1 via Hamiltonian isotopies.

5.2. **The action functional.** Let (M^{2n}, ω) be a symplectic manifold with compact Lagrangian submanifolds L_0 and L_1. Consider the space of paths $\mathcal{V}$ from L_0 to L_1,

$$\mathcal{V} = \{\gamma : [0,1] \to M \mid \gamma(0) \in L_0, \gamma(1) \in L_1\}.$$

Let's assume that all paths in $\mathcal{V}$ are smooth.

A path in $\mathcal{V}$ is a map

$$u : [0,1] \times [0,1] \to M^{2n}$$

such that for all $t \in [0,1]$, $u(t,0) \in L_0$ and $u(t,1) \in L_1$.

Let's first discuss the action functional in the case when our symplectic manifold M is *exact*, meaning that $\omega = d\alpha$ for some 1-form α, and the Lagrangain submanifolds are exact. A Lagrangian submanifold L of an exact symplectic manifold M is called *exact* if there exists a function $f : L \to \mathbb{R}$ such that $\alpha|_L = df$. In our case, we have two exact Lagrangian submanifolds L_0 and L_1, so choose f_i for $i = 0, 1$ such that $\alpha|_{L_i} = df_i$.

5.2.1. *Action functional in exact case.* Define the *action functional* $\mathcal{A}$ on $\mathcal{V}$ by the following formula on a path $\gamma : [0,1] \to M$ in $\mathcal{V}$:

$$\mathcal{A}(\gamma) = f_0(\gamma(0)) - f_1(\gamma(1)) + \int_{[0,1]} \gamma^*(\alpha).$$

There is another interpretation of the action functional, which determines it up to an additive constant, that will be useful to us when generalizing to the non-exact case. Suppose γ_0, γ_1 are in the same path component of $\mathcal{V}$, so there exists a path $u : [0,1] \times [0,1] \to M$ such that for all $t \in [0,1]$, $u(t,0) \in L_0$, $u(t,1) \in L_1$, $u(0,s) = \gamma_0(s)$, and $u(1,s) = \gamma_1(s)$.

Lemma 5.1. *The action functional satisfies the following formula:*

$$\mathcal{A}(\gamma_1) - \mathcal{A}(\gamma_0) = \int_{(t,s)\in[0,1]\times[0,1]} u^*(\omega). \tag{10}$$

Proof. Since $u^*(\omega) = du^*(\alpha)$, by Stokes' theorem,

$$\int_{[0,1]^2} u^*(\omega) = \left(\int_{[0,1]\times 0} + \int_{1\times[0,1]} + \int_{[1,0]\times 1} + \int_{0\times[1,0]} \right) u^*(\alpha). \tag{11}$$

Since L_0 is exact, by Stokes' theorem,

$$\int_{[0,1]\times 0} u^*(\alpha) = f_0(\gamma_1(0)) - f_0(\gamma_0(0)),$$

and since L_1 is exact, by Stokes' theorem,

$$\int_{[1,0]\times 1} u^*(\alpha) = f_1(\gamma_0(1)) - f_1(\gamma_1(1)).$$

Using the fact that $\int_{1\times[0,1]} u^*(\alpha) = \int_{[0,1]} \gamma_1^*(\alpha)$ and $\int_{0\times[1,0]} u^*(\alpha) = -\int_{[0,1]} \gamma_0^*(\alpha)$, and collecting all terms on the right-hand side of (11),

$$\begin{aligned}\int_{[0,1]^2} u^*(\omega) &= \left(f_0(\gamma_1(0)) - f_1(\gamma_1(1)) + \int_{[0,1]} \gamma_1^*(\alpha)\right) - \left(f_0(\gamma_0(0)) - f_1(\gamma_1(1)) + \int_{[0,1]} \gamma_0^*(\alpha)\right)\\ &= \mathcal{A}(\gamma_1) - \mathcal{A}(\gamma_0).\end{aligned}$$

□

Proposition 5.2. *The critical points of $\mathcal{A}$ are the constant paths $\gamma : [0,1] \to L_0 \cap L_1$.*

Proof. A path $\gamma : [0,1] \to M$ with $\gamma(0) \in L_0$ and $\gamma(1) \in L_1$ is a critical point for the action functional $\mathcal{A}$ if for every smooth extension of γ,

$$u : [-\epsilon, \epsilon] \times [0,1] \to M$$

such that $u(t,0) \in L_0$, $u(t,1) \in L_1$, and $\gamma(s) = u(0,s)$, the value $t = 0$ is a critical point for the real valued function

$$t \mapsto \mathcal{A}(u|_{\{t\}\times[0,1]}).$$

The path u is also called a *variation* of γ. Let's use the shorthand u_t for $u|_{\{t\}\times[0,1]}$, so that the real valued function above becomes

$$t \mapsto \mathcal{A}(u_t).$$

Note that by Lemma 5.1, $\mathcal{A}(u_t) - \mathcal{A}(u_0) = \int_{[0,t]\times[0,1]} u^*(\omega)$. So

$$\begin{aligned}\left.\frac{d}{dt}\right|_{t=0} \mathcal{A}(u_t) &= \left.\frac{\partial}{\partial t}\right|_{t=0} \int_{[0,t]\times[0,1]} u^*(\omega)\\ &= \left.\frac{\partial}{\partial t}\right|_{t=0} \int_{[0,t]\times[0,1]} \omega\left(\frac{\partial u}{\partial t}, \frac{\partial u}{\partial s}\right) dt \wedge ds\\ &= \int_{[0,1]} \omega\left(\frac{\partial u}{\partial t}(0,s), \frac{\partial u}{\partial s}(0,s)\right) ds.\end{aligned}$$

Since $\frac{\partial u}{\partial t}(0,s)$ can be chosen to be any arbitrary smooth function with compact support in $(0,1)$ and ω is non-degenerate, at each critical point we have that $\frac{\partial u}{\partial s} \equiv 0$. This implies that $\gamma(s) = u_0(s)$ is a constant path. Also it is clear that if γ is the constant path, then $\frac{\partial u}{\partial s} \equiv 0$, so

$$\left.\frac{d}{dt}\right|_{t=0} \mathcal{A}(u_t) = 0$$

for all variations u. □

We have computed the critical points of $\mathcal{A}$, now onto the gradient of $\mathcal{A}$. First we need to discuss the notion of a tangent vector at $\mathcal{V}$. A tangent vector at a path γ in $\mathcal{V}$ is a vector field

$$v : [0,1] \to TM$$

lifting γ, i.e. the following commutes,

$$\begin{tikzcd}
{[0,1]} \arrow[r, "v"] \arrow[dr, "\gamma"'] & TM \arrow[d, "\pi"] \\
& M
\end{tikzcd}$$

where $\pi : TM \to M$ is the projection map, and v satisfies the boundary conditions

$$v(0) \in T_{\gamma(0)}L_0 \subseteq T_{\gamma(0)}M, \qquad v(1) \in T_{\gamma(1)}L_1 \subseteq T_{\gamma(1)}M.$$

Since we have fixed ω in our symplectic manifold, there exists an ω-tame almost complex structure J, giving a Riemannian metric g defined by the formula

$$g_p(v,w) = \frac{1}{2}(\omega_p(v, J_p w) + \omega(w, J_p v)),$$

for each $p \in M$ and $v, w \in T_pM$, see (9). The Riemannian metric g gives rise to a Riemannian metric on the path space $\mathcal{V}$, given by the formula

$$\langle v, w\rangle = \int_{[0,1]} \langle v(s), w(s)\rangle \, ds$$

for two vector fields v, w along γ.

We claim that the gradient of $\mathcal{A}$ at γ, denoted $\nabla\mathcal{A}_\gamma \in T_\gamma\mathcal{V}$, is the vector field along γ given by

$$s \mapsto -J_{\gamma(s)}\frac{d\gamma}{ds}.$$

Let's again adopt the notation $u_t = u(t,s)$. Note that

$$\left.\frac{\partial}{\partial t}\right|_{t=0} \mathcal{A}(u_t) = \int_{[0,1]} \omega\left(v, \frac{\partial u}{\partial s}\right) ds,$$

where $v(s) = \frac{\partial u}{\partial t}(0,s)$. Since $\omega(v, \frac{\partial u}{\partial s}) = -g(v, J\frac{\partial u}{\partial s})$,

$$\left.\frac{\partial}{\partial t}\right|_{t=0} \mathcal{A}(u_t) = \left\langle v, -J\frac{\partial u}{\partial s}(0,s)\right\rangle = \left\langle v, -J\frac{\partial \gamma}{\partial s}\right\rangle.$$

Having defined the gradient, we formulate an upward gradient flowlines. Fix $\mathbf{x}, \mathbf{y} \in L_0 \cap L_1$, which are critical points of the action functional $\mathcal{A}$ by Proposition 5.2. An *upward gradient flowline* for $\mathcal{A}$ is a map

$$u : \mathbb{R} \times [0,1] \to M^{2n}$$

satisfying the "gradient flow" equation

$$\frac{\partial u}{\partial t} + J_{u(t,s)}\frac{\partial u}{\partial s} = 0 \tag{12}$$

and having boundary conditions $u(t,0) \in L_0$, $u(t,1) \in L_1$ for all $t \in \mathbb{R}$, and $\lim_{t\to-\infty} u(t,s) = \mathbf{x}$ and $\lim_{t\to+\infty} u(t,s) = \mathbf{y}$. The map u satisfying the previous conditions is called a *pseudo-holomorphic strip*. The term "pseudo-holomorphic strip" is meant to emphazie the fact that the complex structure J on M is not necessarily integrable, i.e. it is not induced from a complex manifold by multiplication by i on the tangent bundle.

5.2.2. *Action functional in the non-exact case.* Define the action functional $\mathcal{A}$ on path components of the path space $\mathcal{V}$ using equation (10):

$$\mathcal{A}(\gamma_1) - \mathcal{A}(\gamma_0) = \int_{(t,s)\in[0,1]\times[0,1]} u^*(\omega).$$

There are a couple of issues. First of all, $\mathcal{A}$ is defined up to an additive constant so far. Second, and more importantly, $\mathcal{A}(\gamma_1)-\mathcal{A}(\gamma_0)$ depends on the choice of pseudo-holomorphic u up to homotopy. However, we can fix the second problem if we consider γ_0 and γ_1 sufficiently close, so that u is sufficiently short, the above equation makes sense.

5.2.3. *Hessian of the action functional.* If $\gamma : [0,1] \to M$ is the constant path at $\mathbf{x} \in L_0 \cap L_1$, then we prove that the Hessian at γ is formally the operator

$$\mathrm{Hess}_{\mathcal{A}} : v \mapsto -J\frac{dv}{ds} \tag{13}$$

for $v \in T_\gamma\mathcal{V}$. So, the nullspace of the Hessian is $T_{\mathbf{x}}L_0 \cap T_{\mathbf{x}}L_1$.

Recall coordinate-free description the Hessian at a critical point of a finite dimension in equation (1). With this in mind, suppose

$$u : [-\epsilon, \epsilon] \times [-\epsilon, \epsilon] \times [0,1] \to M$$

is a two-parameter family of paths indexed by $(\tau, t) \in [-\epsilon,\epsilon] \times [-\epsilon,\epsilon]$, and $u(0,0,s) = \gamma(s)$. Denote the two tangent vectors in $T_\gamma\mathcal{V}$ corresponding to $\frac{\partial}{\partial t}|_{t=0}u(0,t,s)$ and $\frac{\partial}{\partial \tau}|_{\tau=0}u(\tau,0,s)$ by v and w respectively.

Then

$$\begin{aligned}\frac{\partial}{\partial \tau}\int_{[0,1]}\left\langle -J\frac{\partial u}{\partial s}, \frac{\partial u}{\partial t}\right\rangle ds &= \frac{\partial}{\partial \tau}\int_{[0,1]}\left\langle J\frac{\partial u}{\partial t}, \frac{\partial u}{\partial s}\right\rangle ds = \int_{[0,1]}\left\langle J\frac{\partial u}{\partial t}, \frac{\partial^2 u}{\partial s \partial \tau}\right\rangle ds \\ &= \int_{[0,1]}\left\langle -J\frac{\partial^2 u}{\partial s\partial\tau}, \frac{\partial u}{\partial t}\right\rangle ds = \int_{[0,1]}\left\langle -J\frac{\partial w}{\partial s}, v\right\rangle ds.\end{aligned}$$

This shows that the Hessian of $\mathcal{A}$ is of the form in (13). In particular, $\mathcal{A}$ is a non-degenerate Morse function if and only if L_0 and L_1 intersect each other transversely.

It is important to keep in mind that the Hessian need not have finite index, as the following example illustrates

Example 5.3. Suppose $M = \mathbb{C}$ with two Lagrangian submanifolds $L_0 = \mathbb{R}$ and $L_1 = e^{2\pi i\theta}\cdot\mathbb{R}$. The two Lagrangians intersect at 0, so let γ be the constant path at 0. Then $T_\gamma\mathcal{V}$ consists of vector fields $v : I \to TM$ such that $\pi \circ v = \gamma$, where $\pi : TM \to M$ is the projection map. Since γ is a constant path, this means that v maps into $T_0\mathbb{C} = \mathbb{C}$. Thus we can interpret v as a path in $\mathbb{C}$.

To compute the eigenvectors and eigenvalues of the Hessian, we need to solve the equation

$$\lambda v = \mathrm{Hess}_{\mathcal{A}}(v) = -i\frac{dv}{ds}.$$

The solutions of this equation are $v(s) = re^{2\pi i s(\theta+n)}$ for $n \in \frac{1}{2}\mathbb{Z}$. Thus the eigenvalues are $2\pi(\theta+n)$ for $n \in \frac{1}{2}\mathbb{Z}$. In particular, the Hessian has infinitely many positive and negative eigenvalues.

However, there is a well-defined relative quantity, the *Maslov index*, which plays the role of the difference of the indices.

5.3. **Whitney disks.** Fix $\mathbf{x}, \mathbf{y} \in L_0 \cap L_1$. A *Whitney strip* from $\mathbf{x}$ to $\mathbf{y}$ is a continuous map

$$u : \mathbb{R} \times [0,1] \to M^{2n}$$

satisfying the boundary conditions $u(\mathbb{R} \times \{0\}) \subseteq L_0$, $u(\mathbb{R} \times \{1\}) \subseteq L_1$, and the asymptotics

$$\lim_{t \to \infty} u(t,s) = \mathbf{x}, \qquad \lim_{t \to -\infty} u(t,s) = \mathbf{y}.$$

It is natural to reformulate the condition of a Whitney strip in terms of disks. Let $\mathbb{D}$ denote the standard disk in $\mathbb{C}$ and fix $\mathbf{x}, \mathbf{y} \in L_0 \cap L_1$. A *Whitney disk from* $\mathbf{x}$ *to* $\mathbf{y}$ is a continuous map

$$u : \mathbb{D} \to M^{2n}$$

such that for $z \in \mathbb{D}$,

- $u(z) \in L_0$ if $|z| = 1$ and $\mathrm{Re}(z) > 0$ and $u(z) \in L_1$ if $|z| = 1$ and $\mathrm{Re}(z) < 0$,
- $u(-i) = \mathbf{x}$ and $u(-i) = \mathbf{y}$.

In particular, any pseudo-holomorphic strip gives rise to a Whitney disk using the conformal diffeomorphism $\mathbb{R} \times [0,1] \cong \mathbb{D} \setminus \{\pm i\}$.

It will useful for us to consider homotopy classes of disks, which are defined as follows. Fix $\mathbf{x}, \mathbf{y} \in L_0 \cap L_1$. and two Whitney disks u_0 and u_1 from $\mathbf{x}$ to $\mathbf{y}$. A *homotopy* from u_0 to u_1 is a map

$$u : \mathbb{D} \times [0,1] \to M$$

such that for $z \in \mathbb{D}$ and $t \in [0,1]$:

- $u(z,t) \in L_0$ if $|z| = 1$ and $\mathrm{Re}(z) > 0$, and $u(z,t) \in L_1$ if $|z| = 1$ and $\mathrm{Re}(z) < 0$,
- $u(-i,t) = \mathbf{x}$ and $u(i,t) = \mathbf{y}$, and
- $u(z,0) = u_0(z)$ and $u(z,1) = u_1(z)$.

The set of homotopy classes of Whitney disks from $\mathbf{x}$ to $\mathbf{y}$ is denoted $W(\mathbf{x}, \mathbf{y})$.

Given $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \in L_0 \cap L_1$ and Whitney disks u_1 from $\mathbf{x}_1$ to $\mathbf{x}_2$ and u_2 from $\mathbf{x}_2$ to $\mathbf{x}_3$, the *juxtaposition* $u_1 * u_2$ is defeined as follows. Consider the quotient map $q : \mathbb{D} \to \mathbb{D} \vee \mathbb{D}$, where $\mathbb{D} \vee \mathbb{D}$ is the disk with the real interval collapsed to a point, and the homeomorphism $\mathbb{D} \vee \mathbb{D} = \mathbb{D}_1 \sqcup \mathbb{D}_2 / (i \in \mathbb{D}_1) \sim (-i \in \mathbb{D}_2)$. The map $u_1 * u_2$ is the composite map

$$\mathbb{D} \xrightarrow{q} \mathbb{D} \vee \mathbb{D} \xrightarrow{u_1 \vee u_2} M.$$

The map $*$ only depends on the homotopy class of the Whitney disks, giving rise to a map $* : W(\mathbf{x}_1, \mathbf{x}_2) \times W(\mathbf{x}_2, \mathbf{x}_3) \to W(\mathbf{x}_1, \mathbf{x}_3)$. A homotopy class of Whitney disks moreover determines a relative homotopy class in $\pi_2(M, L_0 \cup L_1)$ and hence a relative homology class in $H_2(M, L_1 \cup L_1)$.

5.4. **The Maslov index.**

5.4.1. *The Lagrangian Grassmannian.* If (V^{2n}, ω) is a symplectic vector space, call the subspace $\Lambda \subseteq V^{2n}$ *Lagrangian* if $\dim \Lambda = n$ and $\omega|_\Lambda = 0$. The *Lagrangian Grassmannian* $\mathcal{L}(V, \omega)$ is the space of all Lagrangian subspaces of (V^{2n}, ω).

The Lagrangian Grassmannian can be identified with the quotient space $U(n)/O(n)$ by the following procedure. Fix a compatible complex structure J on V (such a structure exists because there exists a complex structure on $(\mathbb{R}^{2n}, \omega_{st})$ and there is a symplectic isomorphism of vector spaces $(V^{2n}, \omega) \cong (\mathbb{R}^{2n}, \Omega_0))$, and let g be the induced positive definite symmetric form

$$g(u,v) = \frac{1}{2}(\omega(u, Jv) + \omega(v, Ju)).$$

A Hermitian form on V is specified by the equation

$$\langle v, w\rangle = g(v, w) + i\omega(v, w).$$

The Gram-Schmidt processes tells us that any n-dimensional subspace Λ of V^{2n} can be given an orthonormal basis $e_1, \dots, e_n$ with respect to g. In this framework, Λ is Lagrangian if and only if $\omega(e_i, e_j) = 0$ for all $i, j \in \{1, \dots, n\}$, i.e. $e_1, \dots, e_n$ is a unitary orthonormal basis for V. Two bases specify the same subspace if and only if they can be transformed into one another by an element of the orthogonal group $O(n)$. This gives the desired identification between $\mathcal{L}(V, \omega)$ and $U(n)/O(n)$.

There is a special cycle Σ in the $\mathcal{L}(\mathbb{R}^{2n}, \omega_{st})$, defined by

$$\Sigma = \{\Lambda \in \mathcal{L}(\mathbb{R}^{2n}, \omega_{st}) \mid \Lambda \cap \mathbb{R}^n \neq 0\} \subseteq \mathcal{L}(\mathbb{R}^{2n}, \omega_{st}) \tag{14}$$

which will be useful when defining the Maslov index.

5.4.2. *Construction of the Maslov index.* Suppose $u : \mathbb{R} \times [0, 1] \to M^{2n}$ is a Whitney strip. The bundle $u^*(TM)$ is a bundle of symplectic vector spaces over $\mathbb{R} \times [0, 1]$ with subbundles

$$(u|_{\mathbb{R}\times\{0\}})^*(TL_0) \text{ and } (u|_{\mathbb{R}\times\{1\}})^*(TL_1)$$

of Lagrangian subspaces over $\mathbb{R} \times \{0\}$ and $\mathbb{R} \times \{1\}$ respectively. Consider the symplectic vector space $V_t = u^*(TM)_{(t,0)}$ with Lagrangian subspace $\Lambda_0^t = (u|_{\mathbb{R}\times\{0\}})^*(TL_0)_{(t,0)}$. Parallel transport identifies $u^*(TM)_{(t,1)}$ with $u^*(TM)_{(t,0)}$ and the Lagrangian $(u|_{\mathbb{R}\times\{1\}})^*(TL_1)_{(t,1)}$ is transported to a Lagrangian Λ_t inside V_t. Then we can identify (V_t, ω) with $(\mathbb{R}^{2n}, \omega_{st})$ such that the Lagrangian Λ_0^t maps to the Lagrangian $\mathbb{R}^n \subseteq \mathbb{R}^{2n}$ consisting of the first n coordinates. The image of Λ^t under this identification is an element of $\mathbb{R}^{2n}$, which can be thought of as an element $A_t \in U(n)/O(n)$. The intersection number of $\{A_t\}_{t\in\mathbb{R}} \subseteq U(n)/O(n)$ with the cycle Σ from (14) is the *Maslov index* of u, denoted $\mu(u)$.

A priori the Maslov index might depend on the choice of parallel transport used to trivialize the symplectic vector bundle $u^*(TM)$. It can be shown that there is no such dependence, compare [10, p. 137].

Example 5.4. In the two pictures in Figure 2, the ambient manifold is $\mathbb{C}$, and there are two curves L_0, L_1 contained in the plane, intersecting at two points x and y. As L_0 and L_1 are one-dimensional, they are Lagrangian. The cycle Σ consists of all Lagrangians which have a nontrivial intersection with the real axis. In our case, this is just the real axis, so $\Sigma = \{\mathbb{R}\}$. The element Λ_t/Λ_t^0 in $U(n)/O(n)$ is equal to $\mathbb{R}$ only when the tangent spaces to the Lagrangians are parallel. Thus, the Maslov index is the number of parallel pairs of tangent subspaces, as we do parallel transport over all $t \in \mathbb{R}$. In the left figure this means that $\mu = 1$ and in the right it means that $\mu = 2$.

The juxtaposition of Whitney disks behaves well with respect to the Maslov index. Compare [10, Proposition 7.5.3].

Proposition 5.5. *Homotopic Whitney disks have the same Maslov index. Moreover, if $\phi \in W(\mathbf{x}, \mathbf{y})$ and $\psi \in W(\mathbf{y}, \mathbf{z})$, then*

$$\mu(\phi * \psi) = \mu(\phi) + \mu(\psi).$$

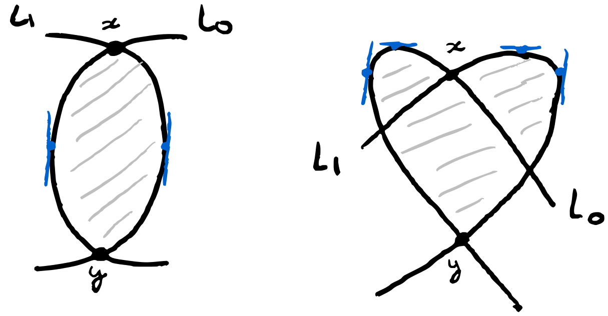

FIGURE 2. In the two pictures, there are two Lagrangians L_0 and L_1 in $\mathbb{C}$ intersecting at two points x and y. The blue tangent lines are where the element Λ_t/Λ_t^0 intersects the cycle Σ.

5.5. **Pseudo-holomorphic strips.** Recall that upward gradient flowlines of the action functional can be formulated as pseudo-holomorphic strips.

Definition 5.6. Let $\{J^s\}_{s\in[0,1]}$ be a one-parameter family of ω-tame almost-complex structures on (M^{2n}, ω). A $\{J^s\}$*-pseudo-holomorphic strip* $u : \mathbb{R} \times [0,1] \to M^{2n}$ is a Whitney strip (see Section 5.3) such that

$$\frac{\partial u}{\partial t} + J^s \frac{\partial u}{\partial s} = 0;$$

i.e. at each $(t,s) \in \mathbb{R} \times [0,1]$,

$$\frac{\partial u}{\partial t} + J^s_{u(t,s)} \frac{\partial u}{\partial s} = 0,$$

where $J^s_p : T_pM \to T_pM$ is the endomorphism determined by $\{J^s\}$. For fixed $\phi \in W(\mathbf{x}, \mathbf{y})$, let $\mathcal{M}_{\{J^s\}}(\phi)$ denote the set of pseudo-holomorphic representatives of ϕ.

Since for any $\tau \in \mathbb{R}$, the map $\mathbb{R} \times [0,1] \to \mathbb{R} \times [0,1]$ given by $(t,s) \mapsto (t+\tau, s)$ is holomorphic, it follows that if $u : \mathbb{R} \times [0,1] \to M$ is pseudo-holomorphic, so is the map $(t,s) \mapsto u(t+\tau, s)$. The following is an analogue of Theorem 3.2 in Morse theory. Compare [10, Theorem 7.6.2]

Theorem 5.7. *Let (M^{2n}, ω) be a symplectic manifold, equipped with Lagrangians L_0 and L_1. If $\{J^s\}$ is a suitably generic one-parameter family of ω-tame almost complex structures, then for any non-constant homotopy class $\phi \in W(\mathbf{x}, \mathbf{y})$ with $\mu(\phi) \leq 2$, the space $\widehat{\mathcal{M}}_{\{J^s\}}(\phi)$ is a smooth manifold with dimension given by*

$$\dim \widehat{\mathcal{M}}_{\{J^s\}}(\phi) = \mu(\phi) - 1.$$

In particular, if ϕ is a non-constant homotopy class with $\mu(\phi) \leq 0$, then $\widehat{\mathcal{M}}_{\{J^s\}}(\phi)$ is empty. Further, if ϕ is the homotopy class represented by a constant Whitney disk, then $\widehat{\mathcal{M}}_{\{J^s\}}(\phi)$ is a single point, i.e. the constant flowline.

5.6. **Definition of the Lagrangian Floer complex.** From now on, we assume that our manifold M is compact. Define $\mathrm{CF}(L_0, L_1)$ to be the $\mathbb{F}$-vector space generated by $L_0 \cap L_1$ with endomorphism ∂ given by

$$\partial \mathbf{x} = \sum_{\mathbf{y} \in L_0 \cap L_1} \sum_{\{\phi \in W(\mathbf{x},\mathbf{y}) | \mu(\phi) = 1\}} \#\widehat{\mathcal{M}}(\phi) \cdot \mathbf{y}. \tag{15}$$

This turns out to be a correct definition only when (M,ω) is an exact symplectic manifold with exact and L_0 and L_1 are exact Lagrangians. Just like Morse–Smale case, we have arrived at the definition, but there are a number of components to resolve:

(1) The definition of $\partial\mathbf{x}$ given in (15) is a finite sum. The moduli spaces $\widehat{\mathcal{M}}(\phi)$ for $\mu(\phi)=1$ are zero dimensional by Theorem 5.7. So, to make sense of the sum (15), we would like the union of $\widehat{\mathcal{M}}(\phi)$ over all $\phi\in W(\mathbf{x},\mathbf{y})$ to be compact. This turns out not to be true in general, but is true if M is exact and L_0 and L_1 are exact. The correction will involve an enlargement of the ring. This is outlined in Section 5.7

(2) The endomorphism ∂ satisfies $\partial^2=0$, making $\mathrm{CF}(L_0,L_1)$ into a chain complex with differential ∂. In this case, we will see that we are analyzing moduli spaces $\widehat{\mathcal{M}}(\phi)$ with $\mu(\phi)=2$, and the boundary of this moduli space will be the terms in the sum $\partial^2\mathbf{x}$. This is outlined in Section 5.8.

(3) In the moduli spaces $\widehat{\mathcal{M}}(\phi)$, there is an implicit dependence on $\{J^s\}$. It would be better to write the moduli spaces as $\widehat{\mathcal{M}}_{\{J^s\}}(\phi)$. That said, the Lagrangian Floer complex $\mathrm{CF}(L_0,L_1)$ is independent of the choice of almost complex structures. The proof of this is outlined in Section 5.9.

5.7. **Compactness.**

5.7.1. *Energy functional.* Let (M^{2n},ω) be a symplectic manifold with a pair of compact Lagrangian submanifolds L_0 and L_1. The *energy* of a pseudo-holomorphic strip u representing a homotopy class $\phi\in W(\mathbf{x},\mathbf{y})$ is defined to be

$$E(u)=\int_{\mathbb{R}\times[0,1]}\frac{1}{2}\left(\left|\frac{\partial u}{\partial t}\right|^2+\left|\frac{\partial u}{\partial s}\right|^2\right)dt\,ds=\int_{\mathbb{R}\times[0,1]}\frac{1}{2}\left(g_s\left(\frac{\partial u}{\partial t},\frac{\partial u}{\partial t}\right)+g_s\left(\frac{\partial u}{\partial s},\frac{\partial u}{\partial s}\right)\right)dt\,ds,$$

where g_s denotes the metric on TM associated to ω and the ω-tame (but not necessarily compatible) almost-complex structure J^s, compare equation (9).

Since L_0 and L_1 are Lagrangian submanifolds, the symplectic form ω provides a cohomology class in $H^2(M,L_0\cup L_1;\mathbb{R})$. Thus ω can be evaluated on a Whitney disk from $\mathbf{x}$ to $\mathbf{y}$, and this value will depend only on the homotopy class of the Whitney disk. Compare [10, Lemma 7.7.3]

Lemma 5.8. *If $u:\mathbb{D}\to M^{2n}$ is a $\{J^s\}$-holomorphic disk with respect to a 1-parameter family of almost complex structures $\{J^s\}_{s\in[0,1]}$ which are ω-tame, then*

$$E(u)=\int_{\mathbb{D}}u^*(\omega),$$

where ω is thought of as a relative cohomology class in $H^2(M,L_0\cup L_1;\mathbb{R})$.

Proof. By definition

$$2E(u)=\int_{\mathbb{R}\times[0,1]}\left(\left|\frac{\partial u}{\partial s}\right|^2+\left|\frac{\partial u}{\partial t}\right|^2\right)dt\,ds,$$

and by the relation $g(u,u)=\omega(u,Ju)$, we have that

$$2E(u)=\int_{\mathbb{R}\times[0,1]}\left(\omega\left(\frac{\partial u}{\partial s},J^s\frac{\partial u}{\partial s}\right)+\omega\left(\frac{\partial u}{\partial t},J^s\frac{\partial u}{\partial t}\right)\right)dt\,ds.$$

Since u is a $\{J^s\}$-holomorphic strip and $J^s \circ J^s = -\operatorname{id}$, we have the equation $J^s \frac{\partial u}{\partial t} = \frac{\partial u}{\partial s}$. So

$$2E(u) = \int_{\mathbb{R}\times[0,1]} \left(-\omega\left(\frac{\partial u}{\partial s}, \frac{\partial u}{\partial t}\right) + \omega\left(\frac{\partial u}{\partial t}, \frac{\partial u}{\partial s}\right)\right) dt\, ds = 2\int_{\mathbb{R}\times[0,1]} u^*(\omega) = 2\int_{\mathbb{D}} u^*(\omega)$$

with the conformal identification $\mathbb{R}\times[0,1] \cong \mathbb{D}\setminus\{\pm i\}$ □

5.7.2. *Gromov compactness.* A *broken holomorphic strip* from $\mathbf{x}$ to $\mathbf{y}$ is a sequence $\mathbf{x}_0, \dots, \mathbf{x}_{n+1}$ of intersection points between L_0 and L_1 such that $\mathbf{x}_0 = \mathbf{x}$ and $\mathbf{x}_{n+1} = \mathbf{y}$, and a sequence $u_0, \dots, u_n$ of non-constant $\{J^s\}$-holomorphic strips, modulo translation. A broken holomorphic strip represents a fixed homotopy class $\phi \in W(\mathbf{x}, \mathbf{y})$ if $\phi = [u_0] * \cdots * [u_n]$

The following comactness result is due to Gromov [6], and in its simplest form is stated as follows. Compare [10, Theorem 7.7.5].

Theorem 5.9 (Gromov compactness)**.** *Assume that $\pi_2(M, L_i) = 0$ for $i = 0, 1$, $\pi_2(M) = 0$, and L_0 and L_1 are compact Lagrangian submanifolds. Further assume that M is compact. Let $\mathbf{x}$ and $\mathbf{y}$ be two intersection points of L_0 and L_1. Then any sequence of $\{J^s\}$-pseudo-holomorphic strips from $\mathbf{x}$ to $\mathbf{y}$ with a fixed energy bound has a $C^{\infty,\mathrm{loc}}$ convergent subsequence to a broken holomorphic strip from $\mathbf{x}$ to $\mathbf{y}$.*

5.7.3. *Exact symplectic manifolds.* Let $(M^{2n}, d\alpha)$ be an exact symplectic manifold. Recall that a Lagrangian submanifold L in M is called exact if there exists $f : L \to \mathbb{R}$ such that $\alpha|_L = df$. Also recall the action functional $\mathcal{A}$ on the space of paths $\mathcal{V}$ from L_0 to L_1. For a path $\gamma : [0,1] \to M$ in $\mathcal{V}$,

$$\mathcal{A}(\gamma) = f_0(\gamma(0)) - f_1(\gamma(1)) + \int_{[0,1]} \gamma^*(\alpha).$$

For a constant path $\mathbf{x}$, the action functional is given by

$$\mathcal{A}(\mathbf{x}) = f_0(\mathbf{x}) - f_1(\mathbf{x}).$$

Thus for a pseudo-holomorphic u in $W(\mathbf{x}, \mathbf{y})$, the energy of u is given by

$$E(u) = \int_{\mathbb{D}} u^*(\omega) = \mathcal{A}(\mathbf{x}) - \mathcal{A}(\mathbf{y}) = f_0(\mathbf{x}) - f_1(\mathbf{x}) - f_0(\mathbf{y}) + f_1(\mathbf{y}), \tag{16}$$

where we used Lemma 5.8 followed by Lemma 5.1 in the left-most two equalities. Consider the union

$$\bigcup_{\{\phi\in W(\mathbf{x},\mathbf{y})|\mu(\phi)=1\}} \widehat{\mathcal{M}}(\phi).$$

It is a zero-dimensional manifold by Theorem 5.7. Thus, equation (16) tells us that there is an energy bound on the pseudo-holomorphic strips in the above union, so by Gromov compactness Theorem 5.9, the above union is finite. This implies that ∂ in equation (15) is a finite sum.

However, in general, we cannot ensure that $\sum \#\widehat{\mathcal{M}}(\phi)$ is finite, even in the case when M is compact. We introduce the Novikov ring to alleviate this issue.

5.7.4. *The Novikov ring.* The *Novikov ring* $N_{\mathbb{Z}/2\mathbb{Z}}$ over $\mathbb{Z}/2\mathbb{Z}$ is defined as the collection of formal sums

$$x_A = \sum_{a \in A} x_a t^a,$$

where A is a discrete set, $x_a \in \mathbb{Z}/2\mathbb{Z}$, and t is a formal variable. We can define ring multiplication to be $x_A \cdot x_B = x_{A+B}$, where $A + B$ is the Minkowski sum $A + B = \{a + b \mid a \in A, b \in B\}$.

Definition 5.10. For a compact manifold M, define the *Lagrangian Floer complex* for a compact manifold $\mathrm{CF}(L_0, L_1; N_{\mathbb{Z}/2\mathbb{Z}})$ be the module over the Novikov ring $N_{\mathbb{Z}/2\mathbb{Z}}$ generated by the intersection points of L_0 and L_1 (of which there are finitely many). For a homotopy class ϕ, let $a(\phi)$ be the integral of ω on any representative of ϕ. Define the endomorphism

$$\partial \mathbf{x} = \sum_{\mathbf{y} \in L_0 \cap L_1} \sum_{\{\phi \in W(\mathbf{x},\mathbf{y}) | \mu(\phi)=1\}} \#\widehat{\mathcal{M}}(\phi) t^{a(\phi)} \cdot \mathbf{y}. \tag{17}$$

For a fixed real number a, the set

$$\bigcup_{\{\phi \in W(\mathbf{x},\mathbf{y}) | \mu(\phi)=1, a(\phi)=a\}} \widehat{\mathcal{M}}(\phi)$$

is a compact one-dimensional manifold by Theorem 5.7 and Theorem 5.9.

If M is compact, first note that $H_2(M)$ and $H_1(L_0 \cup L_1)$ are finitely generated, which by the following part of the exact sequence of the pair $(M, L_0 \cup L_1)$

$$H_2(M) \longrightarrow H_2(M, L_0 \cup L_1) \longrightarrow H_1(L_0 \cup L_1)$$

implies that $H_2(M, L_0 \cup L_1)$ is finitely generated. Since $a(\phi)$ can be thought of as the evaluation of $[\omega] \in H^2(M, L_0 \cup L_1; \mathbb{R})$ on the relative homology class in $H_2(M, L_0 \cup L_1)$ associated to ϕ, there are finitely many values of $a(\phi)$ in the sum in (17). Thus, $\partial \mathbf{x}$ is a finite sum.

5.8. **Gluing.** This section is dedicated to outlining the idea behind the proof that $\partial^2 = 0$. We are working under the assumptions that $\pi_2(M) = 0$, $\pi_2(M, L_i) = 0$ for $i = 0, 1$, and M is compact.

Theorem 5.11. *For generic choices of $\{J^s\}$, and for each $\phi \in W(\mathbf{x}, \mathbf{y})$ with $\mu(\phi) = 2$, the moduli space $\mathcal{M}_{\{J^s\}}(\phi)$ has a compactification to a one-manifold with boundary, and the boundary is identified with*

$$\bigcup_{\left\{\phi_1,\phi_2 \middle| \substack{\phi_1 * \phi_2 = \phi \\ \mu(\phi_1)=\mu(\phi_2)=1}\right\}} \widehat{\mathcal{M}}_{\{J^s\}}(\phi_1) \times \widehat{\mathcal{M}}_{\{J^s\}}(\phi_2). \tag{18}$$

Omit the $\{J^s\}$ from the subscript in the moduli space. We have that

$$\partial^2 \mathbf{x} = \sum_{\mathbf{z} \in L_0 \cap L_1} \sum_{\{\phi_1 \in W(\mathbf{x},\mathbf{y}), \phi_2 \in W(\mathbf{y},\mathbf{z}) | \mu(\phi_1)=\mu(\phi_2)=1\}} \#\widehat{\mathcal{M}}(\phi_1)\#\widehat{\mathcal{M}}(\phi_2) t^{a(\phi_1)} t^{a(\phi_2)} \cdot \mathbf{z}.$$

Fix $\mathbf{z}$ in the above sum, and group the inner summand by the value of $a(\phi_1)+a(\phi_2)$ and then $\phi=\phi_1 * \phi_2$. Since $a(\phi)=a(\phi_1)+a(\phi_2)$ by Proposition 5.5, the coefficient of $\mathbf{z}$ is

$$\sum_{\phi\in W(\mathbf{x},\mathbf{z})}\left(\sum_{\left\{\phi_1,\phi_2\middle|\substack{\phi_1*\phi_2=\phi\\ \mu(\phi_1)=\mu(\phi_2)=1}\right\}}\#\widehat{\mathcal{M}}(\phi_1)\#\widehat{\mathcal{M}}(\phi_2)\right)t^{a(\phi)}.$$

Since $\widehat{\mathcal{M}}(\phi)$ is a one-dimensional manifold by Theorem 5.7 and the points of the boundary of its compactification are given by Theorem 5.11, the term in the inner parenthesis is zero modulo two. The above sum is also finite because there are finitely many $a(\phi)$. Thus, $\partial^2=0$, implying that $(\mathrm{CF}(L_0,L_1;N_{\mathbb{Z}/2\mathbb{Z}},\partial)$ is a chain complex.

The homology of this complex is *Lagrangian Floer homology* $HF(L_0,L_1;N_{\mathbb{Z}/2\mathbb{Z}})$.

5.9. **Independence and invariance.** In this section, we go over the main ideas behind showing that the Lagrangian Floer complex is independent of the choice of almost-complex structures appearing in the differential, and also independent under Hamiltonian isotopies, compare property (3) in Section 5.1.

5.9.1. *Almost-complex structure invariance.* Suppose $\{J_0^s\}$ and $\{J_1^s\}$ are two one-parameter families of almost-complex structures that are suitably generic for the complexes

$$\mathrm{CF}_{\{J_0^s\}}(L_0,L_1;N_{\mathbb{Z}/2\mathbb{Z}}) \text{ and } \mathrm{CF}_{\{J_1^s\}}(L_0,L_1;N_{\mathbb{Z}/2\mathbb{Z}})$$

to be defined. Connect $\{J_0^s\}$ and $\{J_1^s\}$ by a one-parameter family of paths of almost complex structures $\{J_t^s\}$.

Given $\mathbf{x},\mathbf{y}\in L_0\cap L_1$ and $\phi\in W(\mathbf{x},\mathbf{y})$, there is a parametrized moduli space $\mathcal{M}_{\{J_t^s\}_{s,t\in[0,1]}}(\phi)$ of Whitney strips u representing ϕ such that

$$\frac{\partial u}{\partial t}+J^s_{\psi(t)}\frac{\partial u}{\partial s}=0,$$

where $\psi:\mathbb{R}\to[0,1]$ is given as before in Section 3.6, a smooth monotone function such that $\psi(t)=0$ for $t\le 0$ and $\psi(t)=1$ for $t\ge 1$. Note that the almost-complex structure $J^s_{\psi(t)}$ is evaluated at $u(t,s)$, and these moduli spaces no longer have an $\mathbb{R}$-action, just like in the Morse–Smale case.

Theorem 3.11 has the following analogue. Compare [10, Theorem 7.9.1].

Theorem 5.12. *Let (M^{2n},ω) be a closed symplectic manifold with two compact Lagrangians L_0 and L_1, and fix two paths of ω-tame almost-complex structures $\{J_0^s\}_{s\in[0,1]}$ and $\{J_1^s\}_{s\in[0,1]}$. Then for any sufficiently generic two-parameter family of ω-tame almost-complex structures $\{J_t^s\}_{s,t\in[0,1]^2}$ connecting $\{J_0^s\}_{s\in[0,1]}$ and $\{J_1^s\}_{s\in[0,1]}$, and for all homotopy classes $\phi\in W(\mathbf{x},\mathbf{y})$ with $\mu(\phi)\le 1$, the space $\mathcal{M}_{\{J_t^s\}_{s,t\in[0,1]}}(\phi)$ is a smooth manifold of dimension*

$$\dim\mathcal{M}_{\{J_t^s\}_{s,t\in[0,1]}}(\phi)=\mu(\phi).$$

In particular, if ϕ is a nonconstant homotopy class with $\mu(\phi)<0$, then $\mathcal{M}_{\{J_t^s\}_{s,t\in[0,1]}}(\phi)$ is empty.

Define the continuation map

$$\Phi=\Phi_{\{J_t^s\}_{s,t\in[0,1]}}:\mathrm{CF}_{\{J_0^s\}}(L_0,L_1;N_{\mathbb{Z}/2\mathbb{Z}})\to\mathrm{CF}_{\{J_1^s\}}(L_0,L_1;N_{\mathbb{Z}/2\mathbb{Z}})$$

by the formula

$$\Phi(\mathbf{x}) = \sum_{\mathbf{y}\in L_0\cap L_1} \sum_{\{\phi\in W(\mathbf{x},\mathbf{y})|\mu(\phi)=0\}} \#\mathcal{M}_{\{J_t^s\}_{s,t\in[0,1]}}(\phi)t^{a(\phi)}\cdot \mathbf{y}.$$

First we show that Φ is a chain map. Gromov compactness and gluing can be used to show the following proposition. Compare [10, Proposition 7.9.1].

Proposition 5.13. *If $\{J_t^s\}_{s,t\in[0,1]}$ is a generic two-parameter family of ω-tame almost-complex structures, then for all $\mathbf{x},\mathbf{y}\in L_0\cap L_1$ and all homotopy classes of Whitney disks $\phi\in W(\mathbf{x},\mathbf{y})$ with $\mu(\phi)=0$, the moduli space $\#\mathcal{M}_{\{J_t^s\}_{s,t\in[0,1]}}(\phi)$ is a compact zero-dimensional manifold. If $\mu(\phi)=1$, the compactification of $\#\mathcal{M}_{\{J_t^s\}_{s,t\in[0,1]}}(\phi)$ is a compact one-dimensional manifold whose boundary is identified with*

$$\begin{aligned}
&\bigcup_{\left\{\phi_1*\phi_2=\phi\,\middle|\,\substack{\mu(\phi_1)=1\\ \mu(\phi_2)=0}\right\}} \widehat{\mathcal{M}}_{\{J_0^s\}_{s\in[0,1]}}(\phi_1)\times\mathcal{M}_{\{J_t^s\}_{s,t\in[0,1]}}(\phi_2)\\
\cup &\bigcup_{\left\{\phi_1*\phi_2=\phi\,\middle|\,\substack{\mu(\phi_1)=0\\ \mu(\phi_2)=1}\right\}} \mathcal{M}_{\{J_t^s\}_{s,t\in[0,1]}}(\phi_1)\times\widehat{\mathcal{M}}_{\{J_1^s\}_{s\in[0,1]}}(\phi_2).
\end{aligned}$$

The $\mathbf{y}$ component of

$$(\partial\circ\Phi+\Phi\circ\partial)(\mathbf{x})$$

counts the points in the boundary given in Proposition 5.13. Since this boundary has an even number of points, we have that $(\partial\circ\Phi+\Phi\circ\partial)(\mathbf{x})=0$, so Φ is a chain map.

Now we show that $\Phi=\Phi_{\{J_t^s\}_{s,t\in[0,1]}}$ induces an isomorphism on homology. To do so, we show that $\Phi_{\{J_{1-t}^s\}_{s,t\in[0,1]}}$ is the homotopy inverse of Φ by constructing a homotopy operator

$$H:\mathrm{CF}_{\{J_0^s\}_{s\in[0,1]}}(L_0,L_1)\to\mathrm{CF}_{\{J_0^s\}_{s\in[0,1]}}(L_0,L_1)$$

such that

$$\partial\circ H+H\circ\partial=\mathrm{id}+\Phi_{\{J_{1-t}^s\}_{s,t\in[0,1]}}\circ\Phi_{\{J_t^s\}_{s,t\in[0,1]}} \tag{19}$$

The homotopy operator H is defined by counting holomorphic disks using a three-parameter family of almost-complex structures $\{J_{r,t}^s\}$.

Start with $\{J_t^s\}$ and choose $\{J_{r,t}^s\}_{\{s\in[0,1],r\in[0,\infty),t\in\mathbb{R}\}}$ such that

- $J_{r,t}^s=J_t^s$ if $r=0$,
- for $t>1$, $J_{r,t}^s=J_{\psi(r+t)}^s$,
- for $t<-1$, $J_{r,t}^s=J_{\psi(1-r-t)}^s$.

Given $\phi\in W(\mathbf{x},\mathbf{y})$, consider the moduli space $\mathcal{M}_H(\phi)$ of pairs (r,u), where $r\in(0,\infty)$ and u is a Whitney strip representing ϕ such that

$$\frac{\partial u}{\partial t}+J_{r,t}^s\frac{\partial u}{\partial s}=0.$$

An analogue of Theorem 5.13 is that $\mathcal{M}_H$ is a smooth manifold of dimension $\mu(\phi)-1$. When the dimension of $\mathcal{M}_H$ is one, a suitable compactification can have three types of boundary, depending on the convergence of r:

(1) $r\to\rho$ for some real number $\rho\in(0,\infty)$. In this case, the boundary point is a $\{J_{r,t}^s\}$-holomorphic disk which breaks off.
(2) $r\to 0$. In this case, the boundary point is a constant flowline for $\{J_0^s\}_{s\in[0,1]}$.

(3) $r \to \infty$. In this case, the boundary point consists of a $\{J_t^s\}_{s,t\in[0,1]}$-holomorphic disk juxtaposed with a $\{J_{1-t}^s\}_{s,t\in[0,1]}$-holomorphic disk.

Counting boundary points gives us the equation (19). A simple adaptation of the above construction also shows that $\Phi_{\{J_t^s\}_{s,t\in[0,1]}} \circ \Phi_{\{J_{1-t}^s\}_{s,t\in[0,1]}}$ is homotopic to the identity map. Thus, $\Phi_{\{J_t^s\}_{s,t\in[0,1]}}$ induces on homology, so there is no dependence on the path of almost-complex structures.

5.9.2. *Hamiltonian isotopy invariance.* Suppose that $\mathcal{H} : M \times [0,1] \to \mathbb{R}$ is a bounded Hamiltonian function and $\{\Psi_t\}_{t\in[0,1]} : M \to M$ is a corresponding family of Hamiltonian diffeomorphisms with $\Psi_0(x) = x$, i.e. for any vector field Y and $x \in M$,

$$\omega\left(\frac{d\Psi_t}{dt}(x), Y\right) = Y_x \mathcal{H}_t.$$

We wish to construct continuation maps

$$\Phi_{\{\Psi_t\}_{t\in[0,1]}} : \mathrm{CF}_{\{J^s\}_{s\in[0,1]}}(L_0, L_1) \to \mathrm{CF}_{\{J^s\}_{s\in[0,1]}}(L_0, \Psi_1(L_1)),$$

which are isomorphisms on homology, as this will imply the Hamiltonian isotopy invariance given in (3) in Section 5.1.

Consider pseudo-holomorphic strips $u : \mathbb{R} \times [0,1] \to M^{2n}$ with the following property with the following moving boundary conditions: $u(t,0) \in L_0$, $u(t,1) \in \Phi_t(L_1)$ for all $t \in \mathbb{R}$, and $\lim_{t\to-\infty} u(t,s) = \mathbf{x}$ and $\lim_{t\to+\infty} u(t,s) = \mathbf{y}$, and the usual Cauchy–Riemann equations

$$\frac{\partial u}{\partial t} + J^s \frac{\partial u}{\partial s} = 0.$$

Such maps u can be assempled into homotopy classes $W'(\mathbf{x}, \mathbf{y})$. The space of pseudo-holomorphic representatives for a given homotopy class $\phi \in W'(\mathbf{x}, \mathbf{y})$ is denoted $\mathcal{M}_{\{\Psi_t\}_{t\in[0,1]}}(\phi)$. There is an appropriate analogue of Theorem 5.13 is that the moduli spaces $\mathcal{M}_{\{\Psi_t\}_{t\in[0,1]}}(\phi)$ with $\mu(\phi) \leq 1$ are smooth manifolds of dimension $\mu(\phi)$.

For $\mathbf{x} \in L_0 \cap L_1$, define the continuation maps

$$\Phi_{\{\Psi_t\}_{t\in[0,1]}}(\mathbf{x}) = \sum_{\mathbf{y}\in L_0\cap\Psi_1(L_1)} \#\mathcal{M}_{\{\Psi_t\}_{t\in[0,1]}}(\phi) t^{a(\phi)} \cdot \mathbf{y}.$$

To show that $\Phi_{\{\Psi_t\}_{t\in[0,1]}}$ is a chain map, the following variant of Gromov compactness adapted to $W'(\mathbf{x}, \mathbf{y})$ is used. Compare [10, Lemma 7.9.3].

Lemma 5.14. *For each $\phi \in W'(\mathbf{x}, \mathbf{y})$, there exists a constant $C(\phi)$ such that for all pseudo-holomorphic representatives u of ϕ,*

$$E(u) \leq C(\phi).$$

Further constructions go into showing that $\Phi_{\{\Psi_t\}_{t\in[0,1]}}$ is a chain map and proving that $\Phi_{\{\Psi_t\}_{t\in[0,1]}}$ induces an isomorphism on homology.

References

[1] V. I. Arnold. *Mathematical methods of classical mechanics*, volume 60 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Heidelberg, 1978. Translated from the Russian by K. Vogtmann and A. Weinstein.

[2] Michèle Audin and Mihai Damian. *Morse theory and Floer homology*. Universitext. Springer, London; EDP Sciences, Les Ulis, 2014. Translated from the 2010 French original by Reinie Erné.

[3] Raoul Bott. The stable homotopy of the classical groups. *Ann. of Math. (2)*, 70:313–337, 1959.

[4] Andreas Floer. Morse theory for Lagrangian intersections. *J. Differential Geom.*, 28(3):513–547, 1988.
[5] Andreas Floer. Symplectic fixed points and holomorphic spheres. *Comm. Math. Phys.*, 120(4):575–611, 1989.
[6] M. Gromov. Pseudo holomorphic curves in symplectic manifolds. *Invent. Math.*, 82(2):307–347, 1985.
[7] Dusa McDuff and Dietmar Salamon. *Introduction to symplectic topology.* Oxford Graduate Texts in Mathematics. Oxford University Press, Oxford, third edition, 2017.
[8] J. Milnor. *Morse theory.* Annals of Mathematics Studies, No. 51. Princeton University Press, Princeton, N.J., 1963. Based on lecture notes by M. Spivak and R. Wells.
[9] John Milnor. *Lectures on the h-cobordism theorem.* Princeton University Press, Princeton, N.J., 1965. Notes by L. Siebenmann and J. Sondow.
[10] Peter Ozsváth, András Stipsicz, and Zoltán Szabó. Heegaard Floer homology. In preparation.
[11] Stephen Smale. On gradient dynamical systems. *Ann. of Math. (2)*, 74:199–206, 1961.
[12] René Thom. Sur une partition en cellules associée à une fonction sur une variété. *C. R. Acad. Sci. Paris*, 228:973–975, 1949.
[13] Edward Witten. Supersymmetry and Morse theory. *J. Differential Geometry*, 17(4):661–692 (1983), 1982.