Quantitative Helly-type theorems via sparse approximation

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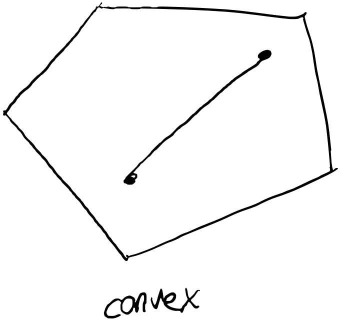
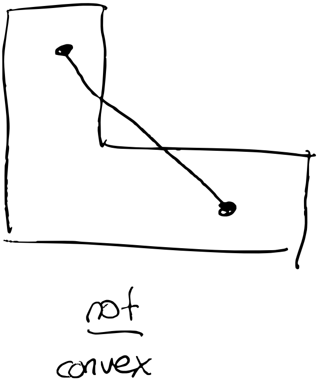
Convex sets

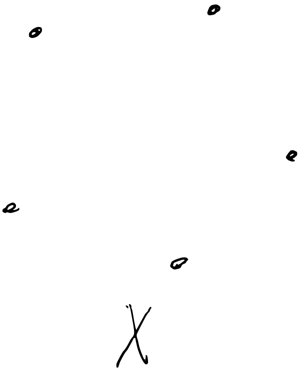
A point set $K \subseteq \mathbb{R}^d$ is *convex* if for any two points $x, y \in K$, the straight line segment from x to y is in K:

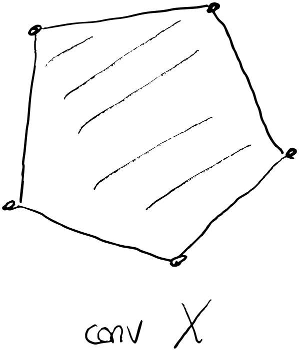




Convex hull

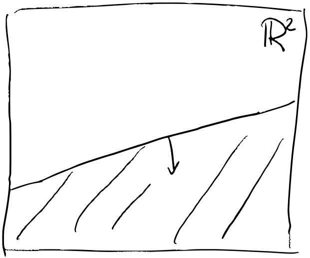
For a point set $X \subseteq \mathbb{R}^d$, the *convex hull* $\operatorname{conv} X$ is the "smallest" convex set containing X, or the intersection of all convex sets containing X.



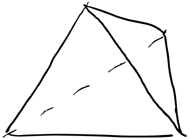


Useful convex sets: halfspaces, simplices

A *hyperplane* in $\mathbb{R}^d$ is a $d-1$ dimensional affine subspace (e.g. line in $\mathbb{R}^2$, plane in $\mathbb{R}^3$).
A (closed) *halfspace* is the set of points "on one side" of the hyperplane.
A *simplex* is the convex hull of $d+1$ points not all lying on the same hyperplane (e.g. triangle in $\mathbb{R}^2$, tetrahedron in $\mathbb{R}^3$).



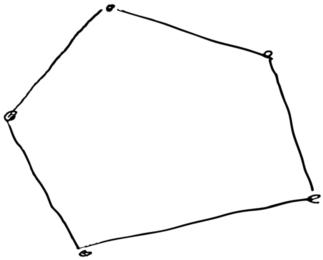
A halfspace in $\mathbb{R}^2$

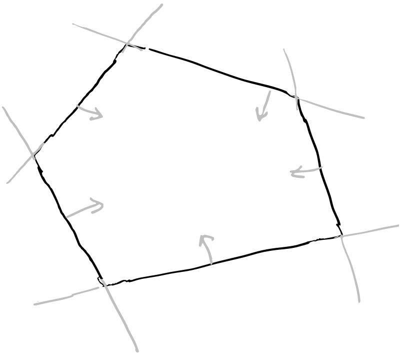


A simplex in $\mathbb{R}^3$

Useful convex sets: polytopes

A *polytope* is the convex hull of a finite set of points. Equivalently, a polytope is the bounded intersection of a finite number of closed halfspaces.





Helly's theorem

Helly's theorem (1913)

Let $K_1, \ldots, K_n$ be convex sets in $\mathbb{R}^d$. If the intersection of any $d+1$ of the sets is nonempty, then the intersection of all the sets is nonempty.

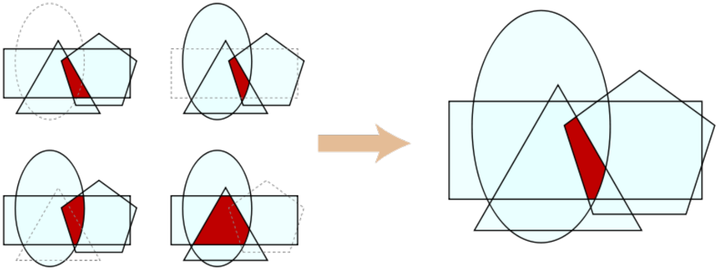


Figure: Helly's theorem in the plane

Quantitative Helly: general scheme

Helly's theorem

Let $K_1, \ldots, K_n$ be convex sets in $\mathbb{R}^d$. If the intersection of any $d+1$ of the sets is nonempty, then the intersection of all the sets is nonempty.

Quantitative version ??

If the intersection of any $d+1$ sets is large, then the intersection of all the sets is not too small.

- NOT TRUE!

General quantitative Helly's theorem

If the intersection of any $2d$ sets is large, then the intersection of all the sets is not too small.

$2d$ necessary

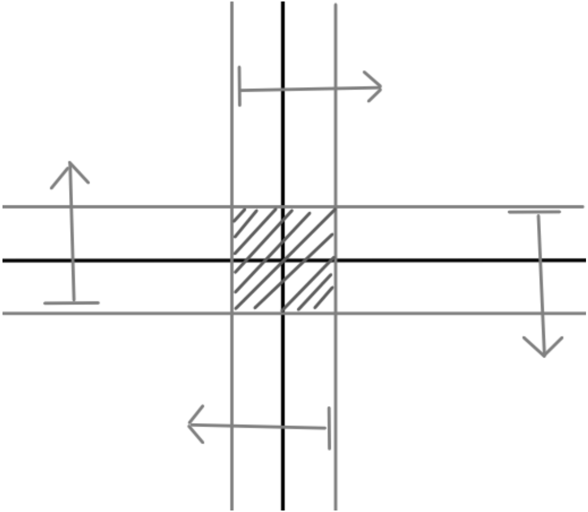


Figure: $2d$ halfspaces intersecting in small cube

Quantitative Helly-type theorem

Two "measures" of size:

- The *volume* $\mathrm{vol}_d(K)$ of a convex set K is its standard Lebesgue measure in $\mathbb{R}^d$.
- The *diameter* $\mathrm{diam}\, K$ of a convex (compact) set K is $\max_{x,y\in K} |x-y|$.

Bárány-Katchalski-Pach (1982, 1984)

Let $\mathcal{F}$ be a finite family of convex sets in $\mathbb{R}^d$. If the intersection of any $2d$ of the sets has volume at least 1, then the volume of $\bigcap \mathcal{F}$ is at least a positive constant.

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Theorem reformulation

Bárány-Katchalski-Pach (1982, 1984)

Let $\mathcal{F} = \{K_1, \ldots, K_n\}$ be a finite family of convex sets in $\mathbb{R}^d$ with $\mathrm{vol}_d(\bigcap \mathcal{F}) = 1$. Then *there exists* a subfamily $\mathcal{F}'$ of at most $2d$ convex sets such that the volume of $\bigcap \mathcal{F}'$ is at most $v(d)$.

Let $\mathcal{F} = \{K_1, \ldots, K_n\}$ be a finite family of convex sets in $\mathbb{R}^d$ with $\mathrm{diam}(\bigcap \mathcal{F}) = 1$. Then *there exists* a subfamily $\mathcal{F}'$ of at most $2d$ convex sets such that the diameter of $\bigcap \mathcal{F}'$ is at most $c(d)$.

Problem history

- Bárány-Katchalski-Pach (1982, 1984)
  - First bounds: $v(d) \leq d^{2d^2}$ and $c(d) \leq c' d^{2d}$.
  - **Conjecture:** $v(d) \approx d^{c_1 d}$, $c(d) \approx c_2 d^{1/2}$.
- Naszódi (2015)
  - Proved BKP volume conjecture: $v(d) \leq d^{2d}$.
  - Lower bound on exponent for volume: $v(d) \geq (cd)^{d/2}$.
- Brazitikos (2017)
  - $v(d) \leq (cd)^{3d/2}$, **current best**.
- Brazitikos (2018)
  - First polynomial bound on diameter: $c(d) \leq (cd)^{11/2}$.
- Ivanov, Naszódi (2021)
  - $c(d) \leq (2d)^3$, **current best**.
  - Lower bound on exponent for diameter: $c(d) \geq (cd)^{1/2}$.

Many other directions! (fractional Helly, large subfamilies, ...)

Our main result

Theorem (A-H., A., K. 2021)

Let $\{K_1, \ldots, K_n\}$ be a family of convex sets and $K = \bigcap K_i$.
There exists a subset of indices $\sigma = \{i_1, \ldots, i_{2d}\}$ such that
$K_\sigma = K_{i_1} \cap \cdots \cap K_{i_{2d}}$ satisfies

$$\operatorname{vol}_d K_\sigma \leq (c_1 d)^{3d/2} \operatorname{vol}_d K \quad \text{and} \quad \operatorname{diam} K_\sigma \leq (c_2 d)^2 \operatorname{diam} K.$$

Thus,

$$v(d) \leq (c_1 d)^{3d/2} \quad \text{and} \quad c(d) \leq (c_2 d)^2.$$

Previously: $v(d) \leq (c_1' d)^{3d/2}$ (Brazitikos '17) and $c(d) \leq (c_2' d)^3$ (Ivanov, Naszódi '21)

Polytope approximation

Let $Q \subset \mathbb{R}^d$ be a polytope. There exists a subset of at most $2d$ vertices of Q such that their convex hull Q' satisfies

$$Q \subseteq cQ'.$$

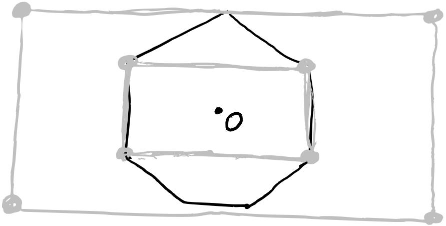

Figure: Polytope approximation in $\mathbb{R}^2$

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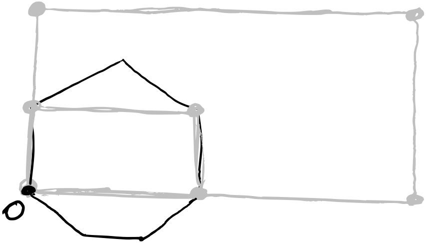


Figure: Polytope approximation in $\mathbb{R}^2$, different origin

Polytope approximation, revised

Lemma

Let $Q \subset \mathbb{R}^d$ be a polytope such that $Q \subset -dQ$. There exists a subset of at most $2d$ vertices of Q such that their convex hull Q' satisfies

$$Q \subseteq -2d^2 Q'.$$

- Natural assumption: for a convex set K, there exists an affine image K' such that $K' \subset -dK'$ (e.g. the centroid of K is the origin).
- Ivanov, Naszódi '21 proved:

$$Q \subseteq -8d^3 Q'.$$

- This statement may be interesting to study in its own right!
    - This is different from other polytope approximation problems because *we are only allowed to choose vertices of the original polytope*

Proof of key lemma I

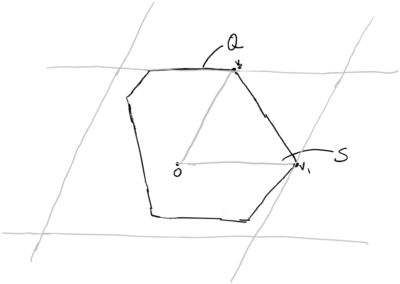


Figure: Planar case

- Among all simplices with d vertices in Q and one vertex *fixed* at the origin, choose a simplex S of *maximal volume*.
- If we replace v_1 with another point $q \in Q$, then q must lie in a "strip". Similar with $v_2, \ldots, v_d$.
- P: intersection of those strips, P is a parallelotope.
  - $Q \subseteq P$

Proof of key lemma II

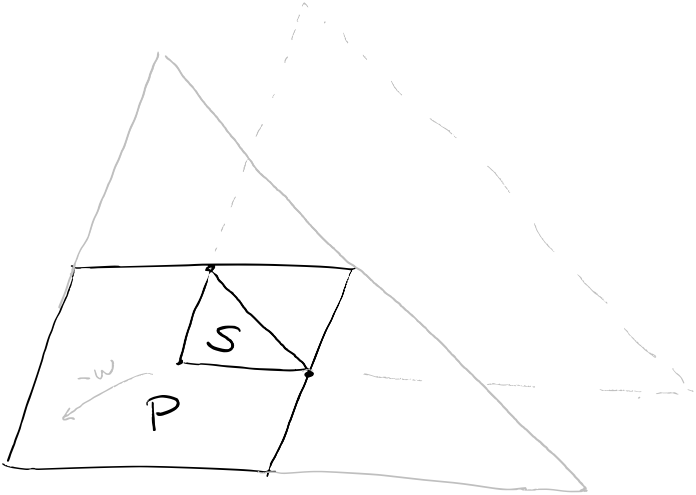


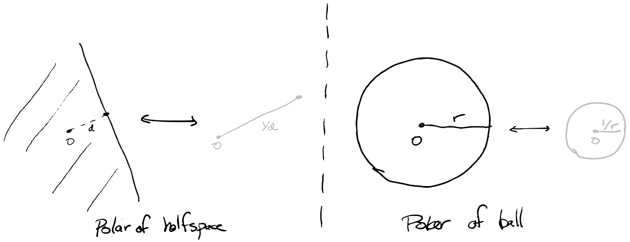
Figure: Including P in a scaled and translated copy of S

Polarity

To use our lemma, we need to build a connection between *vertices* and *convex sets*.

We define the *polar* K° of a convex set K to be $K^\circ = \{x \in \mathbb{R}^d : \langle x, y \rangle \leq 1 \text{ for all } y \in K\}$.

- Polar of a halfspace is a segment in the direction of the normal.
- Polar of a ball is another ball with reciprocal radius.

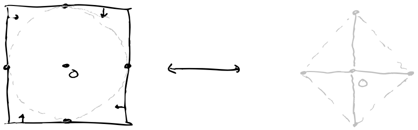


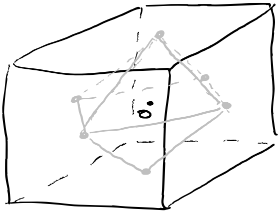
- $(P^\circ)^\circ = P$.

Polarity

- Polars of finite intersections/unions:

$$\left(\bigcap K_i\right)^\circ = \operatorname{conv}\left(\bigcup K_i^\circ\right), \quad \text{and} \quad \left(\operatorname{conv}\left\{\bigcup K_i\right\}\right)^\circ = \bigcap K_i^\circ$$





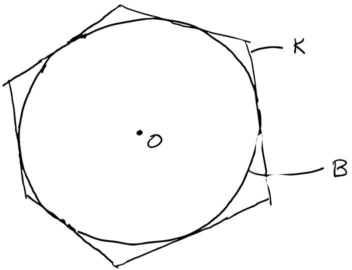
Sketch of proof

Let $\{K_1, \dots, K_n\}$ be a family of convex sets and $K = \bigcap K_i$.
There exists a subset of indices $\sigma = \{i_1, \dots, i_{2d}\}$ such that $K_\sigma = K_{i_1} \cap \dots \cap K_{i_{2d}}$ satisfies

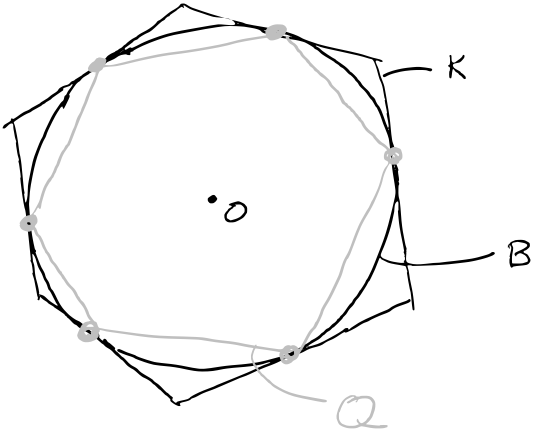
$$\mathrm{vol}_d\, K_\sigma \leq (c_1 d)^{3d/2}\, \mathrm{vol}_d\, K \quad \text{and} \quad \mathrm{diam}\, K_\sigma \leq (c_2 d)^2\, \mathrm{diam}\, K.$$

Simplifications:

- The K_i are halfspaces
- The K_i are tangent to the unit ball

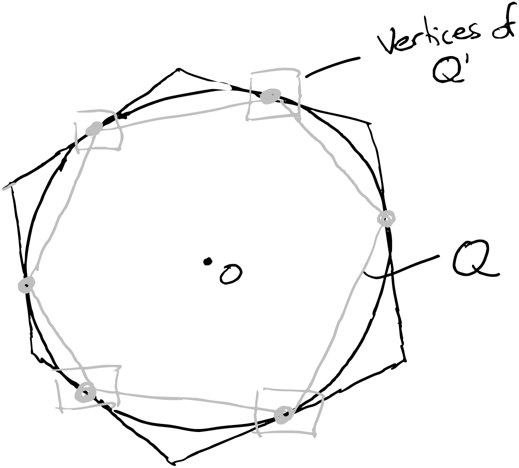


Sketch of proof II



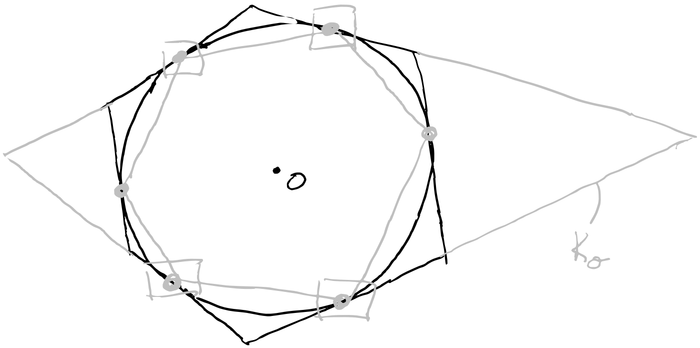
To choose $2d$ halfspaces, we instead choose $2d$ vertices from the polar $Q = K^\circ$.

Sketch of proof III



Obtain $2d$ vertices by applying approximation lemma on $Q = K^{\circ}$.

Sketch of proof IV



Take the polar once more to get the desired $2d$ halfspaces.

Thank you for your attention!