I hate geometry, change my mind

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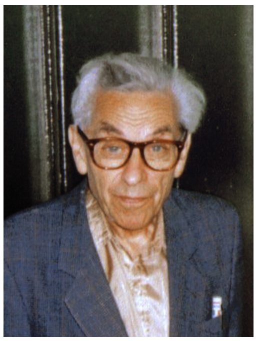

Figure 1. Paul Erdös (1913-1996)

Figure 2. George Szekeres (1911-2005)

Topics

Ramsey Theory. Suppose we have five people in a room. Can we always find 3 people who are all friends with each other or all don't know each other? Assume friendship is mutual: if A is friends with B, then B is friends with A. What if we instead have six people in the room?

We need 18 people to have a group of *four* people who either all are friends or all don't know each other. This will lead to Ramsey theory. I'll tell Paul Erdös's story about aliens and $R(6,6)$.

Polygon separation. Suppose I draw two (convex) polygons P and Q in the plane. Can I always draw a line that separates them?

- Try separating two squares; guess whether it is true for general polygons; what is a convex polygon?

A door to a colorful world. First question, if I give you a 10-sided polygon and a point x inside the polygon, can you choose 3 vertices of the polygon whose triangle contains x? This is called *Caratheodory's Theorem.*

Colorful version: If a red pentagon, a blue square, and a green triangle overlap at a point x, can you choose a vertex of each shape such that x lies in the resulting triangle?

Erdös-Szekeres Theorem. If I draw 4 points on the plane, will they necessarily be the vertices of a *convex* quadrilateral? What if I draw 5 points in the plane? How many points do you need so that you can find 5 of them that are the vertices of a pentagon?

- This problem is called the "happy ending problem" (by Erdös) because Szekeres ended up marrying his future wife Esther Klein after solving versions of this problem.

Definition. Call $f(n)$ to be the minimum number of points so that we can find n of them that form the vertices of a convex n-gon.

Conjecture. $f(n) = 2^{n-2} + 1$.

Our best estimate is by Holmsen, Mojarrad, Pach and Tardos in 2017:

$$f(n) \le 2^{n+6\sqrt{n \log n}}.$$

Incidences. Call a point-line pair (p, ℓ) an *incidence* if p lies on ℓ. Let $I(m, n)$ be the maximum number of incidences between m points and n lines in the plane. What is $I(1, m)$; what is $I(n, 1)$; prove that

$$I(m, n) < mn.$$

Theorem (Szemerédi-Trotter, 1983)**.** *There exists a positive constant c so that for every pair of integers $m, n \geq 1$,*

$$I(m, n) \leq c(m^{2/3}n^{2/3} + m + n).$$

Let $u(n)$ be the maximum number of unit distances among n points. For example, $u(3) = 3$. What is $u(4)$?

Conjecture (Erdös unit distance problem)**.** *Show that $u(n) \approx c \cdot n$ for some constant $c > 0$.*

This is a huge open problem in *discrete geometry*; we are not at all close to solving it.

Covering. Suppose we want to *cover* the plane with a finite number of copies of a polygon P.

Definition. A *covering* of the plane is a collection of copies of P that only overlap on their edges.

Can you cover the plane with triangles? With squares/pentagons/hexagons? How about n-gons for $n > 6$?

Helly's theorem.

Theorem. *If n segments are drawn in the plane such that any 2 intersect, then they all intersect.*

Theorem. *If n polygons are drawn in the plane such that any 3 intersect, then they all intersect.*

There are many variants of this theorem!