Convex Optimisation: Assignment 3

Solutions should be typeset in LaTeX
and submitted via email to `matthew.tam@unimelb.edu.au`

Due: May 13th, 2021 at 5pm

1. (2 points) Use Theorem 2.3.4 to prove Corollary 2.3.8.

2. In this exercise, we show that the proximity operator of a function f can be interpreted as the gradient of another convex function which is differentiable. This is holds independently of whether f is differentiable or not. To this end, we first introduce the following definition.

   Definition. *Let* $f\colon \mathbb{R}^n \to (-\infty, +\infty]$ *and* $\gamma > 0$. *The* Moreau envelope *of* f *with index* γ *is the function* ${}^{\gamma}f\colon \mathbb{R}^n \to [-\infty, +\infty]$ *given by*

   $$ {}^{\gamma}f(x) := \inf_{y\in\mathbb{R}^n} \left\{ f(y) + \frac{1}{2\gamma}\|x-y\|^2 \right\}. $$

   Let $f\colon \mathbb{R}^n \to (-\infty, +\infty]$ be proper, lsc, convex and let $\gamma > 0$.

   (a) (1 point) Explain why

   $$ {}^{\gamma}f(x) = f\big(\operatorname{prox}_{\gamma f}(x)\big) + \frac{1}{2\gamma}\|x - \operatorname{prox}_{\gamma f}(x)\|^2 \quad \forall x \in \mathbb{R}^n. $$

   (b) (1 point) Let $x, y \in \mathbb{R}^n$, and set $p = \operatorname{prox}_{\lambda f}(x)$ and $q = \operatorname{prox}_{\lambda f}(y)$. Explain why

   $$ f(q) - f(p) \geq \frac{1}{\gamma}\langle q - p, x - p\rangle. $$

   (c) (1 point) Using (a) and (b), show that

   $$ {}^{\gamma}f(y) - {}^{\gamma}f(x) \geq \frac{1}{\gamma}\langle y - x, x - p\rangle. $$

   (d) (2 points) With the help of (c), show that

   $$ 0 \leq {}^{\gamma}f(y) - {}^{\gamma}f(x) - \frac{1}{\gamma}\langle y - x, x - p\rangle \leq \frac{1}{\gamma}\|y - x\|^2 \quad \forall x, y \in \mathbb{R}^n. $$

   (e) (2 points) Show that ${}^{\gamma}f$ is differentiable with gradient given by

   $$ \nabla({}^{\gamma}f)(x) = \gamma^{-1}\big(x - \operatorname{prox}_{\gamma f}(x)\big) \quad \forall x \in \mathbb{R}^n. $$

3. Consider the functions $f, g\colon \mathbb{R} \to (-\infty, +\infty]$ given by

$$f(x) := \iota_{[0,1]}(x), \quad g(x) = -\sqrt{x}, \quad h(x) := \begin{cases} x \log x & x > 0 \\ 0 & x = 0 \\ +\infty & x < 0. \end{cases}$$

   (a) (2 points) Explain why f, g and h are proper, lsc, convex functions.

   (b) (3 points) Give expressions for the subdifferentials of f, g and h.

4. (3 points) Let $f\colon \mathbb{R}^n \to (-\infty, +\infty]$ be a convex function. Show that the the following three assertions are equivalent.

   (a) f has a local minimum at $\bar{x}$

   (b) f has a global minimum at $\bar{x}$.

   (c) $0 \in \partial f(\bar{x})$.

5. (1 point) Let $f\colon \mathbb{R}^n \to (-\infty, +\infty]$, let $v \in \mathbb{R}^n$ and set $h(x) := f(x) + \langle v, x\rangle$. Show that $\partial h(x) = \partial f(x) + v$ for all $x \in \mathbb{R}^n$.

6. Given a set $C \subseteq \mathbb{R}^n$, its *normal cone* at a point $\bar{x} \in \mathbb{R}$ is defined by

$$N_C(\bar{x}) := \begin{cases} \partial \iota_C(\bar{x}) & \text{if } \bar{x} \in C, \\ \emptyset & \text{otherwise.} \end{cases}.$$

   (a) (1 point) Show that $\phi \in \partial f(\bar{x})$ if and only if $(\phi, -1) \in N_{\operatorname{epi} f}(\bar{x}, f(\bar{x}))$.

   (b) (1 point) Compute the normal cone to the singleton set $\{\bar{x}\}$ at $\bar{x} \in \mathbb{R}^n$.

   (c) (1 point) Let $C \subseteq \mathbb{R}^n$ and let $\bar{x} \in \operatorname{int} C$. Compute the normal cone of C at $\bar{x}$.

   (d) (1 point) Compute the normal cone to the set

$$C = \{x \in \mathbb{R}^2 : \|x - (1,0)\| \leq 1\}$$

   at the point $(0, 0)$.

   (e) (2 points) By considering the set C from (d) and its translate

$$D = \{x \in \mathbb{R}^2 : \|x + (1,0)\| \leq 1\},$$

   give an example of two convex functions $f, g\colon \mathbb{R}^2 \to (-\infty, +\infty]$ and a point $\bar{x} \in \mathbb{R}^2$ such that

$$\partial(f + g)(\bar{x}) \neq \partial f(\bar{x}) + \partial g(\bar{x}).$$

   You must fully justify your answer.

   (f) (1 point) Draw a picture to illustrate your example from (e).