Convex Optimisation: Assignment 2

Solutions should be typeset in LaTeX
and submitted via email to `matthew.tam@unimelb.edu.au`

Due: April 15th, 2021 at 5pm

1. In this exercise, we will complete the proof of Proposition 2.2.3(b). In other words, we will establish the following result:

   Let $U \subseteq \mathbb{R}^n$ is an open, convex set, and suppose $f\colon U \to \mathbb{R}$ is twice differentiable on U. Then f is convex if and only if $\nabla^2 f(x)$ is positive semi-definite for all $x \in U$.

   In this exercise, you may find the following expression involving the Hessian of a twice differentiable function useful:

   $$\nabla^2 f(x)v = \lim_{t\to 0} \frac{\nabla f(x+tv) - \nabla f(x)}{t} \qquad \forall x \in U,\, v \in \mathbb{R}^n.$$

   (a) (1 point) Give the definition of a *(real) positive semi-definite* matrix.

   (b) Let $x \in U$ and $v \in \mathbb{R}^n$, and suppose f is convex.

      i. (1 point) Show that

      $$f(x+tv) \geq f(x) + t\langle v, \nabla f(x)\rangle$$

      for all sufficiently small $t > 0$

      ii. (2 points) Show that

      $$\left\langle v, \frac{\nabla f(x+tv) - \nabla f(x)}{t} \right\rangle \geq 0$$

      for all sufficiently small $t > 0$.

      iii. (1 point) Deduce that $\nabla^2 f(x)$ is positive semi-definite for all $x \in U$.

   (c) (1 point) Suppose that $\nabla f^2(z)$ is positive semi-definite for all $z \in U$. Let $x, y \in U$ and define $\phi(t) := f(x + t(y-x))$ for $t \in \mathbb{R}$.

      i. (1 point) Explain why ϕ is twice differentiable on $(0,1)$.

      ii. (1 point) By considering an appropriate Taylor expansion of ϕ, explain why there exists $t_0 \in (0,1)$ such that

      $$\phi(1) - \phi(0) - \phi'(0) = \frac{1}{2}\phi''(t_0).$$

iii. (1 point) Show $\phi''(t_0) = \langle y - x, H(x - y)\rangle$ where $H = \nabla^2 f(x + t_0(y - x))$.

iv. (2 points) Deduce that f is convex.

2. (2 points) Let $f\colon \mathbb{R}^n \to (-\infty, +\infty]$. Show that f is lsc if and only if $\operatorname{epi} f$ is closed.

3. (2 points) Let $C \subseteq \mathbb{R}^n$. Show that ι_C is lsc if and only if C is closed.

4. Consider the function $f\colon \mathbb{R}^n \to (-\infty, +\infty]$ given by

$$f(x) = \sum_{i=1}^{n} f_i(x_i),$$

where $f_i\colon \mathbb{R} \to (-\infty, +\infty]$ is proper, lsc and convex for all $i = 1, \ldots, n$.

(a) (1 point) Show that $\operatorname{prox}_f(x) = \left(\operatorname{prox}_{f_1}(x_1), \ldots, \operatorname{prox}_{f_n}(x_n)\right)^T$ for all $x \in \mathbb{R}^n$.

(b) (1 point) Let $g(t) = |t|$ where $t \in \mathbb{R}$ and $\lambda > 0$. Show that

$$\operatorname{prox}_{\lambda g}(t) = \begin{cases} 0 & |t| \leq \lambda \\ t - \lambda \operatorname{sign}(t) & |t| > \lambda \end{cases}$$

where sign denotes the *sign function*.

(c) (1 point) Let $\lambda > 0$. Give an expression for the proximity operator of the function $f(x) = \lambda \|x\|_1$ where $\|x\|_1 = \sum_{i=1}^{n} |x_i|$.

5. (2 points) Let $C = \mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x \geq 0\}$. Show that $P_C(x) = \{0, x\}$ where the maximum is understood elementwise.

6. (3 points) Let $S = \{x \in \mathbb{R}^n : Ax = b\}$ where $A \in \mathbb{R}^{m \times n}$ with $m < n$ and $\operatorname{rank} A = m$, and $b \in \mathbb{R}^m$. Show that

$$P_C(y) = y - A^T(AA^T)^{-1}(Ay - b).$$

7. (2 points) Let $H = \{x \in \mathbb{R}^n : \langle a, x\rangle = b\}$ where $a \in \mathbb{R}^n \setminus \{0\}$ and $b \in \mathbb{R}$. Show that

$$P_H(y) = y + \frac{b - \langle a, y\rangle}{\|a\|^2}\, a.$$