Convex Optimisation: Assignment 1

Solutions should be typeset in LaTeX
and submitted via email to `matthew.tam@unimelb.edu.au`

Due: March 25th, 2021 at 5pm

1. Consider m points $x^1, x^2, \ldots, x^m \in \mathbb{R}^2$.
   (a) (1 point) Formulate the problem of finding the smallest circle containing $x^1, \ldots, x^m$ as an optimisation problem.
   (b) (1 point) Describe the mathematical properties of the optimisation problem from (a).
2. (5 points) Prove parts (a), (b) and (c) of Exercise 2.1.4.
3. (2 points) Prove parts (b) and (c) of Proposition 2.1.5.
4. Show that the following sets/functions are convex.
   (a) (1 point) $S := \{x \in \mathbb{R}^n : Ax \leq b\}$ where $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$.
   (b) (1 point) The *second-order (or ice-cream) cone* given by
   $$K := \{(x,t) \in \mathbb{R}^n \times \mathbb{R} : \|x\| \leq t\}.$$
   (c) (1 point) $f\colon \mathbb{R}^n \to \mathbb{R}$ given by $f(x) := \frac{1}{2}\|Ax - b\|^2$ where $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$.
5. (2 points) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function which is bounded above, that is, there exists a constant $M \in \mathbb{R}$ such that $f(x) \leq M$ for all $x \in \mathbb{R}^n$. Show that f is constant.
   Hint: If $x, y \in \mathbb{R}^n$ and $t \in (0,1)$, then $x = ty + (1-t)\left(\frac{1}{1-t}x - \frac{t}{1-t}y\right)$.
6. Recall that f is locally Lipschitz around z if there exists $\delta > 0$ and $L > 0$ such that
   $$\|f(x) - f(y)\| \leq L\|x - y\|,$$
   for all $x, y \in \mathbb{B}_\delta(z) = \{x \in \mathbb{R}^n : \|x - z\| \leq \delta\}$. In this exercise, will establish the following result.

   Let $f\colon \mathbb{R}^n \to (-\infty, +\infty]$ be a convex function and $z \in \operatorname{dom} f$. Then f is locally Lipschitz around z if and only if f is bounded above on a neighbourhood of z.

(a) (1 point) In order to establish the result, it suffices to consider the case where

$$z = 0, \quad f(0) = 0 \text{ and } f(x) \leq 1 \quad \forall x \in \mathbb{B}_2(0).$$

Briefly explain why we can do this without loss of generality. (For the remainder of this exercise, you should assume that this simplifying assumption holds).

(b) (1 point) Prove the forward implication: If f is locally Lipschitz around z, then f is bounded above on a neighbourhood of z.

(c) (1 point) Show that $-f(x) \leq f(-x)$ for all $x \in \mathbb{R}^n$. *Hint:* $0 = \frac{1}{2}x + \frac{1}{2}(-x)$.

(d) (1 point) Suppose $x, y \in \mathbb{B}_1(z)$ with $x \neq y$. Show that

$$w := y + \frac{1}{\alpha}(y - x) \in \mathbb{B}_2(z) \text{ where } \alpha = \|x - y\|.$$

(e) (1 point) Combine (c) and (d) to deduce

$$f(y) - f(x) \leq \frac{\alpha}{\alpha + 1} f(-x) + \frac{\alpha}{1 + \alpha} f(w).$$

(f) (2 points) Use (e) to deduce the reverse implication: If f is bounded above on a neighbourhood of z, then f is locally Lipschitz around z.

7. Let Δ denote the *(unit) simplex* given by

$$\Delta := \{x \in \mathbb{R}^n : \sum_{i=1}^{n} x_i \leq 1, \, x \geq 0\}$$

and suppose $g\colon \Delta \to \mathbb{R}$ is convex.

(a) (1 point) Let $e_1, e_2, \ldots, e_n \in \mathbb{R}^n$ denote the standard basis vector. Show that

$$g(x) \leq \max\{g(e_1), g(e_2), \ldots, g(e_n), g(0)\} \quad \forall x \in \Delta.$$

(b) (1 point) Deduce that g is continuous on $\operatorname{int} \Delta$.

8. (2 points) Let $f\colon \mathbb{R}^n \to (-\infty, +\infty]$ be a convex function. Then f is continuous on the interior of its domain. (Hint: Use Question 7).