Using LuaLaTeX for Linear Algebra

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Example: elementary Gauss-Jordan

```
\[ A =
\directlua{

M = {
    {Rational:new(nil, 4), Rational:new(nil,12),Rational:new(nil,4),Rational:new(nil,0)},
    {Rational:new(nil, 6), Rational:new(nil,3),Rational:new(nil,-6),Rational:new(nil,9)},
    {Rational:new(nil,6),Rational:new(nil,-7),Rational:new(nil,-14),Rational:new(nil,15)},
    {Rational:new(nil,-9),Rational:new(nil,13),Rational:new(nil,23),Rational:new(nil,-24)}
    }
    tex.print(RationalAlg.MatrixToTeX(M,true))}
\]
\directlua{
    A, R = RationalAlg.GaussJordanRowReduce(M)
    tex.print(RationalAlg.RowOpListToTeX(R,2,true))
}
```

$$A = \begin{pmatrix} 4 & 12 & 4 & 0 \\ 6 & 3 & -6 & 9 \\ 6 & -7 & -14 & 15 \\ -9 & 13 & 23 & -24 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 12 & 4 & 0 \\ 6 & 3 & -6 & 9 \\ 6 & -7 & -14 & 15 \\ -9 & 13 & 23 & -24 \end{pmatrix} \xrightarrow{R_1 \leftarrow 1/4R_1} \begin{pmatrix} 1 & 3 & 1 & 0 \\ 6 & 3 & -6 & 9 \\ 6 & -7 & -14 & 15 \\ -9 & 13 & 23 & -24 \end{pmatrix}$$

$$\xrightarrow{R_2 \leftarrow R_2 - 6R_1} \begin{pmatrix} 1 & 3 & 1 & 0 \\ 0 & -15 & -12 & 9 \\ 6 & -7 & -14 & 15 \\ -9 & 13 & 23 & -24 \end{pmatrix} \xrightarrow{R_3 \leftarrow R_3 - 6R_1} \begin{pmatrix} 1 & 3 & 1 & 0 \\ 0 & -15 & -12 & 9 \\ 0 & -25 & -20 & 15 \\ -9 & 13 & 23 & -24 \end{pmatrix}$$

$$\xrightarrow{R_4 \leftarrow R_4 + 9R_1} \begin{pmatrix} 1 & 3 & 1 & 0 \\ 0 & -15 & -12 & 9 \\ 0 & -25 & -20 & 15 \\ 0 & 40 & 32 & -24 \end{pmatrix} \xrightarrow{R_2 \leftarrow -1/15R_2} \begin{pmatrix} 1 & 3 & 1 & 0 \\ 0 & 1 & 4/5 & -3/5 \\ 0 & -25 & -20 & 15 \\ 0 & 40 & 32 & -24 \end{pmatrix}$$

$$\xrightarrow{R_1 \leftarrow R_1 - 3R_2} \begin{pmatrix} 1 & 0 & -7/5 & 9/5 \\ 0 & 1 & 4/5 & -3/5 \\ 0 & -25 & -20 & 15 \\ 0 & 40 & 32 & -24 \end{pmatrix} \xrightarrow{R_3 \leftarrow R_3 + 25R_2} \begin{pmatrix} 1 & 0 & -7/5 & 9/5 \\ 0 & 1 & 4/5 & -3/5 \\ 0 & 0 & 0 & 0 \\ 0 & 40 & 32 & -24 \end{pmatrix}$$

$$\xrightarrow{R_4 \leftarrow R_4 - 40R_2} \begin{pmatrix} 1 & 0 & -7/5 & 9/5 \\ 0 & 1 & 4/5 & -3/5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Example: finding an inverse using Gauss-Jordan

```
M = {
    {Rational:new(nil,2), Rational:new(nil,3), Rational:new(nil,-2)},
    {Rational:new(nil,1), Rational:new(nil,0), Rational:new(nil,4)},
    {Rational:new(nil,5), Rational:new(nil,2), Rational:new(nil,3)}
}
```

To find the inverse of the matrix

```
\[ A = \directlua{tex.print(RationalAlg.MatrixToTeX(M))}\]
```

$$A = \begin{pmatrix} 2 & 3 & -2 \\ 1 & 0 & 4 \\ 5 & 2 & 3 \end{pmatrix}$$

First we define the augmented matrix by including a 3x3 identity matrix,

```
\directlua{MA = RationalAlg.Augment(M, RationalAlg.IdentityMatrix(3))}
\[\directlua{tex.print(RationalAlg.MatrixToTeX(MA,true))}\]
```

$$\begin{pmatrix} 2 & 3 & -2 & 1 & 0 & 0 \\ 1 & 0 & 4 & 0 & 1 & 0 \\ 5 & 2 & 3 & 0 & 0 & 1 \end{pmatrix}$$

We then row reduce this matrix using Gauss-Jordan

```
\directlua{
    _, R = RationalAlg.GaussJordanRowReduce(MA)
    tex.print(RationalAlg.RowOpListToTeX(R,2,true))
}
```

$$\begin{pmatrix} 2 & 3 & -2 & 1 & 0 & 0 \\ 1 & 0 & 4 & 0 & 1 & 0 \\ 5 & 2 & 3 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_1 \leftarrow 1/2R_1} \begin{pmatrix} 1 & {}^{3}/_{2} & -1 & {}^{1}/_{2} & 0 & 0 \\ 1 & 0 & 4 & 0 & 1 & 0 \\ 5 & 2 & 3 & 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{R_2 \leftarrow R_2 - 1R_1} \begin{pmatrix} 1 & {}^{3}/_{2} & -1 & {}^{1}/_{2} & 0 & 0 \\ 0 & -{}^{3}/_{2} & 5 & -{}^{1}/_{2} & 1 & 0 \\ 5 & 2 & 3 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_3 \leftarrow R_3 - 5R_1} \begin{pmatrix} 1 & {}^{3}/_{2} & -1 & {}^{1}/_{2} & 0 & 0 \\ 0 & -{}^{3}/_{2} & 5 & -{}^{1}/_{2} & 1 & 0 \\ 0 & -{}^{11}/_{2} & 8 & -{}^{5}/_{2} & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{R_2 \leftarrow -2/3R_2} \begin{pmatrix} 1 & {}^{3}/_{2} & -1 & {}^{1}/_{2} & 0 & 0 \\ 0 & 1 & -{}^{10}/_{3} & {}^{1}/_{3} & -{}^{2}/_{3} & 0 \\ 0 & -{}^{11}/_{2} & 8 & -{}^{5}/_{2} & 0 & 1 \end{pmatrix} \xrightarrow{R_1 \leftarrow R_1 - 3/2R_2} \begin{pmatrix} 1 & 0 & 4 & 0 & 1 & 0 \\ 0 & 1 & -{}^{10}/_{3} & {}^{1}/_{3} & -{}^{2}/_{3} & 0 \\ 0 & -{}^{11}/_{2} & 8 & -{}^{5}/_{2} & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{R_3 \leftarrow R_3 + 11/2R_2} \begin{pmatrix} 1 & 0 & 4 & 0 & 1 & 0 \\ 0 & 1 & -{}^{10}/_{3} & {}^{1}/_{3} & -{}^{2}/_{3} & 0 \\ 0 & 0 & -{}^{31}/_{3} & -{}^{2}/_{3} & -{}^{11}/_{3} & 1 \end{pmatrix} \xrightarrow{R_3 \leftarrow -3/31R_3} \begin{pmatrix} 1 & 0 & 4 & 0 & 1 & 0 \\ 0 & 1 & -{}^{10}/_{3} & {}^{1}/_{3} & -{}^{2}/_{3} & 0 \\ 0 & 0 & 1 & {}^{2}/_{31} & {}^{11}/_{31} & -{}^{3}/_{31} \end{pmatrix}$$

$$\xrightarrow{R_1 \leftarrow R_1 - 4R_3} \begin{pmatrix} 1 & 0 & 0 & -{}^{8}/_{31} & -{}^{13}/_{31} & {}^{12}/_{31} \\ 0 & 1 & -{}^{10}/_{3} & {}^{1}/_{3} & -{}^{2}/_{3} & 0 \\ 0 & 0 & 1 & {}^{2}/_{31} & {}^{11}/_{31} & -{}^{3}/_{31} \end{pmatrix} \xrightarrow{R_2 \leftarrow R_2 + 10/3R_3} \begin{pmatrix} 1 & 0 & 0 & -{}^{8}/_{31} & -{}^{13}/_{31} & {}^{12}/_{31} \\ 0 & 1 & 0 & {}^{17}/_{31} & {}^{16}/_{31} & -{}^{10}/_{31} \\ 0 & 0 & 1 & {}^{2}/_{31} & {}^{11}/_{31} & -{}^{3}/_{31} \end{pmatrix}$$

It follows that the inverse is given by

```
\directlua{l, r = RationalAlg.Split(A, 3)}
\[ A^{-1} = \directlua{tex.print(RationalAlg.MatrixToTeX(r,true))}\]
```

$$A^{-1} = \begin{pmatrix} -{}^{8}/_{31} & -{}^{13}/_{31} & {}^{12}/_{31} \\ {}^{17}/_{31} & {}^{16}/_{31} & -{}^{10}/_{31} \\ {}^{2}/_{31} & {}^{11}/_{31} & -{}^{3}/_{31} \end{pmatrix}$$

Random matrices

```
\directlua{
    M = RationalAlg.RandomMatrix(3,4,true)
    tex.print(
        "\\[ A = " .. RationalAlg.MatrixToTeX(M) .. "\\]"
    )
    _, R = RationalAlg.GaussJordanRowReduce(M)
    tex.print(RationalAlg.RowOpListToTeX(R,2,true))
}
```

$$A = \begin{pmatrix} -4 & 3 & -1 & 4 \\ 5 & -4 & -3 & -4 \\ -3 & 2 & 3 & -3 \end{pmatrix}$$

$$\begin{aligned}
&\begin{pmatrix} -4 & 3 & -1 & 4 \\ 5 & -4 & -3 & -4 \\ -3 & 2 & 3 & -3 \end{pmatrix} \xrightarrow{R_1 \leftarrow -1/4R_1} \begin{pmatrix} 1 & -3/4 & 1/4 & -1 \\ 5 & -4 & -3 & -4 \\ -3 & 2 & 3 & -3 \end{pmatrix} \\
&\xrightarrow{R_2 \leftarrow R_2 - 5R_1} \begin{pmatrix} 1 & -3/4 & 1/4 & -1 \\ 0 & -1/4 & -17/4 & 1 \\ -3 & 2 & 3 & -3 \end{pmatrix} \xrightarrow{R_3 \leftarrow R_3 + 3R_1} \begin{pmatrix} 1 & -3/4 & 1/4 & -1 \\ 0 & -1/4 & -17/4 & 1 \\ 0 & -1/4 & 15/4 & -6 \end{pmatrix} \\
&\xrightarrow{R_2 \leftarrow -4R_2} \begin{pmatrix} 1 & -3/4 & 1/4 & -1 \\ 0 & 1 & 17 & -4 \\ 0 & -1/4 & 15/4 & -6 \end{pmatrix} \xrightarrow{R_1 \leftarrow R_1 + 3/4R_2} \begin{pmatrix} 1 & 0 & 13 & -4 \\ 0 & 1 & 17 & -4 \\ 0 & -1/4 & 15/4 & -6 \end{pmatrix} \\
&\xrightarrow{R_3 \leftarrow R_3 + 1/4R_2} \begin{pmatrix} 1 & 0 & 13 & -4 \\ 0 & 1 & 17 & -4 \\ 0 & 0 & 8 & -7 \end{pmatrix} \xrightarrow{R_3 \leftarrow 1/8R_3} \begin{pmatrix} 1 & 0 & 13 & -4 \\ 0 & 1 & 17 & -4 \\ 0 & 0 & 1 & -7/8 \end{pmatrix} \\
&\xrightarrow{R_1 \leftarrow R_1 - 13R_3} \begin{pmatrix} 1 & 0 & 0 & 59/8 \\ 0 & 1 & 17 & -4 \\ 0 & 0 & 1 & -7/8 \end{pmatrix} \xrightarrow{R_2 \leftarrow R_2 - 17R_3} \begin{pmatrix} 1 & 0 & 0 & 59/8 \\ 0 & 1 & 0 & 87/8 \\ 0 & 0 & 1 & -7/8 \end{pmatrix}
\end{aligned}$$

Surds

We can also use surds and complex numbers.

```
\directlua{
    require "Surd"
    z = Surd:new(nil, 2, 3, -1)
    w = Surd:new(nil, Rational:new(nil, 2, 3), Rational:new(nil, 1, 3), -1)
    tex.print("\\[ z = " .. z:totexstring(false) .. ", \\qquad w = " .. w:totexstring(false) ..
    "\\]")
}
```

$$z = 2 + 3i, \qquad w = \frac{2}{3} + \frac{1}{3}i$$

- $z + w = \frac{8}{3} + \frac{10}{3}i$
- $z - w = \frac{4}{3} + \frac{8}{3}i$
- $zw = \frac{1}{3} + \frac{8}{3}i$
- $z/w = \frac{21}{5} + \frac{12}{5}i$

The Surd library will automatically ensure the ‘base’ is square-free. For example the following code

```
\directlua{
    a = Surd:new(nil, 2, 5, -12)
    tex.print("\\[ a = " .. a:totexstring() .. "\\]")
}
```

produces the output:

$$a = 2 + 20i\sqrt{3}$$