Von Neumann Theory of Measurement

Measurement outcomes $\{\alpha\} \leftarrow$ $\{|\alpha\rangle\}$

$\Rightarrow$ One dimensional Projection operators $\{\hat{\mathbb{P}}_\alpha = |\alpha\rangle\langle\alpha|\} \Leftarrow$

Orthonormal basis in Hilbert space $\langle\alpha|\alpha'\rangle = \delta_{\alpha\alpha'}$ $\quad \hat{\mathbb{P}}_\alpha \hat{\mathbb{P}}_{\alpha'} = \hat{\mathbb{P}}_\alpha \delta_{\alpha\alpha'}$

"Complete" $\sum_\alpha \hat{\mathbb{P}}_\alpha = \hat{\mathbb{1}}$ resolution of identity

- Born rule: Probality of outcome α
- Pure state $|\psi\rangle$: $p_\alpha = |\langle\alpha|\psi\rangle|^2 = \langle\psi|\alpha\rangle\langle\alpha|\psi\rangle = \langle\psi|\hat{\mathbb{P}}_\alpha|\psi\rangle = \langle\hat{\mathbb{P}}_\alpha\rangle$

Mixed state: $p_\alpha = \langle\alpha|\hat{\rho}|\alpha\rangle = \mathrm{Tr}(\hat{\rho}\,|\alpha\rangle\langle\alpha|) = \mathrm{Tr}(\hat{\rho}\,\hat{\mathbb{P}}_\alpha) = \langle\hat{\mathbb{P}}_\alpha\rangle$

Post-Measurement State: "Quantum Bayes' Rule"

$$\Rightarrow |\psi\rangle|_a = \frac{\hat{\mathbb{P}}_a|\psi\rangle}{\|\hat{\mathbb{P}}_a|\psi\rangle\|} = \frac{\hat{\mathbb{P}}_a|\psi\rangle}{\sqrt{\langle\psi|\hat{\mathbb{P}}_a^{+}\hat{\mathbb{P}}_a|\psi\rangle}} = \frac{\hat{\mathbb{P}}_a|\psi\rangle}{\sqrt{\langle\psi|\hat{\mathbb{P}}_a|\psi\rangle}} = \frac{\hat{\mathbb{P}}_a|\psi\rangle}{\sqrt{p_a}} \Leftarrow$$

$$= \frac{|a\rangle\langle a|\psi\rangle}{\sqrt{p_a} \to |\langle a|\psi\rangle|} = \cancel{e^{i\phi}}|a\rangle$$

$\to$ irrelevant

$$\Rightarrow \hat{\rho}|_a = \frac{\hat{\mathbb{P}}_a\hat{\rho}\hat{\mathbb{P}}_a^{+}}{\mathrm{Tr}(\quad)} = \frac{\hat{\mathbb{P}}_a\hat{\rho}\hat{\mathbb{P}}_a}{\mathrm{Tr}(\hat{\mathbb{P}}_a\hat{\rho})} = \frac{\hat{\mathbb{P}}_a\hat{\rho}\hat{\mathbb{P}}_a}{p_a} = \frac{|a\rangle\langle a|\hat{\rho}|a\rangle\langle a|}{p_a} = |a\rangle\langle a|$$

Bipartite: $\mathcal{H}_{AB} = h_A \otimes h_B$.

Born rule: $p_\alpha = \langle\Psi_{AB}|\hat{\mathbb{P}}_\alpha^{AB}|\Psi_{AB}\rangle$, Q. Bayes: $|\Psi_{AB}\rangle|_\alpha = \dfrac{\hat{\mathbb{P}}_\alpha^{AB}|\Psi_{AB}\rangle}{\sqrt{p_\alpha}}$

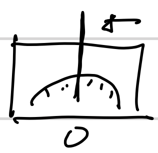
Partial Measurement on only one subsystem $\hat{\mathbb{P}}_\beta^{AB} = \hat{\mathbb{1}}^A \otimes \hat{\mathbb{P}}_\beta^B$ $\{|\beta\rangle_B\}$

$$\Rightarrow |\Psi_{AB}\rangle = \sum_{\alpha\beta} c_{\alpha\beta}|\alpha\rangle_A \otimes |\beta\rangle_B, \quad \hat{\mathbb{P}}_\beta^{AB}|\Psi_{AB}\rangle = \underbrace{\Big(\sum_\alpha c_{\alpha\beta}|\alpha\rangle_A\Big)}_{|\tilde{\psi}\rangle_A|_\beta} \otimes |\beta\rangle$$

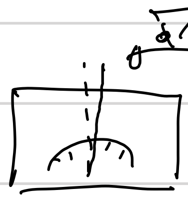
orthonormal

$$p_\beta = \langle\Psi_{AB}|\hat{\mathbb{P}}_\alpha^{AB}|\Psi_{AB}\rangle = \langle\tilde{\psi}|_\beta|\tilde{\psi}\rangle|_\beta = \sum_\alpha |c_{\alpha\beta}|^2$$

Von Neumann Measurement



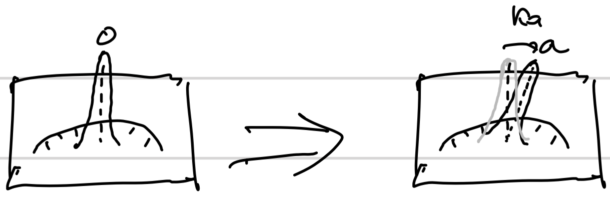
← Meter initialized at $|x=0\rangle$: Wave packet with zero width (eigenstate of $\hat{x}$)



meter at $|x=\kappa a\rangle$: Distinguishable for and a

$\Rightarrow$ Measurement backaction on system is a projection onto eigenstate of a

Suppose this is not the case. Suppose the meter is prepared in a Gaussian wave packet with finite width σ

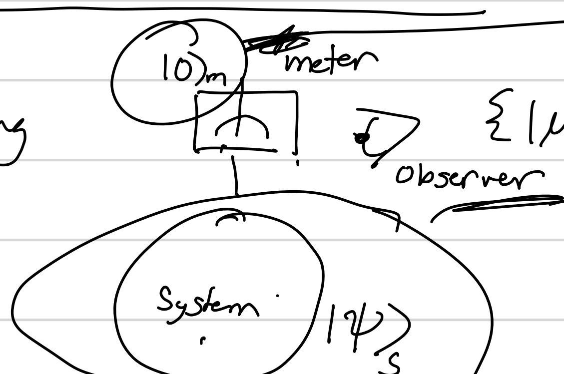


No longer distinguishable for arbitrary a

$\Rightarrow$ Measurement backaction on system will not be projection onto a eigenstate $|a\rangle$. Nonprojective measurement.

Generalized Measurement Model

Entangling Unitary $\hat{U}_{SM}$



$\{|\mu\rangle_M\}$ ←

Fiducial state $|0\rangle_M$

$$|\Psi\rangle_{SM} = \hat{U}_{SM}|\psi\rangle_S \otimes |0\rangle_M$$

Observed performs projective measurement of meter $\{|\mu\rangle_M\}$ ← Basis

$$\hat{P}^M_\mu |\Psi\rangle_{SM} = \hat{I}_S \otimes |\mu\rangle_M\langle\mu| \hat{U}_{SM}|\psi\rangle_S \otimes |0\rangle_M = \underbrace{{}_M\langle\mu|\hat{U}_{SM}|0\rangle_M}_{\hat{M}_\mu} |\psi\rangle_S \otimes |\mu\rangle_M$$

$\hat{M}_\mu$ ← Kraus operator

$$\| \hat{P}^M_\mu |\Psi\rangle_{SM}\|^2 = \|\hat{M}_\mu |\psi\rangle_S \otimes |\mu\rangle_M\|^2$$

$\Rightarrow$ Born rule $\quad p_\mu = \langle \Psi_{SM} | \hat{\mathbb{P}}^M_\mu | \Psi_{SM} \rangle = \langle \psi_s | \hat{M}^\dagger_\mu \hat{M}_\mu | \psi_s \rangle$

$\Rightarrow$ Quantum Bayes' Rule: $\quad |\Psi_{SM}\rangle|_\mu = \dfrac{\hat{\mathbb{P}}^M_\mu |\Psi_{SM}\rangle}{\sqrt{p_\mu}} = \dfrac{\hat{M}_\mu |\psi_s\rangle \otimes |\mu_M\rangle}{\sqrt{p_\mu}}$

Generalized Measurement on System: Outcomes $\{\mu\} \Rightarrow$ Kraus operator $\hat{M}_\mu$

$\Rightarrow$ Measurement Backaction: $\quad |\psi_s\rangle|_\mu = \dfrac{\hat{M}_\mu |\psi\rangle_s}{\|\hat{M}_\mu |\psi_s\rangle\|}$ $\qquad \hat{M}_\mu \Leftrightarrow \langle 0|_M \hat{U}_{SM} |\mu\rangle_M$

Born rule $\boxed{p_\mu = \langle \psi_s | \hat{E}_\mu | \psi_s \rangle}$, where $\hat{E}_\mu = \hat{M}^\dagger_\mu \hat{M}_\mu$

$\Uparrow$

Mixed states $\quad \hat{\rho}|_\mu = \dfrac{\hat{M}_\mu \hat{\rho} \hat{M}^\dagger_\mu}{p_\mu}$ $\qquad$ POVM elements

Said another way, through interaction the system and meter become entangled. $|\Psi_{SM}\rangle = \hat{U}_{SM} |\psi\rangle_s \otimes |0\rangle_M = \sum_\mu \hat{M}_\mu |\psi\rangle_s \otimes |\mu\rangle_M$

Observing the meter in $|\mu\rangle_M$, act $\hat{M}_\mu$ on $|\psi\rangle_s$. But if meter not initially in eigenstate of observable, $\hat{M}_\mu$ is not a projection operator.

Aside: A generalized measurement in quantum mechanics is one in which we can assign probability to measurement outcomes $\{\mu\} \Rightarrow \{p_\mu\}$. We can generally do this via the generalized Born rule. Given state $\hat{\rho}$, for each μ assign an operator $\hat{E}_\mu$ such that $p_\mu = \langle \hat{E}_\mu \rangle = Tr(\hat{\rho}\hat{E}_\mu)$.

$p_\mu \geq 0 \Rightarrow \hat{E}_\mu$ is a "positive operator" $\qquad \{\hat{E}_\mu \geq 0\}$

Normalization $\sum_\mu p_\mu = 1 \Rightarrow \sum_\mu \hat{E}_\mu = \hat{\mathbb{1}}$, Resolution of identity

$\Rightarrow$ POVM (Positive operator - Valued Measure)

Projective: $\hat{M}_\mu = \hat{\mathbb{P}}_\mu$, $\hat{M}_\mu^\dagger \hat{M}_\mu = \hat{\mathbb{P}}_\mu$ $\{\hat{E}_\mu\}$ $\sum_\mu \hat{E}_\mu = \hat{\mathbb{1}}$

Note: A POVM describes the most general measurements in quantum mechanics. Given a POVM, we can determine the probability of measurement outcomes. However, the POVM doest not give us the Quantum Bayes rule. For that we need a measurement model, and the corresponding Kraus operators

Then $\hat{E}_\mu = \hat{M}_\mu^\dagger \hat{M}_\mu$, $\hat{\rho}|_\mu = \dfrac{\hat{M}_\mu \hat{\rho} \hat{M}_\mu^\dagger}{P_\mu}$

CP-Map Recall a CP-Map follows from a "measurement model" when we through away the measurement result!

$$\hat{\rho}|_{out} = \mathcal{A}[\hat{\rho}] = \sum_\mu P_\mu \, \hat{\rho}|_\mu = \sum_\mu \hat{M}_\mu \hat{\rho} \hat{M}_\mu^\dagger = \mathrm{Tr}_M \left(|\Psi_{SM}\rangle\langle\Psi_{SM}| \right)$$

This is the Kraus (operator sum) representation of the CP-map. Thus, we can think about the CP-map as the "enviroment measures" the system a doesn't tell us the result. Because we don't know the result, we must take the statistical mixture of all the states conditioned on the measurement outcomes $\hat{\rho}_\mu$, weighted by the the probability that that outcome occured $P_\mu = \langle \hat{M}_\mu^\dagger \hat{M}_\mu \rangle$.

This gives us a numerical method to simulate the CP map. Given the probability distribution P_μ, we can use "Monte Carlo" to pick a random. Given that outcome μ occurs, the state is updated $\hat{\rho}|_\mu = \hat{M}_\mu \hat{\rho} \hat{M}_\mu^\dagger / P_\mu$. We run multiple times and average the results. This should converge to the CP-map

$\Rightarrow$ Quantum-Montecarlo Wave functions

|

Of course, the enivornment does not actually measure the system in any definite basis. The system and the enviroment are entangled. This is related to the fact that the Kraus decomposition of the CP-map is not unique.

$$\mathcal{A}[\hat\rho] = \sum \hat M_\mu \hat\rho \hat M_\mu^\dagger = \sum_\nu \hat N_\nu \hat\rho \hat N_\nu^\dagger = \mathrm{Tr}_{Env}\left(|\Psi\rangle_{SE}\langle\Psi|\right)$$

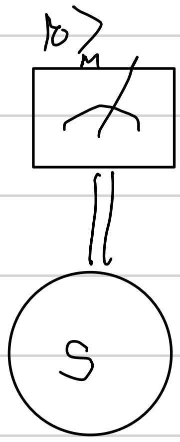
$$\Rightarrow \hat N_\nu = \sum_\mu u_{\nu\mu} \hat M_\mu$$ isometry

The enivornment can have a different record of the system in a different basis, with different backaction.

Thus, we can simulate the CP with a different measurement model

$P_\nu = \langle \hat N_\nu^\dagger \hat N_\nu \rangle$, $\hat\rho_\nu = \hat N_\nu \hat\rho \hat N_\nu^\dagger / P_\nu$. While the individual measurement records a different, the statistical average is the same. This is known as different unravelings of the CP-map

<u>Last Time:</u> General theory of measurment



$\{\mu\}$ Observer

"measures" meter, outcomes $\{\mu\}$ $\hat{\mathbb{P}}_\mu$

$$\Rightarrow |\Psi_{SM}\rangle = \hat{U}_{SM} |\psi\rangle_s \otimes |0\rangle_M = \sum_\mu \hat{M}_\mu |\psi_s\rangle \otimes |\mu\rangle$$

$$\hat{M}_\mu = \langle \mu_M | \hat{U}_{SM} | 0_M \rangle \quad \text{Kraus operator (system op.)}$$

- Probabity of finding meter in state μ: (Born Rule)

$$p_\mu = \langle \psi_s | \hat{M}_\mu^\dagger \hat{M}_\mu | \psi_s \rangle = \langle \hat{E}_\mu \rangle, \quad \underbrace{\hat{E}_\mu \equiv \hat{M}_\mu^\dagger \hat{M}_\mu \geq 0, \quad \sum_\mu \hat{E}_\mu = \mathbb{1}}_{\text{POVM}}$$

- Post Measurement State: Quantum Bayes Rule

$$|\psi_s\rangle\Big|_\mu = \frac{\hat{M}_\mu |\psi\rangle_s}{\| \hat{M}_\mu |\psi_s\rangle \|} = \frac{\hat{M}_\mu |\psi_s\rangle}{\sqrt{p_\mu}}$$

Mixed state of system $p_\mu = Tr(\hat{E}_\mu \hat{\rho})$, $\quad \hat{\rho}\big|_\mu = \dfrac{\hat{M}_\mu \hat{\rho} \hat{M}_\mu^\dagger}{p_\mu}$

What is actually being measured about this system: Observable?

Measurement model: $\hat{U}_{SM} = e^{-i\chi \hat{A}_s \otimes \hat{P}_M}$, $\quad \langle x | 0_M \rangle = \dfrac{e^{-x^2/4\sigma^2}}{(2\pi\sigma^2)^{1/4}}$

$\hat{A}_s$ "observable" on system, $\hat{P}_M$ = meter momentum

↑ Gausian wave packet, width σ

Observer measures position of meter needle: $\{|x_M\rangle\}$

$$\hat{M}_{x_M} = \langle x_M| e^{-i\chi \hat{A}\otimes \hat{P}_M} |0_M\rangle = \sum_a \langle x_M| e^{-i\chi a \hat{P}_M} |0_M\rangle\, |a\rangle_s\langle a|$$

$$= \sum_a \langle x_M - \chi a|0_M\rangle\, |a\rangle_s\langle a| = \sum_a \frac{1}{(2\pi\sigma^2)^{1/4}}\, e^{\frac{(x_M-\chi a)^2}{4\sigma^2}}\, |a\rangle_s\langle a|$$

$$\int dx_M\, \hat{M}^\dagger_{x_M} \hat{M}_{x_M} = \hat{\mathbb{1}}$$

Let $a_M = \frac{\sigma_M}{\chi}$, $\quad \hat{M}_{a_M} = \frac{\hat{M}_{x_M}}{\sqrt{\chi}} \Rightarrow \int da_M\, \hat{M}^\dagger_{a_M} \hat{M}_{a_M} = \hat{\mathbb{1}}$

$$\hat{M}_{a_M} = \sum_a \frac{1}{(2\pi\sigma_M^2)^{1/4}}\, e^{-\frac{(a-a_m)^2}{4\sigma_m^2}}\, |a\rangle_s\langle a|, \quad \sigma_M = \frac{\sigma}{\chi}$$

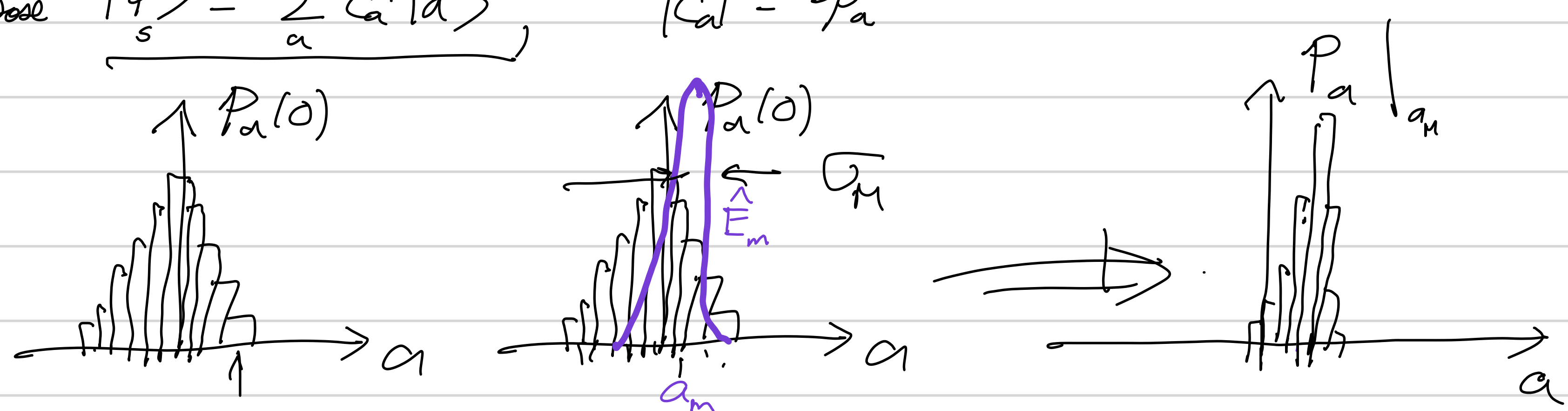
Measurement resolution

Limit $\sigma_M \to 0 \qquad e^{-\frac{(a-a_m)^2}{4\sigma_m^2}} \Rightarrow \delta(a-a_m) \qquad \hat{M}_{a_M} \Rightarrow |a_m\rangle\langle a_m|$

Projective Measurement!

Limit $\sigma_M \to \infty \qquad \hat{M}_{a_M} \Rightarrow \hat{\mathbb{1}}$, no backaction: "Weak measurement"

Suppose $|\psi\rangle_s = \sum_a c_a |a\rangle$, $\qquad |c_a|^2 = P_a$



Nonprojective measurement

Our ability to project into an eigenstate of $\hat{A}$ depends on the ability of the measuring apparatus to distinguish different outcomes $\{a\}$. Fundamental quantum uncertainty in meter degrees of freedom will limit resolution.

CP Map and Measurement

$$\hat{\rho}_{in} = |\psi\rangle_s\langle\psi| \qquad |\Psi\rangle_{SE} = \hat{U}_{SE}|\psi\rangle_s \otimes |0\rangle_E$$

$$\hat{\rho}_{out} = \mathcal{A}[\hat{\rho}_{in}] = \mathrm{Tr}_E\left(|\Psi\rangle_{SE}\langle\Psi|\right) = \sum_\mu \hat{M}_\mu^\dagger |\psi\rangle_s\langle\psi| \hat{M}_\mu$$

$$= \sum_\mu P_\mu |\psi_\mu\rangle\langle\psi_\mu| \qquad P_\mu = \langle\psi|\hat{M}_\mu^\dagger \hat{M}_\mu|\psi\rangle, \quad |\psi_\mu\rangle = \frac{\hat{M}_\mu|\psi\rangle_s}{\sqrt{P_\mu}}$$

$\Rightarrow$ Open quantum system CP-map: Enviroment does a (generalized) measurement on the system, but does not tell us the result. The output state is prepared in $|\psi_\mu\rangle$ conditioned on the measurement outcome, which occurs with probability $P_\mu = \| |\psi_\mu\rangle \|^2 = \langle\psi|\hat{M}_\mu^\dagger \hat{M}_\mu|\psi\rangle$.

Nonuniqueness of ensemble decomposition

$$\hat{\rho}_{out} = \sum_\mu P_\mu |\psi_\mu\rangle\langle\psi_\mu| = \sum_\nu q_\nu |\phi_\nu\rangle\langle\phi_\nu|$$

$\Rightarrow$ Enviroment could preform different measurements, but since we do not know the measurement outcome, our state assignment must be the same, independent of this measurement. CP map same once we average over measurement outcomes with the proper probability

$\equiv$ Nonuniques of Kraus rep: $\mathcal{A}[\rho] = \sum_\mu \hat{M}_\mu \hat{\rho} \hat{M}_\mu^\dagger = \sum_\nu \hat{N}_\nu \hat{\rho} \hat{N}_\nu$

$$\hat{N}_\nu = \sum_\mu U_{\nu\mu} \hat{M}_\mu \qquad \text{(isometry)}$$

Monte-Carlo Simulations. CP Map on $|\psi\rangle$

Pick random number according to $P_\mu = \langle\psi|\hat{M}_\mu^\dagger \hat{M}_\mu|\psi\rangle$

$$|\psi\rangle_\mu = \hat{M}_\mu|\psi\rangle / \sqrt{P_\mu}$$

Repeat N times $\hat{\rho} = \lim_{N\to\infty} \frac{1}{N}\sum_\mu |\psi_\mu^{(i)}\rangle\langle\psi_\mu^{(i)}| \Rightarrow \sum_\mu P_\mu |\psi_\mu\rangle\langle\psi_\mu|$

$$|\psi(t)\rangle = c_g(t)|g\rangle + c_e(t)|e\rangle$$

$$|\tilde{\psi}(t+dt)\rangle = e^{-\frac{i}{\hbar}H_{eff}\,dt}|\psi(t)\rangle = c(t)|g\rangle + e^{-i\omega_{eg}dt - \frac{\Gamma}{2}dt}c_e(t)|e\rangle$$

$$\| |\tilde{\psi}(t+dt)\rangle \| = \sqrt{|c_g|^2 + e^{-\Gamma dt}|c_e|^2} \simeq \sqrt{|c_g|^2 + (1-\Gamma dt)|c_e|^2}$$
$$= \sqrt{1-\Gamma|c_e|^2 dt} = 1 - \frac{\Gamma}{2}|c_e|^2 dt$$

$$|\psi(t+dt)\rangle = \frac{|\tilde{\psi}(t+dt)\rangle}{\| |\tilde{\psi}(t+dt)\rangle \|}$$

$$= \left(1+\frac{\Gamma}{2}|c_e|^2 dt\right)c_g(t)|g\rangle + e^{-i\omega_{eg}dt}\left(1-\frac{\Gamma}{2}dt+\frac{\Gamma}{2}|c_e|^2 dt\right)c_e(t)|e\rangle$$

$$|\psi(t+dt)\rangle = \left(1+\frac{\Gamma}{2}|c_e|^2 dt\right)c_g(t)|g\rangle + e^{i\omega_{eg}dt}\left(1-\frac{\Gamma}{2}|c_g|^2 dt\right)c_e(t)|e\rangle$$

No jump is important information!

Generally: $\| e^{-i\hat{H}_{eff}dt}|\psi(t)\rangle\|^2$ = Probability no jump $t \to t+dt$

Let $P(t)$ = Probability no jump $0 \to t$

$P(t+dt) = P(t) * $(Probability no jump $t \to t+dt$ | no jump $0 \to t$)

$= P(t)\| e^{-iH_{eff}dt}|\psi(t)\rangle\|^2$ given $|\psi(t)\rangle = \dfrac{e^{-iH_{eff}t}|\psi(0)\rangle}{\| e^{-iH_{eff}t}|\psi(0)\rangle\|}$

$$\Rightarrow P(t+dt) = P(t)\frac{\| e^{-iH_{eff}(t+dt)}|\psi(0)\rangle\|^2}{\| e^{-iH_{eff}t}|\psi(0)\rangle\|^2}$$

$$\Rightarrow \quad \frac{P(t+dt)}{\| e^{-iH_{eff}(t+dt)} |\psi(0)\rangle \|^2} = \frac{P(t)}{\| e^{-iH_{eff} t} |\psi(0)\rangle \|}$$

$\Rightarrow$ Probability no jump $0 \to t$

$$\boxed{P(t) = \| e^{-iH_{eff} t} |\psi(0)\rangle \|^2}$$

Eg. $|\psi(0)\rangle = c_g |g\rangle + c_e |e\rangle$

$$P(t) = \left| c_g |g\rangle + e^{-i\omega_{eg} t} e^{-\frac{\Gamma t}{2}} c_e |e\rangle \right|^2$$

$$= |c_g|^2 + e^{-\Gamma t} |c_e|^2$$

As $t \to \infty$ prob no jump $= |c_g|^2$

$\to$ Probability atom was in the ground state!

Without updating state, conditioned on the null result,

Prob to jump in any interval $dp = \Gamma dt\, |c_e|^2$ $\to$ Would always eventually jump!