MINVO Basis: Finding Simplexes with Minimum Volume Enclosing Polynomial Curves

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***Abstract*—Applications such as computer graphics rendering, path planning for robotics and finite element simulations greatly benefit from having an n-simplex that encloses a given polynomial curve. Bézier curves (which use the Bernstein basis) are probably the most common choice for this, since it is guaranteed that each interval of the curve is contained inside the convex hull of its control points. However, it is known that this convex hull is not the one with smallest volume, causing therefore great conservatism in all the applications mentioned before. This project derives and presents the MINVO basis, a polynomial basis that generates the smallest n-simplex that encloses any given polynomial curve. We show that the MINVO basis is able to obtain a volume that is 2.36 times smaller when $n = 3$, and 903 times smaller when $n = 7$. Global optimality is discussed and proven for some n using Sum-Of-Squares (SOS) Programming and moment relaxations.**

SUPPLEMENTARY MATERIAL

The MATLAB code used for this project can be downloaded from the following link:

https://www.dropbox.com/s/lro345qygg7bd7j/code_project_Jesus_Tordesillas_Torres.zip?dl=0

I. INTRODUCTION

Bézier curves are extensively used in many different applications, which include computer aided design ([1], [2], [3], [4], [5]), simulations and animations ([6], [7]), and robotics ([8], [9], [10], [11], [12], [13], [14]), just to mention some. The ability to approximate the space occupied by a given polynomial curve with a polyhedron is crucial for many of these applications, specially those that need to check the intersection between two objects in real time. For instance, many path planning algorithms that use Bézier curves can easily ensure safety by ensuring that the control points of the trajectory are inside the sequence of overlapping polyhedra that define the free space (see [12], [13], [14] for instance). This allows to impose a finite number of constraints (of the form $c(x) \leq 0$), instead of an infinite number of constraints (of the form $c(x, t) \leq 0 \; \forall t$).

The basis for the Bézier curves are the Bernstein polynomials, but, while having some other useful properties, they are not designed to minimize the volume of the simplex that encloses a given interval of the curve. This directly translates into an unnecessary conservatism in many of the applications mentioned above. The goal of this project is to obtain a polynomial basis that can be used to generate the simplex with minimum volume that encloses a polynomial curve.

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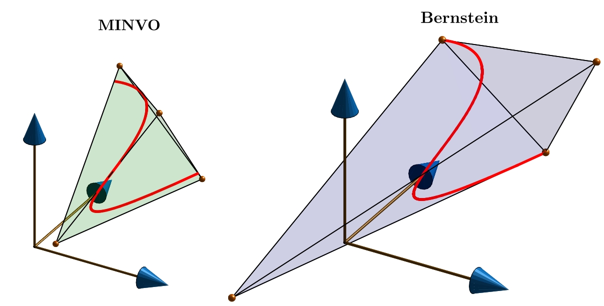

Figure 1: Comparison between the MINVO basis and the Bernstein basis for $n = 3$. The simplex found using the MINVO basis (left) encloses the 3^{rd}-order polynomial curve in red. For any 3^{rd}-order polynomial curve, the MINVO basis generates a simplex that is 2.36 times smaller that the one found using the Bernstein basis (right).

The contributions of this project are summarized as follows:

- Formulation of the optimization problem that solves, without loss of generality, for the smallest n-simplex that encloses a given polynomial curve. Global optima for $n = 1$ is obtained using moment relaxations and semidefinite programming. Local optima for $n = 2, 3, 4$ and feasible solutions for $n = 5, 6, 7$ are also obtained.
- A more tractable formulation that imposes a specific structure in the polynomials. With this formulation, global optima are obtained for $n = 1, 2, 3$ (using also moment relaxations and semidefinite programming), and local optima are obtained for $4, 5, 6, 7$.

II. RELATED WORK

A work that attempted to solve exactly the same problem proposed in this was the work by Gary Herron [15]. In this work, the author imposed a specific structure in the form for the polynomials of the basis, and then solved the associated nonconvex optimization problem over the roots of those polynomials. For this specific form of the polynomials, a global minimizer was found for $n = 2$, and local minimizer was found for $n = 3$, but no global optimality was proven for this. These project goes further, and using SOS programming, it first proposes the most general formulation that does not impose any specific structure in the form of the polynomials (apart from being SOS), proving global and local optimality for some n. Then, by imposing a structure in the polynomials, this project is able to find global optima for $n = 1, 2, 3$ and local optima $n = 4, 5, 6, 7$.

A similar problem has also been studied in the hyperspectral unmixing problem inside the remote sensing community, which consists of trying to find, in a given image pixel, the proportions (or abundances) of each macroscopic materials (endmembers) contained in that pixel [16]. One way to address this problem is to find the simplex with minimum volume that encloses a set of points (see [17] for instance). However, the fact of dealing with many data points, and not necessarily distributed along a polynomial curve, leads to the need of different iterative algorithms to solve it ([18], [19], [20]).

III. Notation and Definitions

This project will use the following notation:

Symbol	Meaning
a	Scalar
$\boldsymbol{a}$	Column vector
$\boldsymbol{A}$	Matrix
A	Set of points
$conv(A)$	Convex hull of the set A
$\lvert\boldsymbol{A}\rvert$	Determinant of $\boldsymbol{A}$
$abs(a)$	Absolute value of a
$\propto$	Proportional to
$\cdot_{m\times n}$	Size of a matrix (m rows $\times$ n columns)
$\boldsymbol{a} \geq \boldsymbol{b}$	Element-wise inequality
$\mathbf{1}$	Column vector of ones
$\mathbf{0}$	Column vector of zeros
$\boldsymbol{e}$	$\begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix}^T$
$\boldsymbol{t}$	$\begin{bmatrix} t^n & t^{n-1} \cdots & 1 \end{bmatrix}^T$ (n given by the context)
$\hat{\boldsymbol{t}}$	$\begin{bmatrix} 1 & \cdots & t^{n-1} & t^n \end{bmatrix}^T$ (n given by the context)
$\mathbb{S}_+^m$	Positive semidefinite cone (set of all symmetric positive semidefinite $m \times m$ matrices)
$vol(\cdot)$	Volume of a polyhedron
$dist(\boldsymbol{a}, \boldsymbol{\pi})$	Distance between the point $\boldsymbol{a}$ and the plane $\boldsymbol{\pi}$

Let us also introduce the two following common definitions and their respective notations:

n-simplex: Convex hull of $n+1$ points $\boldsymbol{v}_0, ..., \boldsymbol{v}_n \in \mathbb{R}^n$. These points are called the **vertexes** of the simplex. The letter S will denote simplex, while S^n will denote the set of all possible n-simplexes. A simplex S is often characterized by its **matrix of vertexes** $\boldsymbol{V} := \begin{bmatrix} \boldsymbol{v}_0 & \cdots & \boldsymbol{v}_n \end{bmatrix}$. A simplex whose vertexes are $\{\mathbf{0}, \boldsymbol{e}_1, \boldsymbol{e}_2, ..., \boldsymbol{e}_n\}$, where $\boldsymbol{e}_i$ are the vectors of the standard basis, is called a **standard simplex**.

Polynomial curve of degree n and dimension k: Parametric curve $\boldsymbol{p}(t) := \begin{bmatrix} p_0(t) \\ \vdots \\ p_{k-1}(t) \end{bmatrix} := \boldsymbol{P}\boldsymbol{t}$, where $p_i(t) = \boldsymbol{c}_i^T \boldsymbol{t}$ is a polynomial of degree n. In this project, we will use the polynomial curves of the form $k = n$ and we will refer to them simply as n-th order polynomial curves. The matrix $\boldsymbol{P}$ is the coefficient matrix, and will be denoted as $\boldsymbol{P} := \begin{bmatrix} \boldsymbol{c}_0^T \\ \vdots \\ \boldsymbol{c}_{k-1}^T \end{bmatrix}_{k\times n} \equiv \begin{bmatrix} \boldsymbol{p}_n & \cdots & \boldsymbol{p}_0 \end{bmatrix}_{k\times n}$.

IV. Problem definition

As discussed in the introduction, the goal of this project is to investigate the following optimization problem:

Problem 1: Given an n-th order polynomial curve $\boldsymbol{p}(t)$, find the n-simplex S with minimum volume that contains $\boldsymbol{p}(t)$ in the interval $t \in [-1, 1]$. In other words:

$$\min_{S\in S^n} \quad f_1 := vol(S) \propto abs\left(\left|\begin{bmatrix} \boldsymbol{V}^T & \mathbf{1} \end{bmatrix}\right|\right)$$

$$s.t. \quad \boldsymbol{p}(t) \in S \quad \forall t \in [-1, 1]$$

For $n = 2$, Problem 1 tries to find the triangle with the smallest area that contains the curve $\begin{bmatrix} x(t) & y(t) \end{bmatrix}^T$, where $x(t)$ and $y(t)$ are 2^{nd}-degree polynomials. For $n = 3$, it tries to find the tetrahedron with the smallest volume that contains the curve $\begin{bmatrix} x(t) & y(t) & z(t) \end{bmatrix}^T$(all 3^{rd}-degree polynomials). Without loss of generality, we will assume throughout the paper that the polynomial curve $\boldsymbol{p}(t)$ is not contained in a subspace $\mathbb{R}^m$, with $m < n$. For $n = 3$, for instance, this means that the curve is not contained in a plane. If $\boldsymbol{p}(t)$ is contained in a subspace $\mathbb{R}^m$ ($m < n$), then we can solve the problem by simply doing a change of variables and solve the problem directly in $\mathbb{R}^m$.

Another way to understand Problem 1 is using the convex hull of the curve $\boldsymbol{p}(t)$: the solution of Problem 1 will give the smallest simplex that completely contains the convex hull of the polynomial curve $\boldsymbol{p}(t)$ (see Fig. 2)

As we will prove later in the report, Problem 1 is closely connected this other optimization problem:

Problem 2: Given the vertexes $\boldsymbol{v}_0, ..., \boldsymbol{v}_n$ of an n-simplex S, find the n-th order polynomial curve $\boldsymbol{p}(t)$ contained in S, whose coefficients vectors $\boldsymbol{p}_n, ..., \boldsymbol{p}_1$ span a parallelogram with largest volume. In other words:

$$\min_{\boldsymbol{p}(t)} \quad f_2 := -abs\left(\left|\begin{bmatrix} \boldsymbol{p}_n & \cdots & \boldsymbol{p}_1 \end{bmatrix}\right|\right) = -abs\left(\left|\begin{bmatrix} \boldsymbol{P} \\ \boldsymbol{e}^T \end{bmatrix}\right|\right)$$

$$s.t. \quad \boldsymbol{p}(t) \in S \quad \forall t \in [-1, 1]$$

Similar to the previous case, without loss of generality we will assume that the simplex S is nondegenerate (it is not contained in $\mathbb{R}^m$, with $m < n$).

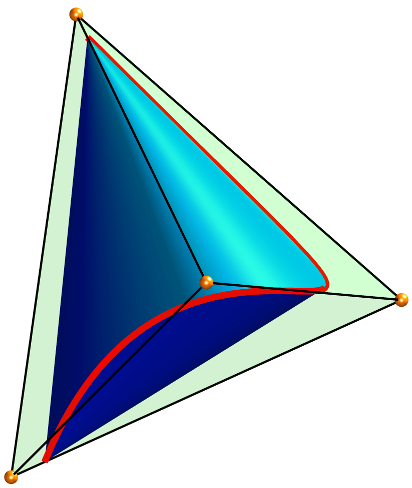

Figure 2: The smallest simplex that encloses a polynomial curve (shown in red) is also the smallest one that completely contains its convex hull (shown in blue).

V. Equivalent Formulation

Let us now study the constraints and the objective functions of Problems 1 and 2.

A. Constraints of Problem 1 and 2:

Both problems share the same constraint $\boldsymbol{p}(t) \in S \quad \forall t \in [-1,1]$, which is equivalent to $\boldsymbol{p}(t)$ being a convex combination of the vertexes $\boldsymbol{v}_i$ of the simplex for $t \in [-1,1]$:

$$\left.\begin{array}{c} \boldsymbol{p}(t) = \sum_{i=0}^{n} \lambda_i(t)\boldsymbol{v}_i \\ \sum_{i=0}^{n} \lambda_i(t) = 1 \quad \forall t \\ \lambda_i(t) \geq 0 \quad \forall i = 0,..,n \quad \forall t \in [-1,1] \end{array}\right\}$$

Taking $\lambda_i(t)$ as an n-th degree polynomial $\lambda_i(t) := \boldsymbol{\lambda}_i^T \boldsymbol{t}$, and matching the coefficients we have that:

$$\boldsymbol{P} = \boldsymbol{V} \underbrace{\begin{bmatrix} \boldsymbol{\lambda}_0^T \\ \boldsymbol{\lambda}_1^T \\ \vdots \\ \boldsymbol{\lambda}_n^T \end{bmatrix}}_{:=\boldsymbol{A}}$$

where we have also defined the $(n+1) \times (n+1)$ matrix $\boldsymbol{A}$. The i-th row of this matrix $\boldsymbol{A}$ contains the coefficients of the polynomial $\lambda_i(t)$. We can also write $\boldsymbol{V}^T = \left(\boldsymbol{P}\boldsymbol{A}^{-1}\right)^T = \boldsymbol{A}^{-T}\boldsymbol{P}^T$.

The geometric interpretation of each $\lambda_i(t)$ is as follows. First note that

$$\boldsymbol{p}(t) = \sum_{i=0}^{n} \lambda_i(t)\boldsymbol{v}_i = \boldsymbol{v}_n + \sum_{i=0}^{n-1} \lambda_i(t)\left(\boldsymbol{v}_i - \boldsymbol{v}_n\right)$$

Defining now $\boldsymbol{n}$ as the normal vector of the hyperplane $\boldsymbol{\pi}$ formed by the points $\boldsymbol{v}_1$,$\boldsymbol{v}_2$,...,$\boldsymbol{v}_n$, and satisfying $\boldsymbol{n}^T\boldsymbol{p}(t) \geq 0$ (i.e. pointing towards the interior of the simplex), we have that:

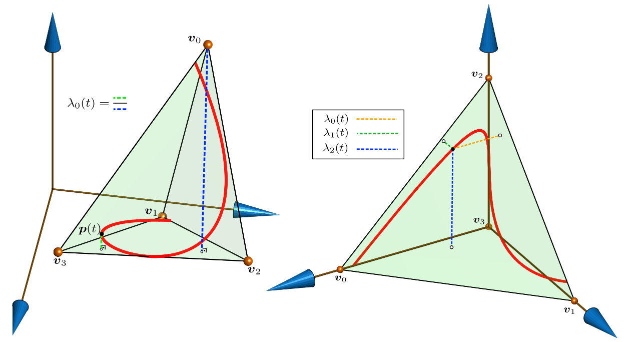

Figure 3: Geometric interpretation of $\lambda_i(t)$. Each $\lambda_i(t)$ represents the distance between the vertex $\boldsymbol{v}_i$ and the plane formed by the vertexes $\boldsymbol{v}_i$, $i \neq j$, normalized with the distance from $\boldsymbol{v}_i$ to that plane (left). For a standard simplex in 3D (right), the curve in red is $\boldsymbol{p}(t) = \begin{bmatrix} \lambda_0(t) & \lambda_1(t) & \lambda_2(t) \end{bmatrix}^T$.

$$dist\left(\boldsymbol{p}(t), \boldsymbol{\pi}\right) := \left(\boldsymbol{p}(t) - \boldsymbol{v}_n\right)^T \boldsymbol{n} =$$

$$= \sum_{i=0}^{n-1} \lambda_i(t)\left(\boldsymbol{v}_i - \boldsymbol{v}_n\right)^T \boldsymbol{n} = \lambda_0(t)\; dist\left(\boldsymbol{v}_0, \boldsymbol{\pi}\right)$$

And therefore:

$$\lambda_0(t) = \frac{dist\left(\boldsymbol{p}(t), \boldsymbol{\pi}\right)}{dist\left(\boldsymbol{v}_0, \boldsymbol{\pi}\right)}$$

A similar reasoning applies for the other $\lambda_i(t)$. Hence, each λ_i represents the ratio between the distance of the curve to the hyperplane $\boldsymbol{\pi}$ (formed by the points $\boldsymbol{v}_j$, $j \neq i$) and the distance from $\boldsymbol{v}_i$ to that hyperplane $\boldsymbol{\pi}$ (see Fig. 3 for the case $n = 3$).

With respect to the second constraint $\sum_{i=0}^{n} \lambda_i(t) = 1$, it can be rewritten as $\boldsymbol{A}^T\boldsymbol{1} = \boldsymbol{e}$ (or equivalently $\boldsymbol{e}^T\boldsymbol{A}^{-1} = \boldsymbol{1}^T$).

B. Objective Function of Problem 1

Given the previous relationships, we have that

$$\left|\begin{bmatrix} \boldsymbol{V}^T & \boldsymbol{1} \end{bmatrix}\right| = \left|\begin{bmatrix} \boldsymbol{A}^{-T}\boldsymbol{P}^T & \boldsymbol{1} \end{bmatrix}\right| =$$

$$= \left|\boldsymbol{A}^{-T}\right| \left|\begin{bmatrix} \boldsymbol{P}^T & \underbrace{\boldsymbol{A}^T\boldsymbol{1}}_{=\boldsymbol{e}} \end{bmatrix}\right| \propto \left|\boldsymbol{A}^{-1}\right| = \frac{1}{|\boldsymbol{A}|}$$

where we have used the fact that everything inside $\begin{bmatrix} \boldsymbol{P}^T\boldsymbol{e} \end{bmatrix}$ is given (i.e. not a decision variable of the optimization problem), and the fact that $|\boldsymbol{A}| = \left|\boldsymbol{A}^T\right|$. Therefore, we conclude that $f_1 \propto \frac{1}{abs(|\boldsymbol{A}|)}$. We can therefore maximize $abs(|\boldsymbol{A}|)$, or, equivalently, minimize $-abs\left(|\boldsymbol{A}|\right)$.

C. Objective Function of Problem 2

Similarly to the previous subsection, we have that

$$\left|\begin{bmatrix} \boldsymbol{P} \\ \boldsymbol{e}^T \end{bmatrix}\right| = \left|\begin{bmatrix} \boldsymbol{V} \\ \underbrace{\boldsymbol{e}^T \boldsymbol{A}^{-1}}_{=\boldsymbol{1}^T} \end{bmatrix} \boldsymbol{A}\right| \propto |\boldsymbol{A}|$$

And therefore, $f_2 \propto -abs\left(|\boldsymbol{A}|\right)$

D. Equivalent Formulation for Problem 1 and 2

Therefore, we can solve Problem 1 (or Problem 2 respectively) for **any** given polynomial curve (**any** given simplex respectively) by solving the following problem:

Problem 3:

$$\min_{\boldsymbol{A}\in\mathbb{R}^{n+1}} \quad f_3 := -abs\left(|\boldsymbol{A}|\right)$$

$$s.t. \quad \boldsymbol{A}\boldsymbol{t} \geq \boldsymbol{0} \quad \forall t \in [-1,1]$$
$$\boldsymbol{A}^T\boldsymbol{1} = \boldsymbol{e}$$

As the objective function of Problem 3 is the determinant of a nonsymmetric matrix, it is clearly a nonconvex problem. For the first constraint of Problem 3, we can use the following result of Sum-Of-Squares programming, to rewrite that constraint and leave positive semidefinite matrices as decision variables [21]:

- If n is odd, $\lambda_i(t) \geq 0 \;\; \forall t \in [-1,1]$ if and only if
$$\begin{cases} \lambda_i(t) = \hat{\boldsymbol{t}}^T\left((t+1)\boldsymbol{W}_i + (1-t)\boldsymbol{V}_i\right)\hat{\boldsymbol{t}} \\ \boldsymbol{W}_i \in \mathbb{S}_+^{\frac{n+1}{2}}, \boldsymbol{V}_i \in \mathbb{S}_+^{\frac{n+1}{2}} \end{cases}$$
- If n is even, $\lambda_i(t) \geq 0 \;\; \forall t \in [-1,1]$ if and only if
$$\begin{cases} \lambda_i(t) = \hat{\boldsymbol{t}}^T\boldsymbol{W}_i\hat{\boldsymbol{t}} + \hat{\boldsymbol{t}}^T(t+1)(1-t)\boldsymbol{V}_i\hat{\boldsymbol{t}} \\ \boldsymbol{W}_i \in \mathbb{S}_+^{\frac{n}{2}+1}, \boldsymbol{V}_i \in \mathbb{S}_+^{\frac{n}{2}} \end{cases}$$

Note that the *if and only if* condition applies because it is a univariate polynomial [21]. We end up therefore with a nonconvex finite-dimensional optimization problem, where the decision variables are the positive semidefinite matrices $\boldsymbol{W}_i$ and $\boldsymbol{V}_i$, $i = 0, ..., n$.

No generality has been lost in Problem 3. However, this problem easily becomes intractable for large n. To make it more tractable, we can try to reduce the number of decision variables of Problem 3 by imposing a structure in $\lambda_i(t)$. As Problem 1 is trying to minimize the volume of the simplex, we can impose that the planes of the n-simplex be tangent to the curve [22], [23]. Using the geometric interpretation of the $\lambda_i(t)$ given in subsection V-A, this means that each $\lambda_i(t)$ should have either real double roots (where the curve is tangent to the plane), and/or roots at $t = \pm 1$. This translates into the formulation shown in Problem 4.

The relationship between Problems 1, 2, 3 and 4 is given in Fig. 4. First note that the structure imposed for $\lambda_i(t)$ in Problem 4 guarantees that they are nonnegative for $t \in [-1, 1]$. Hence the feasible set of Problem 4 is contained

Problem 4:

$$\min_{\boldsymbol{A}} \quad f_4 := -abs\left(|\boldsymbol{A}|\right)$$

$$s.t:$$

<u>If n is odd:</u>

$$\lambda_i(t) = -b_i(t-1)\prod_{j=1}^{\frac{n-1}{2}}(t-t_j)^2 \quad i = 0, 2, ..., n-1$$
$$\lambda_i(t) = \lambda_{n-i+2}(-t) \qquad i = 1, 3, ..., n$$
$$b_i \geq 0 \quad i = 0, 2, ..., n-1$$
$$\textstyle\sum_{i=0}^{i=n}\lambda_i(t) = 1$$

<u>If n is even:</u> define $\mathcal{I}_a$ the set of odd integers $\in \left[0, \frac{n}{2}-1\right]$, and $\mathcal{I}_b$ the set of even integers $\in \left[0, \frac{n}{2}-1\right]$:

$$\lambda_i(t) = -b_i(t+1)(t-1)\prod_{j=1}^{\frac{n-2}{2}}(t-t_j)^2 \quad i \in \mathcal{I}_a$$
$$\lambda_i(t) = b_i\prod_{j=1}^{\frac{n}{2}}(t-t_j)^2 \qquad i \in \mathcal{I}_b$$
$$\lambda_j(t) = \lambda_{n-i+1}(-t) \qquad i = \tfrac{n}{2}+1, ..., n$$
$$b_i \geq 0 \quad i \leq \tfrac{n}{2}$$
$$\textstyle\sum_{i=0}^{i=n}\lambda_i(t) = 1$$

- If $\frac{n}{2}$ is odd ($n = 2, 6, 10...$):
$$\lambda_i(t) = -b_i(t+1)(t-1)\prod_{j=1}^{\frac{n-2}{4}}(t-t_j)^2(t+t_j)^2 \qquad i = \frac{n}{2}$$
- If $\frac{n}{2}$ is even ($n = 4, 8, 12...$):
$$\lambda_i(t) = b_i\prod_{j=1}^{\frac{n}{4}}(t-t_j)^2(t+t_j)^2 \qquad i = \frac{n}{2}$$

in the feasible set of Problem 3 or, in other words, any feasible solution of Problem 4 is feasible for Problem 3 (the converse is not true in general). Therefore, $f_3^* \leq f_4^*$ holds. The matrix $\boldsymbol{A}^*$ found in Problem 3 or 4 can be used to find the vertexes of the simplex in Problem 1 (by simply using $\boldsymbol{V}^* = \boldsymbol{P}\left(\boldsymbol{A}^*\right)^{-1}$ where $\boldsymbol{P}$ is the coefficient matrix of the polynomial given), or to find the polynomial matrix in Problem 2 (by using $\boldsymbol{P}^* = \boldsymbol{V}\boldsymbol{A}^*$, where $\boldsymbol{V}$ contains the vertexes of the simplex given).

VI. Results

The optimization Problems 3 and 4 have been solved using the nonconvex solvers *IPOPT* [24] and *Knitro* [25], [26] (for Problem 3), the *fmincon* nonconvex solver [27] (for Problem 4), and the *Yalmip* interface [28], [29].

We were able to find local minima for Problem 4 for $n = 1, 2, 3, 4, 5, 6, 7$, and the same local minima were found in Problem 3 for $n = 1, 2, 3, 4$ (Problem 3 becomes

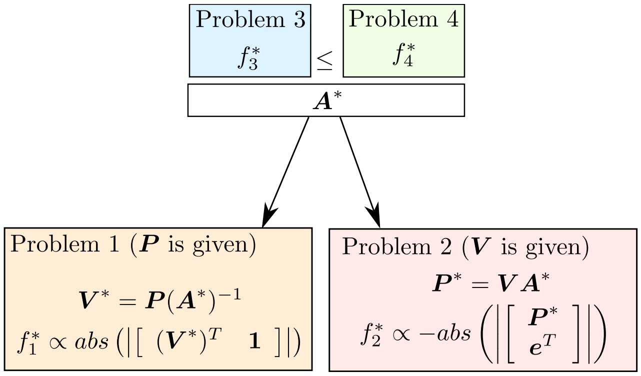

Figure 4: Relationship between Problems 1, 2, 3 and 4: Problem 4 and Problem 3 have the same objective function, but the feasible set of Problem 4 is contained in the feasible set of Problem 3, and therefore $f_3^* \leq f_4^*$. Both Problem 3 and 4 generate a (potentially locally) optimal solution $\boldsymbol{A}^*$, which can be applied to any polynomial curve to find the simplex in Problem 1, or to any simplex to find the polynomial curve in Problem 2.

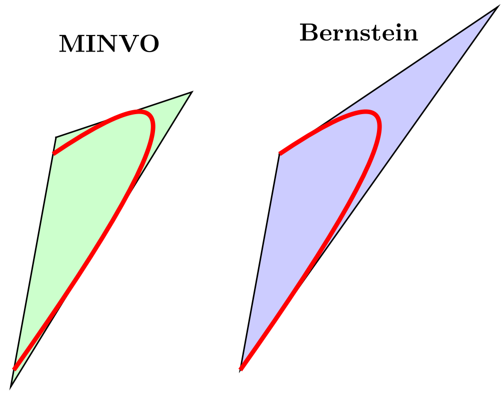

Figure 5: Comparison between the MINVO basis and the Bernstein basis for $n = 2$. The simplex found using the MINVO basis (left) encloses the 2^{nd}-order polynomial curve in red, and, for any 2^{nd}-order polynomial curve, the MINVO basis generates a simplex that is ≈ 1.3 times smaller that the one found using the Bernstein basis (right).

intractable for higher n). The optimal matrices $\boldsymbol{A}$ found are shown in Table I. Note that any permutation in the rows of $\boldsymbol{A}$ will also be a minimizer, since only the sign of the determinant is affected.

The corresponding plots of the MINVO basis functions are shown in Fig. 6, together with the Bernstein and Lagrange bases for comparison.

One natural question to ask is whether the basis found constitutes a global minimizer either for Problem 3 or Problem 4. To answer this, first note that both Problem 3 and Problem 4 are polynomial optimization problems. Therefore, we can make use of Lasserre's moment method [30], and increase the order of the moment relaxation to find tighter lower bounds of the original nonconvex polynomial optimization problem. Using this technique, we were able to obtain the same objective value (proving therefore global optimality) for $n = 1, 2, 3$ (in Problem 4), and for $n = 1$ (in Problem 3). These results lead us to the following conclusions (also summarized in Table I):

- The matrix $\boldsymbol{A}$ found for $n = 1$ is a global minimizer of both Problem 3 and Problem 4.
- The matrices $\boldsymbol{A}$ found for $n = 2, 3$ are global minimizers for Problem 4, and at least local minimizers for Problem 3.
- The matrix $\boldsymbol{A}$ found for $n = 4$ is at least a local minimizer of both Problem 3 and Problem 4.
- The matrices $\boldsymbol{A}$ found for $n = 5, 6, 7$ are at least local minimizers for Problem 4, and are at least feasible solutions for Problem 3.

In Table I, it is also shown the ratio

$$\frac{abs\left(|\boldsymbol{A}_B|\right)}{abs\left(|\boldsymbol{A}|\right)}$$

where $\boldsymbol{A}$ and $\boldsymbol{A}_B$ denote the coefficient matrices for the MINVO and Bernstein bases respectively. Let us now analyze the meaning of these results in both for the Problems 1 and 2.

A. From the point of view of Problem 1

Problem 1 attempts to find the simplex with smallest volume the encloses a given polynomial curve. In Fig. 5 it is shown the simplex found for $n = 2$, and in Fig. 1 for $n = 3$. The simplex found using the Bernstein basis is also shown for comparison. Let S and S_B denote the simplexes that the MINVO and Bernstein bases generate respectively, we have that:

$$\frac{vol\left(S\right)}{vol\left(S_B\right)} = \frac{abs\left(|\boldsymbol{A}_B|\right)}{abs\left(|\boldsymbol{A}|\right)}$$

This means, for instance, that for $n = 3$, the MINVO basis finds a simplex that is ≈ 2.36 smaller than the one the Bernstein basis finds. For $n = 7$, the MINVO basis is able to obtain a simplex that is ≈ 903 times smaller than the one achieved using the Bernstein basis.

B. From the point of view of Problem 2

In problem 2, the n-simplex S is given, and we obtain the polynomial curve that is contained in S, whose coefficients vectors $\boldsymbol{p}_n, ..., \boldsymbol{p}_1$ span a parallelogram with largest volume. The results for a regular triangle ($n = 2$) and a tetrahedron ($n = 3$) are shown in Figs. 7 and 8 respectively. Denoting P and P_B as the parallelograms that the MINVO and Bernstein bases generate respectively, the ratio of their volumes is:

$$\frac{vol\left(P_B\right)}{vol\left(P\right)} = \frac{abs\left(|\boldsymbol{A}_B|\right)}{abs\left(|\boldsymbol{A}|\right)}$$

This means that, for $n = 3$, the MINVO basis finds a parallelogram ≈ 2.36 times bigger than the Bernstein basis (≈ 903 times bigger for $n = 7$).

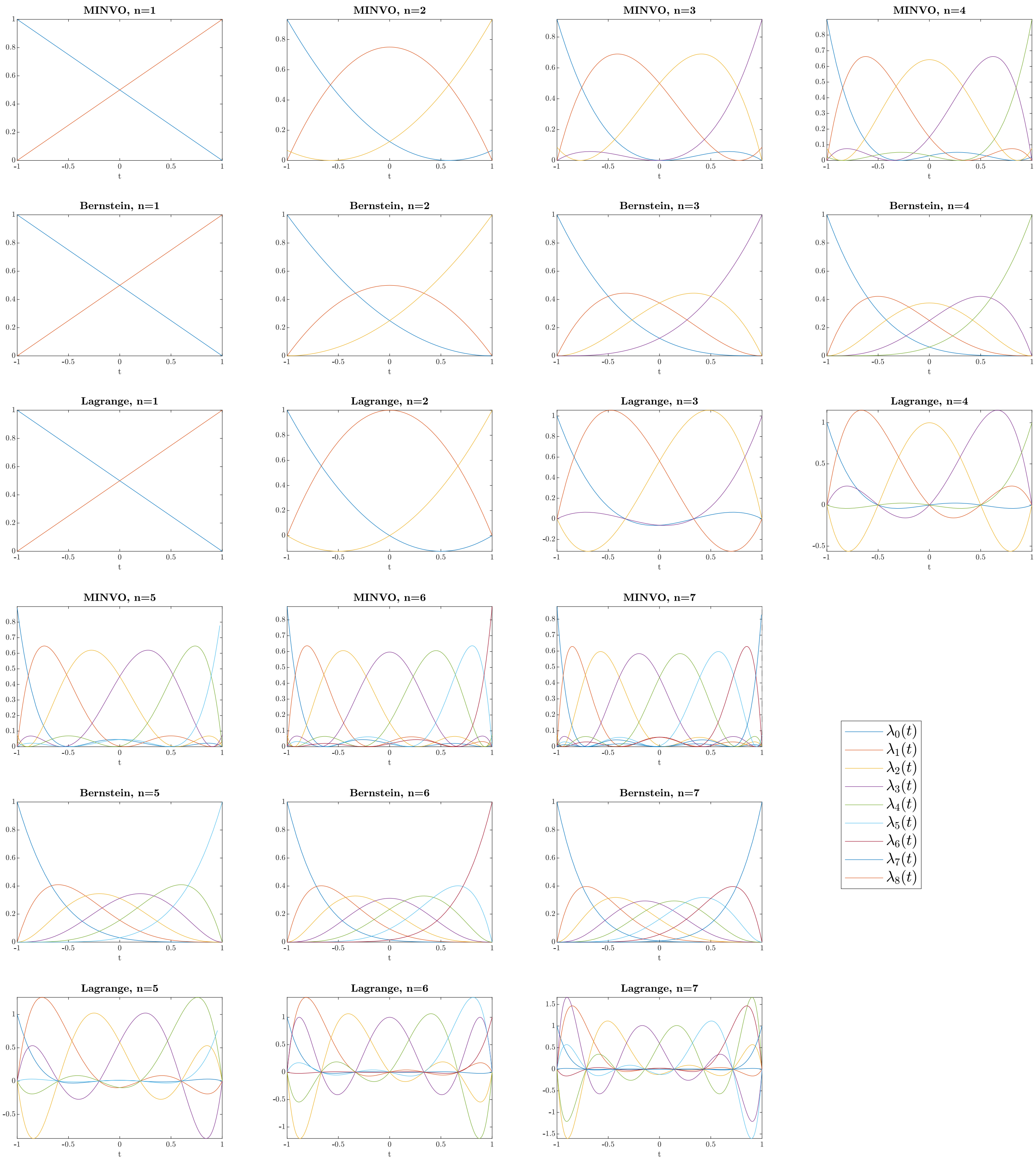

Figure 6: Comparison between the MINVO, and Bernstein and Lagrange bases for $n = 1, 2, 3, 4, 5, 6, 7$. All these bases satisfy $\sum_{i=0}^{i=n} \lambda_i(t) = 1$, and the MINVO and Bernstein bases also satisfy $\lambda_i(t) \geq 0 \; \forall t \in [-1, 1]$

$\boldsymbol{n}$	$\boldsymbol{A}$	$abs\left(\lvert\boldsymbol{A}\rvert\right)$	$\frac{abs(\lvert\boldsymbol{A}_B\rvert)}{abs(\lvert\boldsymbol{A}\rvert)}$	**Problem 3**	**Problem 4**
1	$\frac{1}{2}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$	$\frac{1}{2}=0.5$	$=1$	Global Opt.	Global Opt.
2	$\frac{1}{8}\begin{bmatrix} 3 & -2\sqrt{3} & 1 \\ -6 & 0 & 6 \\ 3 & 2\sqrt{3} & 1 \end{bmatrix}$	$\frac{3\sqrt{3}}{16}\approx 0.3248$	≈ 0.7697	Local Opt. (at least)	Global Opt.
3	$\begin{bmatrix} -0.4302 & 0.4568 & -0.02698 & 0.0004103 \\ 0.8349 & -0.4568 & -0.7921 & 0.4996 \\ -0.8349 & -0.4568 & 0.7921 & 0.4996 \\ 0.4302 & 0.4568 & 0.02698 & 0.0004103 \end{bmatrix}$	≈ 0.3319	≈ 0.4237	Local Opt. (at least)	Global Opt.
4	$\begin{bmatrix} -1.108 & -0.8108 & 0.9602 & 0.8108 & 0.1483 \\ -1.108 & 0.8108 & 0.9602 & -0.8108 & 0.1483 \\ 0.5255 & -0.5758 & -0.09435 & 0.1381 & 0.03023 \\ 0.5255 & 0.5758 & -0.09435 & -0.1381 & 0.03023 \\ 1.166 & 0 & -1.732 & 0 & 0.643 \end{bmatrix}$	≈ 0.5678	≈ 0.1651	Local Opt. (at least)	Local Opt. (at least)
5	$\begin{bmatrix} -0.7392 & 0.7769 & 0.3302 & -0.3773 & -0.0365 & 0.04589 \\ 1.503 & -1.319 & -1.366 & 1.333 & -0.121 & 0.002895 \\ -1.75 & 0.5424 & 2.777 & -0.9557 & -1.064 & 0.4512 \\ 1.75 & 0.5424 & -2.777 & -0.9557 & 1.064 & 0.4512 \\ -1.503 & -1.319 & 1.366 & 1.333 & 0.121 & 0.002895 \\ 0.7392 & 0.7769 & -0.3302 & -0.3773 & 0.0365 & 0.04589 \end{bmatrix}$	≈ 1.6987	≈ 0.0449	Feasible (at least)	Local Opt. (at least)
6	$\begin{bmatrix} 1.06 & -1.134 & -0.7357 & 0.8348 & 0.1053 & -0.1368 & 0.01836 \\ -2.227 & 2.055 & 2.281 & -2.299 & -0.08426 & 0.2433 & 0.0312 \\ 2.59 & -1.408 & -4.27 & 2.468 & 1.58 & -1.081 & 0.152 \\ -2.844 & 0 & 5.45 & 0 & -3.203 & 0 & 0.5969 \\ 2.59 & 1.408 & -4.27 & -2.468 & 1.58 & 1.081 & 0.152 \\ -2.227 & -2.055 & 2.281 & 2.299 & -0.08426 & -0.2433 & 0.0312 \\ 1.06 & 1.134 & -0.7357 & -0.8348 & 0.1053 & 0.1368 & 0.01836 \end{bmatrix}$	≈ 9.1027	≈ 0.00848	Feasible (at least)	Local Opt. (at least)
7	$\begin{bmatrix} -1.637 & 1.707 & 1.563 & -1.682 & -0.3586 & 0.4143 & -0.006851 & 2.854\cdot 10^{-5} \\ 3.343 & -3.285 & -3.947 & 4.173 & 0.6343 & -0.9385 & -0.02111 & 0.05961 \\ -4.053 & 2.722 & 6.935 & -4.96 & -2.706 & 2.269 & -0.2129 & 0.00535 \\ 4.478 & -1.144 & -9.462 & 2.469 & 6.311 & -1.745 & -1.312 & 0.435 \\ -4.478 & -1.144 & 9.462 & 2.469 & -6.311 & -1.745 & 1.312 & 0.435 \\ 4.053 & 2.722 & -6.935 & -4.96 & 2.706 & 2.269 & 0.2129 & 0.00535 \\ -3.343 & -3.285 & 3.947 & 4.173 & -0.6343 & -0.9385 & 0.02111 & 0.05961 \\ 1.637 & 1.707 & -1.563 & -1.682 & 0.3586 & 0.4143 & 0.006851 & 2.854\cdot 10^{-5} \end{bmatrix}$	≈ 89.0191	≈ 0.00110	Feasible (at least)	Local Opt. (at least)

Table I: Results for the MINVO basis. $\boldsymbol{A}$ denotes the coefficient matrix of the MINVO basis while $\boldsymbol{A}_B$ represents the matrix of the Bernstein basis ($t\in[-1,1]$ for both). The bigger the absolute value of the determinant, the smaller the associated simplex (for Problem 1) and the smaller the parallelogram (for Problem 3). These matrices $\boldsymbol{A}$ are independent of the polynomial curve given (in Problem 1), or of the simplex given (in Problem 2).

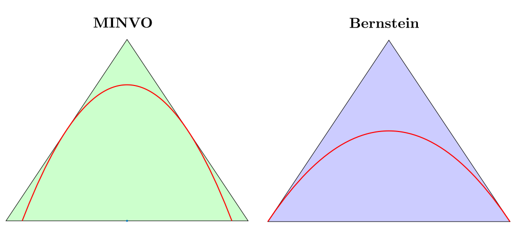

Figure 7: Optimal solution for Problem 2 with $n=2$. The triangle is given, and the red polynomial curve using the MINVO basis is the one whose coefficients vectors $\boldsymbol{p}_2,\boldsymbol{p}_1$ span a rectangle with largest area, while being contained in the triangle. On the right, the curve obtained using the Bernstein basis.

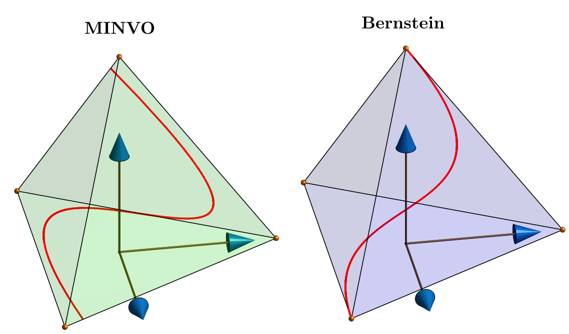

Figure 8: Optimal solution for Problem 2 with $n=3$. The tetrahedron is given, and the red polynomial curve using the MINVO basis is the curve inside the tetrahedron whose coefficients vectors $\boldsymbol{p}_3,\boldsymbol{p}_2,\boldsymbol{p}_1$ span a paralellogram with largest volume. On the right, the curve obtained using the Bernstein basis.

VII. Conclusions and Future Work

This work derived and presented the MINVO basis. The key features of this basis is that it finds the smallest n-simplex that encloses a given polynomial curve (Problem 1), and also finds the polynomial curve that is inside a given simplex, and whose coefficient vectors $\boldsymbol{p}_n, ..., \boldsymbol{p}_1$ span a parallelogram with largest volume (Problem 2). Global optimality was also proven for some n.

The exciting results of this project naturally lead to the following questions and conjectures:

- Is the global optimum of Problem 3 the same as the global optimum of Problem 4? In other words, are we losing optimality by imposing the specific structure in $\lambda_i(t)$? The results seem to indicate that it is likely that no optimality is lost.
- Does there exist an analytical solution (i.e. not numerical) for Problem 3 for any degree n?
- Does there exist a recursive formula to obtain the solution of problem 3 for n given the previous solutions $1, ..., n-1$? Would this recursive formula allow to easily prove global optimality for all n of Problem 3?

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