The Two Process Model
Exploring Sleep-Wake Dynamics

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Objective: to better understand the transition from monophasic to polyphasic sleep-wake patterns in the two process model.

The Two Process Model

The two-process model [1,2] accounts for essential aspects of sleep-wake regulation. By changing parameters in the model one can emulate monophasic and polyphasic sleep patterns, the transition of sleep patterns from childhood to adulthood and even create sleep schedules for shift workers.

The system consists of a homeostatic sleep pressure $H(t)$ that increases during wake and decreases during sleep. Switches between wake and sleep occur at threshold values that are modulated by a circadian process, $C(t) = \sin(2\pi t)$.

- The homeostat is given by

$$H(t) = \begin{cases} H_{0_s} e^{\frac{t_0 - t}{\chi_s}} & \text{(Sleep)}, \\ 1 + (H_{0_w} - 1) e^{\frac{t_0 - t}{\chi_w}} & \text{(Wake)}. \end{cases}$$

- The process C determines the switching:
  - wake to sleep is at

$$H^+(t) = H_0^+ + aC(t);$$

  - sleep to wake is at

$$H^-(t) = H_0^- + aC(t).$$

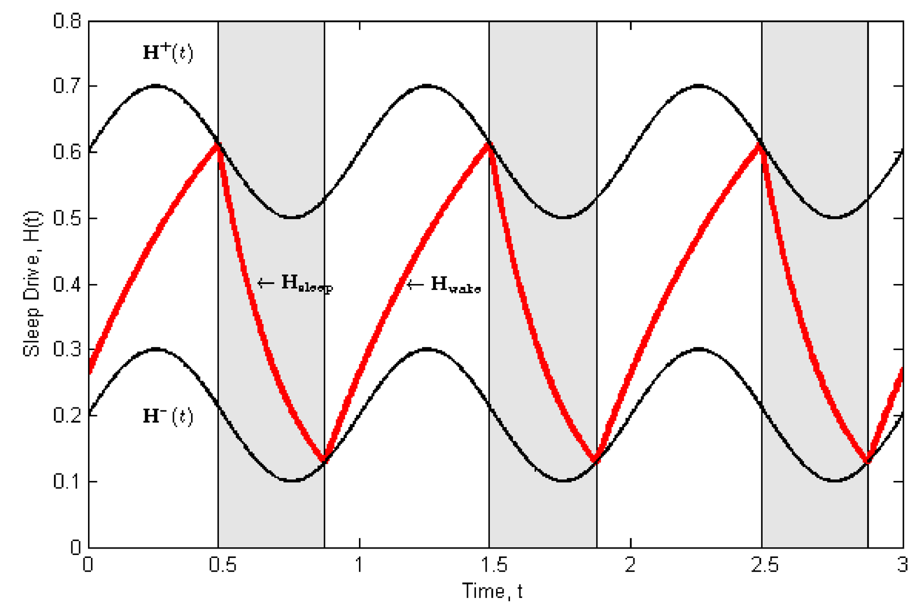

Figure 1: **The mathematical model.** Here we see how the homeostat dynamically moves between the upper and lower circadian thresholds.

In Figure 2 the role of different circadian amplitudes (a) and upper thresholds (H_0^+) is explored [2]. We can see that these changes in parameters create very different behaviour in the model.

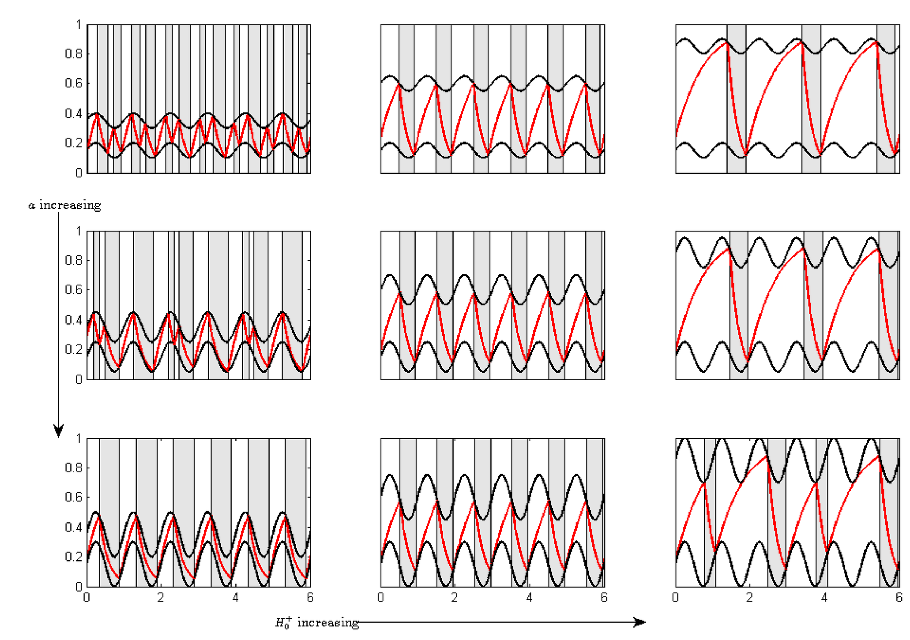

Figure 2: **Model behaviour.** Nine sample simulations as in [2], showing 3 amplitude values ($a = 0.05, 0.10,$ and 0.15) with 3 levels of the upper threshold ($H_0^+ = 0.35, 0.60,$ and 0.85). Low circadian amplitudes usually lead to desynchrony. At high levels of H_0^+ patterns are observed. Intermediate values of H_0^+ and high circadian amplitudes both lead to synchrony.

Since our upper and lower asymptotes are 0 and 1 respectively our model can only have switching when

$$H_0^- + |a| > 0 \quad \text{and} \quad H_0^+ - |a| < 1.$$

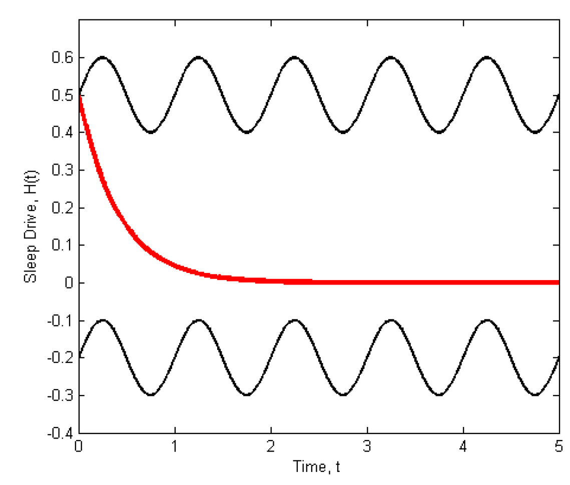

Figure 3: **Understanding switching.** Here we see that if $H_0^- + a \leq 0$ then no switching can occur.

One-Dimensional Maps

The two process model can be viewed as giving a sequence of sleep onset times: if you fall asleep at time t_s^0 on day zero, then the two process model predicts sleep times of t_s^1 (modulo a day) on day one, of t_s^2 on day two etc. So the two process model can be represented as a rule (map) that takes t_s^n to t_s^{n+1}. This can be drawn graphically by plotting t_s^n against t_s^{n+1} as in Figure 4.

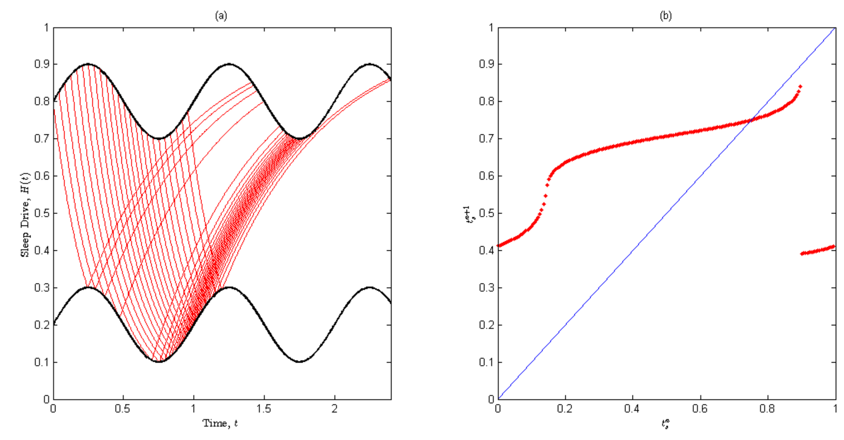

Figure 4: **Creating the one-dimensional map.** The map that takes t_s^n to t_s^{n+1} can be constructed by taking all possible values t_s^0 and finding the corresponding t_s^1 (modulo one day), see (a). The resulting map is shown in (b). The diagonal line intersects the map when $t_s^n = t_s^{n+1}$. This corresponds to the sleep onset time that the model converges to.

The discontinuity in Figure 4(b) is due to a tangency in the orbit of the two process map (as seen in Figure 5) [3]. By grazing the lower boundary the homeostat yields large changes in t_s^1 for small changes in t_s^0.

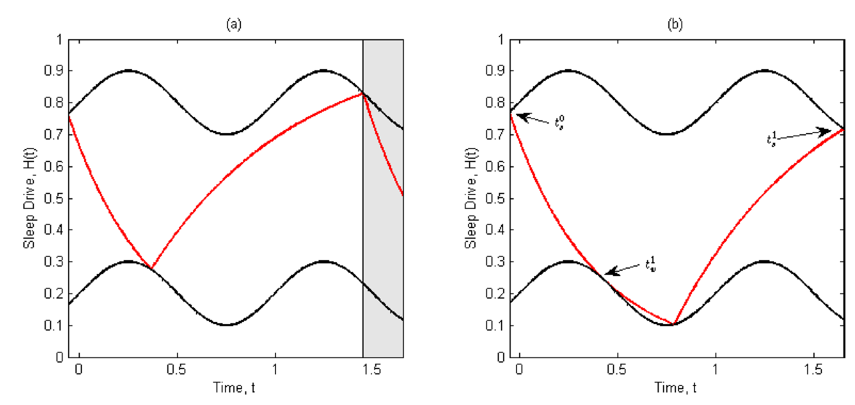

Figure 5: **Sleep tangency.** (a) Sleep-wake cycle giving behaviour just before a tangency occurs. (b) The model just after the tangency occurs, where t_w^1 is the time of the tangency. The shaded region in (a) shows the difference in sleep times after one period, t_s^1, with only a slight perturbation in inital sleep times, t_s^0.

What happens when the identity line (blue) passes through this discontinuity in Figure 4(b)?

Discontinuities and Bifurcations

- Our aim is to understand when tangencies can occur in the two process map and examine the effect of the tangency on the behaviour of the model.
- By decreasing the homeostatic time constant χ_s we see the discontinuity moves towards the $t_s^n = t_s^{n+1}$ identity line which leads to a grazing bifurcation and a transition from one sleep episode a day to two episodes a day. This repeats, leading to a sequence of transitions.

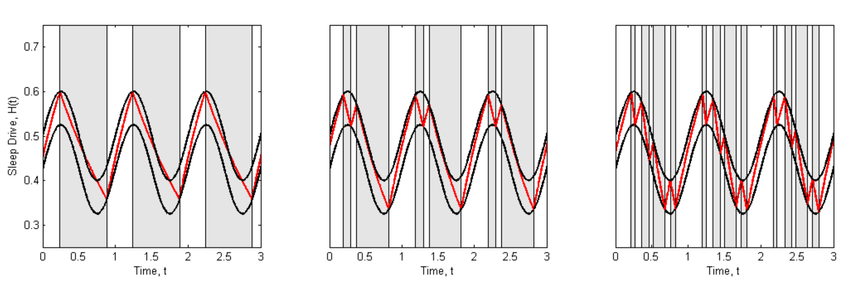

Figure 6: **Decreasing the homeostatic time constant.** Sleep cycles for decreasing $\chi_s = 30, 20, 10$ which add more sleep periods per day. At $\chi_s = 30$ there is one sleep a day, $\chi_s = 20$ gives two and $\chi_s = 10$ gives four.

Figure 7 gives an overview of the sleep regions when varying the parameter χ_s. We observe that as χ_s decreases we have more sleep periods per day.

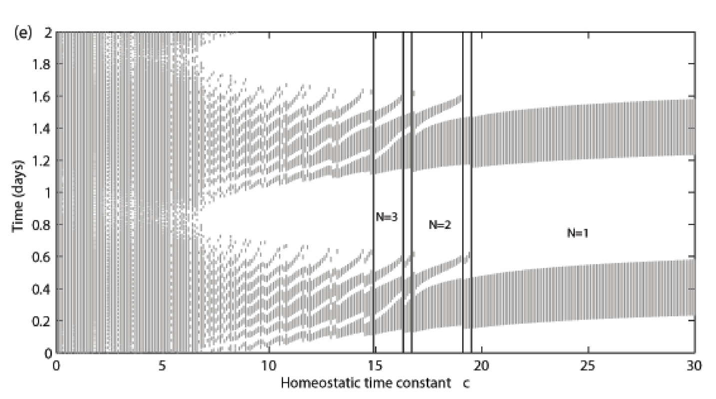

Figure 7: **Bifurcation Diagram.** Sleep regions (shaded) as a function of χ_s [4]. Transition regions exist between N and $N+1$ periods a day. In these regions sleep varies between N and $N+1$ daily sleep episodes.

Parameter Constraints on t_w^1

In the two-process model tangencies occur on the lower bound at time t_w^1. When are these tangencies possible with respect to the parameters in the model?

- Since H_{sleep} has a negative gradient, tangencies can only occur on the lower bound for $t_w^1 \in (\frac{1}{4}, \frac{3}{4})$ where $\frac{d}{dt}H^-(t) < 0$.
- For our tangency on the lower bound to be relevant the homeostat must approach from above, therefore satisfying

$$H''(t_w^1) + 4\pi^2 a \sin(t_w^1) > 0.$$

- Also, tangencies can only exist when

$$-a \leq H_0^- \leq a\sqrt{4\pi^2\chi_s^2 + 1}.$$

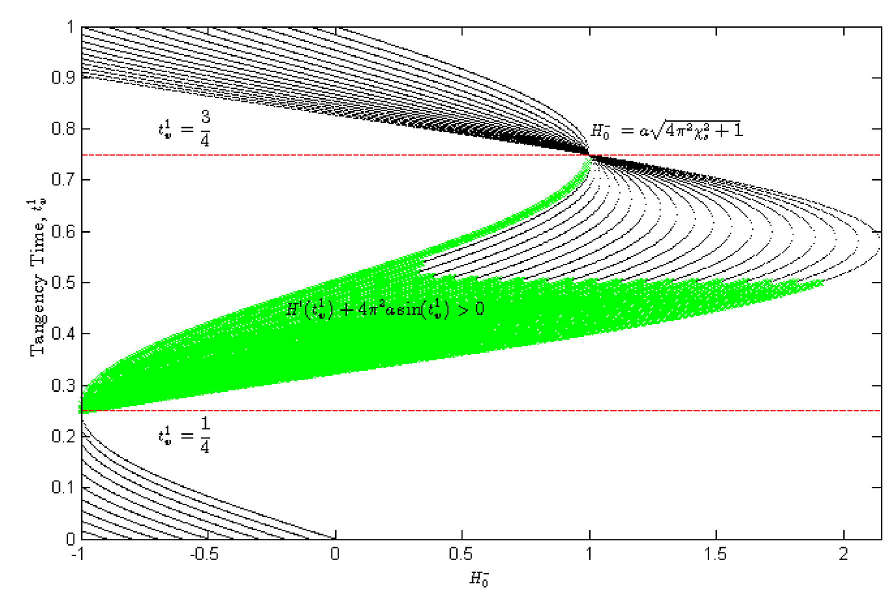

Figure 8: **Parameter values allowing for tangencies.** Each black curve is given by a fixed value of χ_s. The green region shows the parameter space which gives allowed tangencies on the lower bound.

Conclusion

We have shown that the two process model leads to a one-dimensional map [3,4]. This allows us to understand the occurrence of discontinuities with respect to the parameters within the model. Therefore we can

- predict whether a parameter set will give a tangency and when this tangency will occur;
- begin to understand the model's behaviour as various parameters are varied.

Future Work

Extend this work to include the upper circadian boundary and continue analysis on the one-dimensional map to get a better understanding of the bifurcations observed.

References

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[4] Skeldon A, Dijk D-J & Derks G. 2014 *Mathematical Models for Sleep-Wake Dynamics: Comparison of the Two-Process Model and a Mutual Inhibition Neuronal Model.* (PLoS ONE 9(8): e103877)

Acknowledgements

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