Sieve Estimators

N. Joesphs, M. Wiens, B. Draves

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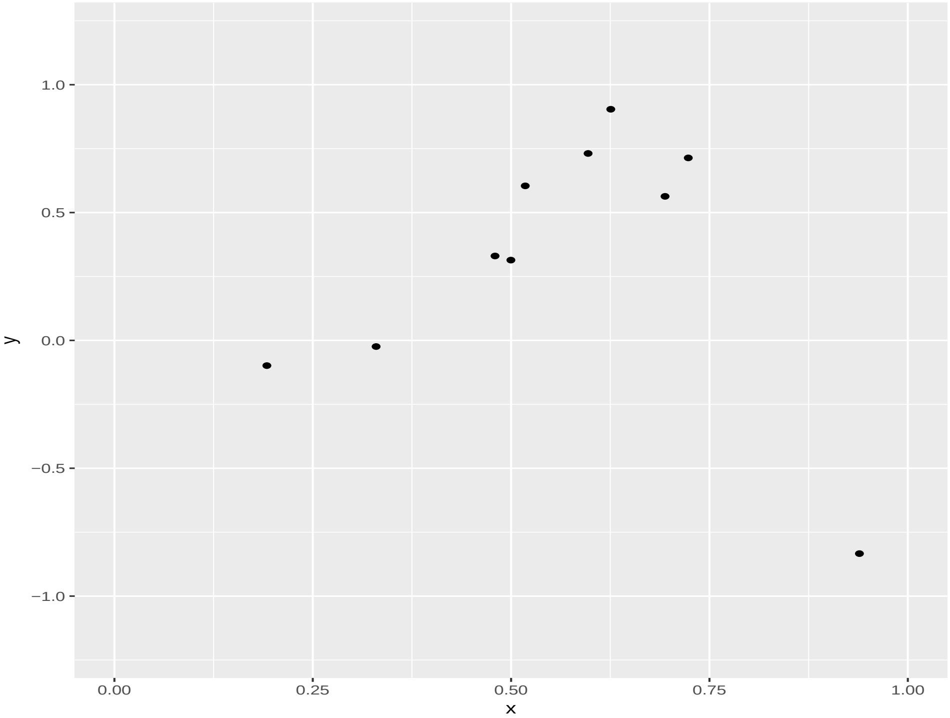
Outline

1. Define *Sieves Estimator*
2. Properties of the Estimator
3. Derive Smoothing Selection Algorithm
4. Simulations and Applications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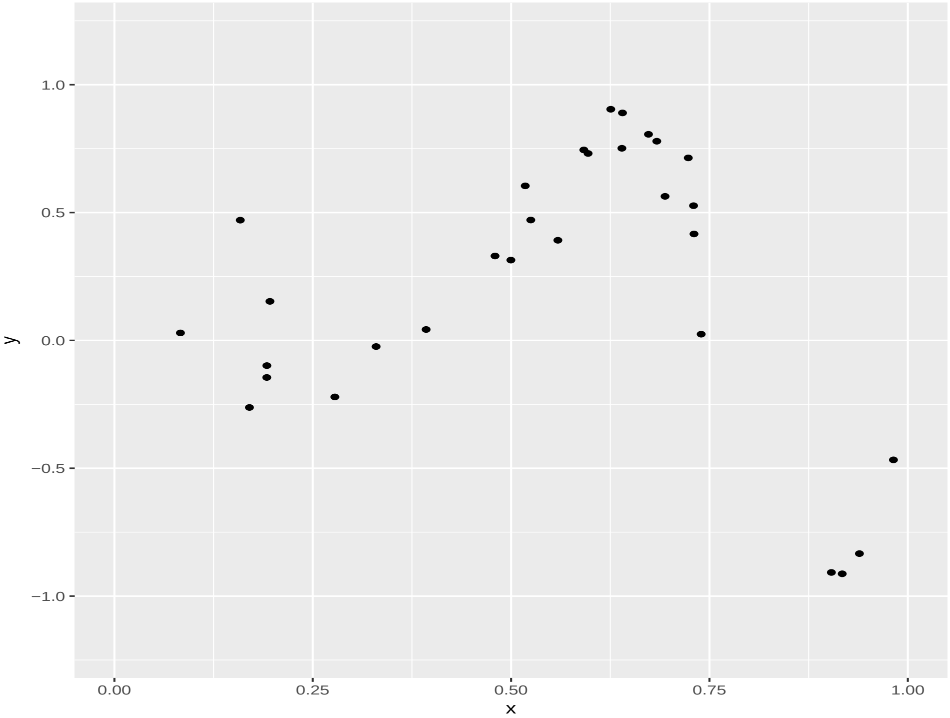
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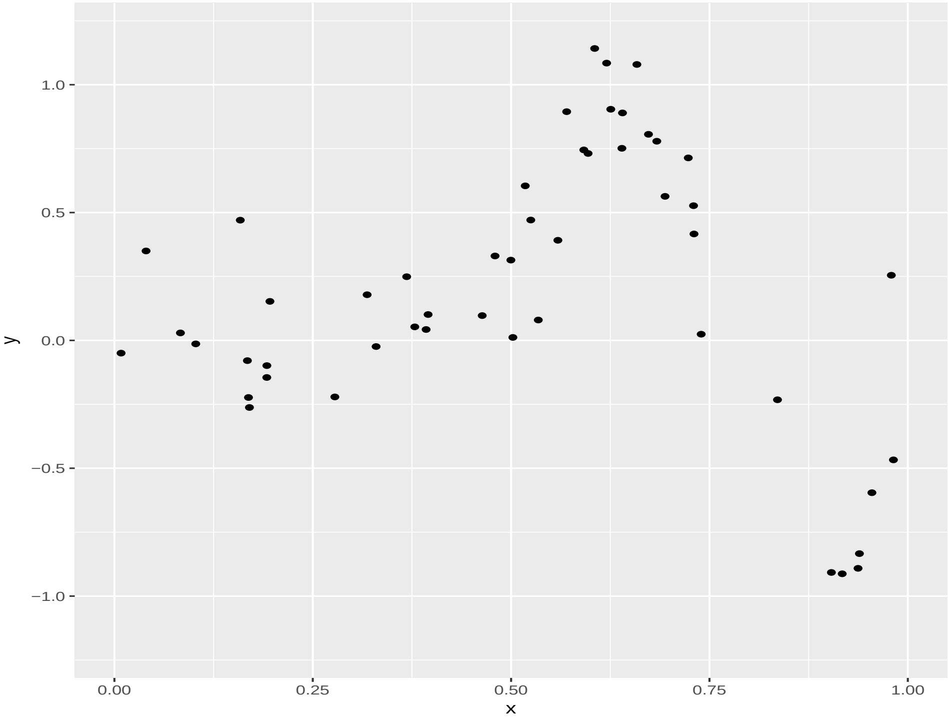

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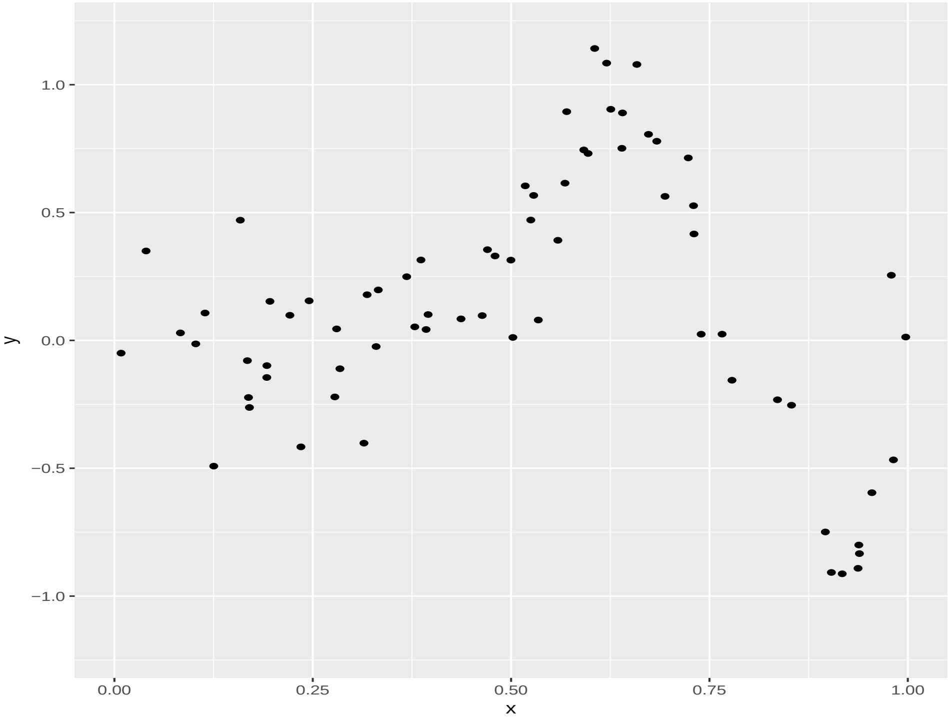

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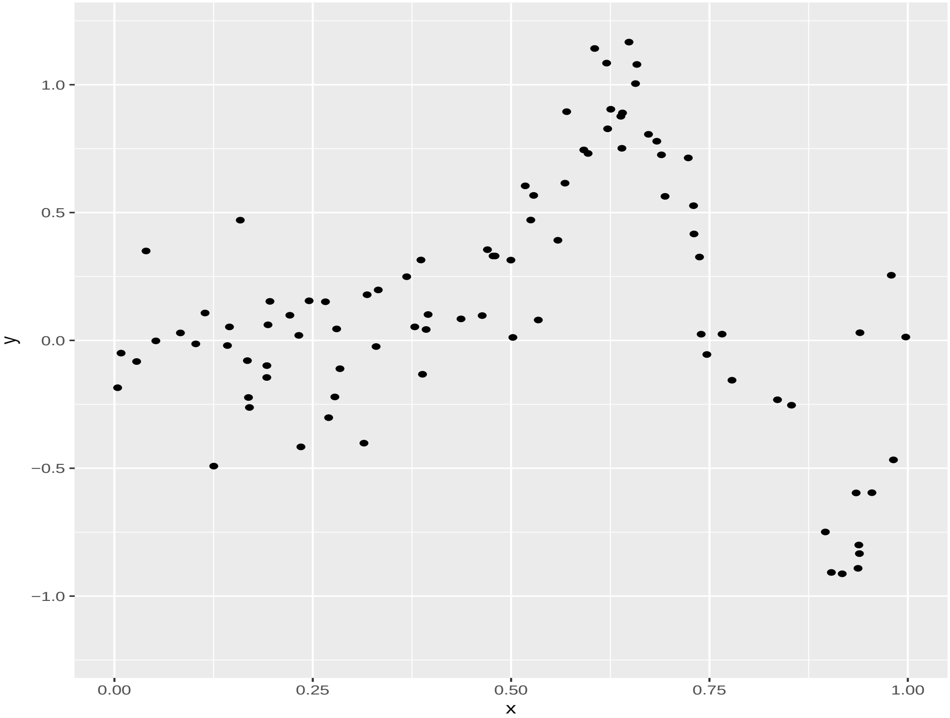
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Sieves

- *Sieve Estimation* constructs a sequence of function spaces $\{\mathcal{F}_n\}_{n=1}^{\infty}$ such that
$$\mathcal{F}_1 \subset \mathcal{F}_2 \subset \mathcal{F}_3 \subset \dots$$
- Then, depending on the sample size n, the estimation procedure is performed over the optimal space $\mathcal{F}_{opt}$
- Heuristically, as n increases, we attain a more robust understanding of the data and should allow the modeling procedure to consider more complex functions

Series Estimators

1. **Goal**: Estimate unknown regression function for $Y = f(X)$
2. We consider the *series estimators* of f

$$\mathcal{F} = \left\{ g(x) : g(x) = \sum_{d=0}^{D} \alpha_d \phi_d(x) \right\}$$

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   - $\phi_d(x)$ is the *dth* basis function
   - α_d is the corresponding coefficient

3. The complexity of $\mathcal{F}$ increases as n increases

Basis Functions

1. Choosing basis function is analogous to choosing kernel
2. Basis functions should represent belief about the underlying function
3. Some popular choices include wavelets and splines and others given below

Basis Functions	Functional Form
Polynomials	$\sum_{i=0}^{d} c_d x^d$
Fourier	$a\ cos(\pi dx) + b\ sin(\pi dx)$
Gaussian	$\varphi^{(d)}(x)$

Table: Series estimators with basis function ϕ for $\hat{g}(x)$

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2. Using the following matrices

$$Y = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix}, P_D = \begin{bmatrix} \phi_0(x_1) & \dots & \phi_D(x_1) \\ \vdots & \ddots & \vdots \\ \phi_0(x_n) & \dots & \phi_D(x_n) \end{bmatrix}, \alpha = \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_D \end{bmatrix}, e = \begin{bmatrix} e(x_1) \\ e(x_2) \\ \vdots \\ e(x_n) \end{bmatrix}$$

   we can write our model as

$$Y = P_D \alpha + e$$

3. Bias and variance depend on choosing D correctly
4. Note that D is our smoothing parameter - analogous to the bandwidth in kernel density estimation

MISE

1. We choose our optimal dimension D_{opt} by minimizing MISE
2. We can decompose the error of the estimate as $e(x_i) = \epsilon(x_i) + r(x_i)$
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3. MISE can be written as

$$MISE(D) = E[r^2(x)] + tr\left[E[\phi(x)\phi(x)^T]E\left((\hat{\alpha} - \alpha)(\hat{\alpha} - \alpha)^T\right)\right]$$

4. Even under the assumption of homoscedasticity

$$MISE^*(D) \simeq E[r^2(x)] + \frac{\sigma^2 D}{n}$$

which is still a function of the unknown $r(x_i)$

Connecting MISE and PSE

1. While MISE is infeasible, MISE is connected to PSE
2. Let x^* be a new data point. The sieve estimate of x^* is given by

$$y^* = \hat{g}(x^*)$$

   with error e^*

3. Then we can show the relation

$$PSE\Big(\hat{g}(x^*)\Big) = Var(e^*) + MISE\Big(\hat{g}(x)\Big)$$

4. Therefore, minimizing MISE is equivalent to minimizing PSE

Connecting PSE to CV

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5. Hansen (2012) show that choosing D based on CV is asymptotically equivalent to choosing D based on MISE

Simulation Design

1. Recall the model from HW5 given by

$$Y = \sin^3(2\pi X^3) + \epsilon$$

where

$$X \sim N(0, 1)$$

$$\epsilon \sim N(0, .2)$$

2. Our previous solution: LLR and Kernel Regression
3. Applying polynomial sieves
   - $N = 150$ data points
   - From $n = 10$ to N in increments of 10, we fit polynomial series estimator with degrees of complexity varying from 1 to 8
   - Repeated this on 200 separate datasets and plotted the PSE against n
4. Simulation Results

Geyser Data Application

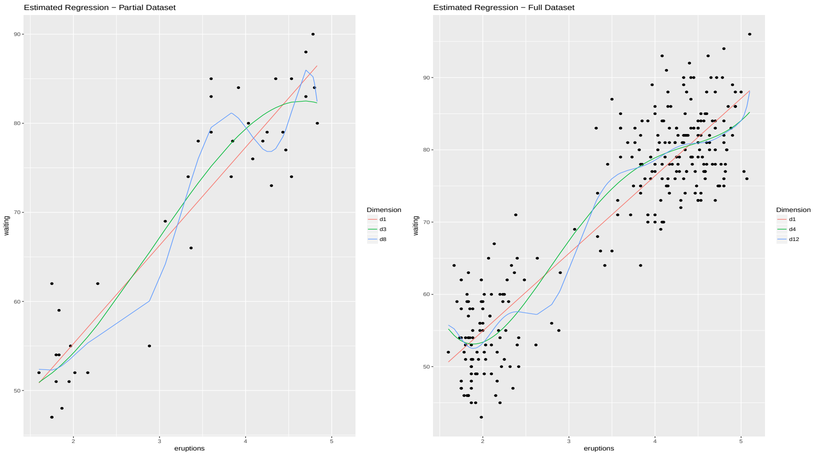

Figure: Sieves Analysis of Geyser dataset. We see that increasing our sample size leads to differing optimal models

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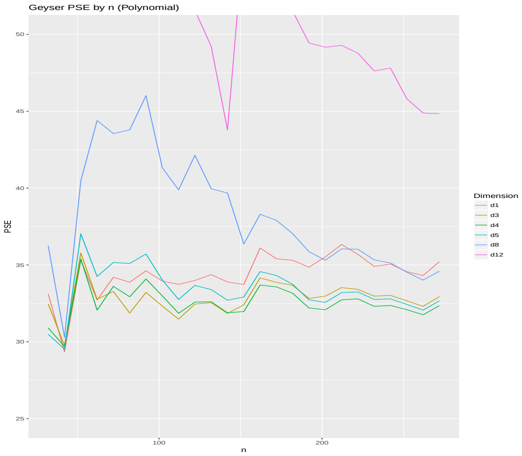

Figure: Geyser data PSE by n

Sieve Estimators and Monetary Policy

1. Sa and Portugal (2015) attempt to estimate a multi-dimensional loss function with a sieve estimator
2. The authors use a polynomial basis to estimate derivatives
3. Based on tests for third derivative they conclude there is asymmetry in the loss function
4. This corresponds to additional loss for low inflation (deflation) with regard to deciding policy

Conclusion

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3. Showed a data driven process for selecting D
4. Illustrated the estimators in simulation and real data applications