Introduction to Partial Differential Equations (PDEs)

Fundamentals, Applications, and Classifications

Your Name

Your Institution

May 6, 2025

Outline

1. What Are Partial Differential Equations (PDEs)?
2. Applications of PDEs in Real Life
3. Types of PDEs
4. Important PDEs in This Course

Table of Contents

1. What Are Partial Differential Equations (PDEs)?
2. Applications of PDEs in Real Life
3. Types of PDEs
4. Important PDEs in This Course

Ordinary Differential Equations (ODEs)

- An n-th order ODE for unknown function $y(x)$:

$$F(y^{(n)}(x), y^{(n-1)}(x), \cdots, y'(x), y(x), x) = 0$$

- Involves derivatives with respect to a single variable
- Two main problem types:
  - **Initial Value Problem (IVP):** Specify function values and derivatives at a point

$$y^{(n-1)}(x_0) = y_{n-1}, \quad y^{(n-2)}(x_0) = y_{n-2}, \quad \cdots \quad y(x_0) = y_0$$

  - **Boundary Value Problem (BVP):** Specify function values at multiple points

Partial Differential Equations (PDEs)

- PDEs involve an unknown function of multiple variables and their partial derivatives
- General form:

$$F(x, y, \cdots, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \cdots, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial x \partial y}, \frac{\partial^2 u}{\partial y^2} \cdots) = 0$$

- PDEs describe how functions change with respect to multiple variables
- Appear naturally in many physical systems where quantities depend on position, time, etc.

Table of Contents

1. What Are Partial Differential Equations (PDEs)?
2. Applications of PDEs in Real Life
3. Types of PDEs
4. Important PDEs in This Course

Heat Transfer

Heat Equation:

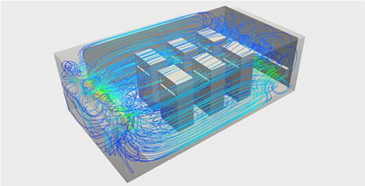
$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + s(x)$$

Where:

- $u(x, t)$ is temperature distribution
- k is thermal conductivity
- $s(x)$ is heat source

Requires:

- Initial condition: $u(x, 0) = g(x)$
- Boundary conditions at domain endpoints



Heat conduction in a metal rod

Structural Mechanics

Wave Equation:

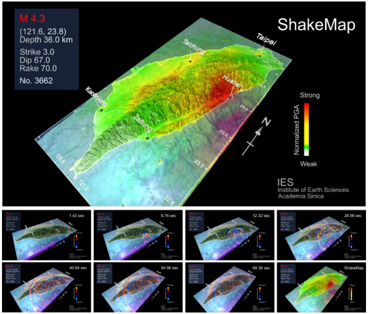
$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Where:

- $u(x, t)$ is displacement
- c is wave propagation speed

Real-World Applications:

- Earthquake simulations
- Structural vibrations in buildings
- Musical instrument vibrations



Seismic wave propagation

Fluid Flow

Navier-Stokes Equations:

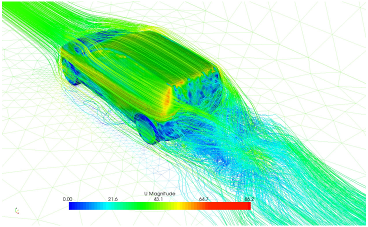
$$\rho \left(\frac{\partial u}{\partial t} + u \cdot \nabla u \right) = -\nabla p + \mu \nabla^2 u + f$$

With continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0$$

Applications:

- Weather prediction
- Aerodynamics
- Oil recovery
- Blood flow in vessels



Fluid flow over a car

Electromagnetism

Maxwell's Equations:

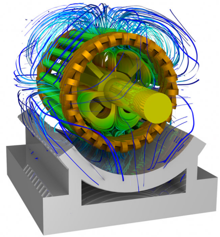
$$\nabla \times E = -\frac{\partial B}{\partial t}$$

$$\nabla \times B = \mu_0 J + \mu_0 \epsilon_0 \frac{\partial E}{\partial t}$$

Where $\mu_0 \epsilon_0 = 1/c^2$ and c is the velocity of light in vacuum.

Applications:

- Wireless communication
- MRI scanners
- Electromagnetic radiation



Electromagnetic fields in an electric motor.

Table of Contents

1 What Are Partial Differential Equations (PDEs)?

2 Applications of PDEs in Real Life

3 Types of PDEs

4 Important PDEs in This Course

General Form of Second-Order PDEs

- A general second-order linear PDE for an unknown function $u(x, y)$:

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} = F(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y})$$

- Classification depends on the discriminant $B^2 - AC$:
  - **Elliptic:** $B^2 - AC < 0$
  - **Parabolic:** $B^2 - AC = 0$
  - **Hyperbolic:** $B^2 - AC > 0$

Classifications of PDEs

PDEs are classified based on their mathematical properties:

Type	**General Form**	**Example**	**Physical Meaning**
Elliptic	$\nabla^2 u = f(x, y)$	Poisson's equation	Steady-state behavior
Parabolic	$u_t = k\nabla^2 u$	Heat equation	Diffusion processes
Hyperbolic	$u_{tt} = c^2\nabla^2 u$	Wave equation	Wave propagation

Type	**Initial condition (time)**	**Boundary condition (space)**
Elliptic	—	✓✓(all boundaries)
Parabolic	✓(1 condition)	✓(for each time step)
Hyperbolic	✓✓(2 conditions)	✓(usually required)

Table 1: Typical initial and boundary conditions for second-order PDEs

Convection-Diffusion Equation

A general PDE that combines transport and diffusion:

$$\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} = k\frac{\partial^2 u}{\partial x^2} + f$$

Special Cases:

- **Heat Equation** ($U = 0, f = 0$):

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

- **Linear Advection Equation** ($k = 0, f = 0$):

$$\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} = 0$$

- **Poisson's Equation** ($U = 0, t \to \infty$):

$$k\frac{\partial^2 u}{\partial x^2} + f = 0$$

Table of Contents

1. What Are Partial Differential Equations (PDEs)?
2. Applications of PDEs in Real Life
3. Types of PDEs
4. Important PDEs in This Course

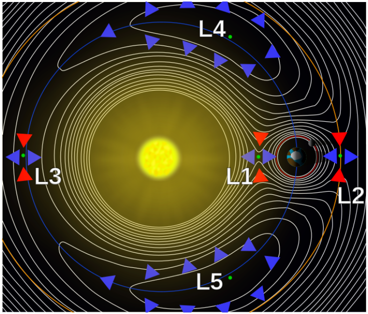
Poisson's Equation

Steady-State Heat and Electrostatics:

$$-\nabla^2 u = f(x)$$

Applications:

- Electrostatic potential
- Steady-state temperature distribution
- Gravitational potential



Solution to Poisson's equation

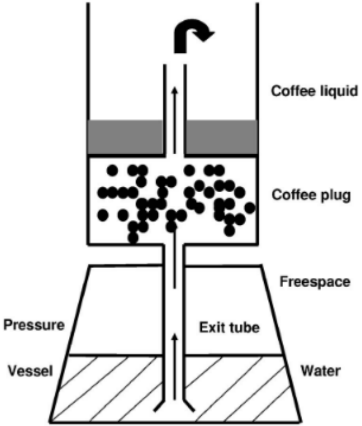
Darcy's Law

Flow in Porous Media:

$$-\nabla \cdot (k \nabla p) = f$$

Applications:

- Groundwater flow
- Oil reservoir simulation
- Soil contaminant transport
- Biological tissue perfusion



Fluid flow through porous media

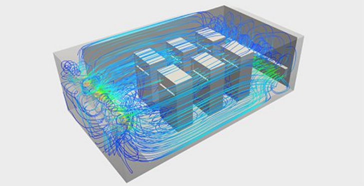
Heat Equation

Transient Heat Transfer:

$$\frac{\partial u}{\partial t} = k\nabla^2 u$$

Applications:

- Heat conduction in materials
- Thermal insulation performance
- Cooling systems design
- Cooking and food processing



Transient heat diffusion

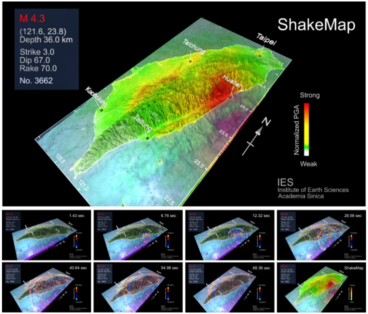
Wave Equation

Vibrations and Acoustics:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$$

Applications:

- Structural vibrations
- Sound wave propagation
- Seismic analysis
- String instruments



Wave propagation visualization