A Brief Introduction to Partial Differential Equations

June 23, 2025

0.1 What Are Partial Differential Equations (PDEs)?

0.1.1 Ordinary Differential Equations (ODEs)

An n-th order ordinary differential equation is an equation for an unknown function $y(x)$ that expresses a relationship between the unknown function and its first n derivatives. One could write this generally as

$$F\left(y^{(n)}(x), y^{(n-1)}(x), \cdots, y'(x), y(x), x\right) = 0,$$

where $y^{(n)}(x)$ represents the n-th derivative of $y(x)$.

- **Initial Value Problem (IVP):** The solution is determined by specifying the function values and its first $n-1$ derivatives at a given point x_0:

$$y^{(n-1)}(x_0) = y_{n-1}, \quad y^{(n-2)}(x_0) = y_{n-2}, \quad \cdots \quad y(x_0) = y_0$$

- **Boundary Value Problem (BVP):** The solution is determined by specifying function values at more than one point.

0.1.2 Partial Differential Equations (PDEs)

A partial differential equation (PDE) involves an unknown function of multiple variables and their partial derivatives. The general form is:

$$F\left(x, y, \cdots, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \cdots, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial x \partial y}, \frac{\partial^2 u}{\partial y^2}, \cdots\right) = 0,$$

where $u(x, y, \cdots)$ is the unknown function.

0.2 Applications of PDEs in Engineering Physics

PDEs play a crucial role in modeling real-world systems. Below are some key applications:

0.2.1 Heat Transfer (Heat Equation)

The governing PDE: Predicting temperature distribution $u(x, t)$ in materials over time (e.g., cooling of a hot metal rod) is governed by the following PDE:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + s(x)$$

where k is the conductivity of the rod and $s(x)$ indicates the heat source. To determine u, we must specify the temperature at every point in the bar when $t = 0$, say

$$u(x, 0) = g(x).$$

We call this the **initial condition**. We must also specify **boundary conditions** that u must satisfy at the ends of the bar for all $t > 0$. We'll call this problem an **initial-boundary value problem**.

Real-World Example: Heat conduction in buildings and thermal insulation.

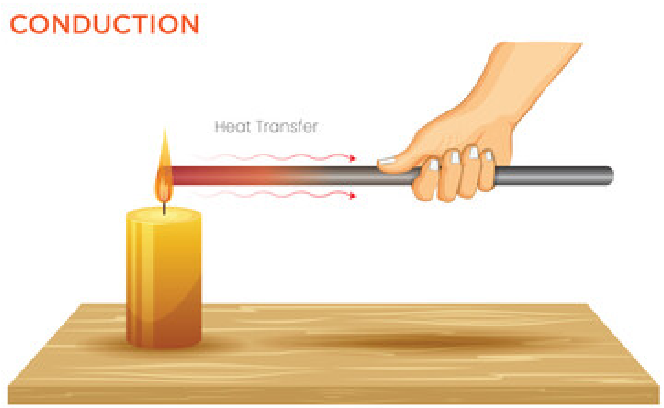

Figure 1: Heat conduction in a metal rod. (from https://stock.adobe.com/)

(2.2) Structural Mechanics (Wave Equation)

The governing PDE: The wave equation is used to describe vibrations in structures like bridges and buildings, and has the following form:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}.$$

where c represents the speed of the wave in the medium.

Real-World Example: Earthquake simulations, musical instrument vibrations.

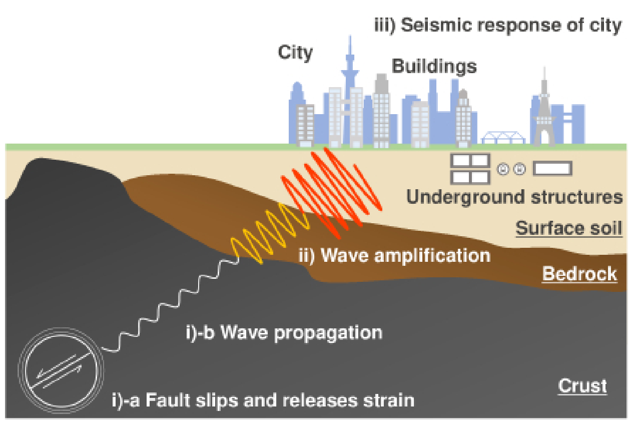

Figure 2: Seismic wave propagation (from https://dl.acm.org/doi/fullHtml/10.1145/3492805.3492814)

0.2.2 Fluid Flow (Navier-Stokes & Darcy's Law)

The governing PDE: The following PDE is used to describe airflow over airplane wings, water flow in porous media, and blood circulation:

$$\rho\left(\frac{\partial u}{\partial t} + u \cdot \nabla u\right) = -\nabla p + \mu \nabla^2 u + f$$

The PDE is always solved together with the continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0.$$

The Navier-Stokes equations represent the conservation of momentum, while the continuity equation represents the conservation of mass.

Real-World Example: Weather prediction, oil recovery, and aerodynamics.

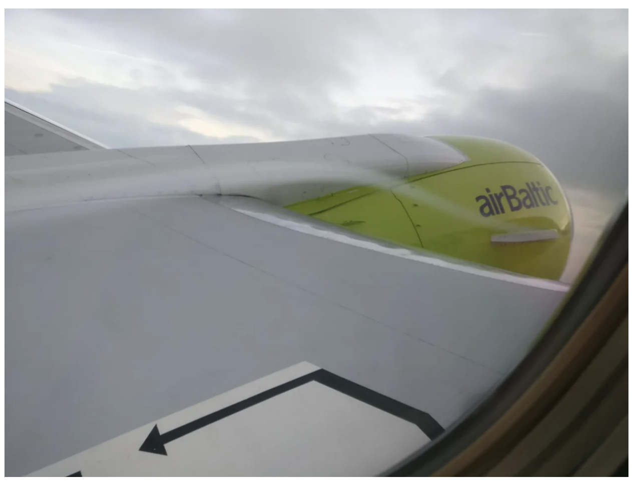

Figure 3: Fluid flow over an airplane wing. (from https://www.reddit.com/)

0.2.3 Electromagnetism (Maxwell's Equations)

The governing PDE: The following Maxwell's equations govern electromagnetic waves (light, radio signals, etc.):

$$\nabla \times E = -\frac{\partial B}{\partial t}, \quad \nabla \times B = \mu_0 J + \mu_0 \epsilon_0 \frac{\partial E}{\partial t}$$

where $\mu_0 \epsilon_0 = 1/c^2$ and c is the velocity of light in vacuum.

Real-World Example: Wireless communication, MRI scanners.

Figure 4: Electromagnetic fields and optical radiation in hospitals (from https://www.rivm.nl/en/electromagnetic-fields/emf-optical-radiation-hospitals)

0.3 Types of PDEs

PDEs are classified into three main types based on their mathematical properties. Understanding these classifications helps us determine the appropriate solution methods.

Type	General Form	Example	Physical Meaning
Elliptic	$\nabla^2 u = f$	Poisson's equation	Steady-state behavior (not time-dependence).
Parabolic	$u_t = k\nabla^2 u$	Heat equation	Diffusion processes (time-dependent but smooth evolution).
Hyperbolic	$u_{tt} = c^2\nabla^2 u$	Wave equation	Oscillations and wave propagation.

0.3.1 General Form of Second-Order PDEs

A general second-order linear PDE for an unknown function $u(x, y)$ can be written as:

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} = F\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right)$$

where A, B, and C are constants. The classification of the PDE depends on the discriminant $B^2 - AC$:

- **Elliptic:** $B^2 - AC < 0$
- **Parabolic:** $B^2 - AC = 0$
- **Hyperbolic:** $B^2 - AC > 0$

Each type exhibits different characteristics and requires specific solution techniques.

0.3.2 Example: Convection-Diffusion Equation

The Convection-Diffusion Equation describes a process involving both transport (convection) and spreading (diffusion):

$$\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} = k\frac{\partial^2 u}{\partial x^2} + f$$

By choosing different values for parameters U, k, and f, we can recover several important PDEs:

Case 1: $U = 0,\ f = 0$ (Heat Equation)

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

- Represents diffusion and heat conduction.

Case 2: $k = 0,\ f = 0$ (Linear Advection Equation)

$$\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} = 0$$

- Describes wave-like transport of a quantity without diffusion.

Case 3: $U = 0,\ t \to \infty$ (Poisson's Equation)

$$k\frac{\partial^2 u}{\partial x^2} + f = 0$$

- Models steady-state heat distribution and electrostatics.