CS103 Problem Set 5 (Written)

Maria Shan Wang, Matthew Phonchay Vilaysack

TOTAL POINTS

57 / 63

QUESTION 1

Induction Proof Critiques 8 pts

1.1 Sums of Natural Numbers 2 / 4

+ **1 pts** P(n) is a number/function rather than a predicate

✓ + **1 pts** **P(1) unnecessary**

✓ + **1 pts** **Inductive step says for all natural number k**

+ **1 pts** Inductive step starts with unknown equality and simplifies to known equality

+ **0 pts** Incorrect

1 You wrote yourself out of points with this: "n(n-1)/2 holds true" doesn't make sense, just as "3 holds true" doesn't.

2 Don't define n in the predicate

1.2 Acyclic Directed Graphs 4 / 4

✓ + **2 pts** **P(n) says for any n, shadowing parameter n**

✓ + **2 pts** **Build-up vs build-down**

+ **0 pts** Incorrect

QUESTION 2

2 Recurrence Relations 3 / 3

✓ + **3 pts** **All correct**

+ **2 pts** Incomplete/incorrect base case, but everything else correct

Intuition Scale

+ **1 pts** Correct

+ **0 pts** Incorrect

Execution Scale

+ **2 pts** Correct

+ **1 pts** Major execution error

+ **0 pts** Incorrect

QUESTION 3

Stacking Cans 12 pts

3.1 Hexagon Sizes 5 / 5

Intuition Scale

✓ + **2 pts** **Correct**

+ **1 pts** Partially correct intuition

+ **0 pts** Incorrect

Execution Scale

✓ + **3 pts** **Correct**

+ **2 pts** Minor execution error

+ **1 pts** Major execution error

+ **0 pts** Incorrect

3.2 Counting Cans 1 / 1

✓ + **1 pts** **Correct**

+ **0 pts** Incorrect

3.3 The General Formula 1 / 1

✓ + **1 pts** **Correct**

+ **0 pts** Incorrect

3.4 Towers of Cans 5 / 5

Intuition Scale

✓ + **2 pts** **Correct**

+ **1 pts** Partially correct intuition

+ **0 pts** Incorrect

Execution Scale

✓ + **3 pts** **Correct**

+ **2 pts** Minor execution error

+ **1 pts** Major execution error

+ **0 pts** Incorrect

QUESTION 4

The Circle Game 11 pts

4.1 Up or Down? 1 / 1

✓ + **1 pts** **Correct**

+ **0.5 pts** Correctly identifies P(n) as a universally-quantified statement, but does not say whether the induction should "induct up" or "induct down".

+ **0 pts** Incorrect

4.2 Winning the Circle Game 10 / 10

Intuition Scale

✓ + **4 pts** **Correct**

+ **2 pts** Partially correct intuition

+ **0 pts** Incorrect

Execution Scale

✓ + **6 pts** **Correct**

+ **4 pts** Minor execution error

+ **2 pts** Major execution error

+ **0 pts** Incorrect

QUESTION 5

5 Regular Graphs 10 / 10

✓ + **10 pts** **Correct**

Intuition Scale

+ **4 pts** Correct

+ **2 pts** Partially correct intuition

+ **0 pts** Incorrect

Execution Scale

+ **6 pts** Correct

+ **4 pts** Minor execution error

+ **2 pts** Major execution error

+ **0 pts** Incorrect

+ **0 pts** Incorrect

QUESTION 6

It'll All Even Out 6 pts

6.1 The Predicate P(n) 1 / 1

✓ + **1 pts** **Correct**

+ **0 pts** Incorrect

6.2 P(0) 1 / 1

✓ + **1 pts** **Correct**

+ **0 pts** Incorrect

6.3 P(1) 4 / 4

✓ + **4 pts** **Correct**

+ **2 pts** Mostly correct

+ **1 pts** Partially correct

+ **0 pts** Incorrect

QUESTION 7

Contractions 13 pts

7.1 Identifying Contractions 1 / 1

✓ + **1 pts** **Correct**

+ **0 pts** Incorrect

7.2 Iterated Contractions 2 / 2

✓ + **2 pts** **All correct**

+ **1 pts** One answer incorrect

+ **0 pts** Incorrect

+ **0 pts** Only one answer is correct

7.3 Iterated Contractions are Contractions 6 / 10

+ **10 pts** All correct

Intuition Scale

+ **4 pts** Correct

✓ + **2 pts** **Partially correct intuition**

+ **0 pts** Incorrect

Execution Scale

+ **6 pts** Correct

✓ + **4 pts** **Minor execution error**

+ **2 pts** Major execution error

+ **0 pts** Incorrect

+ **0 pts** Incorrect

+ **0 pts** Check again

3 No question of "if" - you have either already

assumed that or should do it.

4 This claim is incorrect - this assumes that $$f^*$$ is nondecreasing, which isn't necessarily true.

5 Should use complete induction instead of arguing about maximums.

6 Observe how you don't actually use your complete induction hypothesis (and just a weak induction hypothesis)

QUESTION 8

8 Optional Fun Problem: Egyptian Fractions

0 / 0

+ **0 pts** Correct!! Congratulations!!

✓ + **0 pts** **Incorrect / Not attempted**

Problem One: Induction Proof Critiques

i.

- They did not define P(n) as a predicate. They defined it as a function instead. A better way of introducing P(n) is to say: Let $P(n)$ be the statement "for a natural number n, $\frac{n(n-1)}{2}$ holds true."
- For the inductive step, you cannot assume all natural numbers k. You must pick some arbitrary number k instead.
- The base case P(1) is unnecessary and redundant. P(0) is sufficient. P(1) can be reached via the inductive step.

ii.

- The scope of the P(n) predicate is incorrect. It should not be "for all n" and should be for a specific, arbitrary n instead.
- The theorem itself is incorrect. It is possible for directed graphs to contain cycles.
- The creation of the new directed graph G' has faulty logic. Because $P(n)$ is universally quantified, there are no requirements for how many new edges and what direction the new edges are pointing. For all we know, G' could have no new edges, or all new edges could be going towards v instead of away from v.
- Since this is a universally-quantified consequent, they should go from a graph with $k+1$ nodes to a graph with k nodes instead. They should be "building down". They did not do this, they built up instead.
- Their base case is proven based on the logic that the graph has no nodes and therefore doesn't have any cycles (because the cycle must contain a node). This means they need more base cases with graphs that contain nodes.
- They used a proof by contradiction but didn't state they were.

1.1 Sums of Natural Numbers **2 / 4**

+ **1 pts** P(n) is a number/function rather than a predicate

✓ + **1 pts** **P(1) unnecessary**

✓ + **1 pts** **Inductive step says for all natural number k**

+ **1 pts** Inductive step starts with unknown equality and simplifies to known equality

+ **0 pts** Incorrect

1 You wrote yourself out of points with this: "n(n-1)/2 holds true" doesn't make sense, just as "3 holds true" doesn't.

2 Don't define n in the predicate

Problem One: Induction Proof Critiques

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- They did not define P(n) as a predicate. They defined it as a function instead. A better way of introducing P(n) is to say: Let $P(n)$ be the statement "for a natural number n, $\frac{n(n-1)}{2}$ holds true."
- For the inductive step, you cannot assume all natural numbers k. You must pick some arbitrary number k instead.
- The base case P(1) is unnecessary and redundant. P(0) is sufficient. P(1) can be reached via the inductive step.

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- The scope of the P(n) predicate is incorrect. It should not be "for all n" and should be for a specific, arbitrary n instead.
- The theorem itself is incorrect. It is possible for directed graphs to contain cycles.
- The creation of the new directed graph G' has faulty logic. Because $P(n)$ is universally quantified, there are no requirements for how many new edges and what direction the new edges are pointing. For all we know, G' could have no new edges, or all new edges could be going towards v instead of away from v.
- Since this is a universally-quantified consequent, they should go from a graph with $k+1$ nodes to a graph with k nodes instead. They should be "building down". They did not do this, they built up instead.
- Their base case is proven based on the logic that the graph has no nodes and therefore doesn't have any cycles (because the cycle must contain a node). This means they need more base cases with graphs that contain nodes.
- They used a proof by contradiction but didn't state they were.

1.2 Acyclic Directed Graphs 4 / 4

- ✓ **+ 2 pts** P(n) says for any n, shadowing parameter n
- ✓ **+ 2 pts** Build-up vs build-down
- **+ 0 pts** Incorrect

Problem Two: Recurrence Relations

Fill in the blanks to Problem Two below.

Theorem: For all natural numbers n, we have $a_n = 2^n$.
Proof: Let $P(n)$ be the statement "$a_n = 2^n$." We will prove by induction that $P(n)$ holds for all $n \in \mathbb{N}$, from which the theorem follows.
As our base case, we prove $P(0)$, that $a_0 = 2^0$. To see this, $2^0 = 1$, which is what a_0 is defined as. So we see that $P(0)$ is true.
For our inductive step, assume for some arbitrary $k \in \mathbb{N}$ that $P(k)$ is true, meaning that $a_k = 2^k$. We need to show $P(k+1)$, meaning that $a_{k+1} = 2^{k+1}$. To see this, note that

$$\begin{aligned} a_{k+1} &= 2(a_k) \\ &= 2(2^k)(\text{by our IH}) \\ &= 2^{k+1}. \end{aligned}$$

Therefore, we see that $a_{k+1} = 2(a^k) = 2^{k+1}$, so $P(k+1)$ is true, completing the induction. ■

2 Recurrence Relations 3 / 3

✓ + **3 pts** **All correct**

+ **2 pts** Incomplete/incorrect base case, but everything else correct

Intuition Scale

+ **1 pts** Correct
+ **0 pts** Incorrect

Execution Scale

+ **2 pts** Correct
+ **1 pts** Major execution error
+ **0 pts** Incorrect

+ **0 pts** Incorrect

Problem Three: Stacking Cans

i.

Theorem: For all natural numbers $n \geq 1$, we have $h_n = 3n(n-1)+1$.
Proof: Let $P(n)$ be the statement "$h_n = 3n(n-1)+1$." We will prove by induction that $P(n)$ holds for all $n \in \mathbb{N}$ and $n \geq 1$, from which the theorem follows.
As our base case, we prove $P(1)$, that $h_1 = 3(1)(1-1)+1$. To see this, $3(1)(1-1)+1 = 1$, which is what h_n is defined as. So we see that $P(1)$ is true.
For our inductive step, assume for some arbitrary $k \in \mathbb{N}$ and $k \geq 1$ that $P(k)$ is true, meaning that $h_k = 3k(k-1)+1$. We need to show $P(k+1)$, meaning that $h_{k+1} = 3(k+1)(k)+1 = 3k^2+3k+1$. To see this, note that

$$\begin{aligned} h_{k+1} &= h_k + 6k \text{ (from the recurrence relation)} \\ &= 3k(k-1)+1+6k \text{ (by our IH)} \\ &= 3k^2 - 3k + 1 + 6k. \\ &= 3k^2 + 3k + 1 \\ &= 3(k+1)(k)+1 \end{aligned}$$

Therefore, we see that $h_{k+1} = h_k + 6k = 3(k+1)(k)+1$, so $P(k+1)$ is true, completing the induction. ■

ii. Fill in the blanks to Problem Three, part ii. below.

- A 1-layer tower has 1 can in it.
- A 2-layer tower has 8 cans in it.
- A 3-layer tower has 27 cans in it.
- A 4-layer tower has 64 cans in it.
- A 5-layer tower has 125 cans in it.
- A 6-layer tower has 216 cans in it.
- A 7-layer tower has 343 cans in it.
- A 8-layer tower has 512 cans in it.
- A 9-layer tower has 729 cans in it.
- A 10-layer tower has 1000 cans in it.

iii. Fill in the blank to Problem Three, part iii. below.

An n-layer tower has n^3 cans in it.

iv.

3.1 Hexagon Sizes 5 / 5

Intuition Scale

✓ + **2 pts** Correct

+ **1 pts** Partially correct intuition

+ **0 pts** Incorrect

Execution Scale

✓ + **3 pts** Correct

+ **2 pts** Minor execution error

+ **1 pts** Major execution error

+ **0 pts** Incorrect

Problem Three: Stacking Cans

i.

Theorem: For all natural numbers $n \geq 1$, we have $h_n = 3n(n-1) + 1$.
Proof: Let $P(n)$ be the statement "$h_n = 3n(n-1) + 1$." We will prove by induction that $P(n)$ holds for all $n \in \mathbb{N}$ and $n \geq 1$, from which the theorem follows.
As our base case, we prove $P(1)$, that $h_1 = 3(1)(1-1) + 1$. To see this, $3(1)(1-1) + 1 = 1$, which is what h_n is defined as. So we see that $P(1)$ is true.
For our inductive step, assume for some arbitrary $k \in \mathbb{N}$ and $k \geq 1$ that $P(k)$ is true, meaning that $h_k = 3k(k-1) + 1$. We need to show $P(k+1)$, meaning that $h_{k+1} = 3(k+1)(k) + 1 = 3k^2 + 3k + 1$. To see this, note that

$$\begin{aligned}
h_{k+1} &= h_k + 6k \text{ (from the recurrence relation)} \\
&= 3k(k-1) + 1 + 6k \text{ (by our IH)} \\
&= 3k^2 - 3k + 1 + 6k. \\
&= 3k^2 + 3k + 1 \\
&= 3(k+1)(k) + 1
\end{aligned}$$

Therefore, we see that $h_{k+1} = h_k + 6k = 3(k+1)(k) + 1$, so $P(k+1)$ is true, completing the induction. ■

ii. Fill in the blanks to Problem Three, part ii. below.

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- A 9-layer tower has 729 cans in it.
- A 10-layer tower has 1000 cans in it.

iii. Fill in the blank to Problem Three, part iii. below.

An n-layer tower has n^3 cans in it.

iv.

3.2 Counting Cans 1 / 1

✓ **+ 1 pts** **Correct**

+ 0 pts Incorrect

Problem Three: Stacking Cans

i.

Theorem: For all natural numbers $n \geq 1$, we have $h_n = 3n(n-1)+1$.
Proof: Let $P(n)$ be the statement "$h_n = 3n(n-1)+1$." We will prove by induction that $P(n)$ holds for all $n \in \mathbb{N}$ and $n \geq 1$, from which the theorem follows.
As our base case, we prove $P(1)$, that $h_1 = 3(1)(1-1)+1$. To see this, $3(1)(1-1)+1 = 1$, which is what h_n is defined as. So we see that $P(1)$ is true.
For our inductive step, assume for some arbitrary $k \in \mathbb{N}$ and $k \geq 1$ that $P(k)$ is true, meaning that $h_k = 3k(k-1)+1$. We need to show $P(k+1)$, meaning that $h_{k+1} = 3(k+1)(k)+1 = 3k^2+3k+1$. To see this, note that

$$\begin{aligned} h_{k+1} &= h_k + 6k \text{ (from the recurrence relation)} \\ &= 3k(k-1)+1+6k \text{ (by our IH)} \\ &= 3k^2 - 3k + 1 + 6k. \\ &= 3k^2 + 3k + 1 \\ &= 3(k+1)(k)+1 \end{aligned}$$

Therefore, we see that $h_{k+1} = h_k + 6k = 3(k+1)(k)+1$, so $P(k+1)$ is true, completing the induction. ■

ii. Fill in the blanks to Problem Three, part ii. below.

- A 1-layer tower has 1 can in it.
- A 2-layer tower has 8 cans in it.
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- A 9-layer tower has 729 cans in it.
- A 10-layer tower has 1000 cans in it.

iii. Fill in the blank to Problem Three, part iii. below.

An n-layer tower has n^3 cans in it.

iv.

3.3 The General Formula 1 / 1

✓ **+ 1 pts** **Correct**

+ 0 pts Incorrect

Theorem: For all natural numbers n, the total number of cans in a hexagon pyramid with n levels is n^3.

Proof: Let c_n represent the total number of cans in some hexagon pyramid with n levels. Let $P(n)$ be the statement "$c_n = n^3$." We will prove by induction that $P(n)$ holds for all $n \in \mathbb{N}$, from which the theorem follows.

As our base case, we prove $P(0)$, that $c_0 = 0^3$. To see this, $0^3 = 0$, which is the number of cans that a pyramid with no levels has. So we see that $P(0)$ is true.

For our inductive step, assume for some arbitrary $k \in \mathbb{N}$ that $P(k)$ is true, meaning that $c_k = k^3$. We need to show $P(k+1)$, meaning that $c_{k+1} = (k+1)^3 = k^3 + 3k^3 + 3k + 1$. To see this, note that

$$\begin{aligned}
c_{k+1} &= c_k + h_{k+1} \\
&= c_k + 3(k+1)(k) + 1 \text{ (from Part i. of Problem Three)} \\
&= k^3 + 3(k+1)(k) + 1 \text{ (by our inductive hypothesis)} \\
&= k^3 + 3k^2 + 3k + 1 \\
&= (k+1)^3.
\end{aligned}$$

Therefore, we see that $c_{k+1} = c_k + h_{k+1} = (k+1)^3$, so $P(k+1)$ is true, completing the induction.
■

3.4 Towers of Cans 5 / 5

Intuition Scale

✓ + **2 pts** **Correct**

+ **1 pts** Partially correct intuition

+ **0 pts** Incorrect

Execution Scale

✓ + **3 pts** **Correct**

+ **2 pts** Minor execution error

+ **1 pts** Major execution error

+ **0 pts** Incorrect

Problem Four: The Circle Game

i.

P(n) is universally quantified so we will "induct down".

ii.

Theorem: For any natural number n, if n points labeled $+1$ are placed on the boundary of the circle and n points labeled -1 are placed on the boundary of the circle, there's always some starting position on the circle from which you can start and win the circle game.
Proof: Let $P(n)$ be the statement "for any circle with n points labeled $+1$ and n points labeled -1 boundary, there is a starting position on the circle from which you can win the circle game." We will prove by induction that $P(n)$ holds for any natural number n, from which the theorem follows.

For our base case, we prove $P(0)$, that is when there are 0 labeled points on the circle, there is a winning starting point. Moving clockwise from the starting point, the game is never lost because we never pass through any -1 points and therefore cannot have more -1 points than $+1$ points. So we see that $P(0)$ is true.

For our inductive step, assume for some arbitrary $k \in \mathbb{N}$ that $P(k)$ is true, meaning that for a circle with $2k$ labeled points with k points labeled $+1$ and k points labeled -1, there exists a winning starting point. We need to show $P(k+1)$, meaning that for a circle (let us call it C_{k+1}) with $2(k+1)$ labeled points with $k+1$ points labeled $+1$ and $k+1$ points labeled -1, there exists a winning starting point.

To see this, consider C_{k+1}. On C_{k+1}, there must exist the $+1$ point a and the -1 point b that is directly after a when moving clockwise. Suppose we remove a and b from C_{k+1}. This creates the circle C_k with k points labeled $+1$ and k points labeled -1. By our inductive hypothesis, $P(k)$ is true, therefore there exists a winning starting point in C_k.

Now, let us add back a and b to C_{k+1}. Let the integer s be the running sum in the game for C_{k+1}. Recall that a winning starting point is a point where s is always greater than or equal to 0 when summing s and the next consecutive points moving clockwise along the circle. So, if we start at a position p that is a $+1$ point on C_{k+1}, and continue until we reach a (but not including a), up until then, the winning path is identical to a winning path in C_k. Now, if we consider a and b concurrently, adding a to s first keeps $s \geq 1$ and adding b nullifies the effect of adding a. In other words, adding a to s and then adding b to s has no net impact on the winning path because $a + b = 0$. Thus, the rest of the remaining path after b is unaffected and continues as follows. So, starting from position p on C_{k+1} also yields a winning path.

Therefore, there exists a winning starting position p on a circle with $2(k + 1)$ labeled points, so $P(k + 1)$ is true, completing the induction. ■

4.1 Up or Down? 1 / 1

✓ **+ 1 pts** **Correct**

+ 0.5 pts Correctly identifies P(n) as a universally-quantified statement, but does not say whether the induction should "induct up" or "induct down".

+ 0 pts Incorrect

Problem Four: The Circle Game

i.

P(n) is universally quantified so we will "induct down".

ii.

Theorem: For any natural number n, if n points labeled $+1$ are placed on the boundary of the circle and n points labeled -1 are placed on the boundary of the circle, there's always some starting position on the circle from which you can start and win the circle game.
Proof: Let $P(n)$ be the statement "for any circle with n points labeled $+1$ and n points labeled -1 boundary, there is a starting position on the circle from which you can win the circle game." We will prove by induction that $P(n)$ holds for any natural number n, from which the theorem follows.

For our base case, we prove $P(0)$, that is when there are 0 labeled points on the circle, there is a winning starting point. Moving clockwise from the starting point, the game is never lost because we never pass through any -1 points and therefore cannot have more -1 points than $+1$ points. So we see that $P(0)$ is true.

For our inductive step, assume for some arbitrary $k \in \mathbb{N}$ that $P(k)$ is true, meaning that for a circle with $2k$ labeled points with k points labeled $+1$ and k points labeled -1, there exists a winning starting point. We need to show $P(k+1)$, meaning that for a circle (let us call it C_{k+1}) with $2(k+1)$ labeled points with $k+1$ points labeled $+1$ and $k+1$ points labeled -1, there exists a winning starting point.

To see this, consider C_{k+1}. On C_{k+1}, there must exist the $+1$ point a and the -1 point b that is directly after a when moving clockwise. Suppose we remove a and b from C_{k+1}. This creates the circle C_k with k points labeled $+1$ and k points labeled -1. By our inductive hypothesis, $P(k)$ is true, therefore there exists a winning starting point in C_k.

Now, let us add back a and b to C_{k+1}. Let the integer s be the running sum in the game for C_{k+1}. Recall that a winning starting point is a point where s is always greater than or equal to 0 when summing s and the next consecutive points moving clockwise along the circle. So, if we start at a position p that is a $+1$ point on C_{k+1}, and continue until we reach a (but not including a), up until then, the winning path is identical to a winning path in C_k. Now, if we consider a and b concurrently, adding a to s first keeps $s \geq 1$ and adding b nullifies the effect of adding a. In other words, adding a to s and then adding b to s has no net impact on the winning path because $a + b = 0$. Thus, the rest of the remaining path after b is unaffected and continues as follows. So, starting from position p on C_{k+1} also yields a winning path.

Therefore, there exists a winning starting position p on a circle with $2(k+1)$ labeled points, so $P(k+1)$ is true, completing the induction. ■

4.2 Winning the Circle Game 10 / 10

Intuition Scale

✓ + **4 pts** **Correct**

+ **2 pts** Partially correct intuition

+ **0 pts** Incorrect

Execution Scale

✓ + **6 pts** **Correct**

+ **4 pts** Minor execution error

+ **2 pts** Major execution error

+ **0 pts** Incorrect

Problem Five: Regular Graphs

Theorem: For all natural number n, there exists an n-regular graph containing exactly 2^n nodes.
Proof: Let $P(n)$ be the statement "for the natural number n, there exists an n-regular graph containing exactly 2^n nodes." We will prove by induction that $P(n)$ holds for all $n \in \mathbb{N}$, from which the theorem follows.

For our base case, we prove $P(0)$, that is when $n = 0$. To see this, we must find a 0-regular graph with $2^0 = 1$ nodes. Let $G_0 = (V_0, E_0)$ be a graph with 1 node where $V_0 = \{v_0\}$ and $E_0 = \emptyset$. We see that the degree of v_0 is 0, satisfying the definition of a 0-regular graph. So we see that $P(0)$ is true.

For our inductive step, assume for some arbitrary $k \in \mathbb{N}$ that $P(k)$ is true, meaning that for the natural number k, there exists a k-regular graph with 2^k nodes. We need to show $P(k+1)$, meaning that for the natural number $k+1$, there exists a $(k+1)$-regular graph with 2^{k+1} nodes.

To see this, start by obtaining (via the inductive hypothesis) a k-regular graph $G_k = (V_k, E_k)$ with $|V_k| = 2^k$. Then, obtain a duplicate graph $G_{k'} = (V_{k'}, E_{k'})$ identical to G_k. For every unique $v_k \in V_k$, draw exactly one edge to a unique $v_{k'} \in V_{k'}$ such that each vertex in V_k and $V_{k'}$ has a degree of $k+1$. This creates a $(k+1)$-regular graph $G_{k+1} = (V_{k+1}, E_{k+1})$ where $|V_{k+1}| = 2(2^k) = 2^{k+1}$.

Therefore there exists a $(k+1)$-regular graph with 2^{k+1} nodes for the natural number $k+1$, so $P(k+1)$ is true, completing the induction. ■

5 Regular Graphs 10 / 10

✓ **+ 10 pts** **Correct**

Intuition Scale

- **+ 4 pts** Correct
- **+ 2 pts** Partially correct intuition
- **+ 0 pts** Incorrect

Execution Scale

- **+ 6 pts** Correct
- **+ 4 pts** Minor execution error
- **+ 2 pts** Major execution error
- **+ 0 pts** Incorrect

- **+ 0 pts** Incorrect

Problem Six: Itâ€™ll All Even Out

i.

$P(n)$ is a predicate. If $P(n)$ is true for all $n \in \mathbb{N}$, then the theorem would be true. (However the theorem itself isn't even true so $P(n)$ shouldn't be true for all $n \in \mathbb{N}$.

ii.

Yes it's correct.

iii.

$P(1)$ is not the inductive hypothesis. This should be another base case (and if they attempted to prove this base case, they would realize this theorem is incorrect). They cannot assume $P(1)$ is true, however if they proved $P(1)$ correctly in the base case, then they could use this logic. The inductive hypothesis is actually that $P(k)$ is true.

6.1 The Predicate P(n) 1 / 1

✓ + 1 pts Correct

+ 0 pts Incorrect

Problem Six: Itâ€™ll All Even Out

i.

$P(n)$ is a predicate. If $P(n)$ is true for all $n \in \mathbb{N}$, then the theorem would be true. (However the theorem itself isn't even true so $P(n)$ shouldn't be true for all $n \in \mathbb{N}$.

ii.

Yes it's correct.

iii.

$P(1)$ is not the inductive hypothesis. This should be another base case (and if they attempted to prove this base case, they would realize this theorem is incorrect). They cannot assume $P(1)$ is true, however if they proved $P(1)$ correctly in the base case, then they could use this logic. The inductive hypothesis is actually that $P(k)$ is true.

6.2 P(0) **1 / 1**

✓ **+ 1 pts** Correct

+ 0 pts Incorrect

Problem Six: Itâ€™ll All Even Out

i.

$P(n)$ is a predicate. If $P(n)$ is true for all $n \in \mathbb{N}$, then the theorem would be true. (However the theorem itself isn't even true so $P(n)$ shouldn't be true for all $n \in \mathbb{N}$.

ii.

Yes it's correct.

iii.

$P(1)$ is not the inductive hypothesis. This should be another base case (and if they attempted to prove this base case, they would realize this theorem is incorrect). They cannot assume $P(1)$ is true, however if they proved $P(1)$ correctly in the base case, then they could use this logic. The inductive hypothesis is actually that $P(k)$ is true.

6.3 P(1) **4 / 4**

- ✓ **+ 4 pts** Correct
- **+ 2 pts** Mostly correct
- **+ 1 pts** Partially correct
- **+ 0 pts** Incorrect

Problem Seven: Contractions

i.

1. This is not a contraction.
2. This is a contraction
3. This is a contraction.
4. This is a contraction.
5. This is not a contraction.

ii.

2. 136
3. 0
4. 7

iii.

Theorem: If f is a contraction, then $f^\star(n)$ is also a contraction.
Proof: Let $f : \mathbb{N} \to \mathbb{N}$ be a contraction. We will prove that $f^\star$ is a contraction. To do so, we need to prove that $f^\star(0) = 0$ and for all natural numbers $n \geq 1$, $f^\star(n) \leq n - 1$.

First, we will prove $f^\star(0) = 0$. We want to show that $f^\star(0) = 0$. Since $f(0) = 0$, $f^\star(0) = 0$ by the definition of $f^\star$, as needed.

Now, we will prove for all natural numbers $n \geq 1$, $f^\star(n) \leq n - 1$. Let $P(n)$ be the following statement "$f^\star(n) \leq n - 1$." We will prove by induction that $P(n)$ holds for all natural numbers $n \geq 1$, from which the theorem follows.

As a base case, we will prove that P(1) is true, namely that for $n = 1$, $f^\star(1) \leq 1 - 1 = 0$. For f, it is worth noting that the codomain is the set of natural numbers, so $f(1) = 0$ because $f(1) \leq 1 - 1$ by the definition of a contraction. $f(1) = 0$, then by the definition of $f^\star$, we know $f^\star(1) = 0$. Thus, P(1) holds true.

For our inductive case, we will assume for an arbitrary natural number $k \geq 1$ that $P(1), P(2), ..., P(k)$ is true. We need to prove $P(k + 1)$ is true, meaning $f^\star(k + 1) \leq k$.

There are two cases that arise.

Case 1: If $f^\star(k + 1) = 0$, then we know that $f^\star(k + 1) \leq k$ when $k \geq 0$, as required.
Case 2: If $f^\star(k + 1) \neq 0$, then $f^\star(k + 1) = 1 + f^\star(f(k + 1))$. Since f is a contraction, then $f(k + 1) \leq k$. This means that $f^\star(k + 1) \leq 1 + f^\star(k)$. We know $f^\star(k) \leq k - 1$ from our IH and by rearranging the inequalities, we get $f^\star(k+1) \leq 1 + f^\star(k) \leq k - 1 \leq k$, as required.

Therefore, $P(k + 1)$ holds, completing the induction.
■

7.1 Identifying Contractions 1 / 1

- ✓ + 1 pts Correct
- + 0 pts Incorrect

Problem Seven: Contractions

i.

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5. This is not a contraction.

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Theorem: If f is a contraction, then $f^{\star}(n)$ is also a contraction.
Proof: Let $f : \mathbb{N} \to \mathbb{N}$ be a contraction. We will prove that $f^{\star}$ is a contraction. To do so, we need to prove that $f^{\star}(0) = 0$ and for all natural numbers $n \geq 1$, $f^{\star}(n) \leq n - 1$.

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Now, we will prove for all natural numbers $n \geq 1$, $f^{\star}(n) \leq n - 1$. Let $P(n)$ be the following statement "$f^{\star}(n) \leq n - 1$." We will prove by induction that $P(n)$ holds for all natural numbers $n \geq 1$, from which the theorem follows.

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For our inductive case, we will assume for an arbitrary natural number $k \geq 1$ that $P(1), P(2), ..., P(k)$ is true. We need to prove $P(k+1)$ is true, meaning $f^{\star}(k+1) \leq k$.

There are two cases that arise.

Case 1: If $f^{\star}(k+1) = 0$, then we know that $f^{\star}(k+1) \leq k$ when $k \geq 0$, as required.
Case 2: If $f^{\star}(k+1) \neq 0$, then $f^{\star}(k+1) = 1 + f^{\star}(f(k+1))$. Since f is a contraction, then $f(k+1) \leq k$. This means that $f^{\star}(k+1) \leq 1 + f^{\star}(k)$. We know $f^{\star}(k) \leq k - 1$ from our IH and by rearranging the inequalities, we get $f^{\star}(k+1) \leq 1 + f^{\star}(k) \leq k - 1 \leq k$, as required.

Therefore, $P(k+1)$ holds, completing the induction.
■

7.2 Iterated Contractions **2 / 2**

- ✓ **+ 2 pts** **All correct**
- **+ 1 pts** One answer incorrect
- **+ 0 pts** Incorrect
- **+ 0 pts** Only one answer is correct

Problem Seven: Contractions

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5. This is not a contraction.

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3. 0
4. 7

iii.

Theorem: If f is a contraction, then $f^\star(n)$ is also a contraction.
Proof: Let $f : \mathbb{N} \to \mathbb{N}$ be a contraction. We will prove that $f^\star$ is a contraction. To do so, we need to prove that $f^\star(0) = 0$ and for all natural numbers $n \geq 1$, $f^\star(n) \leq n - 1$.

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As a base case, we will prove that P(1) is true, namely that for $n = 1$, $f^\star(1) \leq 1 - 1 = 0$. For f, it is worth noting that the codomain is the set of natural numbers, so $f(1) = 0$ because $f(1) \leq 1 - 1$ by the definition of a contraction. $f(1) = 0$, then by the definition of $f^\star$, we know $f^\star(1) = 0$. Thus, P(1) holds true.

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There are two cases that arise.

Case 1: If $f^\star(k + 1) = 0$, then we know that $f^\star(k + 1) \leq k$ when $k \geq 0$, as required.
Case 2: If $f^\star(k + 1) \neq 0$, then $f^\star(k + 1) = 1 + f^\star(f(k + 1))$. Since f is a contraction, then $f(k + 1) \leq k$. This means that $f^\star(k + 1) \leq 1 + f^\star(k)$. We know $f^\star(k) \leq k - 1$ from our IH and by rearranging the inequalities, we get $f^\star(k+1) \leq 1 + f^\star(k) \leq k - 1 \leq k$, as required.

Therefore, $P(k + 1)$ holds, completing the induction.
■

7.3 Iterated Contractions are Contractions 6 / 10

+ **10 pts** All correct

Intuition Scale

+ **4 pts** Correct

✓ + **2 pts** **Partially correct intuition**

+ **0 pts** Incorrect

Execution Scale

+ **6 pts** Correct

✓ + **4 pts** **Minor execution error**

+ **2 pts** Major execution error

+ **0 pts** Incorrect

+ **0 pts** Incorrect

+ **0 pts** Check again

3 No question of "if" - you have either already assumed that or should do it.

4 This claim is incorrect - this assumes that $$f^*$$ is nondecreasing, which isn't necessarily true.

5 Should use complete induction instead of arguing about maximums.

6 Observe how you don't actually use your complete induction hypothesis (and just a weak induction hypothesis)

8 Optional Fun Problem: Egyptian Fractions **0 / 0**

+ **0 pts** Correct!! Congratulations!!

✓ + **0 pts** **Incorrect / Not attempted**

CS 103: Mathematical Foundations of Computing
Problem Set #5

Maria Wang and Matthew Vilaysack

May 6, 2022

Due Friday, October 29 at 2:30 pm Pacific

This Problem Set has no coding questions; all answers to the Problem Set go in this file.
Some notation you might find useful here:

- Subscripts can be written as a_{index}. Remember to use curly braces if you have a multicharacter expression as a subscript.
- The notation $f^{\star}(n)$ comes up in the last problem.