semicircledistr: An R Package to Simulate the Semicircle Distribution

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Abstract
This paper introduces semicircledistr, an R package designed to simulate and analyze the Wigner semicircle distribution, a fundamental concept in random matrix theory. Despite its theoretical importance, the distribution—defined by a semicircular density over a bounded interval—has no native implementation on CRAN. The package offers essential distribution functions, including the density, cumulative distribution, quantile, and random sampling. Its implementation is validated through simulations of large symmetric random matrices, illustrating Wigner's semicircle law, and through the convergence behavior of a transformed beta distribution.

Key Words: Semicircle distribution; CRAN; R Package; Software; Random matrix; Simulation

1. Introduction

The semicircle distribution was first proposed by physicist Eugene Wigner who observed the semicircle law for certain classes of random matrices arising in quantum mechanical investigations (Wigner, 1955). It is called the Wigner semicircle distribution because the probability density function forms a symmetrical semicircle shape with an area of 1. The parameters of this distribution are the radius *R* and an offset parameter, which is a linear shift along the x-axis (by default, the distribution is centered at $X = 0$). The semicircle distribution is a continuous univariate distribution which only has a non-zero probability within its radius.

The Wigner Semicircle distribution has application in the realm of random matrices. Given an $N \times N$ symmetric random matrix in which all values are independent and identically distributed (i.i.d.), the eigenvalues of such a matrix will converge to the semicircle distribution as *N* approaches infinity. This applies regardless of the probability distribution, so long as the values are randomly sampled and are independent. For some distributions such as the uniform distribution, the eigenvalues of the i.i.d. matrix will contain exactly one large value, and the remaining eigenvalues will converge to the semicircle distribution (Weisstein, 2025).

Currently, there does not exist an implementation of the Wigner Semicircle distribution on the Comprehensive R Archive Network (CRAN), and we were unable to find any R implementation of such a package. Therefore, the contribution of this project is to create an R implementation of the standard R distribution functions for the Wigner semicircle distribution. This R package will follow existing conventions of including functions for the probability density function, cumulative distribution function, quantile function, and random generation function.

In addition to providing an implementation of the Wigner semicircle distribution, we provide examples of its usage and demonstrate that random matrices do converge to the

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distribution. Finally, we show that for a beta distribution in the special case $\alpha = \beta = \frac{3}{2}$, random variables converge to the semicircle distribution (Wikimedia Foundation, 2024). This relationship between the beta distribution and the semicircle distribution makes it easier to compute some statistical quantiles for the Wigner distribution in terms of the beta distribution, which is better known.

2. The R Package

The repository for the R package can be found at `https://github.com/matthewwhite1/semicircledistr`.

2.1 Probability Density Function

The probability density function (PDF) of a distribution is a function whose value is the relative probability that the value of a random variable from the distribution is equal to that sample. The PDF of the semicircle distribution can be computed using the following closed-form solution in $[-R + a, R + a]$:

$$f(x) = \frac{2}{\pi R^2}\sqrt{R^2 - (x-a)^2} \tag{1}$$

Outside of that range, $f(x) = 0$ (Wikimedia Foundation, 2024).

In R packages, It is convention that this function is prefixed with *d*, such as the `dnorm` function corresponding to the normal distribution. As such, we created a `dsemicircle` function which is the PDF for the Wigner Semicircle distribution.

The `dsemicircle` function takes 3 arguments: `x` which is a vector of values to calculate the probability density of, `R` which is the radius of the circle, and `a` which is an optional parameter which offsets the distribution on the X axis (by default it is centered at 0). The function makes sure to perform validation on the inputs, such as ensuring that the inputs are numeric values. All the components of the closed form solution of this are provided in base R, making the implementation of the PDF straightforward.

The distribution has a mean, median, and mode at *a*, and has a variance of $\frac{R^2}{4}$.

2.2 Cumulative Distribution Function

The cumulative distribution function of a distribution describes the probability that a value is less than or equal to *x* in a distribution. The CDF of the Wigner semicircle distribution is given using a closed form solution, which is defined from $[-R + a, R + a]$.

$$F(x) = \frac{1}{2} + \frac{(x-a)\sqrt{R^2 - (x-a)^2}}{\pi R^2} + \frac{sin^{-1}(\frac{x-a}{R})}{\pi} \tag{2}$$

Following conventions of existing R packages, the function computing the CDF of the Wigner semicircle distribution is called `psemicircle`. This function takes the same arguments as `dsemicircle`, that is `x` which is a vector of values to calculate the cumulative probability of, `R` which is the radius of the circle, and `a` which is the optional offset parameter. Furthermore, validation is performed to ensure function inputs are valid. Just like the PDF, the closed form solution of the CDF can be implemented in R.

2.3 Quantile Function

The quantile function (also known as the inverse cumulative distribution function) returns the value x such that the cumulative distribution function (CDF) evaluated at x equals a given probability p. That is, for a given $p \in [0, 1]$, the quantile function solves for x in the equation:

$$F(x) = p$$

For the Wigner semicircle distribution, the CDF $F(x)$ involves both a square root and an inverse sine function. While it is possible to write down the CDF in closed form, these components make it unable to algebraically solve for x in terms of p. As a result, the inverse CDF (i.e., the quantile function) does not have a closed-form solution.

To compute quantiles numerically, we use the `uniroot` function from base R. This function performs numerical root-finding to solve equations of the form $f(x) = 0$ for a single variable x. In the context of the `qsemicircle` function, `uniroot` is used to find the value of x such that:

$$F(x) - p = 0$$

This process is performed over the domain $[a - R, a + R]$, the support of the Wigner semicircle distribution. The `uniroot` function is used to iteratively narrow the interval containing the root, guaranteeing convergence as long as the function is continuous and changes sign within the interval.

The `qsemicircle` function accepts `p`, a numeric vector of probabilities between 0 and 1; `R`, the radius of the semicircle; and `a`, an optional numeric parameter that shifts the distribution along the x-axis (defaulting to 0).

Basic input validation is included to ensure that values of `p` lie within $[0, 1]$, `R` is positive, and `a` is numeric. For edge cases where $p = 0$ or $p = 1$, the function returns $a - R$ or $a + R$, respectively. For all other values, `uniroot` is called internally, using the previously defined `psemicircle` function to find the value of x corresponding to the given probability p.

This numerical approach ensures an accurate quantile computation even in the absence of a closed-form inverse function.

2.4 Random Sampling

Random sampling from a probability distribution is a common technique in simulation and probabilistic modeling. For the Wigner semicircle distribution, random samples can be generated using the inverse transform sampling method. This method relies on the fact that if U is a uniform random variable on $[0, 1]$, then $X = F^{-1}(U)$ has the desired distribution, where F^{-1} is the quantile function.

In this package, the random sampling function is called `rsemicircle`, again following standard R naming conventions. It uses the quantile function defined in `qsemicircle` to transform uniform samples into semicircle-distributed values.

The `rsemicircle` function also takes three arguments. The first argument, `n`, is an integer indicating the number of random samples to generate. The second argument, `R`, is the radius of the semicircle and must be a positive number. The third argument, `a`, is the optional numeric parameter that offsets the distribution on the x-axis (with a default value of 0).

The function first checks that the inputs are valid: `n` must be an integer, `R` must be positive, and `a` must be numeric. Then, it generates `n` independent values from the uniform distribution on $[0, 1]$ using `stats::runif(n)`. These uniform values are then passed to `qsemicircle`, which transforms them into random samples from the Wigner semicircle distribution leveraging the previously implemented quantile function.

3. Applying Package Functions

3.1 Verifying Wigner's Semicircle Law

Recalling from earlier in this paper: The distribution of the eigenvalues of an $N \times N$ symmetric random matrix will converge to the semicircle distribution as N goes to infinity.

In order to verify this law using the functions in our package, we can perform the following steps:

(i) Simulate three Gaussian random matrices with different sizes (1000×1000, 2500×2500, and 5000×5000) (Jiang, 2021)

(ii) Symmetrize and scale each matrix, which is performed by adding the original matrix's transpose to itself and dividing by $\sqrt{2N}$

(iii) Compute the eigenvalues of each matrix

(iv) Plot histograms to investigate the distributions

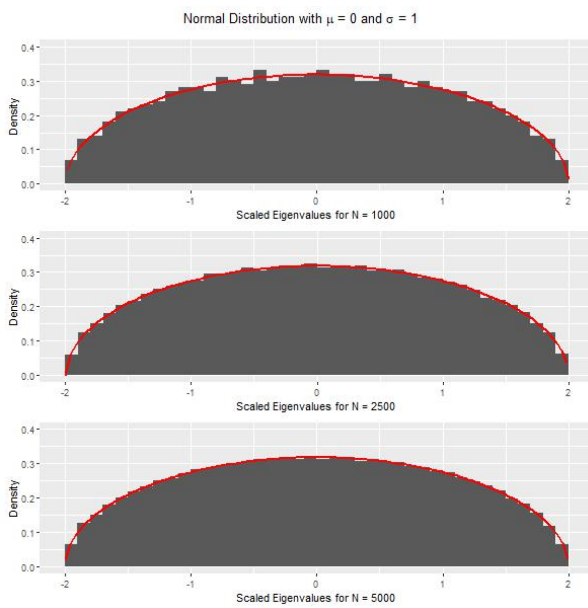


Figure 1: Verification of Wigner's Semicircle Law on normal random matrices

Figure 1 shows the results of following these steps. We can see that as N increases, the distribution more closely follows the red semicircle curve, which is created from our `dsemicircle` function.

While the normal distribution was the first distribution we looked at, it is not the only probability distribution that Wigner's semicircle law applies to. We also applied the same steps to the exponential distribution, the uniform distribution, and our own implementation of the semicircle distribution.

For the semicircle to cleanly be between -2 and 2, the random matrix has to have a mean of 0 and a variance of 1. This required different scaling depending on the distribution - these different scaling techniques can be seen in the "Scripts" folder of the R package repository.

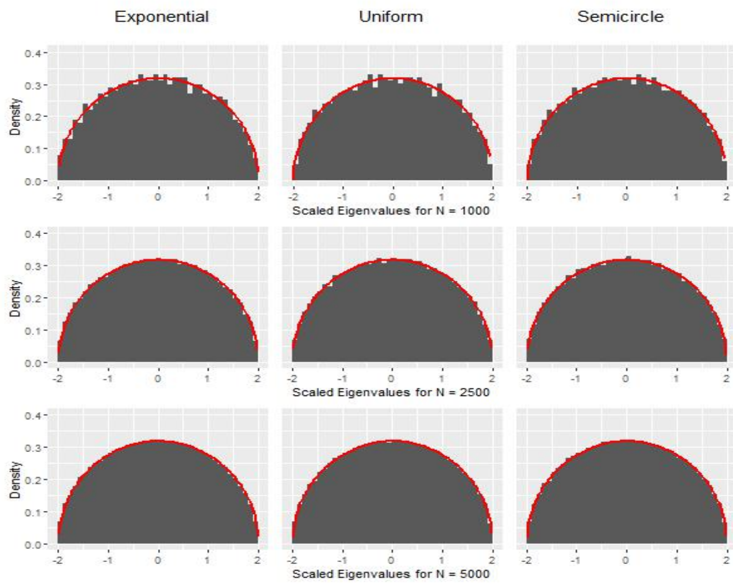


Figure 2: Verification of Wigner's Semicircle Law on random matrices. The left column comes from an exponential distribution with $\lambda = 1$. The middle column comes from a uniform distribution with $a = -\sqrt{3}$ and $b = \sqrt{3}$. The right column comes from our own implementation of the semicircle distribution with $R = 2$ and $a = 0$.

Figure 2 shows that Wigner's semicircle law applies to the exponential distribution, the uniform distribution, and even the semicircle distribution.

3.2 Verifying a Beta Transformation Theory

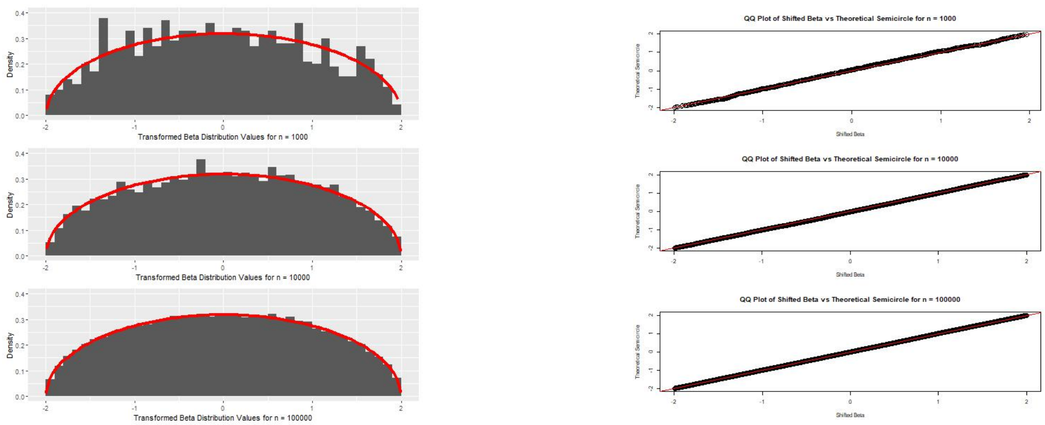
As referenced earlier in this paper: If Y is a beta-distributed random variables with parameters $\alpha = \beta = \frac{3}{2}$, then the random variable $2RY - R$ exhibits a Wigner semicircle distribution with radius R.

In order to verify this theorem using the functions in our package, we can perform the following steps:

(i) Simulate three random vectors from a beta distribution with parameters $\alpha = \beta = \frac{3}{2}$ and different sizes (1000, 10000, 100000)

(ii) Calculate a transformation of each of these vectors given $2RY - R$, choosing a radius value of 2

(iii) Examine the distributions



(a) Histograms (b) QQ Plots

Figure 3: Verification of a beta transformation theory

Figure 3 shows that this theorem seems to hold when compared to the theoretically based functions in our package. The histograms show that the distribution looks more like a semicircle as the size of the vector increases, and the QQ plots corroborate this message.

4. Conclusions and Future Work

This report detailed the R package that was created to simulate the semicircle distribution, which does not have any other current R packages that simulate it. This R package includes the four probability distribution simulation functions, documentation for these functions, a README, and unit tests. We were able to apply our functions to simulated data to determine that both our functions work properly and that the Wigner semicircle law seems to hold.

If we had more time to work on this project, we would have liked to do some more advanced math with the distribution simulation. For example, the Wikipedia page for the Wigner semicircle distribution provides some details about the special properties of the moments of this distribution when $R = 2$ (Wikimedia Foundation, 2024).

Starting in May 2025, we will work on submitting our package to CRAN for public use. Before doing this, we still need to refine our DESCRIPTION file and select an official package maintainer. Fortunately, we have already accomplished most of the necessary tasks for submitting an R package to CRAN. Hopefully, other people can find some meaning in using our package to study ideas like Wigner's semicircle law.

References

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A. R Package Code

```
dsemicircle <- function(x, R, a = 0) {
  if (!is.numeric(x)) {
    stop("x must be numeric.")
  }
  if (R <= 0) {
    stop("R must be positive.")
  }
  if (!is.numeric(a)) {
    stop("a must be numeric.")
  }
  ifelse(abs(x - a) > R, 0, (2 / (pi * R^2)) * sqrt(R^2 - (x - a)^2))
}


psemicircle <- function(x, R, a = 0) {
  if (!is.numeric(x)) {
    stop("x must be numeric.")
  }
  if (R <= 0) {
    stop("R must be positive.")
  }
  if (!is.numeric(a)) {
    stop("a must be numeric.")
  }
  result <- numeric(length(x))
  for (i in 1:length(x)) {
    val <- x[i] - a
    if (abs(val) > R) {
      stop("x must be within radius.")
    }
    result[i] <- 0.5 + (val * sqrt(R^2 - val^2))
    / (pi * R^2) + asin(val / R) / pi
  }
  return(result)
}


qsemicircle <- function(p, R, a = 0) {
  if (!is.numeric(p) || any(p < 0) || any(p > 1)) {
    stop("p must be between 0 and 1.")
  }
  if (R <= 0) {
    stop("R must be positive.")
  }
  if (!is.numeric(a)) {
    stop("a must be numeric.")
  }
  quantile_fn <- function(prob) {
```

```
    sapply(prob, function(p) {
      if (p == 0) return(a - R)
      if (p == 1) return(a + R)
      uniroot(function(x) psemicircle(x, R, a) - p,
              lower = a - R, upper = a + R)$root
    })
  }
  quantile_fn(p)
}


rsemicircle <- function(n, R, a = 0) {
  if (!is.numeric(n) || n != round(n)) {
    stop("n must be an integer.")
  }
  if (R <= 0) {
    stop("R must be positive.")
  }
  if (!is.numeric(a)) {
    stop("a must be numeric.")
  }
  u <- stats::runif(n)
  qsemicircle(u, R, a)
}


library(ggplot2)
library(pracma) # for random matrices
library(patchwork)


# Create 3 N by N random matrices
# Stopping at 5000 instead of 10000 to save on computation time
set.seed(1234)

norm_1000 <- randn(1000)
norm_1000_sym <- (norm_1000 + t(norm_1000)) / sqrt(2 * 1000)
ev_1000 <- eigen(norm_1000_sym)
ev_1000_df <- data.frame(ev = ev_1000$values)

norm_2500 <- randn(2500)
norm_2500_sym <- (norm_2500 + t(norm_2500)) / sqrt(2 * 2500)
ev_2500 <- eigen(norm_2500_sym)
ev_2500_df <- data.frame(ev = ev_2500$values)

norm_5000 <- randn(5000)
norm_5000_sym <- (norm_5000 + t(norm_5000)) / sqrt(2 * 5000)
ev_5000 <- eigen(norm_5000_sym) # This takes a few minutes to run
ev_5000_df <- data.frame(ev = ev_5000$values)
```

```
# Create histograms
g1 <- ggplot(ev_1000_df, aes(ev)) +
  geom_histogram(aes(y = after_stat(density)),
  breaks = seq(-2, 2, by = 0.1)) + xlab("Scaled Eigenvalues for N = 1000")

g2 <- ggplot(ev_2500_df, aes(ev)) +
  geom_histogram(aes(y = after_stat(density)),
  breaks = seq(-2, 2, by = 0.1)) +
  xlab("Scaled Eigenvalues for N = 2500")

g3 <- ggplot(ev_5000_df, aes(ev)) +
  geom_histogram(aes(y = after_stat(density)),
  breaks = seq(-2, 2, by = 0.1)) +
  xlab("Scaled Eigenvalues for N = 5000")

jpeg("Plots/SemiLaw_NoCurve.jpeg", width = 600, height = 600)

g1 /
  g2 /
  g3 + plot_annotation(
    title = expression(paste("Normal Distribution with ",
    mu, " = 0 and ", sigma, " = 1")),
    theme = theme(plot.title = element_text(hjust = 0.5))
  ) &
  scale_y_continuous("Density",
    breaks = seq(0, 0.4, by = 0.1),
    limits = c(0, 0.4)
  )

dev.off()


# Add semicircle distribution densities
g1 <- g1 +
  stat_function(fun = dsemicircle, args = list(R = 2),
  lwd = 1, color = "red")

g2 <- g2 +
  stat_function(fun = dsemicircle, args = list(R = 2),
  lwd = 1, color = "red")

g3 <- g3 +
  stat_function(fun = dsemicircle, args = list(R = 2),
  lwd = 1, color = "red")

jpeg("Plots/SemiLaw.jpeg", width = 600, height = 600)

g1 /
  g2 /
  g3 + plot_annotation(
```

```
    title = expression(paste("Normal Distribution with ", mu, " = 0
    and ", sigma, " = 1")),
    theme = theme(plot.title = element_text(hjust = 0.5))
  ) &
  scale_y_continuous("Density",
                     breaks = seq(0, 0.4, by = 0.1),
                     limits = c(0, 0.4)
  )

dev.off()


library(ggplot2)
library(gridExtra)


# Create random variable transformations
set.seed(1234)

R <- 2

# n = 1000
Y1 <- rbeta(1000, (3 / 2), (3 / 2))
U1 <- 2 * R * Y1 - R
U1_df <- data.frame(U1_vals = U1)

# n = 10000
Y2 <- rbeta(10000, (3 / 2), (3 / 2))
U2 <- 2 * R * Y2 - R
U2_df <- data.frame(U2_vals = U2)

# n = 100000
Y3 <- rbeta(100000, (3 / 2), (3 / 2))
U3 <- 2 * R * Y3 - R
U3_df <- data.frame(U3_vals = U3)

# Create histograms
g1 <- ggplot(U1_df, aes(U1_vals)) +
  geom_histogram(aes(y = after_stat(density)),
  breaks = seq(-2, 2, by = 0.1)) +
  scale_y_continuous("Density", breaks = seq(0, 0.4, by = 0.1),
  limits = c(0, 0.4)) +
  xlab("Transformed Beta Distribution Values for n = 1000")

g2 <- ggplot(U2_df, aes(U2_vals)) +
  geom_histogram(aes(y = after_stat(density)),
  breaks = seq(-2, 2, by = 0.1)) +
  scale_y_continuous("Density", breaks = seq(0, 0.4, by = 0.1),
  limits = c(0, 0.4)) +
  xlab("Transformed Beta Distribution Values for n = 10000")
```

```
g3 <- ggplot(U3_df, aes(U3_vals)) +
  geom_histogram(aes(y = after_stat(density)),
  breaks = seq(-2, 2, by = 0.1)) +
  scale_y_continuous("Density", breaks = seq(0, 0.4, by = 0.1),
  limits = c(0, 0.4)) +
  xlab("Transformed Beta Distribution Values for n = 100000")

jpeg("Plots/BetaTrans_NoCurve.jpeg", width = 600, height = 600)

grid.arrange(g1, g2, g3)

dev.off()

# Add semicircle distribution densities
g1 <- g1 +
  stat_function(fun = dsemicircle, args = R, lwd = 1.5, color = "red")

g2 <- g2 +
  stat_function(fun = dsemicircle, args = R, lwd = 1.5, color = "red")

g3 <- g3 +
  stat_function(fun = dsemicircle, args = R, lwd = 1.5, color = "red")

jpeg("Plots/BetaTrans.jpeg", width = 600, height = 600)

grid.arrange(g1, g2, g3)

dev.off()

jpeg("Plots/BetaTrans_QQ.jpeg", width = 600, height = 600)

# Check with a QQ plot
par(mfrow = c(3, 1))

theoretical1 <- qsemicircle(ppoints(1000), R)
qqplot(U1, theoretical1,
        main = "QQ Plot of Shifted Beta vs
        Theoretical Semicircle for n = 1000",
       xlab = "Shifted Beta", ylab = "Theoretical Semicircle", cex = 2)
abline(a = 0, b = 1, col = "red")

theoretical2 <- qsemicircle(ppoints(10000), R)
qqplot(U2, theoretical2,
        main = "QQ Plot of Shifted Beta vs
        Theoretical Semicircle for n = 10000",
       xlab = "Shifted Beta", ylab = "Theoretical Semicircle", cex = 2)
abline(a = 0, b = 1, col = "red")

theoretical3 <- qsemicircle(ppoints(100000), R)
```

```
qqplot(U3, theoretical3,
        main = "QQ Plot of Shifted Beta vs
        Theoretical Semicircle for n = 100000",
       xlab = "Shifted Beta", ylab = "Theoretical Semicircle", cex = 2)
abline(a = 0, b = 1, col = "red")

dev.off()
```