1. (a) $p(X_2 = Happy) = 0.9$

   (b) $p(Y_2 = Frown) = p(y_2 = Frown|X_2 = Happy)p(X_2 = happy) + p(y_2 = Frown|X_2 = Sad)p(X_2 = Sad) = 0.1 * 0.9 + 0.6 * 0.1 = 0.15$

   (c) $p(X_2 = Happy|Y_2 = Frown) = \frac{p(X_2=happy)p(Y_2=frown|X_2=happy)}{p(Y_2=frown)} = \frac{0.9 \cdot 0.1}{0.15} = 0.6$

   (d) Can show by induction (TODO) that

$$\begin{pmatrix} p(X_j = Happy) \\ p(X_j = Sad) \end{pmatrix} = \begin{pmatrix} 0.9 & 0.1 \\ 0.1 & 0.9 \end{pmatrix}^{j-1} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

   . Thus,

$$\begin{pmatrix} p(X_{80} = Happy) \\ p(X_{80} = Sad) \end{pmatrix} = \begin{pmatrix} 0.9 & 0.1 \\ 0.1 & 0.9 \end{pmatrix}^{79} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \approx \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}$$

   So $p(Y_{80} = yell) = p(y_{80} = yell|X_{80} = Happy)p(X_{80} = happy) + p(y_{80} = yell|X_{80} = Sad)p(X_{80} = Sad) \approx 0.1 * 0.5 + 0.2 * 0.5 = 0.15$.

   (e) Note that

$$\begin{aligned}
&\text{argmax}_{x_1,\cdots,x_5} p(X_1 = x_1, \cdots, X_5 = x_5|Y_1 = ... = Y_5 = frown) = \\
&\text{argmax}_{x_1,\cdots,x_5} \frac{p(X_1 = x_1, \cdots, X_5 = x_5)p(Y_1 = ... = Y_5 = frown|X_1 = x_1, \cdots, X_5 = x_5)}{p(Y_1 = ... = Y_5 = frown)} = \\
&\text{argmax}_{x_1,\cdots,x_5} p(X_1 = x_1, \cdots, X_5 = x_5)p(Y_1 = ... = Y_5 = frown|X_1 = x_1, \cdots, X_5 = x_5)
\end{aligned}$$

   Let n_{happy} be the number of days where Harry is happy. We know that

$$\begin{aligned}
p(Y_1 = ... = Y_5 = frown|X_1 = x_1, \cdots, X_5 = x_5) &= \prod_{i=1}^{5} p(Y_i = frown|X_i = x_i) \\
&= 0.1^{n_{happy}} * 0.2^{(5-n_{happy})}
\end{aligned}$$

   and that for $\max_{x_1,\cdots,x_5} p(X_1 = x_1, \cdots X_5 = x_5|n_{happy})$ there should be at most one transition between states such that

$$x_i = \begin{cases} Happy, & \text{if } i \leq n_{happy} \\ Sad, & \text{otherwise.} \end{cases}$$

   Hence,

$$\max_{x_1,\cdots,x_5} p(X_1 = x_1, \cdots X_5 = x_5|n_{happy}) = \begin{cases} 0.9^5 & \text{if } n_{happy} = 5 \\ 0.9^4 * 0.1 & \text{otherwise} \end{cases}$$

   TODO solve analytically

2. Suppose we have a directed graph G with two nodes u and v, edges $u \to v$ and $v \to u$ and the random variables X_u and X_v each of which can take on the values a and b.

   Consider
   $$f_u(x_u|x_{pa(u)}) = \begin{cases} 0 & \text{if } x_v \neq x_u \\ 1 & \text{if } x_v = x_u \end{cases}$$
   and
   $$f_v(x_v|x_{pa(v)}) = \begin{cases} 0 & \text{if } x_v = x_u \\ 1 & \text{if } x_v \neq x_u \end{cases}$$
   These functions specify distributions over X_u and X_v respectively. Further, note that for any given values of x_u and x_v, $f(x_u, x_v) = f_u(x_u|x_v)f_v(x_v|x_u) = 0$. Hence,
   $$\sum_{x_u, x_v} f(x_u, x_v) = 0 \neq 1$$
   and thus does not a define a valid probability distribution.