Question 3

Poisson Regression

$$f(x_i|\lambda) = \frac{\lambda^{x_i} e^{-\lambda}}{x_i!}$$

Likelihood

$$L(\lambda) = \prod_{i=1}^{n} f(x_i|\lambda)$$

$$= \prod_{i=1}^{n} \frac{e^{-\lambda}\lambda^{x_i}}{x_i!}$$

$\rightarrow$ Log Likelihood

$$\log L = \sum_{i=1}^{n} \log\left(\frac{e^{-\lambda}\lambda^{x_i}}{x_i!}\right)$$

$$= \sum_{i=1}^{n} \left[\log e^{-\lambda} + \log \lambda^{x_i} - \log x_i!\right]$$

$$= \sum_{i=1}^{n} -\lambda + \log\lambda \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} \log(x_i!)$$

$$-n\lambda + \log\lambda \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} \log(x_i!)$$

The liklihood equation for estimating λ is

$$\frac{d}{d\lambda}\log L(\lambda) = -n\lambda + \log\lambda \sum_{i=1}^{n} x_i - \cancel{\sum_{i=1}^{n} \log(x_i!)} \to 0$$

$$\frac{d}{d\lambda}\log L = 0 \rightarrow -n + \frac{1}{\lambda}\sum_{i=1}^{n} x_i$$

$$\lambda = \frac{\sum_{i=1}^{n} x_i}{n} \qquad \lambda = \bar{x}$$

$$\frac{d^2}{d\lambda^2}\log L = -\frac{1}{\lambda^2}\sum_{i=1}^{n} x_i < 0$$

The MLE is

$$\lambda = \frac{\sum_{i=1}^{n} x_i}{n}$$

Question 4 : Another MLE Practice

When $0 \leq X_i \leq \theta$ we have:

$$f(x_i|\theta) = 2 \cdot \frac{(\theta - x_i)}{\theta^2}$$

1) Proving that this equation is a PDF

Check two conditions:
a) all values of pdf are non-negative ✓
b) the integral of the pdf over its entire domain is equal to 1 ✓

condition a is met because:
i) the numerator $(\theta - X_i)$ is always non-negative given the range $0 \leq X_i \leq \theta$
ii) the denominator, θ^2, is a positive constant
Therefore $f(x_i|\theta) \geq 0$ for all x_i within the specified range satisfying condition a

Integration of the PDF

$$= \int_0^\theta 2 \cdot \frac{(\theta - x_i)}{\theta^2} \, dx$$

$$= \frac{2}{\theta^2} \int_0^\theta \frac{(\theta - x_i)}{\theta^2} \, dx$$

$$= \frac{2}{\theta^2} \left[\theta x_i - \frac{1}{2} x_i^2\right]_0^\theta$$

$$= \frac{2}{\theta^2} \left[\left(\theta \cdot \theta - \frac{1}{2}\theta^2\right) - \left(\theta \cdot 0 - \frac{1}{2} \cdot 0\right)\right]$$

$$\frac{2}{\theta^2}\left[\theta^2 - \frac{1}{2}\theta^2\right] = \frac{2}{\theta^2} \cdot \frac{1}{2}\theta^2 = 1$$

2) Finding MLE

likelihood function

$$L(\theta | x_i) = \prod_{i=1}^n f(x_i|\theta) = \prod_{i=1}^n 2 \cdot \frac{(\theta - x_i)}{\theta^2}$$

Log likelihood function

$$\operatorname{Log}(\theta | x_i) = \sum_{i=1}^n \log\left(2 \cdot \frac{(\theta - x_i)}{\theta^2}\right)$$

$$= \sum_{i=1}^n \left[\log(2) + \log(\theta - x_i) - \log(\theta^2)\right]$$

$$= n \cdot \log(2) + \sum_{i=1}^n \log(\theta - x_i) - 2n \cdot \log(\theta)$$

$$\frac{d}{dx} \operatorname{Log}(\theta \mid x_i) = \cancel{n \cdot \log(2)} + \sum_{i=1}^{n} \log(\theta - x_i) - 2n \cdot \log(\theta)$$

This is the derivative:

$$\frac{d}{dx} \operatorname{Log}(\theta \mid x_i) = \sum_{i=1}^{n} \frac{1}{(\theta - x_i)} - \frac{2n}{\theta}$$

$$\frac{d}{dx} \operatorname{Log}(\theta \mid x_i) = 0 = \sum_{i=1}^{n} \frac{1}{(\theta - x_i)} - \frac{2n}{\theta}$$

$$\frac{2n}{\theta} = \sum_{i=1}^{n} \frac{1}{(\theta - x_i)}$$

$$2n \sum_{i=1}^{n} (\theta - x_i) = \sum_{i=1}^{n} \theta$$

$$2n\theta - 2n \sum_{i=1}^{n} x_i = n\theta$$

$$-2n \sum_{i=1}^{n} x_i = -n\theta$$

$$\theta = 2n \sum_{i=1}^{n} x_i$$