

MA2108S Midterms AY25/26 Sem 2

1. Pre-requisites

Well-Ordering Principle of \mathbb{N} Every non-empty subset $S \subseteq \mathbb{N}$ has a least (smallest) element.

2. The Real Numbers

Algebraic Properties, ~

Different types of means

- **Arithmetic Means** $A_n = \frac{1}{n} \sum_{k=1}^n a_k$
- **Geometric Means** $G_n = \left(\prod_{k=1}^n a_k \right)^{1/n}$
- **Harmonic Means** $H_n = n \left(\sum_{k=1}^n a_k^{-1} \right)^{-1}$

, for $n \in \mathbb{N}_{\geq 2}$ and $a_1, a_2, \dots, a_n \in \mathbb{R}$ are positive. For the means, we have the **AM-GM-HM Inequality** :

$$H_n \leq G_n \leq A_n$$

, taking "=" iff. $a_1 = \dots = a_n$.

Bernoulli's Inequality For $x > -1$, we have $(1+x)^n \geq 1+nx$, for any $n \in \mathbb{N}$.

Triangle Inequality $|a+b| \leq |a|+|b|$, for all $a, b \in \mathbb{R}$.

Derived: [1] $||a|-|b|| \leq |a-b|$, [2] $|a-b| \leq |a|+|b|$.

Neighbourhood

For any $a \in \mathbb{R}$ and $\epsilon > 0$, the ϵ -neighbourhood of a is the set:

$$V_\epsilon(a) = \{x \in \mathbb{R} : |x-a| < \epsilon\}$$

Theorem 2.2.8 For $a \in \mathbb{R}$, if $x \in V_\epsilon(a)$ for every $\epsilon > 0$, then $x = a$.

Completeness Properties, ~

For a non-empty $S \subseteq \mathbb{R}$, it is **Bounded Above** (**Bounded Below**) if S has an upper bound (a lower bound). S is **Bounded** if it is bounded above and below, and is **Unbounded**, otherwise.

For a non-empty $S \subseteq \mathbb{R}$, u is the **Supremum** of S if the following conditions are met, and we denote it as **sup** S :

1. u is an upper bound of S .
2. $\forall v \in \mathbb{R}$, if v is an upper bound of S , then $v \geq u$.

For a non-empty $S \subseteq \mathbb{R}$, w is the **Infimum** of S if the following conditions are met, and we denote it as **inf** S :

1. w is a lower bound of S .
2. $\forall v \in \mathbb{R}$, if v is a lower bound of S , then $v \leq w$.

Note: Sup. and Inf. are **uniquely determined**, if they exist.

Alternative Definition (Similarly for Infimum):

Lemma 2.3.4 For u an upper bound of $S \subseteq \mathbb{R}$, $u = \sup S$ iff.

$$\forall \epsilon > 0, \exists s_\epsilon \in S, u - \epsilon < s_\epsilon$$

For a non-empty $S \subseteq \mathbb{R}$, u is the **Maximum** (**Minimum**) of S , if $u = \sup S$ ($u = \inf S$) and $u \in S$.

Note: Sup. and Inf. are not necessarily elements in S (if they exist), but maximum and minimum are.

Supremum Property of \mathbb{R} Every non-empty subset of \mathbb{R} that has an upper bound has a supremum.

The Archimedean Property If $x \in \mathbb{R}$, then $\exists n_x \in \mathbb{N}$ s.t. $x < n_x$.

Corollary 2.4.6 If $x > 0$, then $\exists n \in \mathbb{N}$ such that $n-1 \leq x < n$.

Density Theorems For $x, y \in \mathbb{R}$ with $x < y$, there exists $r \in \mathbb{Q}$ ($z \in \mathbb{R} \setminus \mathbb{Q}$) s.t. $x < r < y$ ($x < z < y$).

Intervals

A sequence of intervals $I_n, n \in \mathbb{N}$ is **Nested** if

$$I_1 \supseteq I_2 \supseteq \dots \supseteq I_n \supseteq I_{n+1} \supseteq \dots$$

Properties: [1] If $I_n = [a_n, b_n], n \in \mathbb{N}$ is a nested seq. of closed bounded intervals, then $\exists \xi \in \mathbb{R}$ s.t. $\xi \in I_n, \forall n \in \mathbb{N}$. [2] If $I_n = [a_n, b_n], n \in \mathbb{N}$ satisfying $\inf\{b_n - a_n : n \in \mathbb{N}\} = 0$, then ξ contained in all I_n is unique.

3. Sequences & Series

Sequence & Convergence

Sequence in \mathbb{R} : a real-valued function $X : \mathbb{R} \rightarrow \mathbb{R}$. We write $x_n = X(n)$ for the n -th term of the sequence, and denote the sequence as $(x_n : n \in \mathbb{N})$.

A sequence $X = (x_n)$ in \mathbb{R} is **Convergent** to $x \in \mathbb{R}$ iff. for every $\epsilon > 0$, there exists $K = K(\epsilon) \in \mathbb{N}$ s.t.

$$n \geq K(\epsilon) \implies |x_n - x| < \epsilon$$

, and we call x the **Limit** of (x_n) , denoted as $\lim_{n \rightarrow \infty} x_n = x$. A sequence is **Divergent** if it is not convergent.

Technique for proving convergence:

1. Express $|x_n - x|$ in terms of n and find a simpler upper bound $L = L(n)$, i.e. $|x_n - x| < L$.
2. Let $\epsilon > 0$ be arbitrary, find $K \in \mathbb{N}$ s.t. for all $n \geq K$, $L = L(n) < \epsilon$, then

$$n \geq K \implies |x_n - x| < L < \epsilon$$

Squeeze Theorem If $x_n \leq y_n \leq z_n$, for all $n \in \mathbb{N}$ and $\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} z_n = a$, then

$$\lim_{n \rightarrow \infty} y_n = a$$

A sequence $X = (x_n)$ is **Bounded** if there exists $M > 0$ such that $|x_n| \leq M$ for all $n \in \mathbb{N}$.

Monotone Convergence Theorem Let (x_n) be a monotone sequence of real numbers, then (x_n) is convergent iff. (x_n) is bounded. If it is bounded and increasing, then $\lim_{n \rightarrow \infty} x_n = \sup\{x_n : n \in \mathbb{N}\}$. (Similarly for decreasing.)

For a sequence (x_n) , it **tends to** $+\infty$, i.e. $\lim_{n \rightarrow \infty} x_n = +\infty$ if for all $\alpha \in \mathbb{R}$, there exists $K = K(\alpha) \in \mathbb{N}$ such that if $n \geq K(\alpha)$, then $x_n > \alpha$. (Similarly for $\lim_{n \rightarrow \infty} x_n = -\infty$.)

A sequence (x_n) is **Properly Divergent** if $\lim_{n \rightarrow \infty} x_n = \pm\infty$.

Subsequences

A **Subsequence** of $X = (x_n)$ is $X' = (x_{n_k})$:

$$X' = (x_{n_1}, x_{n_2}, \dots, x_{n_3})$$

, where $n_1 < n_2 < \dots < n_k < \dots$ is a strictly increasing sequence in \mathbb{N} . **Note:** $n_k \geq n, \forall k$.

Theorem 3.4.2 If (x_n) converges to x , then any subsequence (x_{n_k}) also converges to x ,

$$\lim_{n_k \rightarrow \infty} x_{n_k} = \lim_{k \rightarrow \infty} x_{n_k} = x$$

Theorem 3.4.5 If (x_n) has either of the following properties, it is divergent: [1] (x_n) has two convergent subsequences with different limits. [2] (x_n) is unbounded.

Theorem 3.4.7 Every sequence has a monotone subsequence.

Bolzano-Weierstrass Theorem Every bounded sequence has a convergent subsequence.

Cauchy Sequences

A **Cauchy Sequence** (x_n) is a sequence where for all $\epsilon > 0$, there exists $H = H(\epsilon) \in \mathbb{N}$ such that

$$\forall n, m \in \mathbb{N}, n, m \geq H \implies |x_n - x_m| < \epsilon$$

Cauchy Criterion A sequence is convergent iff. it is Cauchy.

A **Contractive Sequence** (x_n) is a sequence where there exists $C \in (0, 1)$ s.t.

$$|x_{n+2} - x_{n+1}| \leq C|x_{n+1} - x_n|, \forall n \in \mathbb{N}$$

Theorem 3.5.8 Every contractive sequence is Cauchy.

Limit Points

Suppose $\{a_n\}$ bounded from above. The **upper limit** of $\{a_n\}$ is defined as

$$\limsup_{n \rightarrow \infty} a_n := \lim_{n \rightarrow \infty} \sup\{a_k : k \geq n\}.$$

Suppose $\{a_n\}$ bounded from below. The **lower limit** of $\{a_n\}$ is defined as

$$\liminf_{n \rightarrow \infty} a_n := \lim_{n \rightarrow \infty} \inf\{a_k : k \geq n\}.$$

A is called a **limit point** of a sequence $\{a_n\}$ if there exists subsequence $\{b_k = a_{n_k}\}$ such that $\lim_{k \rightarrow \infty} b_k = A$. **Theorem 3.6.1** Let $\{a_n\}$ be a bounded sequence and E denote the set of limit points of $\{a_n\}$. Then both upper limit and lower limit of $\{a_n\}$ are contained in E . Moreover,

$$\limsup_{n \rightarrow \infty} a_n = \max E, \liminf_{n \rightarrow \infty} a_n = \min E$$

Corollary 3.6.2 For any bounded sequence $\{a_n\}$,

$$\liminf_{n \rightarrow \infty} a_n \leq \limsup_{n \rightarrow \infty} a_n$$

$\liminf_{n \rightarrow \infty} a_n = \limsup_{n \rightarrow \infty} a_n$ iff the sequence is convergent.

Infinite Series

For (x_n) , its **(Infinite) Series** is sequence (s_n) , where $s_n = \sum_{k=1}^n x_k$ is called a **Partial Sum** of the series, and x_k is a **Term**.
Tests for infinite series' convergence:

- **n -th Term Test** - If $\sum x_n$ converges, then $\lim_{n \rightarrow \infty} x_n = 0$.
- Cauchy Criterion Test
- **Partial Sum Bounded Test**, for series w. non-negative terms
- Suppose $x_n \geq 0, \forall n \in \mathbb{N}$, then $\sum x_n$ converges iff. (s_n) is bounded.
- **Comparison Test** - For $(x_n), (y_n)$ with some $K \in \mathbb{N}$, s.t. $n \geq K \Rightarrow 0 \leq x_n \leq y_n$. Then [1] $\sum y_n$ converges $\Rightarrow \sum x_n$ converges, and [2] $\sum x_n$ diverges $\Rightarrow \sum y_n$ diverges.
- **Limit Comparison Test** - For **strictly positive** $(x_n), (y_n)$ with limit $r = \lim_{n \rightarrow \infty} (\frac{x_n}{y_n})$. Then [1] if $r = 0$, $\sum y_n$ converges $\Rightarrow \sum x_n$ converges. [2] if $r > 0$, $\sum y_n$ converges iff $\sum x_n$ converges.
- **Cauchy Condensation Test** - Suppose $\{a_n\}$ decreasing and every term is positive. Define $b_k := 2^k a_{2^k}$. $\sum_{n=1}^{\infty} a_n$ convergent iff $\sum_{n=1}^{\infty} b_n$ convergent.

- **Leibniz Test** - Suppose $a_n \geq 0$ decreasing and $\lim_{n \rightarrow \infty} a_n = 0$. Then $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$ convergent.
- **Dirichlet's Test** Let $\{a_n\}$ satisfy that its partial sums $\{A_n = \sum_{k=1}^n a_k\}$ is bounded, and $\{b_n\}$ be a positive decreasing sequence approaching zero. Then $\sum_{n=1}^{\infty} a_n b_n$ is convergent.
- **Abel's Test** Let $\{a_n\}$ satisfy that its partial sums $\sum_{n=1}^{\infty} a_n$ is convergent. Let $\{b_n\}$ be a monotone and bounded sequence. Then $\sum_{n=1}^{\infty} a_n b_n$ is convergent.

Absolute Convergence

Series $\sum x_n$ is **Absolutely Convergent** if series $\sum |x_n|$ is convergent. A series is **Conditionally Convergent** if it is convergent but not absolutely convergent.

Tests for absolute convergence:

- Limit Comparison Test - Consider convergence of positive sequences $|x_n|$ and $|y_n|$ if $(x_n), (y_n)$ non-negative.
- **Root Test** - For (x_n) , [1] if $\exists r \in \mathbb{R}, 0 < r < 1$ and $K \in \mathbb{N}$ s.t. $|x_n|^{1/n} \leq r, \forall n \geq K$, then $\sum x_n$ is abs. convergent. [2] If

$\exists r \in \mathbb{R}, r > 1$ and $K \in \mathbb{N}$ s.t. $|x_n|^{1/n} \geq r > 1, \forall n \geq K$, then $\sum x_n$ is **divergent**.

- **Ratio Test** - For (x_n) nonzero, [1] if $\exists r \in \mathbb{R}, 0 < r < 1$ and $K \in \mathbb{N}$ s.t. $|\frac{x_{n+1}}{x_n}| \leq r, \forall n \geq K$, then $\sum x_n$ is abs. convergent. [2] If $\exists K \in \mathbb{N}$ s.t. $|\frac{x_{n+1}}{x_n}| \geq 1, \forall n \geq K$, then $\sum x_n$ is **divergent**.

Abel's Summation Formula

Let $A_n = \sum_{k=1}^n a_k$. For any $q > p$,

$$\sum_{n=p}^q a_n b_n = A_q b_q - A_{p-1} b_p + \sum_{n=p}^{q-1} A_n (b_n - b_{n+1}).$$

Useful statements

- $\sum_{i=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.
- $\left(1 + \frac{1}{n}\right)^n < e < \left(1 + \frac{1}{n}\right)^{n+1} \Rightarrow \frac{1}{n+1} < \log\left(\frac{n+1}{n}\right) < \frac{1}{n}$
- $\sum_{n=1}^{\infty} \frac{1}{n \log n}$ is divergent