

# MA2108S Midterms AY25/26 Sem 2

## 1. Pre-requisites

Well-Ordering Principle of  $\mathbb{N}$  Every non-empty subset  $S \in \mathbb{N}$  has a least (smallest) element.

## 2. The Real Numbers

### Algebraic Properties, ~

Different types of means

- **Arithmetic Means**  $A_n = \frac{1}{n} \sum_{k=1}^n a_k$
- **Geometric Means**  $G_n = \left( \prod_{k=1}^n a_k \right)^{1/n}$
- **Harmonic Means**  $H_n = n \left( \sum_{k=1}^n a_k^{-1} \right)^{-1}$

, for  $n \in \mathbb{N}_{\geq 2}$  and  $a_1, a_2, \dots, a_n \in \mathbb{R}$  are positive. For the means, we have the AM-GM-HM Inequality :

$$H_n \leq G_n \leq A_n$$

, taking " $=$ " iff.  $a_1 = \dots = a_n$ .

Bernoulli's Inequality For  $x > -1$ , we have  $(1+x)^n \geq 1+nx$ , for any  $n \in \mathbb{N}$ .

Triangle Inequity  $|a+b| \leq |a| + |b|$ , for all  $a, b \in \mathbb{R}$ .

Derived: [1]  $\|a-b\| \leq |a-b|$ , [2]  $|a-b| \leq |a| + |b|$ .

### Neighbourhood

For any  $a \in \mathbb{R}$  and  $\epsilon > 0$ , the  **$\epsilon$ -neighbourhood** of  $a$  is the set:

$$V_\epsilon(a) = \{x \in \mathbb{R} : |x-a| < \epsilon\}$$

Theorem 2.2.8 For  $a \in \mathbb{R}$ , if  $x \in V_\epsilon(a)$  for every  $\epsilon > 0$ , then  $x = a$ .

### Completeness Properties, ~

For a non-empty  $S \subseteq \mathbb{R}$ , it is **Bounded Above (Bounded Below)** if  $S$  has an upper bound (a lower bound).  $S$  is **Bounded** if it is bounded above and below, and is **Unbounded**, otherwise.

For a non-empty  $S \subseteq \mathbb{R}$ ,  $u$  is the **Supremum** of  $S$  if the following conditions are met, and we denote it as  $\sup S$ :

1.  $u$  is an upper bound of  $S$ .
2.  $\forall v \in \mathbb{R}$ , if  $v$  is an upper bound of  $S$ , then  $v \geq u$ .

For a non-empty  $S \subseteq \mathbb{R}$ ,  $w$  is the **Infinum** of  $S$  if the following conditions are met, and we denote it as  $\inf S$ :

1.  $w$  is a lower bound of  $S$ .
2.  $\forall v \in \mathbb{R}$ , if  $v$  is a lower bound of  $S$ , then  $v \leq w$ .

Note: Sup. and Inf. are **uniquely determined**, if they exist.

Alternative Definition (Similarly for Infinum):

Lemma 2.3.4 For  $u$  an upper bound of  $S \subseteq \mathbb{R}$ ,  $u = \sup S$  iff.

$$\forall \epsilon > 0, \exists s_\epsilon \in S, u - \epsilon < s_\epsilon$$

For a non-empty  $S \subseteq \mathbb{R}$ ,  $u$  is the **Maximum (Minimum)** of  $S$ , if  $u = \sup S$  ( $u = \inf S$ ) and  $u \in S$ .

Note: Sup. and Inf. are not necessarily elements in  $S$  (if they exist), but maximum and minimum are.

**Supremum Property of  $\mathbb{R}$**  Every non-empty subset of  $\mathbb{R}$  that has an upper bound has a supremum.

**The Archimedean Property** If  $x \in \mathbb{R}$ , then  $\exists n_x \in \mathbb{N}$  s.t.  $x < n_x$ .

**Corollary 2.4.6** If  $x > 0$ , then  $\exists n \in \mathbb{N}$  such that  $n-1 \leq x < n$ .

**Density Theorems** For  $x, y \in \mathbb{R}$  with  $x < y$ , there exists  $r \in \mathbb{Q}$  ( $z \in \mathbb{R} \setminus \mathbb{Q}$ ) s.t.  $x < r < y$  ( $x < z < y$ ).

### Intervals

A sequence of intervals  $I_n, n \in \mathbb{N}$  is **Nested** if

$$I_1 \supseteq I_2 \supseteq \dots \supseteq I_n \supseteq I_{n+1} \supseteq \dots$$

Properties: [1] If  $I_n = [a_n, b_n], n \in \mathbb{N}$  is a nested seq. of closed bounded intervals, then  $\exists \xi \in \mathbb{R}$  s.t.  $\xi \in I_n, \forall n \in \mathbb{N}$ . [2] If  $I_n = [a_n, b_n], n \in \mathbb{N}$  satisfying  $\inf\{b_n - a_n : n \in \mathbb{N}\} = 0$ , then  $\xi$  contained in all  $I_n$  is unique.

## 3. Sequences & Series

### Sequence & Convergence

**Sequence** in  $\mathbb{R}$ : a real-valued function  $X : \mathbb{R} \rightarrow \mathbb{R}$ . We write  $x_n = X(n)$  for the  $n$ -th term of the sequence, and denote the sequence as  $(x_n : n \in \mathbb{N})$ .

A sequence  $X = (x_n)$  in  $\mathbb{R}$  is **Convergent** to  $x \in \mathbb{R}$  iff. for every  $\epsilon > 0$ , there exists  $K = K(\epsilon) \in \mathbb{N}$  s.t.

$$n \geq K(\epsilon) \implies |x_n - x| < \epsilon$$

, and we call  $x$  the **Limit** of  $(x_n)$ , denoted as  $\lim_{n \rightarrow \infty} x_n = x$ . A sequence is **Divergent** if it is not convergent.

Technique for proving convergence:

1. Express  $|x_n - x|$  in terms of  $n$  and find a simpler upper bound  $L = L(n)$ , i.e.  $|x_n - x| < L$ .
2. Let  $\epsilon > 0$  be arbitrary, find  $K \in \mathbb{N}$  s.t. for all  $n \geq K$ ,  $L = L(n) < \epsilon$ , then

$$n \geq K \implies |x_n - x| < L < \epsilon$$

**Squeeze Theorem** If  $x_n \leq y_n \leq z_n$ , for all  $n \in \mathbb{N}$  and  $\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} z_n = a$ , then

$$\lim_{n \rightarrow \infty} y_n = a$$

A sequence  $X = (x_n)$  is **Bounded** if there exists  $M > 0$  such that  $|x_n| \leq M$  for all  $n \in \mathbb{N}$ .

**Monotone Convergence Theorem** Let  $(x_n)$  be a monotone sequence of real numbers, then  $(x_n)$  is convergent iff.  $(x_n)$  is bounded. If it is bounded and increasing, then  $\lim_{n \rightarrow \infty} x_n = \sup\{x_n : n \in \mathbb{N}\}$ . (Similarly for decreasing.)

For a sequence  $(x_n)$ , it **tends to  $+\infty$** , i.e.  $\lim_{n \rightarrow \infty} x_n = +\infty$  if for all  $\alpha \in \mathbb{R}$ , there exists  $K = K(\alpha) \in \mathbb{N}$  such that if  $n \geq K(\alpha)$ , then  $x_n > \alpha$ . (Similarly for  $\lim_{n \rightarrow \infty} x_n = -\infty$ .)

A sequence  $(x_n)$  is **Properly Divergent** if  $\lim_{n \rightarrow \infty} x_n = \pm\infty$ .

### Subsequences

A **Subsequence** of  $X = (x_n)$  is  $X' = (x_{n_k})$ :

$$X' = (x_{n_1}, x_{n_2}, \dots, x_{n_3})$$

, where  $n_1 < n_2 < \dots < n_k < \dots$  is a strictly increasing sequence in  $\mathbb{N}$ . Note:  $n_k \geq n, \forall k$ .

**Theorem 3.4.2** If  $(x_n)$  converges to  $x$ , then any subsequence  $(x_{n_k})$  also converges to  $x$ ,

$$\lim_{n_k \rightarrow \infty} x_{n_k} = \lim_{k \rightarrow \infty} x_{n_k} = x$$

**Theorem 3.4.5** If  $(x_n)$  has either of the following properties, it is divergent: [1]  $(x_n)$  has two convergent subsequences with different limits. [2]  $(x_n)$  is unbounded.

**Theorem 3.4.7** Every sequence has a monotone subsequence.

**Bolzano-Weierstrass Theorem** Every bounded sequence has a convergent subsequence.

### Cauchy Sequences

A **Cauchy Sequence**  $(x_n)$  is a sequence where for all  $\epsilon > 0$ , there exists  $H = H(\epsilon) \in \mathbb{N}$  such that

$$\forall n, m \in \mathbb{N}, n, m \geq H \implies |x_n - x_m| < \epsilon$$

**Cauchy Criterion** A sequence is convergent iff. it is Cauchy.

A **Contractive Sequence**  $(x_n)$  is a sequence where there exists  $C \in (0, 1)$  s.t.

$$|x_{n+2} - x_{n+1}| \leq C|x_{n+1} - x_n|, \forall n \in \mathbb{N}$$

**Theorem 3.5.8** Every contractive sequence is Cauchy.

### Limit Points

Suppose  $\{a_n\}$  bounded from above. The **upper limit** of  $\{a_n\}$  is defined as

$$\limsup_{n \rightarrow \infty} a_n := \lim_{n \rightarrow \infty} \sup\{a_k : k \geq n\}.$$

Suppose  $\{a_n\}$  bounded from below. The **lower limit** of  $\{a_n\}$  is defined as

$$\liminf_{n \rightarrow \infty} a_n := \lim_{n \rightarrow \infty} \inf\{a_k : k \geq n\}.$$

$A$  is called a **limit point** of a sequence  $\{a_n\}$  if there exists subsequence  $\{b_k = a_{n_k}\}$  such that  $\lim_{k \rightarrow \infty} b_k = A$ . **Theorem 3.6.1** Let  $\{a_n\}$  be a bounded sequence and  $E$  denote the set of limit points of  $\{a_n\}$ . Then both upper limit and lower limit of  $\{a_n\}$  are contained in  $E$ . Moreover,

$$\limsup_{n \rightarrow \infty} a_n = \max E, \liminf_{n \rightarrow \infty} a_n = \min E$$

**Corollary 3.6.2** For any bounded sequence  $\{a_n\}$ ,

$$\liminf_{n \rightarrow \infty} a_n \leq \limsup_{n \rightarrow \infty} a_n$$

$\liminf_{n \rightarrow \infty} a_n = \limsup_{n \rightarrow \infty} a_n$  iff the sequence is convergent.

## Infinite Series

For  $(x_n)$ , its **(Infinite) Series** is sequence  $(s_n)$ , where  $s_n = \sum_{k=1}^n x_k$  is called a **Partial Sum** of the series, and  $x_k$  is a **Term**. Tests for infinite series' convergence:

- **$n$ -th Term Test** - If  $\sum x_n$  converges, then  $\lim_{n \rightarrow \infty} x_n = 0$ .
- Cauchy Criterion Test
- **Partial Sum Bounded Test**, for series w. non-negative terms  
- Suppose  $x_n \geq 0, \forall n \in \mathbb{N}$ , then  $\sum x_n$  converges iff.  $(s_n)$  is bounded.
- **Comparison Test** - For  $(x_n), (y_n)$  with some  $K \in \mathbb{N}$ , s.t.  $n \geq K \implies 0 \leq x_n \leq y_n$ . Then [1]  $\sum y_n$  converges  $\implies \sum x_n$  converges, and [2]  $\sum x_n$  diverges  $\implies \sum y_n$  diverges.
- **Limit Comparison Test** - For strictly positive  $(x_n), (y_n)$  with limit  $r = \lim_{n \rightarrow \infty} \left(\frac{x_n}{y_n}\right)$ . Then [1] if  $r = 0$ ,  $\sum y_n$  converges  $\implies \sum x_n$  converges. [2] if  $r > 0$ ,  $\sum y_n$  converges iff  $\sum x_n$  converges.
- **Cauchy Condensation Test** - Suppose  $\{a_n\}$  decreasing and every term is positive. Define  $b_k := 2^k a_{2^k}$ .  $\sum_{n=1}^{\infty} a_n$  convergent iff  $\sum_{n=1}^{\infty} b_n$  convergent.

- **Leibniz Test** - Suppose  $a_n \geq 0$  decreasing and  $\lim_{n \rightarrow \infty} a_n = 0$ . Then  $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$  convergent.
- **Dirichlet's Test** Let  $\{a_n\}$  satisfy that its partial sums  $\{A_n = \sum_{k=1}^n a_k\}$  is bounded, and  $\{b_n\}$  be a positive decreasing sequence approaching zero. Then  $\sum_{n=1}^{\infty} a_n b_n$  is convergent.
- **Abel's Test** Let  $\{a_n\}$  satisfy that its partial sums  $\sum_{n=1}^{\infty} a_n$  is convergent. Let  $\{b_n\}$  be a monotone and bounded sequence. Then  $\sum_{n=1}^{\infty} a_n b_n$  is convergent.

## Absolute Convergence

Series  $\sum x_n$  is **Absolutely Convergent** if series  $\sum |x_n|$  is convergent. A series is **Conditionally Convergent** if it is convergent but not absolutely convergent.

Tests for absolutely convergence:

- Limit Comparison Test - Consider convergence of positive sequences  $|x_n|$  and  $|y_n|$  if  $(x_n), (y_n)$  non-negative.
- **Root Test** - For  $(x_n)$ , [1] if  $\exists r \in \mathbb{R}, 0 < r < 1$  and  $K \in \mathbb{N}$  s.t.  $|x_n|^{1/n} \leq r, \forall n \geq K$ , then  $\sum x_n$  is abs. convergent. [2] If

$\exists r \in \mathbb{R}, r > 1$  and  $K \in \mathbb{N}$  s.t.  $|x_n|^{1/n} \geq r > 1, \forall n \geq K$ , then  $\sum x_n$  is **divergent**.

- **Ratio Test** - For  $(x_n)$  nonzero, [1] if  $\exists r \in \mathbb{R}, 0 < r < 1$  and  $K \in \mathbb{N}$  s.t.  $|\frac{x_{n+1}}{x_n}| \leq r, \forall n \geq K$ , then  $\sum x_n$  is abs. convergent. [2] If  $\exists K \in \mathbb{N}$  s.t.  $|\frac{x_{n+1}}{x_n}| \geq 1, \forall n \geq K$ , then  $\sum x_n$  is **divergent**.

## Abel's Summation Formula

Let  $A_n = \sum_{k=1}^n a_k$ . For any  $q > p$ ,

$$\sum_{n=p}^q a_n b_n = A_q b_q - A_{p-1} b_p + \sum_{n=p}^{q-1} A_n (b_n - b_{n+1}).$$

Useful statements

- $\sum_{i=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ .
- $\left(1 + \frac{1}{n}\right)^n < e < \left(1 + \frac{1}{n}\right)^{n+1} \Rightarrow \frac{1}{n+1} < \log\left(\frac{n+1}{n}\right) < \frac{1}{n}$
- $\sum_{n=1}^{\infty} \frac{1}{n \log n}$  is divergent