Notes on Alexandrov Spaces

Mengchen Zeng

Last Compile: January 22, 2025

Contents

Preface

This is a highly intensive and quick seminar on metric geometry, particularly on basic Riemannian geometry and Alexandrov spaces. Only prerequisite of this seminar is basic calculus and linear algebra, with adequate mathematical maturity. For simpler typesetting, we do not draw any pictures.

We plan to introduce general topology first, then introduce metric spaces and length spaces, together with Gromov–Hausdorff convergence. After this, we discuss some basic Riemannian manifolds, preparing for later discussion. As we are well-prepared, we begin our tour on Alexandrov spaces, with plenty of examples. (to be continued…)

We follow mainly *A course in metric geometry* of Burago et al. for general metric geometry, and *An introduction of the geometry of Alexandrov spaces* of Shiohama for Alexandrov spaces.

Chapter 1

General Topology

1.1 Topology and Continuity

In this section, we start our tour by introducing basic topology conepts.

Definition 1.1. Let X be a set. A *topology* $\mathcal{T}$ on X is a collection of subsets of X satisfies

(1) $\mathcal{T}$ contains $\varnothing$ and X;

(2) For any collection $\{U_i\}_{i\in I} \subset \mathcal{T}$, we have $\bigcup_{i\in I} U_i \in \mathcal{T}$;

(3) For $U, V \in \mathcal{T}$, we have $U \cap V \in \mathcal{T}$.

A set in $\mathcal{T}$ is called an *open set*. A set together with a topology on it is called a *topological space*.

Remark 1.2. By induction, being open is stable under finite intersection, that is, for any finite many of open sets $U_1, \cdots, U_n$, we have $\bigcup_{i=1}^{n} U_i \in \mathcal{T}$.

Definition 1.3. If $\mathcal{T}_1, \mathcal{T}_2$ are two topologies on X, we say that $\mathcal{T}_1$ is coarser than $\mathcal{T}_2$ (or equivalently $\mathcal{T}_2$ is finer than $\mathcal{T}_1$) if every open set in $\mathcal{T}_1$ is an open set in $\mathcal{T}_2$. Clearly this is a partial order.

Definition 1.4. Let $\mathcal{B} = \{B_i\}_{i\in I}$ is a collection of subsets of a topological space X. If any open set of X is a union of elements in $\mathcal{B}$, then $\mathcal{B}$ is called a *basis* of X, and we say $\mathcal{B}$ generates (the topology of) X.

Definition 1.5. A *continuous mapping*, or simply a *map*, $f : X \to Y$ between topological spaces X, Y is a mapping such that for any open set $U \subset Y$, the inverse image $f^{-1}(U) \subset X$ is an open set.

Lemma 1.6. *Identity is continuous. The composition of continuous mappings is continuous.*

Proof. Obvious. □

Definition 1.7. Let $f : X \to Y$, $g : Y \to X$ are maps between X, Y, if $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$, then X and Y are called to be *homeomorphic*. f, g are called *homeomorphisms*.

Example 1.8. Let X, Y be two topological spaces. The *product space* of X, Y, denoted by $X \times Y$, is the topological spaces on the Cartesian product $X \times Y$ (by abuse of notation), whose topology is generated by

$$\{U \times V : U, V \text{ are open in } X, Y \text{ respectively}\}.$$

Product spaces are characterized as follows: If Z is a topological space with maps $p_X : Z \to X$, $p_Y : Z \to Y$, and have the property that for any topological space Z' with maps $f_X : Z' \to X$, $f_Y : Z' \to Y$, there exists a unique map $f : Z' \to Z$ such that the following diagram commutes

$$\begin{tikzcd}
Z' \arrow[dr, dashed, "f"] \arrow[drr, bend left, "f_X"] \arrow[ddr, bend right, "f_Y"'] & & \\
& Z \arrow[r, "p_X"] \arrow[d, "p_Y"] & X \\
& Y, &
\end{tikzcd}$$

then Z is homeomorphic to $X \times Y$.

Please prove the claim.

Example 1.9. Let $A \subset X$ is a subset of topological space X. Then A carries a *subspace topology* naturally: Let $U \subset A$ be open if and only if $U = A \cap \tilde{U}$ for some $\tilde{U}$ open with respective to X, then this is a topology on A.

Please prove this claim.

Remark 1.10. If you know some category theory, Lemma 1.6 says that topological spaces together with continuous mappings consists a category, Example 1.8 gives the construction of products in the category of topological spaces.

1.2 Open and Closed Sets

Definition 1.11. Let X be a topological space, $Z \subset X$ is a subset. Z is said to be *closed* if $X - Z$ is open.

Definition 1.12. Let X be a topological space, $S \subset X$ and $x \in X$. If there exists an open set $U \ni x$ such that $U \subset S$, then x is called an *interior point* of S. If for any open set $U \ni x$, $U - \{x\}$ contains a point of S, then x is called an *accumulation point* of S.

Proposition 1.13. *Let X be a topological space and $S \subset X$.*

(1) S is open if and only if every point of S is an interior point.

(2) S is closed if and only if every accumulation point of S is in S.

Please prove this proposition.

Definition 1.14. Let X be a topological space and $x \in X$. A *neighborhood* of x is a subset of X such that x is an interior point of it.

Definition 1.15. Let X be a topological space and $S \subset X$.

(1) The *interior* of S is the maximal open set that contained in S.

(2) The *closure* of S is the minimal closed set that contains S.

Remark 1.16. The closure of a nonempty subset is always nonempty, but its interior can be empty.

1.3 Countable and Separable Axioms

We introduce two most important axioms in general topology.

Definition 1.17. A topological space is called to satisfy *second countable axiom* (C2 axiom) if it has a countable basis.

Definition 1.18. A topological space is called to satisfy *Hausdorff axiom* (T2 axiom) if for any p, q, there are neighborhoods U, V of p, q such that $U \cap V = \varnothing$.

Proposition 1.19. *Topological space X is Hausdorff if and only if the diagonal $\Delta = \{(x, x) \in X \times X : x \in X\}$ of product space $X \times X$ is closed.*

Proof. If X is Hausdorff, we show that $X \times X - \Delta$ is open. Let (x, y) with $x \neq y$ be any point in $X \times X - \Delta$, since X is Hausdorff, there exists open sets $U \ni x$, $V \ni y$ with $U \cap V = \varnothing$. Then $U \times V$ is an open set in $X \times X$, and since $U \cap V = \varnothing$, we have $U \times V \cap \Delta = \varnothing$. Thus $(x, y) \in U \times V \subset X \times X - \Delta$ is an interior point, and therefore $X \times X - \Delta$ is open. Conversely, if Δ is closed, let $x \neq y \in X$, then $(x, y) \in X \times X - \Delta$. Thus there exists an open set in $X \times X - \Delta$ that contains (x, y), and since $X \times X$ is generated by $\{U \times V : U, V \text{ open}\}$, $(x, y) \in U \times V$ for some open U, V. Moreover, since $U \times V \subset X \times X - \Delta$, we have $U \cap V = \varnothing$. $\square$

Proposition 1.20. *The product of two Hausdorff spaces is Hausdorff.*

Please prove this claim.

1.4 Compact Spaces

Definition 1.21. An *open cover* of a topological space X is a collection of open sets $\mathcal{U} = \{U_i\}_{i\in I}$ such that $X = \bigcup_{i\in I} U_i$. A (open) *subcover* of $\mathcal{U}$ is a subset of $\mathcal{U}$ that is still an open cover.

Definition 1.22. A topological space X is called a *compact space* if any open cover of X has a finite subcover.

Similarly we have the notion of open covers of a set and compact subsets. Please give these definitions (or you can deduce them from the following lemma).

Lemma 1.23. *Let $K \subset X$, then K is a compact space with respective to subspace topology if and only if K is a compact subset of X.*

Proof. Suppose K is a compact space with respective to subspace topology, then for any open cover $\{\tilde{U}_i\}_{i\in I}$ in X of K, $\{U_i = \tilde{U}_i\}_{i\in I}$ is an open cover of topological space K. $\{U_i\}$ has a finite subcover, say $U_1, \cdots, U_n$, then $\tilde{U}_1, \cdots, \tilde{U}_n$ is a subcover of $\{\tilde{U}_i\}_{i\in I}$. Conversely, suppose K is a compact subset in X, reverse the argument above we can prove K is a compact space with respective to subspace topology. $\square$

We list some important properties of compact spaces.

Proposition 1.24. *Let X be a topological space, $K \subset X$ be a compact subset, $f : X \to Y$ is a continuous mapping. Then $f(K)$ is compact in Y.*

Proof. Let $\{U_i\}_{i\in I}$ be an open cover of $f(K)$ in Y, then $\{f^{-1}(U_i)\}_{i\in I}$ is an open cover of $f^{-1}(f(K)) \supset K$, has a finite subcover $f^{-1}(U_1), \cdots, f^{-1}(U_n)$. Then $U_1, \cdots, U_n$ is a finite subcover of $f(K)$. $\square$

Proposition 1.25. *A closed subset of a compact space is compact.*

Proof. Let X be a compact space and Z be a closed subset. If $\mathcal{U}$ is an open cover of Z, then $\mathcal{U} \cup \{X - Z\}$ is an open cover of X, which has a finite subcover. Exclude $X - Z$ if necessary, we obtain a subcover of $\mathcal{U}$ of Z. $\square$

Proposition 1.26. *A compact subset of a Hausdorff space is closed.*

Proof. Let X be a Hausdorff space, K be a compact subset. Let $x \notin K$, we prove x is an interior point of $X - K$. For any $k \in K$, we associate k with two open sets $U_k \ni k$ and $V_k \ni x$ with $U_k \cap V_k = \varnothing$, this can be achieved since X is Hausdorff. Then $\{U_k\}_{k\in K}$ is an open cover of K. Since K is compact, it has a subcover $\{U_{k_i}\}_{1\le i\le n}$. Then $V = \bigcap_{i=1}^n V_{k_i}$ is disjoint with K, and is open. Thus x is an interior point of $X - K$, $X - K$ is open, and consequencely K is closed. $\square$

Corollary 1.27. *A subset of a compact Hausdorff space is compact if and only if its closed.*

Definition 1.28. A topological space is called *locally compact* if any point has a compact neighborhood.

1.5 Connected and Path Connected Spaces

The title of this section indicates two types of connectedness. The first one is more general.

Definition 1.29. A topological space X is *disconnected* if $X = U \cup V$ for some nonempty open U, V such that $U \cap V = \varnothing$. A subset $A \subset X$ is *disconnected* if A is disconnected with respective to the subspace topology. A space or a subset is *connected* if it is not disconnected.

Lemma 1.30. *Let X be a topological space and $x \in X$, and $\{C_i\}_{i \in I}$ is the collection of all connected subsets contains x. Set $C = \bigcup_{i \in I} C_i$, then C is connected.*

Proof. Suppose $C = U \cup V$ for some disjoint open sets U, V with respective to C, assume without loss of generality that $x \in U$. For each $i \in I$, $C_i = (C_i \cap U) \cup (C_i \cap V)$, then $C_i \cap U$ and $C_i \cap V$ are disjoint. But C_i is connected, $C_i \cap V$ must be empty, since $x \in C_i \cap U$. Thus

$$V = C \cap V = \bigcup_{i \in I} (C_i \cap V) = \varnothing,$$

C is connected. □

Definition 1.31. According to Lemma 1.30, the union of all connected subsets that contains a point x is connected, it is called the *connected component* which x lies in.

Proposition 1.32. *Two connected components are either disjoint or coincide. Thus a topological space is partitioned into connected components.*

Please prove this claim.

Proposition 1.33. *Let X, Y be topological spaces and X is connected, $f : X \to Y$ is a map. Then $f(X)$ is connected.*

Proof. Let $f(X) = (U \cap f(X)) \cup (V \cap f(X))$ for open sets U, V in Y. Then $X = f^{-1}(U) \cup f^{-1}(V)$, and $f^{-1}(U), f^{-1}(V)$ are open. Since X is connected, one of $f^{-1}(U), f^{-1}(V)$ is empty, say $f^{-1}(V)$. Then $f(f^{-1}(V)) = (V \cap f(X))$ is empty, hence $f(X)$ is connected. □

Another type of connectedness is path connected. This is easier to describe.

Definition 1.34. A *path* in space X is a continuous mapping from a closed interval to X.

Definition 1.35. A topological space is *path connected* if any two points in it can be joined with a path. A subset of a topological space is *path connected* if it is path connected with respective to subspace topology.

Lemma 1.36. *Path connected spaces are connected.*

Proof. Suppose not. Let X be a path connected but disconnected space, then $X = U \cup V$ for some nonempty disjoint open sets. Choose $x \in U, y \in V$, join x, y with a path $f : I \to X$, I is a closed interval. Then $f(I)$ is connected by Proposition 1.33, but $f(I) = (f(I) \cap U) \cup (f(I) \cap V)$, which indicates $f(I)$ is disconnected, contradiction. □

Remark 1.37. Lemma 1.36 shows that path connectedness is stronger than connectedness. For an example showing these two notions are not equivalent, please search "topologist's sine curve".

Proposition 1.38. *Let X, Y be topological spaces and X is path connected, $f : X \to Y$ is a map. Then $f(X)$ is path connected.*

Proof. Let $y, y' \in f(X)$ with $f(x) = y$, $f(x') = y'$, and path $i : I \to X$ joins x, x'. Then $f \circ i$ is a path joining y and y'. □

Definition 1.39. Let X be a topological space. Define an equivalence relation $\sim$ on X as follows: two points $x \sim y$ if x and y can be joined with a path (please verify this is an equivalence relation). Then an element in $X/\sim$ is called a *path connected component*.

I think this is enough to be a rapid introduction to the language of general topology.