A.D. ALEXANDROV SPACES WITH CURVATURE BOUNDED BELOW EXPLAINED

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ABSTRACT. This is an extended version of Burago, Gromov and Perel'man's paper *A.D. Alexandrov spaces with curvature bounded below* [BGP92]. We review the proofs in the paper and add the omitted details into our article. We do not intend to contain the original text in our article, however, we will cover the mathematical content of the paper.

1. Basic Notions in Metric Geometry

We start from intrinsic metric spaces. The authors choose the equivalent definition of existence of δ-midpoint, however, we use the more basic definition.

Notation 1.1. Let X be a metric space, the distance between $x, y \in X$ will be denoted by $|xy|$.

Definition 1.2. Let X be a metric space, $\gamma : [a, b] \to X$ be a curve. We define the *length* of γ to be

$$L[\gamma] := \sup \left\{ \sum_{i=1}^{n} |x_{i-1}x_i| \colon a = x_0 < x_1 < \cdots < x_{n_1} < x_n = b \right\}.$$

If $L[\gamma] < +\infty$, we call γ is *rectifiable*. Denote the set of all the (isomorphism classes under linear reparametrization of) rectifiable curves between x, y by Ω_{xy}.

Definition 1.3. Let X be a metric space. If for any $x, y \in X$, we have

$$|xy| = \inf_{\gamma\in\Omega_{xy}} L[\gamma],$$

then we call X an *intrinsic metric space*.

Proposition 1.4. *Let X be a complete metric space, then X is intrinsic if and only if for any $x, y \in X$ and $\delta > 0$, there exists $z \in X$, such that*

$$|xz| < \frac{1}{2}|xy| + \delta,\ |yz| < \frac{1}{2}|xy| + \delta.$$

Proof. The only if part is relatively easy, we only show if part. Fix $\delta > 0$. For x, y, we define a curve $\gamma : [0, 1] \to X$ with $L[\gamma] < |xy| + \delta$. By assumption, we can choose $\gamma(1/2)$ with

$$\left|x\gamma\left(\frac{1}{2}\right)\right| < \frac{1}{2}|xy| + \frac{1}{2^2}\delta,\ \left|y\gamma\left(\frac{1}{2}\right)\right| < \frac{1}{2}|xy| + \frac{1}{2^2}\delta.$$

Then we choose

$$\left|x\gamma\left(\frac{1}{4}\right)\right| < \frac{1}{2}\left|x\gamma\left(\frac{1}{2}\right)\right| + \frac{1}{4^2}\delta,\ \left|\gamma\left(\frac{1}{4}\right)\gamma\left(\frac{1}{2}\right)\right| < \frac{1}{2}\left|x\gamma\left(\frac{1}{2}\right)\right| + \frac{1}{4^2}\delta$$
$$\left|\gamma\left(\frac{3}{4}\right)\gamma\left(\frac{1}{2}\right)\right| < \frac{1}{2}\left|\gamma\left(\frac{1}{2}\right)y\right| + \frac{1}{4^2}\delta,\ \left|\gamma\left(\frac{3}{4}\right)y\right| < \frac{1}{2}\left|x\gamma\left(\frac{1}{2}\right)\right| + \frac{1}{4^2}\delta.$$

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Define inductively to all diadic numbers, add $\gamma(0) = x, \gamma(1) = y$. Notice that γ is Lipschitz, hence γ can be defined on whole $[0, 1]$. Since

$$\begin{aligned}\sum_{i=0}^{2^n-1}\left|\gamma\left(\frac{i}{2^n}\right)\gamma\left(\frac{i+1}{2^n}\right)\right| &< \sum_{i=0}^{2^{n-1}-1}\left(\left|\gamma\left(\frac{i}{2^{n-1}}\right)\gamma\left(\frac{i+1}{2^{n-1}}\right)\right| + \frac{1}{2^{2n-1}}\delta\right)\\ &< \cdots \\ &< |xy| + \delta,\end{aligned}$$

passing to limit, we obtain $L[\gamma] < |xy| + \delta$. □

Definition 1.5. A *geodesic* is a curve whose length is equal to the distance between its ends. An intrinsic metric space is called *geodesic* if any two points can be joined with a geodesic.

Notation 1.6. We use $[xy]$ to denote a geodesic between x and y. There may be several geodesics between x and y, and if we use this notation, we mean we specified a particular geodesic.

Proposition 1.7. *Let X be a complete metric space, then X is geodesic if and only if for any $x, y \in X$, there exists $z \in X$, such that*

$$|xz| = |yz| = \frac{1}{2}|xy|.$$

Proof. We also only show if part, and this can be achieved by taking $\delta = 0$ in the proof of Proposition 1.4. □

We now discuss the Hopf–Rinow theorem. The proof is taken from [AKP24, 2.15].

Definition 1.8. A metric space X is called *proper* if any closed bounded subset of X is compact.

Lemma 1.9. *Proper intrinsic metric spaces are geodesic.*

Proof. Let X be a proper intrinsic metric space. Consider $1/n$-midpoints z_n of two points $x, y \in X$. Since they are contained in the ball $B(x, |xy|)$, they are contained in $\overline{B(x, |xy|)}$, which is bounded and closed, hence compact. Therefore $\{z_n\}$ contains a convergent subsequence, which converge to the midpoint of x and y, that is, the midpoint of x and y exists. By Proposition 1.7, X is geodesic. □

Theorem 1.10 (Hopf–Rinow)**.** *Locally compact complete intrinsic metric spaces are proper.*

Proof. Let X be a locally compact complete intrinsic metric space. For $x \in X$, define $\rho(x)$ to be the supremum of all $R > 0$ such that $\overline{B(x, R)}$ is compact. Since X is locally compact, $\rho(x) > 0$ for any $x \in X$. It's sufficient to show $\rho(x) = +\infty$ for some (and therefore any) x. Suppose not, i.e., $\rho(x) < +\infty$.

First, notice that $B = \overline{B(x, \rho(x))}$ is compact. Indeed, for any $\varepsilon > 0$, $\overline{B(x, \rho(x) - \varepsilon)}$ is compact, and since X is intrinsic, it is an ε-net of B, hence B is compact.

Second, we claim that $|\rho(x) - \rho(y)| \le |xy|$, and in particular, ρ is continuous. If this does not hold, we have $\rho(x) + |xy| < \rho(y)$. Then $\overline{B(x, \rho(x) + \varepsilon)}$ is a closed subset of $\overline{B(y, \rho(y))}$ for sufficiently small $\varepsilon > 0$, hence $\overline{B(x, \rho(x) + \varepsilon)}$ is compact. This contradicts to the definition of ρ.

Now let $\varepsilon = \min_{y \in B} \rho(y)$, since B is compact, the minimum can be reached, and $\varepsilon > 0$. Choose a finite $\varepsilon/10$-net $\{a_1, \cdots, a_n\}$ in B, set

$$W = \bigcup_{i=1}^{n} \overline{B(a_i, \varepsilon)}.$$

Then W is compact. However, $\overline{B(x, \rho(x) + \varepsilon/10)} \subset W$, this means $\overline{B(x, \rho(x) + \varepsilon/10)}$ is compact. This contradicts to the definition of ρ. Hence we must have $\rho(x) = +\infty$. □

Corollary 1.11. *Locally compact complete intrinsic metric spaces are geodesic.*

Remark 1.12. After stating Hopf–Rinow theorem in 2.1, [BGP92] claims that the limit of geodesics is still a geodesic. This is not correct. Think about the surface of a solid cylinder, consider the geodesics connecting antipodal points converging to the upper face. They converge to a semicircle, but it is not a geodesic—now the shortest path is a line segement on the upper face.

Definition 1.13. Let X be an intrinsic metric space. A subset $S \subset X$ is called *convex* if any two points $p, q \in S$ can be joined with a geodesic, and the geodesic $[pq] \subset S$.

Remark 1.14. The definition of convexity in ambiguous in [BGP92], so we adopt the usual definition.

Definition 1.15. A *triangle* on an intrinsic metric space X is the collection of three points $p, q, r \in X$ and three geodesics $[qr], [pr], [pq]$, denoted by $\triangle pqr$.

Notation 1.16. Denote S_k^2 the 2-dimensional complete simply-connected Riemannian manifold of curvature k.

Definition 1.17. Let p, q, r be a triple of points in an intrinsic metric space X. Define their *comparison triangle on S_k^2*, denoted by $\tilde{\triangle}_k pqr$, is the triangle with vertices $\tilde{p}, \tilde{q}, \tilde{r}$ on S_k^2 with $|pq| = |\tilde{p}\tilde{q}|$, $|pr| = |\tilde{p}\tilde{r}|$, $|qr| = |\tilde{q}\tilde{r}$. Comparison triangle always exists up to a rigid motion when $k \leq 0$, and when $k > 0$ we ask the perimeter of $\triangle pqr$ is less than $2\pi/\sqrt{k}$. We denote $\tilde{\angle}_k pqr$ by the angle at vertex $\tilde{q}$ in triangle $\tilde{\triangle}_k pqr$.

2. Basic Concepts

Definition 2.1. A locally complete space X with intrinsic metric is called *space with curvature* $\geq k$ if in some neighborhood U_x of each point $x \in X$ the following condition is satisfied:

(D) for any four distinct points $(a; b, c, d)$ in U_x we have the inequality

$$\tilde{\angle}_k bac + \tilde{\angle}_k bad + \tilde{\angle}_k cad \leq 2\pi.$$

If X is a 1-dimensional manifold and $k > 0$, we require in addition that $\operatorname{diam} X \leq \pi/\sqrt{k}$.

For traditional Toponogov's comparison theorem, we have the following condition.

Theorem 2.2. *A geodesic space X is a space with curvature $\geq k$ if and only if the following condition is satisfied:*

(A) *for any triangle $\triangle pqr$ with vertices in U_x and any point $s \in [qr]$, the inequality $|ps| \geq |\tilde{p}\tilde{s}|$ is satisfied, where $\tilde{s}$ is the point on the side $[\tilde{q}\tilde{r}]$ of the comparison triangle $\tilde{\triangle}_k pqr$ with $|qs| = |\tilde{q}\tilde{s}|$.*

To prove the only if part, we need the following lemma.

Lemma 2.3 (Alexandrov)**.** *Let $\tilde{\triangle}_k pqs, \tilde{\triangle}_k prs$ be given on S_k^2, which are joined to each other in an exterior way along the side $[ps]$. Let there also be given $\tilde{\triangle} bcd$, where $|bc| = |pq|$, $|bd| = |pr|$, $|cd| = |qs| + |rs|$, and $|bc| + |bd| + |cd| < 2\pi/\sqrt{k}$ in the case $k > 0$. Then $\tilde{\angle}_k psq + \tilde{\angle}_k psr \leq \pi$ ($\geq \pi$) if and only if $\tilde{\angle}_k pqs \geq \tilde{\angle}_k bcd$ and $\tilde{\angle}_k prs \geq \tilde{\angle}_k bdc$ (respectively $\tilde{\angle}_k pqs \leq \tilde{\angle}_k bcd$ and $\tilde{\angle}_k prs \leq \tilde{\angle}_k bdc$).*

Proof. Observe that in absolute geometry (Euclidean, spherical or non-Euclidean), when two sides are fixed, bigger angle is opposite to bigger side. Using a rigid motion, we can assume $q = c$ and $s \in [cd]$. Then $\tilde{\angle}_k psq + \tilde{\angle}_k psr \leq \pi$ is equivalent to $\tilde{\angle}_k rsd \leq \tilde{\angle}_k psd$. Using observation, this is equivalent to $|pd| \geq |pr| = |bd|$. Using observation again, this is equivalent to $\tilde{\angle}_k pqs \geq \tilde{\angle}_k bcd$. The other inequality holds similarly. □

Proof of (D) $\implies$ (A). Apply Alexandrov lemma (Lemma 2.3) to comparison triangle $\tilde{\triangle}_k pqs$ and $\tilde{\triangle}_k psr$. By (D), we have $\tilde{\measuredangle}_k psq + \tilde{\measuredangle}_k psr \le \pi$ (since $\tilde{\measuredangle}_k qsr = \pi$), hence $\tilde{\measuredangle}_k pqs \ge \tilde{\measuredangle}_k pqr$, that is, $|ps| \ge |\tilde{p}\tilde{s}|$. □

The proof of converse needs more work. We need the notion of angle between two geodesics with the same origin.

Notation 2.4. Let γ, σ be geodesics with origin p, q, r on γ, σ respectively and $x = |pq|, y = |pr|$. Denote $\omega_k(x, y) = \tilde{\measuredangle}_k qpr$.

Definition 2.5. Let γ, σ be geodesics with origin p. If the limit

$$\lim_{x,y\to 0} \omega_k(x, y)$$

exists, the limit is called the *angle* between γ and σ.

We need to show angle between two geodesics does not depend on the choice of k.

Lemma 2.6. *Let p, q, r be three points in a metric space, then $k \mapsto \tilde{\measuredangle}_k qpr$ is increasing.*

Proof. This follows from the triangle version of the Toponogov's comparison theorem (cf. [Pet16, Theorem 12.2.2]). Since angles on Riemannian manifolds are defined by inner product, this proof does not rely on Definition 2.5. □

Lemma 2.7. *Let $p, q, r \in X$, with $|pq| + |qr| + |rp| < 2\pi/\sqrt{k}$ when $k > 0$. Let $\ell = \max\{|pq|, |pr|\}$, then we have*

$$A[\tilde{\triangle} pqr] \le \pi(\ell^2 + o(\ell^2)).$$

Proof. We adopt geodesic polar coordinate. Since geodesic balls are convex, we have

$$\begin{aligned} A[\tilde{\triangle} pqr] &\le \int_0^\ell \int_0^\theta \operatorname{sn}_k(r)\, ds\, dr \\ &= \theta \operatorname{md}_k(\ell) \\ &\le \pi \operatorname{md}_k(\ell) \\ &= \pi(\ell^2 + o(\ell^2)). \end{aligned}$$

Here sn_k and md_k are solutions to initial value problem

$$\operatorname{sn}_k(t) + k(\operatorname{sn}_k)''(t) = 0, \ \operatorname{sn}_k(0) = 0, \ (\operatorname{sn}_k)'(0) = 1,$$

and

$$\operatorname{md}_k(t) + k(\operatorname{md}_k)''(t) = 1, \ \operatorname{md}_k(0) = 0, \ (\operatorname{md}_k)'(0) = 0.$$ □

Lemma 2.8. *Let $p, q, r \in X$, with $|pq| + |qr| + |rp| < 2\pi/\sqrt{k}$ when $k > 0$. Let $\ell = \max\{|pq|, |pr|\}$, $k < K$, then we have*

$$0 < \tilde{\measuredangle}_k qpr - \tilde{\measuredangle}_k qpr \le \pi(|K| + |k|)(\ell^2 + o(\ell^2)). \tag{2.1}$$

Proof. By previous lemmas and Gauss–Bonnet formula, we have

$$\begin{aligned} 0 &< \tilde{\measuredangle}_k qpr - \tilde{\measuredangle}_k qpr \\ &< \tilde{\measuredangle}_k qpr + \tilde{\measuredangle}_k pqr + \tilde{\measuredangle}_k prq - \tilde{\measuredangle}_k qpr - \tilde{\measuredangle}_k pqr - \tilde{\measuredangle}_k prq \\ &= KA[\tilde{\triangle}_k pqr] - kA[\tilde{\triangle}_k pqr] \\ &\le (|K| + |k|)(\ell^2 + o(\ell^2)). \end{aligned}$$ □

Let $\ell \to 0$ in inequality (2.1), we have thus shown the angle between geodesics is well-defined.

Let us back to our discussion on spaces with curvature bounded below. (A) has the following corollary.

Proposition 2.9. *If* (A) *is satisfied, then for any geodesics γ, σ with origin p, the function $\omega_k(x, y)$ is non-increasing with respective to each variable x, y when x, y are sufficiently small.*

Proof. Let U_p be a neighborhood of p such that (A) holds, $x, x' \in \gamma$, $y \in \sigma$ and x is between p and x'. We must show $\omega_k(x, y) \geq \omega_k(x', y)$. By (A), we have $|yx| \geq |\tilde{y}\tilde{x}|$, then we have

$$\omega_k(x, y) = \tilde{\measuredangle}_k ypx \geq \measuredangle \tilde{y}\tilde{p}\tilde{x} = \tilde{\measuredangle}_k ypx' = \omega_k(x', y).$$

Similarly $\omega_k(x, y)$ is non-increasing in y. ◻

By Proposition 2.9, the angle between two geodesics with same origin is always defined.

Proposition 2.10. *The angles between three geodesics with same origin satisfy triangle inequality.*

Proof. A proof for general metric space can be found in [AKP24, 6.5]. But for spaces with curvature $\geq k$, notice that angles in space form satisfy triangle inequality, and then we can take the limit. ◻

Proposition 2.9 also has the following direct corollaries.

Corollary 2.11. *If* (A) *is satisfied, then*

(C) *For any triangle $\triangle pqr$ contained in U_x, none of its angles is less than the corresponding angle of the comparison triangle $\tilde{\triangle}_k pqr$ in S_k^2.*

(C_1) *If r is an interior point of a geodesic $[pq]$, then for any geodesic $[rs]$, we have $\measuredangle prs + \measuredangle qrs = \pi$.*

(C_1) has a corollary that geodesics do not branch.

Proposition 2.12. *Let $[px], [py]$ be geodesics, such that $[ps] \subset [px] \cap [py]$, then $[px] \subset [py]$ or $[py] \subset [px]$.*

Proof. By (C_1), we have $\measuredangle psy + \measuredangle xsy = \pi$. But $\measuredangle psy = \pi$, since p, s, y are colinear, hence $\measuredangle xsy = 0$, and result follows. ◻

Now we are ready to prove (A) $\implies$ (D).

Proof of (A) $\implies$ (D). Let $(p; a, b, c)$ in U_x. Choose $d \in [pa]$ closed to p, then by triangle inequality (Proposition 2.10) and (C_1) we have

$$\begin{aligned} \measuredangle adb + \measuredangle bdc + \measuredangle cda &\leq \measuredangle adb + \measuredangle bdp + \measuredangle pdc + \measuredangle cda \\ &= 2\pi. \end{aligned}$$

Apply Proposition 2.9, we have

$$\begin{aligned} \tilde{\measuredangle}_k adb + \tilde{\measuredangle}_k bdc + \tilde{\measuredangle}_k cda &\leq \measuredangle adb + \measuredangle bdc + \measuredangle cda \\ &\leq 2\pi. \end{aligned}$$

Since angles vary continuously in S_k^2, let $d \to p$, we obtain

$$\tilde{\measuredangle}_k apb + \tilde{\measuredangle}_k bpc + \tilde{\measuredangle}_k cpa \leq 2\pi.$$ ◻

3. Globalization Theorem

In this section we introduce the *globalization theorem*.

Theorem 3.1 (Globalization theorem)**.** *Let X be a complete geodesic metric space with curvature $\geq k$. Then for any quadruple of points $(a; b, c, d)$, the inequality*

$$\tilde{\measuredangle}_k bac + \tilde{\measuredangle}_k cad + \tilde{\measuredangle}_k dab \leq 2\pi$$

holds.

Globalization theorem is the most important structure theorem of space with curvature $\geq k$. Simpler proofs can be found in [Pla01] or [AKP24] (where without geodesic assumption). However, we shall present the original proof in [BGP92].

Definition 3.2. Define the *size* of the quadruple $(a; b, c, d)$, denoted by $S(a; b, c, d)$, is the greatest number of the perimeters of triangles $\triangle bac$, $\triangle cad$, $\triangle dab$. Define the *excess* of the quadruple $(a; b, c, d)$, denoted by $E(a; b, c, d)$, is $\max\{0, \tilde{\measuredangle}_k bac + \tilde{\measuredangle}_k cad + \tilde{\measuredangle}_k dab - 2\pi\}$.

Lemma 3.3. *Let p, q, r, s be points in an intrinsic metric space, and let t lie on a geodesic joining p and q. Then*

$$S(p; q, r, s) \geq \max\{S(p; t, r, s), S(t; p, q, r), S(t; p, q, s)\}.$$

Proof. This is evident from the triangle inequality. □

Lemma 3.4. *Let $\triangle pqr$, $\triangle prx$, $\triangle qry$ be given on S_k^2 such that $|rx| = |ry|$, $|pq| = |px| + |py|$, $\measuredangle pxr + \measuredangle pyr \geq \pi$. Then we have*

$$\measuredangle qpr - \measuredangle xpr \leq \measuredangle qyr + \measuredangle pxr - \pi.$$

If $k > 0$, then in addition we assume that $\max\{|pr|, |pq|\} + 10|px| < \pi/2\sqrt{k}$.

Proof. We may suppose $\triangle pxr$, $\triangle qyr$ lie in $\triangle pqr$ and do not overlap. Let $z \in [pq]$ with $|pz| = |px$. We calculate the total angle at x, y, z. Denote $\delta(\triangle)$ the excess of a triangle, i.e. the difference between its sum of interior angles and π. Then we have

$$\begin{aligned} 5\pi =& (\measuredangle pxr + \measuredangle qyr) + (\pi + \delta(\triangle xyr) - \measuredangle xry) + (\pi + \delta(\triangle xyz)) \\ &+ (\pi + \delta(\triangle xpz) + \measuredangle qpr - \measuredangle xpr) + (\pi + \delta(\triangle qyz) - \measuredangle yqz). \end{aligned}$$

If $k \leq 0$, then $\delta(\triangle) \leq 0$, the inequality evidently holds. If $k > 0$, the condition $\max\{|pr|, |pq|\} + 10|px| < \pi/2\sqrt{k}$ guarantees all $\delta(\triangle)$'s are not greater than the half of $\min\{\measuredangle xry, \measuredangle yqz\}$, hence the inequality still holds. □

Lemma 3.5. *Let p, q, r, s be points in an intrinsic metric space, and let t lie on a geodesic joining p and q. Then*

$$E(p; q, r, s) \leq E(p; t, r, s) + E(t; p, q, r) + E(t; p, q, s).$$

Proof. By Lemma 3.4, we have the inequalities

$$\begin{aligned} E(t; p, q, r) &\geq \tilde{\measuredangle}_k qpr - \tilde{\measuredangle}_k tpr, \\ E(t; p, q, s) &\geq \tilde{\measuredangle}_k qps - \tilde{\measuredangle}_k tps. \end{aligned}$$

And by definition, we have

$$E(p; t, r, s) = \tilde{\measuredangle}_k tpr + \tilde{\measuredangle}_k rps + \tilde{\measuredangle}_k tps - 2\pi,$$

add all three formulas together, we obtain

$$\begin{aligned} E(p; t, r, s) + E(t; p, q, r) + E(t; p, q, s) &\geq \tilde{\measuredangle}_k qpr + \tilde{\measuredangle}_k rps + \tilde{\measuredangle}_k qps \\ &= E(p; q, r, s). \end{aligned}$$ □

Proof of globalization theorem. Let us assume the theorem is false. Then there exists a point $p \in X$ and $\ell > 0$ ($\ell < 2\pi/\sqrt{k}$ if $k > 0$) such that

(a) The excess of any quadruple of size $\leq 0.99\ell$ lying in $B_p(100\ell)$ is zero.
(b) There is a quadruple of size $\leq \ell$ lying in $B_p(10\ell)$ and having a positive excess.

In fact, fix ℓ_0 with a quadruple $(a; b, c, d)$ such that $S(a; b, c, d) = \ell_0$ and $E(a; b, c, d) > 0$. Let $(a; b, c, d)$ be contained in $B_{p_0}(10\ell_0)$, if all quadruples lying in $B_{p_0}(100\ell_0)$ with size $\leq 0.99\ell_0$ have zero excess, choose $\ell = \ell_0$. Otherwise let $(a_1; b_1, c_1, d_1)$ be another quadruple with positive excess that has size $\ell_1 \leq 0.99\ell_0$, assume the quadruple is contained in $B_{p_1}(10\ell_1)$. Then $|p_0 p_1| \leq 1000\ell_0$. Construct inductively, either we obtain ℓ after finite steps, or we obtain a Cauchy sequence $\{p_n\}$. Since X is complete, $\{p_n\}$ converges to some p. Then in any small neighborhood of p, (D) is violated, contradiction.

We claim that a quadruple $(a;b,c,d)$ of size $\le \ell$ lying in $B_p(20\ell)$ has zero excess if

$$|ab| \le 0.01\ell,\ |cd| \ge 0.1\ell + \max\{|ac|,|ad|\}. \tag{3.1}$$

If $k>0$, we ask in addition that $\max\{|ac|,|ad|\} < \pi/2\sqrt{k}$. We assume there is a quadruple $(a;b,c,d)$ violates this, and let $E(a;b,c,d)=\delta>0$. Let $x\in[ac]$ with $|ax|=\varepsilon\le 0.01\ell$. By the triangle inequality, we have

$$S(x;a,b,c)\le 0.99\ell,\ S(a;x,b,d)\le 0.99\ell.$$

Hence by Lemma 3.5, we have

$$E(a;b,c,d)\le E(x;a,c,d).$$

Now let $y\in[dx]$ such that $|xy|=\varepsilon$. Then similarly, we have

$$E(y;x,c,d)\ge E(x;a,c,d)\ge\delta.$$

Notice that $S(y;x,c,d)\le\ell$, and it satisfies an inequality analogous to (3.1). Finally, we have

$$|yc|+|yd|\le|ac|+|ad|-\frac{\delta^2\varepsilon}{2}. \tag{3.2}$$

In fact, since $E(x;a,b,c)=0$, using Proposition 2.9, we have

$$\tilde\measuredangle_k xab\ge\tilde\measuredangle_k cab.$$

Since $E(a;x,d,b)=0$, we have

$$2\pi\ge\tilde\measuredangle_k xab+\tilde\measuredangle_k dab+\tilde\measuredangle_k xad.$$

Moreover, we have

$$\tilde\measuredangle_k dac+\tilde\measuredangle_k dab+\tilde\measuredangle_k cab=2\pi+\delta,$$

and

$$\pi\ge\tilde\measuredangle_k dac,$$

add all these inequalities, we obtain $\tilde\measuredangle_k xad\le\pi-\delta$. Similarly we have $\tilde\measuredangle yxc\le\pi-\delta$, hence by first variation inequality (cf. [AKP24, 6.7]), we have

$$|dx|\le|da|+\left(1-\frac{\delta^2}{4}\right)\varepsilon,$$
$$|cy|\le|cx|+\left(1-\frac{\delta^2}{4}\right)\varepsilon,$$

and

$$|cx|=|ca|-\varepsilon,$$
$$|dy|=|dx|-\varepsilon.$$

Add all these inequalities, we obtain (3.2). Therefore with a counterexample, we can construct another counterexample with sum of two sides decreases $\delta^2\varepsilon/2$. After no more than $[2\ell/\delta^2\varepsilon]+1$ steps we can reach a contradiction.

Now we consider the general case. Let $(a;b,c,d)$ be a quadruple of size $\le\ell$ lying in $B_p(10\ell)$ with positive excess. Choose $x\in[ab]$ such that $|ax|=\varepsilon\le 0.001\ell$ (if $k>0$, we ask $10\varepsilon<2\pi/\sqrt{k}-\ell$), by Lemma 3.3, the quadruples $(a;x,c,d)$, $(x;a,b,c)$, $(x;a,b,d)$ all have size $\le\ell$. Hence we reduce our study to the case of $|ab|=\varepsilon$. Next we want to reduce our consideration to a quadruple $(x;y,z,t)$ of size $\le\ell$ such that $|xy|\le2\varepsilon$, $|xz|\le2\varepsilon$. Divide $[ac]$ into segements $[aa_1],[a_1a_2],\cdots,[a_{n-1}c]$ with each of which has length $\le\varepsilon$. By Lemma 3.5, at least one of the quadruples $(a;a_1,b,d)$, $(a_1;a,b,c)$, $(a_1;a,c,d)$ has positive excess. If one of the first two has positive excess, then we are done. Otherwise one of the quadruples $(a_1;a_2,a,d)$, $(a_2;a_1,a,c)$, $(a_2;a_1,c,d)$ has positive excess. Continue "thinning out" the quadruples with positive excess until we get a quadruple we want. Finally, let $(a; b, c, d)$ be a quadruple of size $\leq \ell$ with positive excess for which $|ab| \leq 2\varepsilon$, $|ac| \leq 2\varepsilon$. Choose $x \in [ad]$ with $|cx| = |dx|$. Then we have

$$|cx| \leq \frac{\ell}{2} + 2\varepsilon.$$

Lemma 3.5 tells us that at least one of the quadruples $(a; x, b, c)$, $(x; a, b, d)$, $(x; a, c, d)$ has positive excess. But $S(a; x, b, c) \leq 0.99\ell$, hence $E(a; x, b, c) = 0$. Without loss of generality, we assume $E(x; a, c, d) > 0$. Choose $y \in [ax]$ with $|xy| = \varepsilon$. Then Lemma 3.5 tells us that at least one of the quadruples $(x; y, c, d)$, $(y; x, a, c)$, $(y; x, a, d)$ has positive excess. However, $(x; y, c, d)$ is the situation considered in (3.1), $(y; x, a, c)$ is small, $\tilde{\measuredangle}_k ayx = 0$, both of them has zero excess, contradiction! Hence the theorem holds. □

We now give some application of globalization theorem.

Theorem 3.6. *Let X be a complete geodesic space with curvature $\geq k$, $k > 0$, then* $\operatorname{diam} X \leq \pi/\sqrt{k}$.

This proof is partly taken from [BBI01, Theorem 10.4.1] and [Shi93, Theorem 6.2].

Proof. Suppose the theorem is false. Let p, q be points with $|pq| = (\pi + \varepsilon)/\sqrt{k}$, where $0 < \varepsilon < 0.1\pi$, and m be a midpoint of them. Let $U = B_m(\varepsilon)$.

First we show that U contains a point that does not lie on $[pq]$. Suppose not. For every $x \in X$ there is a geodesic γ joining x and m. Our assumption tells that γ coincides with $[pq]$ on a subinterval. By Proposition 2.12, geodesics do not branch, it follows that x belongs to a unique geodesic containing $[pm]$. Hence X is covered by two geodesics starting from m passing through p and q, hence X is a 1-dimensional manifold. By Definition 2.1, $\operatorname{diam} X \leq \pi/\sqrt{k}$, contradiction!

Choose $x \in U \setminus [pq]$, then by triangle inequality, $\triangle pmx$, $\triangle qmx$ have perimeters less than $2\pi/\sqrt{k}$. Let $\tilde{p}, \tilde{q}, \tilde{m}, \tilde{x}$ on S_k^2 satisfy $|pq| = |\tilde{p}\tilde{q}|$ and $\tilde{m}$ be the midpoint, $|mx| = |\tilde{m}\tilde{x}|$, $\measuredangle pmx = \measuredangle \tilde{p}\tilde{m}\tilde{x}$ and $\measuredangle qmx = \measuredangle \tilde{q}\tilde{m}\tilde{x}$. By Corollary 2.11 and globalization theorem (which implies $U_x = X$), we have $|\tilde{p}\tilde{x}| \geq |px|$ and $|\tilde{q}\tilde{x}| \geq |qx|$. On S_k^2, $\tilde{p}, \tilde{m}, \tilde{q}$ are on a great sphere, hence

$$|\tilde{p}\tilde{x}| + |\tilde{q}\tilde{x}| < |\tilde{p}\tilde{m}| + |\tilde{q}\tilde{m}|.$$

Thus

$$\begin{aligned} |pq| &= |\tilde{p}\tilde{m}| + |\tilde{q}\tilde{m}| \\ &> |\tilde{p}\tilde{x}| + |\tilde{q}\tilde{x}| \\ &\geq |px| + |qx|, \end{aligned}$$

contradicting to the triangle inequality. Therefore the theorem holds. □

Theorem 3.7. *Let X be a complete geodesic space with curvature $\geq k$, $k > 0$. Then for any three points $a, b, c \in X$ we have $|ab| + |bc| + |ac| \leq 2\pi/\sqrt{k}$ and condition* (D) *is satisfied for any quadruples of size $2\pi/\sqrt{k}$. (Here we suppose that, if $|ab| = \pi/\sqrt{k}$ and $|ac| + |bc| = \pi/\sqrt{k}$, then $\tilde{\measuredangle}_k bac = \tilde{\measuredangle}_k abc = 0$, $\tilde{\measuredangle}_k acb = \pi$.)*

Proof. We first prove the second assertion. We check for a quadruple $(a; b, c, d)$ of size $2\pi/\sqrt{k}$. We may suppose that $\max\{|ab|, |ac|, |ad|\} < \pi/\sqrt{k}$, and the perimeter of $\triangle abd$ is $2\pi/\sqrt{k}$. We choose a sequence $\{b_i\} \subset [ab]$ converges to b monotonically. Then the perimeters of $\triangle ab_ic, \triangle ab_id$ are all $< 2\pi/\sqrt{k}$. Applying Theorem 2.2 to quadruple $(b_i; a, b, c)$, we have $\tilde{\measuredangle}_k b_iac \geq \tilde{\measuredangle}_k bac$. Moreover, applying Theorem 2.2 to quadruple $(b_i; a, b_{i+1}, c)$, we have $\tilde{\measuredangle}_k b_iad \geq \tilde{\measuredangle}_k b_{i+1}ad$. However, when $b_i \to b$, the perimeter of $\triangle ab_id \to 2\pi/\sqrt{k}$, hence $\tilde{\measuredangle}_k b_iad \downarrow \pi$, which implies $\tilde{\measuredangle}_k b_iad = \pi$ for all $i \in \mathbb{N}$. Therefore, if $E(a;b,c,d) > 0$, we can find a quadruple $(a;b_i,c,d)$ with positive excess but size $< 2\pi/\sqrt{k}$, which is impossible.

Now we turn to the first assertion. Assume the perimeter of $\triangle abc$ is greater than $2\pi/\sqrt{k}$. We may suppose $\max\{|ab|,|bc|,|ac|\} < \pi/\sqrt{k}$ and $|bc| > |ac|$. Now choose a point $x \in [bc]$ such that $|bx| \le |cx| - |ac|$ and

$$\frac{2\pi}{\sqrt{k}} < |ax| + |cx| + |ac| < |ab| + |bc| + |ac|.$$

Let $y \in [ax]$ such that the perimeter of $\triangle cxy$ is equal to $2\pi/\sqrt{k}$. Then

$$|xy| \ge |cx| - |ac| + |ay| > |bx|.$$

Since $|cx| < \pi/\sqrt{k}$ and $|xy| < \pi/\sqrt{k}$, we have $\tilde{\angle}_k cxy = \pi$. Now apply (D) to $(x;c,y,b)$, we have $\tilde{\angle}_k bxy = 0$. But $|xb| < |xy|$, we have $|xy| = |xb| + |xy|$. Thus $|xa| = |xb| + |ab|$, which contradicts to the choice of x. Therefore the theorem holds. □

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