A.D. ALEXANDROV SPACES WITH CURVATURE BOUNDED BELOW EXPLAINED

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Abstract. This is an extended version of Burago, Gromov and Perel'man's paper *A.D. Alexandrov spaces with curvature bounded below* [BGP92]. We review the proofs in the paper and add the omitted details into our article. We do not intend to contain the original text in our article, however, we will cover the mathematical content of the paper.

1. Basic Notions in Metric Geometry

We start from intrinsic metric spaces. The authors choose the equivalent definition of existence of δ-midpoint, however, we use the more basic definition.

Notation 1.1. Let X be a metric space, the distance between $x, y \in X$ will be denoted by $|xy|$.

Definition 1.2. Let X be a metric space, $\gamma : [a, b] \to X$ be a curve. We define the *length* of γ to be

$$L[\gamma] := \sup\left\{\sum_{i=1}^{n} |x_{i-1}x_i| \colon a = x_0 < x_1 < \cdots < x_{n_1} < x_n = b\right\}.$$

If $L[\gamma] < +\infty$, we call γ is *rectifiable*. Denote the set of all the (isomorphism classes under linear reparametrization of) rectifiable curves between x, y by Ω_{xy}.

Definition 1.3. Let X be a metric space. If for any $x, y \in X$, we have

$$|xy| = \inf_{\gamma \in \Omega_{xy}} L[\gamma],$$

then we call X an *intrinsic metric space*.

Proposition 1.4. *Let X be a complete metric space, then X is intrinsic if and only if for any $x, y \in X$ and $\delta > 0$, there exists $z \in X$, such that*

$$|xz| < \frac{1}{2}|xy| + \delta, \ |yz| < \frac{1}{2}|xy| + \delta.$$

Proof. The only if part is relatively easy, we only show if part. Fix $\delta > 0$. For x, y, we define a curve $\gamma : [0, 1] \to X$ with $L[\gamma] < |xy| + \delta$. By assumption, we can choose $\gamma(1/2)$ with

$$\left|x\gamma\left(\frac{1}{2}\right)\right| < \frac{1}{2}|xy| + \frac{1}{2^2}\delta, \ \left|y\gamma\left(\frac{1}{2}\right)\right| < \frac{1}{2}|xy| + \frac{1}{2^2}\delta.$$

Then we choose

$$\left|x\gamma\left(\frac{1}{4}\right)\right| < \frac{1}{2}\left|x\gamma\left(\frac{1}{2}\right)\right| + \frac{1}{4^2}\delta, \ \left|\gamma\left(\frac{1}{4}\right)\gamma\left(\frac{1}{2}\right)\right| < \frac{1}{2}\left|x\gamma\left(\frac{1}{2}\right)\right| + \frac{1}{4^2}\delta$$
$$\left|\gamma\left(\frac{3}{4}\right)\gamma\left(\frac{1}{2}\right)\right| < \frac{1}{2}\left|\gamma\left(\frac{1}{2}\right)y\right| + \frac{1}{4^2}\delta, \ \left|\gamma\left(\frac{3}{4}\right)y\right| < \frac{1}{2}\left|x\gamma\left(\frac{1}{2}\right)\right| + \frac{1}{4^2}\delta.$$

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Define inductively to all diadic numbers, add $\gamma(0) = x, \gamma(1) = y$. Notice that γ is Lipschitz, hence γ can be defined on whole $[0, 1]$. Since

$$\begin{aligned}\sum_{i=0}^{2^n-1} \left|\gamma\left(\frac{i}{2^n}\right)\gamma\left(\frac{i+1}{2^n}\right)\right| &< \sum_{i=0}^{2^{n-1}-1} \left(\left|\gamma\left(\frac{i}{2^{n-1}}\right)\gamma\left(\frac{i+1}{2^{n-1}}\right)\right| + \frac{1}{2^{2n-1}}\delta\right) \\ &< \cdots \\ &< |xy| + \delta,\end{aligned}$$

passing to limit, we obtain $L[\gamma] < |xy| + \delta$. □

Definition 1.5. A *geodesic* is a curve whose length is equal to the distance between its ends. An intrinsic metric space is called *geodesic* if any two points can be joined with a geodesic.

Notation 1.6. We use $[xy]$ to denote a geodesic between x and y. There may be several geodesics between x and y, and if we use this notation, we mean we specified a particular geodesic.

Proposition 1.7. *Let X be a complete metric space, then X is geodesic if and only if for any $x, y \in X$, there exists $z \in X$, such that*

$$|xz| = |yz| = \frac{1}{2}|xy|.$$

Proof. We also only show if part, and this can be achieved by taking $\delta = 0$ in the proof of Proposition 1.4. □

We now discuss the Hopf–Rinow theorem. The proof is taken from [AKP24, 2.15].

Definition 1.8. A metric space X is called *proper* if any closed bounded subset of X is compact.

Lemma 1.9. *Proper intrinsic metric spaces are geodesic.*

Proof. Let X be a proper intrinsic metric space. Consider $1/n$-midpoints z_n of two points $x, y \in X$. Since they are contained in the ball $B(x, |xy|)$, they are contained in $\overline{B(x, |xy|)}$, which is bounded and closed, hence compact. Therefore $\{z_n\}$ contains a convergent subsequence, which converge to the midpoint of x and y, that is, the midpoint of x and y exists. By Proposition 1.7, X is geodesic. □

Theorem 1.10 (Hopf–Rinow)**.** *Locally compact complete intrinsic metric spaces are proper.*

Proof. Let X be a locally compact complete intrinsic metric space. For $x \in X$, define $\rho(x)$ to be the supremum of all $R > 0$ such that $\overline{B(x, R)}$ is compact. Since X is locally compact, $\rho(x) > 0$ for any $x \in X$. It's suffice to show $\rho(x) = +\infty$ for some (and therefore any) x. Suppose not, i.e., $\rho(x) < +\infty$.

First, notice that $B = \overline{B(x, \rho(x))}$ is compact. Indeed, for any $\varepsilon > 0$, $\overline{B(x, \rho(x) - \varepsilon)}$ is compact, and since X is intrinsic, it is an ε-net of B, hence B is compact.

Second, we claim that $|\rho(x) - \rho(y)| \leq |xy|$, and in particular, ρ is continuous. If this does not hold, we have $\rho(x) + |xy| < \rho(y)$. Then $\overline{B(x, \rho(x) + \varepsilon)}$ is a closed subset of $\overline{B(y, \rho(y))}$ for sufficiently small $\varepsilon > 0$, hence $\overline{B(x, \rho(x) + \varepsilon)}$ is compact. This contradicts to the definition of ρ.

Now let $\varepsilon = \min_{y \in B} \rho(y)$, since B is compact, the minimum can be reached, and $\varepsilon > 0$. Choose a finite $\varepsilon/10$-net $\{a_1, \cdots, a_n\}$ in B, set

$$W = \bigcup_{i=1}^{n} \overline{B(a_i, \varepsilon)}.$$

Then W is compact. However, $\overline{B(x, \rho(x) + \varepsilon/10)} \subset W$, this means $\overline{B(x, \rho(x) + \varepsilon/10)}$ is compact. This contradicts to the definition of ρ. Hence we must have $\rho(x) = +\infty$. □

Corollary 1.11. *Locally compact complete intrinsic metric spaces are geodesic.*

Remark 1.12. After stating Hopf–Rinow theorem in 2.1, [BGP92] claims that the limit of geodesics is still a geodesic. This is not correct. Think about the surface of a solid cylinder, consider the geodesics connecting antipodal points converging to the upper face. They converge to a semicircle, but it is not a geodesic—now the shortest path is a line segement on the upper face.

Definition 1.13. A *triangle* on an intrinsic metric space X is the collection of three points $p, q, r \in X$ and three geodesics $[qr], [pr], [pq]$, denoted by $\triangle pqr$.

Notation 1.14. Denote S_k^2 the 2-dimensional complete simply-connected Riemannian manifold of curvature k.

Definition 1.15. Let p, q, r be a triple of points in an intrinsic metric space X. Define their *comparison triangle on* S_k^2, denoted by $\tilde{\triangle}^k pqr$, is the triangle with vertices $\tilde{p}, \tilde{q}, \tilde{r}$ on S_k^2 with $|pq| = |\tilde{p}\tilde{q}|$, $|pr| = |\tilde{p}\tilde{r}|$, $|qr| = |\tilde{q}\tilde{r}|$. Comparison triangle always exists up to a rigid motion when $k \le 0$, and when $k > 0$ we ask the perimeter of $\triangle pqr$ is less than $2\pi/\sqrt{k}$. We denote $\tilde{\angle}^k pqr$ by the angle at vertex $\tilde{q}$ in triangle $\tilde{\triangle}^k pqr$.

2. Basic Concepts

Definition 2.1. A locally complete space X with intrinsic metric is called *space with curvature* $\ge k$ if in some neighborhood U_x of each point $x \in X$ the following condition is satisfied:

(D) for any four distinct points $(a; b, c, d)$ in U_x we have the inequality
$$\tilde{\angle}^k bac + \tilde{\angle}^k bad + \tilde{\angle}^k cad \le 2\pi.$$

If X is a 1-dimensional manifold and $k > 0$, we require in addition that $\operatorname{diam} X \le \pi/\sqrt{k}$.

For traditional Toponogov's comparison theorem, we have the following condition.

Theorem 2.2. *A geodesic space X is a space with curvature $\ge k$ if and only if the following condition is satisfied:*

(A) *for any triangle $\triangle pqr$ with vertices in U_x and any point $s \in [qr]$, the inequality $|ps| \ge |\tilde{p}\tilde{s}|$ is satisfied, where $\tilde{s}$ is the point on the side $[\tilde{q}\tilde{r}]$ of the comparison triangle $\tilde{\triangle}^k pqr$ with $|qs| = |\tilde{q}\tilde{s}|$.*

To prove the only if part, we need the following lemma.

Lemma 2.3 (Alexandrov)**.** *Let $\tilde{\triangle}^k pqs, \tilde{\triangle}^k prs$ be given on S_k^2, which are joined to each other in an exterior way along the side $[ps]$. Let there also be given $\tilde{\triangle} bcd$, where $|bc| = |pq|$, $|bd| = |pr|$, $|cd| = |qs| + |rs|$, and $|bc| + |bd| + |cd| < 2\pi/\sqrt{k}$ in the case $k > 0$. Then $\tilde{\angle}^k psq + \tilde{\angle}^k psr \le \pi$ $(\ge \pi)$ if and only if $\tilde{\angle}^k pqs \ge \tilde{\angle}^k bcd$ and $\tilde{\angle}^k prs \ge \tilde{\angle}^k bdc$ (respectively $\tilde{\angle}^k pqs \le \tilde{\angle}^k bcd$ and $\tilde{\angle}^k prs \le \tilde{\angle}^k bdc$).*

Proof. Observe that in absolute geometry (Euclidean, spherical or non-Euclidean), when two sides are fixed, bigger angle is opposite to bigger side. Using a rigid motion, we can assume $q = c$ and $s \in [cd]$. Then $\tilde{\angle}^k psq + \tilde{\angle}^k psr \le \pi$ is equivalent to $\tilde{\angle}^k rsd \le \tilde{\angle}^k psd$. Using observation, this is equivalent to $|pd| \ge |pr| = |bd|$. Using observation again, this is equivalent to $\tilde{\angle}^k pqs \ge \tilde{\angle}^k bcd$. The other inequality holds similarly. □

Proof of (D) $\implies$ (A). Apply Alexandrov lemma (Lemma 2.3) to comparison triangle $\tilde{\triangle}^k pqs$ and $\tilde{\triangle}^k psr$. By (D), we have $\tilde{\angle}^k psq + \tilde{\angle}^k psr \le \pi$ (since $\tilde{\angle}^k qsr = \pi$), hence $\tilde{\angle}^k pqs \ge \tilde{\angle}^k pqr$, that is, $|ps| \ge |\tilde{p}\tilde{s}|$. □

The proof of converse needs more work. We need the notion of angle between two geodesics with the same origin.

Notation 2.4. Let γ, σ be geodesics with origin p, q, r on γ, σ respectively and $x = |pq|, y = |pr|$. Denote $\omega_k(x, y) = \tilde{\measuredangle}^k qpr$.

Definition 2.5. Let γ, σ be geodesics with origin p. If the limit

$$\lim_{x,y\to 0} \omega_k(x, y)$$

exists, the limit is called the *angle* between γ and σ.

We need to show angle between two geodesics does not depend on the choice of k.

Lemma 2.6. *Let p, q, r be three points in a metric space, then $k \mapsto \tilde{\measuredangle}^k qpr$ is increasing.*

Proof. This follows from the triangle version of the Toponogov's comparison theorem (cf. [Pet16, Theorem 12.2.2]). Since angles on Riemannian manifolds are defined by inner product, this proof does not rely on Definition 2.5. □

Lemma 2.7. *Let $p, q, r \in X$, with $|pq| + |qr| + |rp| < 2\pi/\sqrt{k}$ when $k > 0$. Let $\ell = \max\{|pq|, |pr|\}$, then we have*

$$A[\tilde{\triangle} pqr] \le \pi(\ell^2 + o(\ell^2)).$$

Proof. We adopt geodesic polar coordinate. Since geodesic balls are convex, we have

$$\begin{aligned} A[\tilde{\triangle} pqr] &\le \int_0^\ell \int_0^\theta \mathrm{sn}^k(r)\, ds\, dr \\ &= \theta\, \mathrm{md}^k(\ell) \\ &\le \pi\, \mathrm{md}^k(\ell) \\ &= \pi(\ell^2 + o(\ell^2)). \end{aligned}$$

Here sn^k and md^k are solutions to initial value problem

$$\mathrm{sn}^k(t) + k(\mathrm{sn}^k)''(t) = 0,\ \mathrm{sn}^k(0) = 0,\ (\mathrm{sn}^k)'(0) = 1,$$

and

$$\mathrm{md}^k(t) + k(\mathrm{md}^k)''(t) = 1,\ \mathrm{md}^k(0) = 0,\ (\mathrm{md}^k)'(0) = 0.$$ □

Lemma 2.8. *Let $p, q, r \in X$, with $|pq| + |qr| + |rp| < 2\pi/\sqrt{k}$ when $k > 0$. Let $\ell = \max\{|pq|, |pr|\}$, $k < K$, then we have*

$$0 < \tilde{\measuredangle}^K qpr - \tilde{\measuredangle}^k qpr \le \pi(|K| + |k|)(\ell^2 + o(\ell^2)). \tag{2.1}$$

Proof. By previous lemmas and Gauss–Bonnet formula, we have

$$\begin{aligned} 0 &< \tilde{\measuredangle}^K qpr - \tilde{\measuredangle}^k qpr \\ &< \tilde{\measuredangle}^K qpr + \tilde{\measuredangle}^K pqr + \tilde{\measuredangle}^K prq - \tilde{\measuredangle}^k qpr - \tilde{\measuredangle}^k pqr - \tilde{\measuredangle}^k prq \\ &= KA[\tilde{\triangle}^K pqr] - kA[\tilde{\triangle}^k pqr] \\ &\le (|K| + |k|)(\ell^2 + o(\ell^2)). \end{aligned}$$ □

Let $\ell \to 0$ in inequality (2.1), we have thus shown the angle between geodesics is well-defined.

Let us back to our discussion on spaces with curvature bounded below. (A) has the following corollary.

Proposition 2.9. *If (A) is satisfied, then for any geodesics γ, σ with origin p, the function $\omega_k(x, y)$ is non-increasing with respective to each variable x, y when x, y are sufficiently small.*

Proof. Let U_p be a neighborhood of p such that (A) holds, $x, x' \in \gamma$, $y \in \sigma$ and x is between p and x'. We must show $\omega_k(x, y) \geq \omega_k(x', y)$. By (A), we have $|yx| \geq |\tilde{y}\tilde{x}|$, then we have

$$\omega_k(x, y) = \tilde{\measuredangle}^k ypx \geq \measuredangle \tilde{y}\tilde{p}\tilde{x} = \tilde{\measuredangle}^k ypx' = \omega_k(x', y).$$

Similarly $\omega_k(x, y)$ is non-increasing in y. □

By Proposition 2.9, the angle between two geodesics with same origin is always defined.

Proposition 2.10. *The angles between three geodesics with same origin satisfy triangle inequality.*

Proof. A proof for general metric space can be found in [AKP24, 6.5]. But for spaces with curvature $\geq k$, notice that angles in S_k^2 satisfy triangle inequality, and then we can take the limit. □

Proposition 2.9 also has the following direct corollaries.

Corollary 2.11. *If* (A) *is satisfied, then*

(C) *For any triangle $\triangle pqr$ contained in U_x, none of its angles is less than the corresponding angle of the comparison triangle $\tilde{\triangle}^k pqr$ in S_k^2.*

(C_1) *If r is an interior point of a geodesic $[pq]$, then for any geodesic $[rs]$, we have $\measuredangle prs + \measuredangle qrs = \pi$.*

Now we are ready to prove (A) $\implies$ (D).

Proof of (A) $\implies$ (D). Let $(p; a, b, c)$ in U_x. Choose $d \in [pa]$ closed to p, then by triangle inequality (Proposition 2.10) and (C_1) we have

$$\begin{aligned} \measuredangle adb + \measuredangle bdc + \measuredangle cda &\leq \measuredangle adb + \measuredangle bdp + \measuredangle pdc + \measuredangle cda \\ &= 2\pi. \end{aligned}$$

Apply Proposition 2.9, we have

$$\begin{aligned} \tilde{\measuredangle}^k adb + \tilde{\measuredangle}^k bdc + \tilde{\measuredangle}^k cda &\leq \measuredangle adb + \measuredangle bdc + \measuredangle cda \\ &\leq 2\pi. \end{aligned}$$

Since angles vary continuously in S_k^2, let $d \to p$, we obtain

$$\tilde{\measuredangle}^k apb + \tilde{\measuredangle}^k bpc + \tilde{\measuredangle}^k cpa \leq 2\pi.$$ □

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