A Concise Note on Differential Geometry

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Preface

I had planned to write the preface at the end of my writing, but now I think I need a temporary preface.

Roughly speaking, this note is a combination of notes on differentiable (or equivalently, smooth) manifolds and Riemannian geometry, as the GitHub description indicates. The smooth manifold part comes from an unfinished note of mine, written in Chinese during the summer of 2022, when I began studying differential geometry. The Riemannian geometry part comes from the lecture notes I took during the Riemannian geometry course at the 2024 BICMR summer school on differential geometry, taught by Prof. Chao Xia from Xiamen University. Unfortunately, my note on differential geometry is unfinished, and my note on Riemannian geometry is full of typos and false proofs, so I had the idea to combine–or rather, rewrite–both notes.

We will adopt Einstein's summation convention throughout this note. That is, if an index occurs both in the superscript and subscript, it means to take summation over this index. For instance, we have

$$a^i b_i = \sum_{i=1}^{n} a^i b_i.$$

If an index occurs twice in superscript or subscript, then this means something is wrong, except in the case of orthonormal frames and similar context. For latter case, when indices actually occur twice at same position, we will not omit the summation symbol.

This note is not an encyclopedia, or even a textbook. Hence many proofs will not be provided, such as the proofs to the partition of unity theorem, the rank theorem, Stokes Theorem and so on. However, I will do my best to reference these proofs.

The part on smooth manifolds is kept minimal. I only want to write about what is needed in the Riemannian geometry part, so I omitted many topics such as distributions and Frobenius theorem, de Rham theory, and so on. Also, I have not covered topology in detail, as I assume the readers are familiar with basic point-set topology and covering space theory.

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Chapter 1

Smooth Manifolds

In this chapter we will introduce the basic object of differential geometry: smooth (or differentiable) manifolds and smooth maps. In the meanwhile, we will introduce the main idea of differential geometry: linearization. To be specific, we will introduce tangent spaces and tangent maps, i.e. the linearization of smooth maps. The notion of manifolds with boundary and submanifolds will be discussed as well.

1.1 Category of Smooth Manifolds

Feel free if you know nothing about category theory, the section title just indicates we will define the notions of smooth manifolds and smooth maps. Throughout the note, we will use "smooth" to mean being C^∞.

We first have a topological object.

Definition. Let $n \in \mathbb{N}$, a *topological manifold* M^n is a second countable Hausdorff space that is locally Euclidean, i.e. for any $p \in M$, there is an open neighborhood U of p and a map $\varphi_U : U \to \mathbb{R}^n$ that maps U to an open set in $\mathbb{R}^n$ and homeomorphically onto its image. (U, φ_U) is called a *chart*. n is called the *dimension* of the manifold, denoted by $\dim M$. We will omit the superscript that indicates dimension if the dimension is clear.

By gluing chart smoothly, we can have the notion of a smooth manifold.

Definition. Let M be a topological manifold, two charts (U, φ_U) and (V, φ_V) are called *compatible* if both $\varphi_V \circ \varphi_U^{-1}$ and $\varphi_U \circ \varphi_V^{-1}$ are smooth. If compatible charts $\{(U_i, \varphi_i)\}$ consist an open cover of M, then it is called an *atlas*. A maximal atlas is called a *smooth structure*. A topological manifold together with a smooth structure is called a *smooth manifold*.

Remark 1.1. (1) Not all topological manifolds admit a smooth structure, see for isntance, [4].

(2) Dimension of manifolds are well-defined, that is, there is no diffeomorphisms between open subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ if $m \neq n$. The proof requires topological invariants, such as homology groups. See for instance, [5, Theorem 2.55].

(3) We will assume all the manifolds occur in our note are path connected unless we specially claim.

(4) Notice that we only need to give an atlas of a manifold to define a smooth structure.

We now present some examples of smooth manifolds.

Example 1.2. (1) $\mathbb{R}^n$ itself is a smooth manifold, with single chart id.

(2) Any open subset U of a smooth manifold M. If $\{U_i, \varphi_i\}$ is an atlas of M, then $\{U_i \cap U, \varphi_i|_{U_i \cap U}\}$ is an atlas of U. In particular, any open set of $\mathbb{R}^n$ is a smooth manifold.

Example 1.3. Sphere $\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$. An atlas is given by stereographic projections from north pole and south pole. We give the formula explicitly. Let $U = \mathbb{S}^n - \{(0, \cdots, 0, 1)\}$, $V = \mathbb{S}^n - \{(0, \cdots, 0, -1)\}$, then

$$U \xrightarrow{\varphi} \mathbb{R}^n,\ (x_1, \cdots, x_{n+1}) \mapsto \left(\frac{x_1}{1 - x_{n+1}}, \cdots, \frac{x_n}{1 - x_{n+1}}\right),$$
$$V \xrightarrow{\psi} \mathbb{R}^n,\ (x_1, \cdots, x_{n+1}) \mapsto \left(\frac{x_1}{1 + x_{n+1}}, \cdots, \frac{x_n}{1 + x_{n+1}}\right).$$

It's easy to see (U, φ) and (V, ψ) are compatible.

Example 1.4. (1) If M, N are smooth manifolds, then $M \times N$ is a smooth manifold with atlas $\{U_i \times V_j, \varphi_i \times \psi_j\}$, provided $\{U_i, \varphi_i\}$ and $\{V_j, \psi_j\}$ are atlases of M and N respectively.

(2) The torus $\mathbb{T}^n$ is defined by $\mathbb{S}^1 \times \cdots \times \mathbb{S}^1$ for n times.

To complete the definition of category of smooth manifolds, we need to define smooth maps.

Definition. A map $f : M \to N$ is called *smooth* at $p \in M$, if there exists chart (U, φ) of p and (V, ψ) of $f(p)$, such that $\psi \circ f \circ \varphi$ is smooth on $\varphi(U)$. If f is smooth everywhere, we simply say f is smooth. If $N = \mathbb{R}$, then we call f a smooth *function*, and we denote the set (actually $\mathbb{R}$-algebra) of smooth functions on M by $C^\infty(M)$.

The isomorphisms in the category of smooth manifolds are called diffeomorphisms, and we repeat the definition here.

Definition. A smooth map $f : M \to N$ is called a *diffeomorphism* if there exists a smooth $g : N \to M$, such that $g \circ f = \mathrm{id}_M$ and $f \circ g = \mathrm{id}_N$.

At the end of the section, we discuss partition of unity.

Definition. A collection of subsets $\{X_i\}_{i\in I}$ of topological space X is called *locally finite* if for any $x \in X$, there is a neighborhood U of x such that only finite many $i \in I$ satisfy $U \cap X_i \neq \varnothing$.

Definition. The *support* of a smooth function $f \in C^\infty(M)$ is the closure of $\{p \in M : f(p) \neq 0\}$, denoted by $\operatorname{supp} f$.

Now we can state the partition of unity theorem here, but we will not present the long and complicated proof. For a proof, one can check [6, Theorem 2.23].

Theorem 1.5 (Partition of unity)**.** *Let M be a smooth manifold with an open cover $\{U_i\}_{i\in I}$. Then there exists a family of smooth functions $\{\varphi_i\}_{i\in I}$ satisfying*

(1) $0 \leq \varphi_i \leq 1$;

(2) $\operatorname{supp} \varphi_i \subset U_i$ *for each* $i \in I$;

(3) $\{\operatorname{supp} \varphi_i\}_{i\in I}$ *is locally finite;*

(4) $\sum_{i\in I} \varphi_i(p) = 1$ *for* $p \in M$.

Remark 1.6. By (3), the summation in (4) is a finite sum, so we will not encounter convergence problem here.

Partition of unity plays an important role in defining many geometric objects, such as integral, metric, connection and so on. But any of them is beyond our scope in this section, so we have to postpone the illustration of applications of partition of unity.

1.2 Linearization

In this section we discuss the linearization of smooth maps, the main idea of differential geometry.

Tangent Spaces and Tangent Maps

We first define tangent spaces. In modern textbooks, tangent vectors are defined to be derivatives on the germ of smooth functions. This definition can show directly that tangent vectors consists a vector space. However, we will adopt traditional definition via curves on manifolds, which will make our calculation more convenient.

Definition. The *germ of smooth functions* at $p \in M$ is a ring defined by

$$\{(f,U) |\ U \subset M \text{ open}, f \in C^\infty(U)\}/\sim,$$

where $(f,U) \sim (g,V)$ if there exists a $W \subset U \cap V$ such that $f|_W = g|_W$. Ring operations are defined in natural way. Elements in $C_p^\infty(M)$ are denoted by f_p if it has representative element f.

Definition. Let $\gamma : [0,1] \to M$ be a smooth curve, i.e. a smooth map, with $\gamma(0) = p$. Define the *tangent vector at p along γ* to be a map

$$\begin{aligned} \dot\gamma(0) : C_p^\infty(M) &\to \mathbb{R} \\ f_p &\mapsto (f \circ \gamma)'(0). \end{aligned}$$

Then we define the *tangent space at p* to be

$$T_pM := \{\dot\gamma(0) |\ \gamma : [0,1] \to M \text{ smooth}, \gamma(0) = p\}.$$

Remark 1.7. The assignment of curve to its tangent vector is not a bijection. For instance, on $T_0\mathbb{R}^2$, $y = 0$ and $y - x^2 = 0$ yield same tangent vector at $(0,0)$.

It's easy to see that

Proposition 1.8. *We have* Leibniz rule

$$\dot\gamma(0)(f_p g_p) = f(p)\dot\gamma(0)(g_p) + g(p)\dot\gamma(0)(f_p).$$

Therefore a tangent vector is a derivative *on $C_p^\infty(M)$.*

We will show that T_pM has a real vector space structure. In fact, we have an enhanced conclusion.

Proposition 1.9. *Let M^n be a smooth manifold, (U,φ) be a chart containing p, $\varphi = (x^1, \cdots, x^n)$. Define $\left.\frac{\partial}{\partial x^i}\right|_p$ to be the tangent vector of $\sigma_i = \varphi^{-1}(\varphi(p) + te_i)$, where $e_i = (0, \cdots, 1, \cdots, 0)$ with only i-th component being* 1. *Then we have*

$$T_pM = \operatorname{Span}\left\{ \left.\frac{\partial}{\partial x^i}\right|_p \right\}$$

as sets, so we can equip T_pM a real vector space structure.

Proof. Notice that
$$\left.\frac{\partial}{\partial x^i}\right|_p f = \frac{\partial}{\partial x^i}(f \circ \varphi^{-1})(p),$$
the partial derivative on right hand side is the usual Euclidean partial derivative. Hence by chain rule, we have
$$\begin{aligned}
\dot{\gamma}(0)f &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (f \circ \gamma)(t) \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (f \circ \varphi^{-1}) \circ (\varphi \circ \gamma)(t) \\
&= \left.\frac{\partial}{\partial x^i}\right|_{\varphi(p)} (f \circ \varphi^{-1}) \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} x^i(\gamma(t)) \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} x^i(\gamma(t)) \left.\frac{\partial}{\partial x^i}\right|_p f.
\end{aligned}$$
Therefore we have
$$\dot{\gamma}(0) = \dot{\gamma}(0)(x^i) \left.\frac{\partial}{\partial x^i}\right|_p \in \mathrm{Span}\left\{ \left.\frac{\partial}{\partial x^i}\right|_p \right\},$$
and we obtain
$$T_pM \subset \mathrm{Span}\left\{ \left.\frac{\partial}{\partial x^i}\right|_p \right\}.$$
The reverse inclusion is trivial. □

Remark 1.10. The vector space structure of tangent space does not depend on the choice of chart, since for any charts (U_1, φ_1) and (U_2, φ_2), the Jacobian of transition function $\varphi_1 \circ \varphi_2^{-1}$ is a vector space isomorphism. However, this is again rely on the well-definedness of dimension of manifolds, but we have assumed it automatically.

Example 1.11. Let M, N be smooth manifolds, and $(p, q) \in M \times N$. Then $T_{(p,q)}M \times N = T_pM \oplus T_qN$.

Next we define tangent maps.

Definition. Let $f : M \to N$ be a smooth map, $p \in M$. We define $f_{*p} : T_pM \to T_{f(p)}N$ by
$$f_{*p}(v)(g) = v(g \circ f)$$
for any $v \in T_pM$ and $g \in C^\infty_{f(p)}N$.

Remark 1.12. We often write f_{*p} as $\mathrm{d}f|_p$ if f is a smooth function. This coincides the notation for 1-form we will define later.

Lemma 1.13. *On charts (U, φ) of $p \in M$ and (V, ψ) of $q \in N$, the smooth map $f : M \to N$ that $f(p) = q$ satisfies*

$$f_{*p}\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) = \frac{\partial}{\partial x^i}(\psi \circ f \circ \varphi)^j(\varphi(p)) \left.\frac{\partial}{\partial y^j}\right|_q.$$

The proof is similar to Proposition 1.9.

Moreover, we have the chain rule.

Proposition 1.14. *Let $f : M \to N$, $g : N \to P$ be two smooth maps, then for $p \in M$ we have*

$$(g \circ f)_{*p} = g_{*f(p)} \circ f_{*p}.$$

Proof. Let $v \in T_pM$, then for any $\varphi \in C^\infty_{g(f(p))}(M)$ we have

$$\begin{aligned}
(g \circ f)_{*p}(v)(\varphi) &= v(\varphi \circ (g \circ f)) \\
&= v((\varphi \circ g) \circ f) \\
&= (f_{*p}(v))(\varphi \circ g) \\
&= (g_{*f(p)}(f_{*p}(v)))(\varphi) \\
&= (g_{*f(p)} \circ f_{*p})(v)(\varphi).
\end{aligned}$$

□

Since we linearize smooth maps, we can talk about rank of a map at a point.

Definition. Let $f : M \to N$ be a smooth map and $p \in M$, the *rank* of f at p, denoted by $\operatorname{rank}_p f$, is defined by the rank of f_{*p}.

Definition. Let $f : M^m \to N^n$ be a smooth map.

(1) Assume $m \leq n$, if f_{*p} is injective for any $p \in M$, then we call f an *immersion*.

(2) Assume f is an immersion, if f maps M homeomorphically onto its image, then we call f an *embedding*.

(3) Assume $m \geq n$, if f_{*p} is surjective for any $p \in M$, then we call f a *submersion*.

Remark 1.15. An equivalent description for immersion and submersion is that they are of constant maximal rank. Also notice that a map is both an immersion and submersion if and only if it is a diffeomorphism.

Example 1.16. The definition of embedding $f : M \to N$ asks $f(M)$ is equipped with subspace topology from N, so f being injective is not enough. For example, consider the injective map

$$r : (-\pi, \pi) \to \mathbb{R}^2,\ t \mapsto (2\sin t, \sin(2t)).$$

The image of r is the zero locus of $4y^2 = x^2(4 - x^2)$, it's easy to see it is compact. But $(-\pi, \pi)$ is not compact, hence r does not map the interval homeomorphically onto its image.

Tangent Bundles and Vector Fiels

We want to collect the tangent spaces and tangent maps together to obtain a global linearization.

Definition. Assume smooth manifold M^n has atlas $\{U_\alpha, \varphi_\alpha\}$, define

$$TM := \bigsqcup_{p \in M} T_pM$$
$$\pi : TM \to M,\ (p, v) \mapsto p$$

We give an atlas of TM to make it into a $2n$-dimensional smooth manifold. Let

$$\Phi_\alpha : \bigsqcup_{p \in U_\alpha} T_pM \to \mathbb{R}^{2n}$$
$$(p, v) \mapsto (\varphi_\alpha(p), (v^1, \cdots, v^n))$$

where $v = v^i \left.\frac{\partial}{\partial x^i}\right|_p$. Let's check

$$\Phi_\beta \circ \Phi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \times \mathbb{R}^n \to \varphi_\beta(U_\alpha \cap U_\beta) \times \mathbb{R}^n$$
$$(x^1, \cdots, x^n, v^1, \cdots, v^n) \mapsto \left(\varphi_\beta \circ \varphi_\alpha^{-1}(x), \frac{\partial(\varphi_\beta \circ \varphi_\alpha^{-1})^1}{\partial x^i} v^i, \cdots, \frac{\partial(\varphi_\beta \circ \varphi_\alpha^{-1})^n}{\partial x^i} v^i \right)$$

Clearly it is smooth, then $\{(\pi^{-1}(U_\alpha), \Phi_\alpha)\}$ induces a smooth structure on TM.

We call T_pM a *fiber* over p, and $\pi : TM \to M$ the *projection*.

Definition. A *vector field* is a smooth map $X : M \to TM$ such that $\pi \circ X = \mathrm{id}_M$.

We will use the symbol $\Gamma(TM)$ to denote the set (or in fact, $C^\infty(M)$-module) of vector fields. The explanation for this symbol will be given after we introduced the notion of vector bundles.

Proposition 1.17. *$X \in \Gamma(TM)$ if and only if in any local chart (U, φ), we have $X(p) = X^i(p) \left.\frac{\partial}{\partial x^i}\right|_p$ for $X^i \in C^\infty(U)$, $i = 1, 2, \cdots, n$.*

Remark 1.18. This proposition is equivalent to X can be a mapping $C^\infty(M) \to C^\infty(M)$ defined by $Xf(p) = X(p)f$.

We now provide some approaches to generate new vector fields.

Definition. For $X, Y \in \Gamma(TM)$, define *Lie bracket* of X, Y as $[X, Y] = XY - YX$, then $[X, Y] \in \Gamma(TM)$.

Remark 1.19. We explain the definition more explicitly. If we act two vector fields on the product of two functions, we have

$$\begin{aligned}(XY)_p(fg) &= X_p(Y(fg)) = X_p(gYf + fYg) \\ &= \boxed{X_p g \cdot Y_p f + X_p f \cdot Y_p g} + g(p)X_p Yf + f(p)X_p Yg\end{aligned}$$

The boxed thing is bad, it spoils Leibniz rule. But if we substract $YX_p(fg)$, the boxed thing is cancelled. So $XY - YX \in \Gamma(TM)$.

Proposition 1.20. *On some local chart, we have* $\left[\left.\frac{\partial}{\partial x^i}\right|_p, \left.\frac{\partial}{\partial x^j}\right|_p\right] = 0.$

Proof. This is equivalent to mixed partial derivative is commutative for smooth functions in $\mathbb{R}^n$. □

Moreover, we can pass a vector field to another manifold via a diffeomorphism.

Proposition 1.21. *Let $f : M \to N$ be a diffeomorphism and $X \in \Gamma(TM)$. Then there exists a vector field $Y \in \Gamma(TN)$ such that $Y = f_* X$, called the* pushforward *of X.*

Proof. Just define Y pointwisely. □

Proposition 1.22. *Let $f : M \to N$ be a diffeomorphism and $X, Y \in \Gamma(TM)$, then*

$$f_*[X, Y] = [f_* X, f_* Y].$$

Proof. Clear when considering the corresponding derivatives. □

1.3 Submanifolds

In this section, we discuss submanifolds of an ambient manifold. We first present the definition. Then, we will discuss rank theorem, a powerful tool to discuss submanifolds. We will use rank theorem to derive inverse function theorem and implicit function theorem, and discuss submanifolds and embedded submanifolds.

Definition. Let M be a smooth manifold, $\Sigma \subset M$ be a subset. Σ is called a *(k-dimensional) submanifold* of M if for any $p \in \Sigma$, there exists a chart (U, φ) of p in M such that

$$\varphi(U \cap \Sigma) = \varphi(U) \cap \{(x^1, \cdots, x^k, 0, \cdots, 0) \in \mathbb{R}^n \mid x^1, \cdots, x^k \in \mathbb{R}\}.$$

We define the *codimension* of Σ to be $n - k$, denoted by $\operatorname{codim} \Sigma = n - k$.

We will not linger to give any examples. On the contrary, we will discuss rank theorem at once.

Theorem 1.23 (Rank theorem)**.** *Let $f : M^m \to N^n$ be a smooth map that has constant rank $r (\leq m)$. For any $p \in M$, there exists charts (U, φ) of p and (V, ψ) of $f(p)$, such that*

$$(\psi \circ f \circ \varphi^{-1})(x^1, \cdots, x^r, \cdots, x^m) = (x^1, \cdots, x^r, 0, \cdots, 0)$$

on $\varphi(U)$.

The proof of rank theorem is also complicated, and we just refer to [6, Theorem 4.12]. Rank theorem has important applications. First, as corollary, we have inverse function theorem and implicit function theorem.

Theorem 1.24 (Inverse function theorem)**.** *Let $f : M \to N$ be a smooth map, $p \in M$. Suppose f_{*p} is an isomorphism, then there exists a neighborhood U of p such that $f|_U$ is an diffeomorphism on U.*

Proof. Since determinant function is smooth, $\det f_{*p} \neq 0$ in a neighborhood of p. Then by rank theorem, there exists sufficiently small charts (U, φ) of p and (V, ψ) of $f(p)$ such that $\psi \circ f \circ \varphi^{-1}$ is identity on $\varphi(U)$. Thus $f|_U = \psi^{-1} \circ \varphi$ is a homeomorphism, and it is a diffeomorphism since f is smooth. □

Theorem 1.25 (Implicit function theorem)**.** *Let $f : M^{k+n} \to \mathbb{R}^k$ be a smooth map. Suppose f_{*p} is a submersion for any $p \in f^{-1}(0)$, then $f^{-1}(0)$ is an n-dimensional submanifold of M (may not be connected).*

Proof. For $p \in f^{-1}(0)$, since determinant function is smooth, there exists a neighborhood of p such that the tangent map has constant rank. Hence by rank theorem, there exists sufficiently small charts (U, φ) of p and (V, ψ) of 0 such that

$$(\psi \circ f \circ \varphi^{-1})(x^1, \cdots, x^k, x^{k+1}, \cdots, x^{k+n}) = (x^1, \cdots, x^k).$$

Then $f^{-1}(0)$ restricts on U has coordinate

$$\varphi(f^{-1}(0) \cap U) = \{(0, \cdots, 0, x^{k+1}, \cdots, x^{k+n}) \in \mathbb{R}^{n+k}\} \cap \varphi(U).$$

Thus $f^{-1}(0)$ is an n-dimensional submanifold. □

Remark 1.26. (1) The value 0 plays an unimportant role in the proof. In fact, we can replace 0 by r_0 and consider $g(x) = f(x) - r_0$.

(2) The condition of f_{*p} being a submersion for any $p \in f^{-1}(r_0)$ is seemingly hard to verify. However, we will prove later that such property holds *almost everywhere* in $\mathbb{R}$. (In fact, $\mathbb{R}^k$ for all $k \in \mathbb{N}$. This is Sard's Theorem, but we only need $k = 1$ case in this note.)

We now characterize submanifolds.

Definition. Let $f : \Sigma \to M$ be an immersion, then the image of f is called an *immersed submanifold*. Moreover, if f is an embedding, the image of f is called an *embedded manifold*.

Proposition 1.27. *Any immersed manifold is locally an embedded manifold.*

Proof. Only need to show any immersion is locally an embedding, and this is clear from rank theorem. □

Proposition 1.28. *A subset of a smooth manifold is a submanifold if and only if it is an embedded submanifold.*

Proof. Let M be a smooth manifold, $S \subset M$ be a submanifold, then the inclusion $i : S \to M$ is an embedding. Conversely, let $f : \Sigma^k \to M^n$ be an embedding. Let $p \in f(\Sigma)$, $f(q) = p$, then by rank theorem, there exists charts (U, φ) of q and (V, ψ) of p such that

$$(\psi \circ f \circ \varphi^{-1})(x^1, \cdots, x^k) = (x^1, \cdots, x^k, 0, \cdots, 0).$$

Since $\Sigma \cong f(\Sigma)$ with latter equipped with subspace topology, $f(U) = \tilde{V} \cap f(\Sigma)$ for some open $\tilde{V} \subset M$. By shrinking $\tilde{V}$ (and U), we can assume $\tilde{V} \subset V$, then

$$\psi(\tilde{V} \cap f(\Sigma)) = \{(x^1, \cdots, x^k, 0, \cdots, 0) \in \mathbb{R}^n\} \cap \psi(\tilde{V}),$$

hence $f(\Sigma)$ is a submanifold of M. □

Now we provide some examples of submanifolds.

Example 1.29. (1) Open subsets of a smooth manifold. This is obvious.

(2) *Surfaces* in $\mathbb{R}^3$. This means a submanifold of codimension 1. In classical differential geometry, a surface means a 2-dimensional immersed submanifold. That is, a smooth map $r : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^3$ with $r_{;1} \times r_{;2} \neq 0$. Generally, a submanifold of codimension 1 (no matter what ambient manifold) is called a *hypersurface*.

(3) *Zero locus* of $f : M \to \mathbb{R}$ such that f_{*p} has maximal rank for all $p \in f^{-1}(0)$. In particular, n-sphere $\mathbb{S}^n$ is the zero locus of $|x|^2 - 1 = \sum_{i=1}^n (x^i)^2 - 1$.

(4) *(Affine) algebraic manifolds*, i.e. *smooth algebraic varieties*[1]. Algebraic varieties are common zero locus of a family of polynomials $\{f^i\}_{i\in I}$. By Hilbert's basis theorem, we can assume the index set I is finite (see for example, [8, Theorem B-1.16]). Hence we can have a differential smoothness condition that $f = (f^1, \cdots, f^{|I|})$ has maximal rank everywhere, instead of using commutative algebra.

1.4 Manifolds with Boundary

A manifold with boundary is locally an upper half space. First we difine the upper half space.

Definition. The *upper half space* $\mathcal{H}^n$ is defined as $\{(x^1, \cdots, x^n) \in \mathbb{R}^n | \ x^n \geq 0\}$ equipped with subpace topology. Denote the *boundary* of $\mathcal{H}$, i.e. $\mathbb{R}^{n-1} \times \{0\}$, by $\partial\mathcal{H}^n$.

Now we can define manifolds with boundary as a topological object.

Definition. Let $n \in \mathbb{N}$, a *(topological) manifold with boundary* M^n is a second countable Hausdorff space that is locally $\mathcal{H}^n$, i.e. for any $p \in M$, there is an open neighborhood U of p and a map $\varphi_U : U \to \mathcal{H}^n$ that maps U to an open set in $\mathcal{H}^n$ and homeomorphically onto its image. If p is mapped to $\partial\mathcal{H}^n$, then we call p a *boundary point*, otherwise we call it an *interior point*. We demand that the set of boundary points is nonempty.

Remark 1.30. The invariance of domain theorem [3, Theorem 2B.3] ensures that boundary points and interior points are well-defined.

If we want to equip manifolds with boundary smooth structures, we need to extend our calculus to functions defined on arbitrary subsets of $\mathbb{R}^n$.

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[1]Usually we only discuss algebraic varieties on algebraic closed fields, such as $\mathbb{C}$. But define varieties on arbitrary field does not violate the definition in algebraic geometry.

Definition. Let $S \subset \mathbb{R}^n$ be an arbitrary subset, $f : S \to \mathbb{R}^m$ be a mapping. f is called *smooth* if for any $p \in S$, there is an open neighborhood U of p and $\tilde{f} : U \to \mathbb{R}^m$, such that $\tilde{f}$ is smooth at p and $\tilde{f}|_S = f$.

By this definition, we can carry the definition of smooth structures on manifolds without boundary verbatim to manifolds with boundary.

A simple observation shows

Lemma 1.31. *Let M^n be a smooth manifold with boundary. Then ∂M, i.e. the set of boundary points is an $n-1$-dimensional smooth manifold with induced smooth structure. A chart of ∂M is a chart of M's boundary point restricted to ∂M.*

Remark 1.32. We have some remarks on manifolds with boundary.

(1) Topological boundary may not coincide with manifold boundary. Let $A = \{x \in \mathbb{R}^2 : |x| < 1\}$, then A is a manifold without boundary, but its topological boundary is unit circle $\mathbb{S}^1$.

(2) ∂M may not be connected even if M is connected. For example, the cylinder $\mathbb{S}^1 \times [0,1]$ has two disjoint circles as boundary.

Next we discuss the tangent spaces on manifold with boundary. If $p \in M - \partial M$, then T_pM is the ordinary tangent space. We need to figure out what T_pM is when $p \in \partial M$.

Proposition 1.33. *Let I_pM be the set of all tangent vectors along a curve start from $p \in \partial M$. Then $I_pM = T_p\,\partial M \times \mathbb{R}_{\geq 0}$ as sets.*

Proof. Use same calculation in Proposition 1.9, we have

$$\dot{\gamma}(0) = \dot{\gamma}(0)(x^i)\left.\frac{\partial}{\partial x^i}\right|_p,$$

where $\{\partial/\partial x^i\}$ consists a basis for $T_p\,\partial M$, and $\partial/\partial x^n$ is the tangent vector along $\sigma^n(t) = \varphi^{-1}(\varphi(p) + te_n)$. Since φ maps a neighborhood of p to the neighborhood of $\varphi(p)$ on $\mathcal{H}^n$, we have $x^n \geq 0$. Hence

$$\dot{\gamma}(0)(x^n) = \lim_{t\to 0^+} \frac{x^n(\sigma(t))}{t} \geq 0,$$

and the result follows. □

Definition. Let M be a smooth manifold with boundary, $p \in \partial M$. Then we define the *tangent space* at p to be $T_pM := \operatorname{Span} I_pM$. If a nonzero tangent vector in $T_pM - T_p\,\partial M$ is a tangent vector along some curve, then we call it an *inward-pointing vector*, otherwise we call it an *outward-pointing vector*.

From Proposition 1.33, we have the following lemma.

Lemma 1.34. *A nonzero tangent vector at a boundary point p is either in $T_p \partial M$, or inward-pointing, or outward-pointing.*

1.5 Lie Groups

We first introduce the notion of Lie groups, which lie in the intersection of geometry, algebra and analysis. Moreover, we will use Lie groups to discuss the quotient manifolds.

An abstract nonsense definition for Lie groups is they are group objects in the category of smooth manifolds. We want to write the definition explicitly.

Definition. A group G is called a *Lie group* if G has a (not necessarily connected) smooth manifold structure and the map $G \times G \to G$, $(g, h) \mapsto gh^{-1}$ is smooth.

More generally, we have the notion of Lie algebras.

Definition. A *Lie algebra* is a real vector space V equipped with a bilinear form $[\cdot, \cdot]$, satisfying

(1) $[v, v] = 0$ for $v \in V$;

(2) $[u, [v, w]] + [v, [w, u]] + [w, [v, u]] = 0$ for $u, v, w \in V$ (Jacobi identity).

Example 1.35. The real vector space $\Gamma(TM)$ with Lie bracket is a Lie algebra. We only need to verify Jacobi identity. We have

$$\begin{aligned}
[X, [Y, Z]] &= XYZ - XZY - YZX + ZYX \\
[Y, [Z, X]] &= YZX - YXZ - ZXY + XZY \\
[Z, [X, Y]] &= ZXY - ZYX - XYZ + YXZ,
\end{aligned}$$

add all three equations and we see all derivatives are cancelled.

An important Lie algebra is the Lie algebra associated to a Lie group.

Definition. Let G be a Lie group, a vector field $X \in \Gamma(TG)$ is called *left invariant* if for any left translation $L_g : h \mapsto gh$, we have $(L_g)_* X = X$.

Proposition 1.36. *Left invariant vector fields consist a Lie algebra.*

Proof. Clearly left invariant vector fields consist a real vector space. We need to show $[X, Y]$ is left invariant provided X, Y left invariant, then the Jacobi identity holds automatically. But this is because

$$(L_g)_*[X, Y] = [(L_g)_* X, (L_g)_* Y] = [X, Y]. \qquad \square$$

Proposition 1.37. *The vector space of left invariant vector fields is isomorphic to T_eG. Moreover, we can make T_eG a Lie algebra.*

Proof. Let X be a left invariant vector field. For any $g \in G$, we have

$$(L_{g^{-1}})_{*g}X|_g = X|_e,$$

hence the value of X is determined by its value at e. This gives the isomorphism. With this isomorphism, we can carry the Lie algebra structure to T_eG. $\square$

Definition. We define the *Lie algebra of Lie group* G by T_eG equipped with Lie algebra structure. The Lie algebra of G is denoted by $\mathfrak{g}$.

We give some examples of Lie groups and Lie algebras.

Example 1.38. If a group is at most countable, we often equip it with discrete topology, and regard it as a 0-dimensional manifold. Hence they are Lie groups. $\mathbb{Z}$ and $\mathbb{Z}_n$ are prototypes of these groups.

Example 1.39. *Real general linear group* $\mathrm{GL}(n,\mathbb{R})$ and its subgroups. $\mathrm{GL}(n,\mathbb{R})$ is the preimage of open set $\mathbb{R} - \{0\}$ under determinant function, hence is open in $\mathbb{R}^{n\times n}$. Thus $\mathfrak{gl}(n,\mathbb{R})$ is isomorphic to $\mathbb{R}^{n\times n}$, with Lie bracket to be usual commutator $AB - BA$.

(1) *Complex general linear group* $\mathrm{GL}(n,\mathbb{C})$. It is the preimage of open set $\mathbb{C} - \{0\}$ under determinant function, hence is open in $\mathbb{C}^n$. Thus $\mathfrak{gl}(n,\mathbb{C})$ is isomorphic to $\mathbb{R}^{2n^2}$. It can be embedded into $\mathrm{GL}(2n,\mathbb{R})$ by writing $a + b\sqrt{-1}$ as the matrix
$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

(2) *Real special linear group* $\mathrm{SL}(n,\mathbb{R})$. It is the zero locus of $A \mapsto (\det A - 1)$. To figure out what $\mathfrak{sl}(n,\mathbb{R})$ is, we choose a curve $X(t)$ on $\mathrm{SL}(n,\mathbb{R})$ and $X(0) = I$, then

$$0 = (\det X(t))' = \det X(t)\,\mathrm{tr}(X(t)X'(t)),$$

take $t = 0$ we have $\mathrm{tr}\, X'(0) = 0$, this means $\mathfrak{sl}(n,\mathbb{R})$ is the vector space of matrices with vanishing trace, which has dimension $n^2 - 1$.

(3) *Complex special linear group* $\mathrm{SL}(n,\mathbb{C})$. It is also the zero locus of $A \mapsto (\det A - 1)$, by same argument above we know that $\mathfrak{sl}(n,\mathbb{C})$ is the vector space of complex matrices with vanishing trace, which has real dimension $2n^2 - 2$.

Example 1.40. Automorphisms of bilinear forms.

(1) The automorphism group of standard real inner product on $\mathbb{R}^n$, that is $(v, w) \mapsto v^t w$, is defined to be *orthonormal group*, denoted by $\mathrm{O}(n)$. This group can be characterized by

$$\mathrm{O}(n) = \{X \in \mathrm{GL}(n, \mathbb{R}) : \; X^t X = I\}.$$

To figure out what $\mathfrak{o}(n)$ is, we choose a curve $X(t)$ on $\mathrm{O}(n)$ with $X(0) = I$. Then

$$0 = (X(t)^t X(t))' = X(t)^t X'(t) + X'(t)^t X(t),$$

let $t = 0$, we obtain $X'(0)^t + X'(0) = 0$. This means $\mathfrak{o}(n)$ is the vector space of antisymmetric matrices, hence $\dim \mathfrak{o}(n) = \frac{n(n-1)}{2}$.

(2) The subgroup of $\mathrm{O}(n)$ with determinant 1 is called the *special orthonormal group*, denoted by $\mathrm{SO}(n)$. It is a component of $\mathrm{O}(n)$, hence $\mathfrak{so}(n) = \mathfrak{o}(n)$.

(3) The automorphism group of standard Hermitian inner product on $\mathbb{C}^n$, that is $(v, w) \mapsto \bar{v}^t w$, is defined to be *unitary group*, denoted by $\mathrm{U}(n)$. It can be characterized by

$$\mathrm{U}(n) = \{X \in \mathrm{GL}(n, \mathbb{C}) : \; \bar{X}^t X = 0\}.$$

Use the same argument above we can show that $\mathfrak{u}(n) = \{X \in \mathbb{C}^{n \times n} : \bar{X}^t + X = I\}$. Hence $\dim \mathfrak{u}(n) = n^2$.

(4) The subgroup of $\mathrm{U}(n)$ with determinant 1 is called the *special unitary group*, denoted by $\mathrm{SU}(n)$. By the very definition we know that $\mathrm{SU}(n) = \mathrm{U}(n) \cap \mathrm{SL}(n, \mathbb{C})$, hence $\mathfrak{su}(n)$ is the matrices in $\mathfrak{u}(n)$ with vanishing trace, hence has real dimension $n^2 - 1$.

(5) The automorphism group of Lorentz bilinear form on $\mathbb{R}^{n,1}$, that is the bilinear form of $L = \operatorname{diag}(1, \cdots, 1, -1)$ and preserves upper half space, denoted by $\mathrm{O}^+(n, 1)$. We will not meet the Lie algebra of $\mathrm{O}^+(n, 1)$ generally.

Remark 1.41. The Lie groups in Example 1.39 and 1.40 are called *linear groups*, since they are subgroups of general linear group $\mathrm{GL}(n, \mathbb{R})$.

1.6 Quotient Manifolds

We first review the notion of group actions.

Definition. A *smooth (left) action* of group G on smooth manifolds M is a smooth map

$$\begin{aligned} G \times M &\to M \\ (g, x) &\mapsto g \cdot x, \end{aligned}$$

which satisfies

(1) $e \cdot x = x$;

(2) $(hg) \cdot x = h \cdot (g \cdot x)$ for $g, h \in G$.

Sometimes we omit the dot and simply write gx. The *orbit* of $x \in M$ is the set $\{gx : g \in G\}$. The *isotropy group* (or *stabilizer*) of $x \in M$ is the subgroup $G_x := \{g \in G : gx = x\}$.

Group actions give rise to quotient manifolds.

Definition. Let a Lie group G acts smoothly on a manifold M, we define the *quotient space* M/G by collecting all orbits, and equip M/G the finest topology to make $\pi : M \to M/G$ continuous.

Next we discuss when the quotient space of a smooth manifold admits a smooth manifold structure.

Definition. Let group G has a smooth action of M.

(1) The action is called *proper* if for any $x, y \in M$ with y not in the orbit of x, there exists two neighborhood U and V of x and y such that $gU \cap V = \varnothing$ for all g.

(2) The action is called *free* if $gx = x$ for some $x \in M$ implies $g = \mathrm{id}$.

Theorem 1.42. *Let Lie group G has a smooth, proper and free action of M. Then the quotient space M/G is a topological manifold with* $\dim(M/G) = \dim M - \dim G$. *Moreover, M/G has a unique smooth structure that makes $\pi : M \to M/G$ a smooth submersion.*

The proof is very complicated, please see [6, Theorem 21.10].

We now give some examples of quotient manifolds.

Example 1.43. If G is discrete, then $\pi : M \to M/G$ is a covering.

(1) $\mathbb{RP}^n = \mathbb{S}^n/\mathbb{Z}_2$, where the nontrivial element in $\mathbb{Z}_2$ corresponds to the antipodal map.

(2) $\mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z} \cong \mathbb{R}/\mathbb{Z}$, $2\pi\mathbb{Z}$ acts on $\mathbb{R}$ by addition.

(3) $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$. This is equivalent to the definition of $\mathbb{S}^1 \times \cdots \times \mathbb{S}^1$ by third isomorphism theorem.

Example 1.44. By algebra, if G acts on M transitively, then M equals to G/G_x, $x \in M$.

(1) The isometry group of Euclidean space $\operatorname{Isom}\mathbb{R}^n$ is equal to the semidirect product $\mathrm{O}(n) \ltimes \mathbb{R}^n$, where $\mathbb{R}^n$ represent translations. Notice that $\mathrm{O}(n)$ fixes origin, hence $\mathbb{R}^n \cong \mathrm{O}(n) \ltimes \mathbb{R}^n/\mathrm{O}(n)$.

(2) Let $\mathbb{S}^n$ defined by 1's preimage of quadric form $\sum_{i=1}^{n+1}(x^i)^2$, its automorphism group $\mathrm{O}(n+1)$ acts naturally on $\mathbb{S}^n$. Notice that $\mathrm{O}(n)$ acting on first n coordinates fixes $(0, \cdots, 0, 1)$, hence $\mathbb{S}^n = \mathrm{O}(n+1)/\mathrm{O}(n)$.

(3) Let the hyperboloid defined by -1's preimage of quadric form $\sum_{i=1}^{n}(x^i)^2 - (x^{n+1})^2$ and taking upper component, its automorphism group $\mathrm{O}^+(n,1)$ acts naturally on it. Notice that $\mathrm{O}(n)$ acting on first n coordinates fixes $(0, \cdots, 0, 1)$, hence the upper component of hyperboloid is diffeomorphic to $\mathrm{O}^+(n,1)/\mathrm{O}(n)$.

These three spaces are called *space forms*. We will show in Riemannian geometry part that they have constant sectional curvature.

Example 1.45. We give the example on real and complex Grassmannians.

(1) (Real) Grassmannians $\mathrm{Gr}(n,k)$ are the collection of all k-dimensional subspaces of $\mathbb{R}^n$. Let V be a k-dimensional subspace of $\mathbb{R}^n$, then $\mathbb{R}^n$ splits into $V \oplus V^\perp$. Thus an isometry on V and an isometry on $V^\perp$ acts separately, together fixes V. Hence $\mathrm{Gr}(n,K)$ has a smooth structure induced by $\mathrm{O}(n)/(\mathrm{O}(k) \times \mathrm{O}(n-k))$. In particular, $\mathrm{Gr}(n+1,1)$ is the n-dimensional projective space $\mathbb{RP}^n$.

(2) We can carry the same procedure to complex vector spaces. Hence we have complex Grassmannians are $\mathrm{U}(n)/(\mathrm{U}(k) \times \mathrm{U}(n-k))$. But one notices that even complex Grassmannians have complex structure, $\mathrm{U}(n)$ is never a complex manifold.

Chapter 2

Bundles and Sheaves

In this chapter we generalize the construction of tangent bundle to obtain general vector bundles. This will allow us to define tensors, an important class of functions on smooth manifolds. Moreover, we will discuss sheaves and fiber bundles, which are two generalizations of vector bundles.

2.1 Vector Bundles

Definitions

Definition. A *vector bundle* E of rank k over M consists of two manifolds E and M, a surjective smooth map $\pi : E \to M$ which satisfies:

(1) For each $p \in M$, there exists a neighborhood U of p such that there is a diffeomorphism $\varphi_U : \pi^{-1}(U) \to U \times \mathbb{R}^k$, the pair (U, φ_U) is called a *local trivialization*.

(2) The diffeomorphism above restricted on $\pi^{-1}(p)$ is a vector space isomorphism $\varphi_U|_{\pi^{-1}(p)} : \pi^{-1}(p) \xrightarrow{\sim} \{p\} \times \mathbb{R}^k$.

Example 2.1. (1) We have *trivial bundle* $M \times \mathbb{R}^k$.

(2) The tangent bundle TM of a smooth manifold M is a vector bundle of rank n.

(3) A vector bundle of rank 1 is called a *line bundle*.

(4) Infinite Möbius band $I \times \mathbb{R}$ with $(0, x)$ and $(1, -x)$ identified is a nontrivial line bundle over $\mathbb{S}^1$.

Definition. Let $\pi : E \to M$ and $\pi' : E' \to M$ be two vector bundles, a smooth map $f : E \to E'$ is called a *bundle map* if

(1) the following diagram is commutative

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "\pi"'] & & E' \arrow[dl, "\pi'"] \\ & M, & \end{tikzcd}$$

(2) f restricted on fiber $f|_{\pi^{-1}(p)} : E|_{\pi^{-1}(p)} \to E'|_{(\pi')^{-1}(p)}$ is a linear transformation.

An invertible bundle map is called an *isomorphism* between bundles.

Therefore we obtain the notion of category of vector bundles over M. Moreover, we can define the sub-object in this category as Proposition 1.28.

Definition. Let $\pi : E \to M$ be a vector bundle, a submanifold $F \subset E$ is called a *subbundle* if

(1) $\pi|_F : F \to M$ is a vector bundle;

(2) the inclusion $F \hookrightarrow E$ is a bundle map.

At the end of this subsection, we provide a method to construct a vector bundle.

Definition. Let $\pi : E \to M$ be a vector bundle, and $\{U_i, \varphi_i\}_{i\in I}$ be a cover of local trivializations. The *transition functions* with respective to this cover is a collection of smooth maps $g_{ij} := \varphi_j^{-1} \circ \varphi_i$ for all indices $i, j \in I$. Clearly g_{ij} can be regard as a map $U_i \cap U_j \to \mathrm{GL}(k, \mathbb{R})$.

Proposition 2.2. *Let M be a manifold, $\{U_i\}_{i\in I}$ be an open cover of M. Given a collection of smooth maps $g_{ij} : U_i \cap U_j \to \mathrm{GL}(k, \mathbb{R})$ for any indices $i, j \in I$, which satisfy*

$$g_{ij} \cdot g_{ji} = I,$$
$$g_{ij} \cdot g_{jk} \cdot g_{ki} = I,$$

then the cover together with the collection of maps determine a vector bundle of rank k over M with same transition functions with respective to the cover. The conditions on $\{g_{ij}\}$ is called cocycle condition.

Proof. Let $E_i := U_i \times R^k$, define

$$E = \bigsqcup_{i \in I} E_i \Big/ \sim,$$

where the equivalence relation $(p, v) \in E_i \sim (q, w) \in E_j$ if and only if $p = q$ and $w = g_{ij}(v)$. It's easy to check this construction indeed gives a vector bundle, and clearly g_{ij}'s are the transition functions with respective to $\{U_i\}_{i \in I}$. $\square$

Example 2.3. We revisit the vector bundles in Example 2.1.

(1) Trivial bundle is given by single transition function $U_1 = M$, $g_{11} = I$.

(2) Tangent bundle TM of M. Let $\{U_i, \varphi_i\}$ be an atlas, then the transition functions are $g_{ij} = \varphi_j^{-1} \circ \varphi_i$, where we equip $\mathbb{R}^n$ with natural smooth structure. Then cocycle conditions are immediately verified.

(3) Line bundles. The transition functions of line bundles are nonzero real numbers, this hints us that we can define a group associated to line bundles over M in some way.

(4) Möbius bundle. Let I_1, I_2 be two intervals that cover $\mathbb{S}^1$, then g_{12} is given by -1.

The transition function approach to vector bundle allows us to define algebraic operations on vector bundles. We will turn to this topic now.

Algebraic Operations

We can migrate the methods to construct new vector spaces from old to vector bundles.

Let $E \to M$ and $F \to M$ be two vector bundles, with local trivialization cover $\{U_i\}_{i \in I}$ (up to a common refinement), and transition functions $\{g_{ij}\}$ and $\{h_{ij}\}$ with respective to this cover. Assume g_{ij} and $h_[ij]$ have matrix representation A_{ij} and B_{ij}. Then we have:
Direct sum. The bundle $E \oplus F \to M$ is given by transition functions $g_{ij} \oplus h_{ij}$. In matrix form, the transition functions are

$$\begin{bmatrix} A_{ij} & 0 \\ 0 & B_{ij} \end{bmatrix}.$$

Tensor product. The bundle $E \otimes F \to M$ is given by transition functions $g_{ij} \otimes h_{ij}$. In matrix form, assume that $A_{ij} = [a_{ij}{}^{\alpha}_{\beta}]_{1\le\alpha,\beta\le k}$, then $A_{ij} \otimes B_{ij}$ has blocked form

$$\begin{bmatrix} a_{ij}{}^1_1 B_{ij} & a_{ij}{}^2_1 B_{ij} & \cdots & a_{ij}{}^k_1 B_{ij} \\ a_{ij}{}^1_2 B_{ij} & a_{ij}{}^2_2 B_{ij} & \cdots & a_{ij}{}^k_2 B_{ij} \\ \vdots & \vdots & \ddots & \cdots \\ a_{ij}{}^1_k B_{ij} & a_{ij}{}^2_k B_{ij} & \cdots & a_{ij}{}^k_k B_{ij} \end{bmatrix}.$$

Dual bundle. The bundle E^* is given by transition functions $(g_{ij}^{-1})^t$. We will explain the term "dual" later.

Using above algebraic operations, we can easy to define algebraic operation on the set of equivalence classes of line bundles over M.

Proposition 2.4. *The set of equivalence classes of line bundles over M can be made into a group, where the multiplication is given by $E \otimes F$, and the inverse is given by E^*. This group is called* Picard group, *and denoted by* $\operatorname{Pic} M$.

2.2 Sections

Sections are the main tool to investigate vector bundles.

Definition. Let $\pi : E \to M$ be a vector bundle, a *section* of E is a smooth map $s : M \to E$ with $\pi \circ s = \mathrm{id}_M$. The set of all sections of E is denoted by $\Gamma(E)$.

Remark 2.5. In differential, or Riemannian geometry, we are particularly interested in the sections of tangent bundles, that is, vector fields. The notation above for vector fields coincides with the notation we introduced in Section 1.2.

Clearly we have

Proposition 2.6. *Let $E \to M$ be a vector bundle, then $\Gamma(E)$ is a $C^\infty(M)$-module, the scalar product is given pointwisely.*

Now we use transition functions to characterize sections.

Proposition 2.7. *Let vector bundle $E \to M$ be defined by transition functions $\{g_{ij}\}$, then a section is equivalent to a collection of smooth functions $s_i : U_i \to \mathbb{R}^k$, which satisfies $s_j = g_{ij} s_i$ on each $U_i \cap U_j$.*

The proof is similar to Proposition 2.2, just glue each local section.

We can use the local expression of sections to give a explanation of "dual bundle".

Proposition 2.8. *Let $E \to M$ be a vector bundle, there is a natural bilinear paring $\langle \cdot, \cdot \rangle : E \times E^* \to C^\infty(M)$.*

Proof. Let E and E^* have transition functions $\{g_{ij}\}$ and $\{(g_{ij}^{-1})^t\}$. Let $s = \{s_i\} \in \Gamma(E)$ and $t = \{t_i\} \in \Gamma(E^*)$ be sections, then we define

$$\langle s, t \rangle|_{U_i} = s_i^t t_i.$$

We must show $\langle s, t \rangle$ is well-defined. Consider $\langle s, t \rangle$ restricted on $U_i \cap U_j$, we have another expression

$$\begin{aligned}\langle s, t \rangle|_{U_i \cap U_j} &= s_j^t t_j \\ &= (g_{ij} s_i)^t ((g_{ij}^{-1})^t t_i) \\ &= s_i^t g_{ij}^t (g_{ij}^{-1})^t t_i \\ &= s_i^t t_i,\end{aligned}$$

hence $\langle s, t \rangle$ is well-defined. □

2.3 Fiber Bundles and Principal Bundles

In this and next sections we generalize vector bundle in two ways. First, we replace the fiber of a vector space by arbitrary fiber, obtaining fiber bundle. Moreoever, when the fiber is chosen to be Lie groups, we obtain principal bundles.

Definition. A *fiber bundle* with fiber F consists of two manifolds E and M, a surjective smooth map $\pi : E \to M$ which satisfies

(1) For each $p \in M$, there exists a neighborhood U of p such that there is a diffeomorphism $\varphi_U : \pi^{-1}(U) \to U \times F$, the pair (U, φ_U) is called a *local trivialization*;

(2) The diffeomorphism above restricted on $\pi^{-1}(p)$ is a bijection $\varphi_U|_{\pi^{-1}(p)} : \pi^{-1}(p) \xrightarrow{\sim} \{p\} \times F$.

Example 2.9. (1) Any covering space is a fiber bundle with discrete fiber.

(2) Any vector bundle is a fiber bundle with fiber to be $\mathbb{R}^k$.

(3) We have the *Hopf fibration*:

$$H_n : \mathbb{S}^{2n+1} \to \mathbb{CP}^n$$

$$(z^0, \cdots, z^n) \mapsto [z^0, \cdots, z^n].$$

This is a fiber bundle with fiber $\mathbb{S}^1$. In particular, when $n = 1$, we have an explicit formula

$$\begin{aligned} H_1 : \mathbb{S}^3 &\to \mathbb{S}^2(1/2) \\ (z, w) &\mapsto \left(z\bar{w}, \frac{1}{2}\left(|w|^2 - |z|^2\right)\right), \end{aligned}$$

where we regard $\mathbb{S}^2(1/2) \subset \mathbb{C} \oplus \mathbb{R}$. The radius $1/2$ is chosen to make H_1 a *Riemannian submersion*, which will be explained this later.

Now we discuss principal bundles.

Definition. (1) Let $f : M \to N$ be a smooth map, G acts smoothly on M and N. f is called *G-equivariant* if $f(gx) = gf(x)$.

(2) Let G be a Lie group. A *principal G-bundle* is a fiber bundle $\pi : P \to M$ with fiber G, together with a smooth left action of G on P and each local trivialization $\pi^{-1}(U) \to U \times G$ is G-equivariant, where G acts on $U \times G$ by $g(x, h) = (x, gh)$.

Principal bundles also have cocycle construction. But instead of gluing $U \times \mathbb{R}^k$'S, principal bundles glue $U \times G$'s.

Proposition 2.10. *A principal G-bundle P over M is equivalent to the following data: given an open cover $\{U_i\}$ of M, the collection of smooth maps $g_{ij} : U_i \cap U_j \to G$ which satisfies*

$$g_{ij} \cdot g_{ji} = e$$
$$g_{ij} \cdot g_{jk} \cdot g_{ki} = e$$

determines a principal G-bundle uniquely.

Example 2.11. (1) Frame bundle $F(E)$ of a rank k vector bundle $E \to M$. $F(E)$ is the submanifold of $E^{\oplus k}$ consisting of k-tuples $(e_1, \cdots, e_k)$ which give a basis over the base point. Locally $\pi^{-1}(U)$ is diffeomorphic to $U \times \mathrm{GL}(k, \mathbb{R})$, and has a natural action of $\mathrm{GL}(k, \mathbb{R})$, hence $F(E)$ is a principal $\mathrm{GL}(k, \mathbb{R})$-bundle.

(2) Principal $\mathrm{GL}(k, \mathbb{R})$ bundles. Using same transition functions, one can construct the corresponding rank k vector bundle of the principal bundle. This gives a functor from the category of principal $\mathrm{GL}(k, \mathbb{R})$-bundles to the category of vector bundles of rank k, it is also a categorical equivalence, whose inverse is given by taking frame bundle, although we don't want to explain what all these mean.

2.4 Sheaves

In this section we release the requirement of local trivializations being fiber preserving, obtaining sheaves. But instead of defining sheaves like vector bundles, we adopt a modern definition.

Definition. Let X be a topological space, a *presheaf* $\mathcal{F}$ of sets (abelian groups, rings, A-modules, etc.) over X is an assignment of each open set U a set (abelian group, ring, A-module, etc.) $\Gamma(\mathcal{F}, U)$, called *sections*, together with a collection of mappings (homomorphisms) $r_V^U : \Gamma(\mathcal{F}, U) \to \Gamma(\mathcal{F}, V)$ for open sets $V \subset U$, called *restriction maps*, which satisfies

(1) $r_U^U = \mathrm{id}_U$;

(2) $r_W^U = r_W^V \circ r_V^U$ provided $W \subset V \subset U$.

If $s \in \Gamma(\mathcal{F}, U)$ and $V \subset U$, $r_V^U(s)$ is usually denoted by $s|_V$. The set $\Gamma(\mathcal{F}, X)$ is called *global section*, usually denoted by $\Gamma(\mathcal{F})$.

Definition. Let $\mathcal{F}$ be a presheaf over X. $\mathcal{F}$ is called a *sheaf* if it additionally satisfies the following *sheaf axioms*:

(1) Given an open cover $\{U_i\}$ of $U \subset X$, if $s, t \in \Gamma(\mathcal{F}, U)$ satisfy $s|_{U_i} = t|_{U_i}$ for any i, then $s = t$;

(2) Given an open cover $\{U_i\}$ of $U \subset X$, if $s_i \in \Gamma(\mathcal{F}, U_i)$ satisfy $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for any i, j, then there exists an $s \in \Gamma(\mathcal{F}, U)$ such that $s|_{U_i} = s_i$.

Example 2.12. (1) Let M be a smooth manifold, we define the *sheaf of germs of smooth functions* C_M^∞ by

$$\Gamma(C_M^\infty, U) = \{f : U \to \mathbb{R} : f \text{ is smooth}\}.$$

Then C_M^∞ is clearly a sheaf with evident restriction map.

(2) Let $E \to M$ be a topological vector bundle, that is, the local trivializations are homeomorphisms and sections are merely continuous mappings. Then we have a sheaf associated to E with (use same symbol)

$$\Gamma(E, U) = \{s : U \to E |\ \pi \circ s = \mathrm{id}_U\}.$$

E has evident restriction map. The notation for global section of sheaf E conincides with the notation of vector bundle E.

(3) We give an example of a presheaf which is not a sheaf. Let $X = X_1 \sqcup X_2$ with X_1, X_2 open. Define $\Gamma(\mathcal{F}, U) = \mathbb{Z}_2$ for any $U \neq \varnothing \subset X$, $\Gamma(\mathcal{F}, \varnothing) = \{0\}$, the restriction maps are taken to be identity maps for nonempty open sets, and $r^U_\varnothing$ is the zero homomorphism. Then $\mathcal{F}$ is a presheaf. But take $0 \in \Gamma(\mathcal{F}, X_1)$, $1 \in \Gamma(\mathcal{F}, X_2)$, we have

$$0|_{X_1 \cap X_2} = 0|_\varnothing = 0 = 1|_\varnothing = 1|_{X_1 \cap X_2},$$

then by sheaf axiom, there exists $x \in \mathbb{Z}_2 = \Gamma(\mathcal{F}, X)$ such that $x|_{X_1} = 0$, $x|_{X_2} = 1$, this is impossible. Hence $\mathcal{F}$ is not a sheaf.

Definition. Let $\mathcal{F}$ be a presheaf over X, the *germ* of $\mathcal{F}$ at x, denoted by $\mathcal{F}_x$, is

$$\bigsqcup_{x \in U \text{ open}} \Gamma(\mathcal{F}, U) \Big/ \sim,$$

where $s \in \Gamma(\mathcal{F}, U)$ and $t \in \Gamma(\mathcal{F}, V)$ are equivalent (i.e. $s \sim t$) if there exists $W \subset U \cap V$ such that $s|_W = t|_W$.

We have a natural map $\Gamma(\mathcal{F}, U) \to \mathcal{F}_x$ for $x \in U$, the image of s under this natural map is denoted by s_x.

Remark 2.13. One can show that if $\mathcal{F}$ is a presheaf of abelian groups, rings, A-modules etc., then $\mathcal{F}_x$ carries the corresponding algebraic structure.

Definition. Let $\mathcal{F}$ be a presheaf over X. The *étale space* $\operatorname{et}\mathcal{F}$ of $\mathcal{F}$ is defined as follows: as a set, we have

$$\operatorname{et}\mathcal{F} = \bigsqcup_{x \in X} \mathcal{F}_x,$$

and $\operatorname{et}\mathcal{F}$ carries the coarsest topology making the mapping

$$s : x \mapsto s_x$$

continuous for any $s \in \Gamma(\mathcal{F}, U)$ and $U \subset X$ open. Denote $\pi : \operatorname{et}\mathcal{F} \to X$ to be the evident projection map.

A little observation can tell that

Proposition 2.14. $\operatorname{et}\mathcal{F}$ *has a basis* $\{s(x) \in \mathcal{F}_x \mid x \in U,\ s(x) = s_x \text{ for } s \in \Gamma(\mathcal{F}, U)\}$ *for all* $s \in \Gamma(\mathcal{F}, U)$ *and* $U \subset X$ *open.*

Example 2.15. (1) Let $E \to M$ be a topological vector bundle, then E is the étale space of the associated sheaf of E.

(2) Let $\mathcal{F}$ be a presheaf, we can define a *sheaf* $\tilde{\mathcal{F}}$ by

$$\Gamma(\tilde{\mathcal{F}}, U) = \{s : U \to \operatorname{et}\mathcal{F} \mid \pi \circ s = \mathrm{id}_U,\ s \text{ is continuous}\}.$$

The sheaf is called the *sheafification* of $\mathcal{F}$.

Continuation

Chapter 3

Metric Structure of Riemannian Manifolds

In this chapter we introduce the metric structure on a Riemannian manifold, i.e. using the metric tensor to make the manifold into a metric space. We will prove that the topology induced by metric coincides with the topology carried by the manifold. Then we will study the length-minimizing problem: which curve minimize the distance between two points? The answer is two-sided: we will prove that length-minimizing curves are geodesics, and geodesics are *locally* length-minimizing.

3.1 Metric Structure on Riemannian Manifolds

Let (M, g) be a Riemannian manifold, $\gamma : [0, 1] \to M$ be a regular curve (i.e. γ is an immersion). We define the length functional of regular curves.

Definition. The *length functional* $L[\cdot]$ on the set of regular curves is defined by

$$L[\gamma] = \int_0^1 \sqrt{g(\dot{\gamma}(t), \dot{\gamma}(t))}\, \mathrm{d}t.$$

It is a well-known result from calculus that any regular curve can be reparametrized by arc-length. From now on the word "curve" means a regular curve.

Now we define the distance between two points.

Definition. Let $p, q \in M$ be two points on M, then we define their *distance* by

$$d(p, q) = \inf_{\gamma \in C_{p,q}} L[\gamma],$$

where $C_{p,q}$ is the set of all regular curves joining p and q.

Proposition 3.1. *The distance function $d : M \times M \to \mathbb{R}$ has the following properties:*

(1) $d(p,q) \geq 0$, and $d(p,q) = 0 \iff p = q$;

(2) $d(p,q) = d(q,p)$;

(3) $d(p,r) \leq d(p,q) + d(q,r)$.

Thus the distance function makes M into a metric space.

Proof. Only need to show $d(p,q) = 0 \iff p = q$, all else are trivial. We assume $p \neq q$, need to show $d(p,q) > 0$. Let $\gamma : [0,1] \to M$ be any curve joining p and q. Choose a local chart (U, φ) such that $\varphi(U) = B_r(0)$, $q \neq U$. By Jordan–Brouwer Separation Theorem, γ must intersect ∂U at $s := \gamma(c)$. Then we have

$$L[\gamma] \geq L[\gamma|_{[0,c]}] = \int_0^c \sqrt{g_{ij}\dot{x}^i(\gamma(t))\dot{x}^j(\gamma(t))}\, \mathrm{d}t.$$

Regarding $g : \overline{U} \times \mathbb{S}^{n-1} \to \mathbb{R}$, g is a continuous function on a compact set, thus it attains its minimum $g(x)(v,v) \geq m$, and $m > 0$ since $v \in \mathbb{S}^{n-1} \neq 0$. Thus we have

$$L[\gamma] \geq L[\gamma|_{[0,c]}] \geq m \int_0^c |\dot{x}(\gamma(t))|\, \mathrm{d}t \geq mr > 0. \tag{3.1}$$

mr does not depend on γ, hence $d(p,q) \geq mr > 0$. □

However, the metric space topology is nothing but the original topology carried by the manifold.

Proposition 3.2. *The metric space topology on M coincides with the manifold topology.*

We first need a lemma.

Lemma 3.3. *The distance function to p defined by $r(q) = d(p,q)$ is continuous with respective to the manifold topology.*

Proof. Since manifolds satisfy the second countable axiom, the Sequence Lemma holds. Then it's equivalent to show for any $q_i \to q$ in manifold topology, we have $r(q_i) \to r(q)$. Without loss of generality we can assume $\{q_i\} \subset U$ and (U, φ) is a local chart such that $\varphi(q) = 0$, $\varphi(U) = B_r(0)$. Let δ be the Euclidean metric on $B_r(0)$, then by regarding g as a continuous function on $\overline{U} \times \mathbb{S}^{n-1}$ again, we have $g \leq M\delta$ for some $M > 0$. By assumption, $q_i \to q$ in manifold topology implies $L_\delta[\psi_i] \to 0$, where $\psi_i(t) = t\varphi(q_i)$, the radial line joining $\varphi(q)$ and $\varphi(q_i)$ in $\varphi(U) = B_r(0)$. Let $\varphi^{-1}\psi_i = \gamma_i$, then we have

$$L_g[\gamma_i] = \int_0^1 \sqrt{g(\dot{\gamma}_i(t), \dot{\gamma}_i(t))}\, \mathrm{d}t \leq M \int_0^1 \sqrt{\delta(\dot{\psi}_i(t), \dot{\psi}_i(t))}\, \mathrm{d}t \leq ML_\delta[\psi_i].$$

Since r is Lipschitz, i.e. $|r(q) - r(s)| \leq d(q,s)$, we have

$$d(q_i, q) \leq L_g[\gamma_i] \leq ML_\delta[\psi_i] \to 0,$$

hence r is continuous. $\square$

Proof of Proposition 3.2. Since distance function is continuous, metric balls are open in manifold topology. Now we prove the converse.

Let U be open with respective to manifold topology. Let $p \in U$, V be a neighborhood of p so small that $\varphi(V) = B_r(0)$ for some $r > 0$. The estimate (3.1) shows if $q \notin V$ then $d(p,q) \geq mr$ for some fixed $m > 0$, then by taking contrapositive statement, we have $q \in V$ if $d(p,q) < mr$. Therefore $B_p(mr) \subset U$, then U is open with respective to metric space topology. $\square$

3.2 Length-Minimizing Curves

This section and the next is guided by the following problem:

Problem. Let p, q be two distinct points on a Riemannian manifold (M, g), then what is the curve γ satisfying $L[\gamma] = d(p,q)$?

This section we will show if the length-minimizing curve exists, then it is a *geodesic*. Next section we will show if p, q are sufficiently closed, *the* geodesic joining p and q is length-minimizing. Finally, the existence of length-minimizing curves is related to Hopf-Rinow Theorem, which we will discuss at the last section of the chapter.

To find the length-minimizing curve, we "gather" curves with same initial and end points, which is called a variation.

Definition. Let $\gamma_0 : [0,a] \to M$ be a curve, a *variation* of γ_0 is a smooth map $\gamma : [0,a] \times (-\varepsilon, \varepsilon) \to M$ such that $\gamma(t,0) = \gamma_0(t)$. If $\gamma(0,s) = \gamma_0(0)$ and $\gamma(a,s) = \gamma_0(a)$ for any $s \in (-\varepsilon, \varepsilon)$, then we call the variation a *proper variation*. We call $\frac{\partial}{\partial s}\Big|_{s=0} \gamma(s,t) =: V(t)$ the *variation vector field*.

Now we introduce the energy functional, which is easier to calculate.

Definition. The *energy functional* on the set of curves is defined by

$$E[\gamma] = \int_0^a \frac{1}{2} |\dot{\gamma}(t)|^2 \, \mathrm{d}t,$$

where $\gamma : [0,a] \to M$ is a regular curve.

We will prove that a curve is energy-minimizing if and only if it is length-minimizing.

Lemma 3.4. *For a curve $\gamma : [0, a] \to M$, we have*

$$L^2[\gamma] \leq 2aE[\gamma],$$

with equality holds if and only if $|\dot{\gamma}(t)| = \text{const}$.

Proof. This is Cauchy–Schwarz inequality. □

Proposition 3.5. *If γ is length-minimizing, then it is energy-minimizing.*

Proof. Let $\tilde{\gamma}$ be another curve, then we have

$$2aE[\gamma] = L^2[\gamma] \leq L^2[\tilde{\gamma}] \leq 2aE[\tilde{\gamma}]. \qquad \square$$

Our aim is to prove the converse.

Proposition 3.6. *If γ is an energy-minimizing curve, then it is length-minimizing.*

To prove this, we need to differentiate the variation.

Proposition 3.7 (First variation formula)**.** *Let $\gamma(t, s)$ be a variation, define its energy $E(s) = \int_0^a \frac{1}{2} \left| \frac{\partial}{\partial t} \gamma(t, s) \right|^2 \mathrm{d}t$, then we have*

$$E'(0) = \boxed{\langle V, \dot{\gamma} \rangle |_0^a} - \int_0^a \langle V(t), \nabla_{\dot{\gamma}_0(t)} \dot{\gamma}_0(t) \rangle.$$

The boxed term is called boundary term*, and it vanishes when the variation is proper.*

Proof. This is a calculation. We have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s} E(s) &= \int_0^a \left\langle \frac{\partial \gamma}{\partial t}, \nabla_{\frac{\partial}{\partial s}} \frac{\partial \gamma}{\partial t} \right\rangle \mathrm{d}t \\
&= \int_0^a \left\langle \frac{\partial \gamma}{\partial t}, \nabla_{\frac{\partial}{\partial t}} \frac{\partial \gamma}{\partial s} \right\rangle \mathrm{d}t \\
&= \int_0^a \frac{\partial}{\partial t} \left\langle \frac{\partial \gamma}{\partial t}, \frac{\partial \gamma}{\partial s} \right\rangle - \left\langle \frac{\partial \gamma}{\partial s}, \nabla_{\frac{\partial}{\partial t}} \frac{\partial \gamma}{\partial t} \right\rangle \mathrm{d}t.
\end{aligned}$$

Take $s = 0$, we obtain

$$\begin{aligned}
E'(0) &= \int_0^a \frac{\partial}{\partial t} \langle V(t), \dot{\gamma}_0(t) \rangle - \langle V(t), \nabla_{\dot{\gamma}_0(t)} \dot{\gamma}_0(t) \rangle \, \mathrm{d}t \\
&= \langle V, \dot{\gamma}_0 \rangle |_0^a - \int_0^a \langle V(t), \nabla_{\dot{\gamma}_0(t)} \dot{\gamma}_0(t) \rangle \, \mathrm{d}t. \qquad \square
\end{aligned}$$

Now we can derive the definition of a geodesic.

Definition. A curve $\gamma : [0, a]$ is called a *geodesic* if $\nabla_{\dot{\gamma}(t)}\dot{\gamma}(t) = 0$ for $t \in [0, a]$.

Remark 3.8. Geodesics are constant speed, this can be shown by $\frac{\mathrm{d}}{\mathrm{d}t}|\dot{\gamma}(t)|^2 = 2\langle\dot{\gamma}(t), \nabla_{\dot{\gamma}(t)}\dot{\gamma}(t)\rangle = 0$.

Corollary 3.9. *γ is a critical value for all proper variation if and only if $\nabla_{\dot{\gamma}(t)}\dot{\gamma}(t) = 0$, that is, γ is a geodesic.*

We can give the proof of Proposition 3.6 now.

Proof of Proposition 3.6. Let $\gamma : [0, a] \to M$ be a curve such that for any $\tilde{\gamma} : [0, a] \to M$ with $\tilde{\gamma}(0) = \gamma(0)$, $\tilde{\gamma}(1) = \gamma(1)$, the inequality $E[\gamma] \leq E[\tilde{\gamma}]$ holds, we show that $L[\gamma] \leq L[\tilde{\gamma}]$. Let $\gamma(t, s)$ be any proper variation with $\gamma(t, 0) = \gamma(t)$, then γ is a critical point of $E(s)$. Hence by Corollary 3.9, γ is a geodesic. Now we can reparametrize $\tilde{\gamma}$ into arc-length, obtaining $\hat{\tilde{\gamma}}$. Therefore

$$L^2[\gamma] = 2aE[\gamma] \leq 2aE\left[\hat{\tilde{\gamma}}\right] = L^2\left[\hat{\tilde{\gamma}}\right] = L^2[\tilde{\gamma}],$$

which implies $L[\gamma] \leq L[\tilde{\gamma}]$. □

Combining all results above, we have

Proposition 3.10. *If a curve is length-minimizing, then it is a geodesic.*

3.3 Geodesics and Exponential Maps

To prove the local length-minimizing property of geodesic, we need to introduce the exponential map.

We first need to investigate the equation that determine a geodesic.

Proposition 3.11. *Given $p \in M$ and $v \in T_pM$, there exists a unique geodesic γ (whose domain may not be maximal) such that $\gamma(0) = p$, $\dot{\gamma}(0) = v$.*

Proof. Let (U, φ) be a local chart containing p, compose φ with γ we obtain coordinate curves x^i's. Then the geodesic equation is equivalent to

$$\ddot{x}^k(t) + \Gamma^k_{ij}(\gamma(t))\dot{x}^i(t)\dot{x}^j(t) = 0,\ k = 1, \cdots, n.$$

This is a system of second order ordinary differential equations, by the unique existence theorem of ODE, the solution is completely determined by x^i's and $\dot{x}^i$'s, that is, p and v. □

Since the solution of an ODE relies continuously on its initial value, we have the following proposition.

Proposition 3.12. *For any $p \in M$, there exists a neighborhood V of p, such that there exists $\delta > 0$, $\varepsilon > 0$ and a smooth map $\gamma : (-\delta, \delta) \times \mathscr{U} \to M$, where $\mathscr{U} = \{(q, v) \in TM : q \in V, v \in T_qM, |v| < \varepsilon\}$, such that $\gamma(t; q, v)$ is a geodesic with $\gamma(0) = q$, $\dot{\gamma}(0) = v$.*

A proof can be found in [10, Chapter 3, Lemma 1].

Observe that $\gamma(\lambda t; p, v) = \gamma(t; p, \lambda v)$. Denote $\gamma(t; p, v)$ by $\gamma_v(t)$, then above observation can be written as $\gamma_{\lambda v}(t) = \gamma_v(\lambda t)$. Therefore, we can shorten the initial vector to lengthen the domain of geodesic.

Definition. Let $U \subset T_pM$ be a neighborhood of origin, such that for any $v \in U$, $\gamma_v(1)$ is defined (existence is guaranteed by Proposition 3.12). We define the *exponential map* at p by

$$\begin{aligned} \exp_p : U &\to M \\ v &\mapsto \gamma_v(1). \end{aligned}$$

Remark 3.13. We can scale the initial vector and obtain

$$\exp_p(v) = \gamma_v(1) = \gamma_{v/|v|}(|v|).$$

This means the action of exponential map on v is to move forward the distance $|v|$ along the geodesic with initial direction $v/|v|$.

Proposition 3.14. $\exp_{p*}|_0 : T_0(T_pM) \to T_pM$ *is identity (we identify $T_0(T_pM)$ with T_pM).*

Proof. We have

$$\exp_{p*}|_0(v) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (tv) = v.$$

□

Corollary 3.15. *There exists a ball $B_\varepsilon(0) \subset T_pM$ such that $\exp_p : B_\varepsilon(0) \to M$ is a diffeomorphism onto its image.*

Proof. Since $\exp_{p*}|_0$ is identity, it is nondegenerate, the corollary follows from inverse function theorem. □

Exponential maps enjoys naturality.

Proposition 3.16. *Let $f : M \to N$ be a local isometry, $p \in M$, $q \in N$ and $f(p) = q$. Then we have the following commutative diagram*

$$\begin{array}{ccc} U \subset T_pM & \xrightarrow{f_{*p}} & V \subset T_qN \\ \downarrow{\scriptstyle \exp_p} & & \downarrow{\scriptstyle \exp_q} \\ M & \xrightarrow{\quad f \quad} & N. \end{array}$$

Proof. This is simply because local isometry preserves geodesics. □

Example 3.17. (1) We know that the geodesics on $\mathbb{S}^n$ are great circles, hence $\exp_p$ is defined on the whole T_pM. But $\exp_p$ is not injective, since

$$\exp_p(0) = \exp(2\pi v) = p$$

for unit vector v in T_pM.

(2) Let $M = \mathbb{S}^1 \times \mathbb{R}$ be the cylinder. We know from elementary differential geometry that the geodesics on cylinder are directrix circles, helices and generatrix lines. Then in local charts $(e^{2\pi it}, s) \mapsto (t, s)$, we know $\exp_p$ is not injective in the direction $(1, 0)$, and injective in other directions.

We postpone the discussion on whether the exponential map can be defined on the whole tangent space, the answer is Hopf–Rinow Theorem, which will be discussed in next section.

Now we prove that geodesics are locally length-minimizing. For this, we introduce some local charts. Given a Riemannian manifold (M, g) and $p \in M$, let $\exp_p : B_\varepsilon(0) \to \exp(B_\varepsilon(0)) = B_\varepsilon(p)$ be a diffeomorphism.

Definition. We define *geodesic normal coordinate* as follows: Let $\{e_i\}$ be an orthonormal basis of Euclidean space (T_pM, δ), $\{\alpha^i\}$ be its dual basis. Then we define the coordinate by

$$q \in B_\varepsilon(p) \mapsto (\alpha^1(\exp_p^{-1}(q)), \cdots, \alpha^n(\exp_p^{-1}(q))).$$

Proposition 3.18. *Under geodesic normal coordinate, we have*

$$g_{ij}(p) = \delta_{ij},\ \Gamma_{ij}^k(p) = 0.$$

Proof. Since $\exp_p$ is a diffeomorphism, we have $\left.\frac{\partial}{\partial x^i}\right|_p = \exp_{p*}|_0(e_i) = e_i$, hence $g_{ij} = \delta(e_i, e_j) = \delta_{ij}$. Moreover, let $x(t) = ty$ for $y \in T_pM - \{0\}$, then $x(t)$ is the coordinate of some geodesic in $B_\varepsilon(p)$, thus it satisfies the equation

$$\ddot{x}^k(t) + \Gamma_{ij}^k(x(t))\dot{x}^i(t)\dot{x}^j(t) = 0.$$

Since $\dot{x}^i = y^i \neq 0$, $\ddot{x}^k = 0$, we have $\Gamma_{ij}^k(ty) = 0$. Let $y \to 0$ we obtain the conclusion. □

Next we introduce the geodesic polar coordinate.

Definition. We define *geodesic polar coordinate* as follows: Let $(r, \theta^1, \cdots, \theta^{n-1})$ be a polar coordinate on Euclidean space (T_pM, δ), then we defined the coordinate by

$$q \in B_\varepsilon(p) - \{p\} \mapsto (r(\exp_p^{-1}(q)), \theta^1(\exp_p^{-1}(q)), \cdots, \theta^{n-1}(\exp_p^{-1}(q))).$$

Proposition 3.19. *Under geodesic polar coordinate, we have*

$$\left\langle \frac{\partial}{\partial r}, \frac{\partial}{\partial r} \right\rangle = 1, \ \left\langle \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta^i} \right\rangle = 0.$$

Proof. To make things clear, we write the inverse of geodesic polar coordinate as

$$F : (r, \omega) \mapsto \exp_p(r\omega)$$

for $r \in (0, \varepsilon)$, $\omega \in \mathbb{S}^{n-1}$. Then we use $\partial_0, \partial_1, \cdots, \partial_{n-1}$ to denote the tangent vectors in $(0, \varepsilon) \times \mathbb{S}^{n-1}$, we have

$$\begin{aligned} \frac{\partial}{\partial r} &= F_*(\partial_0), \\ \frac{\partial}{\partial \theta^i} &= F_*(\partial_i), \ i = 1, \cdots, n-1. \end{aligned}$$

First we know that ∂_0 is the tangent vector of radial line $r\omega$, hence $\partial/\partial r$ is the tangent vector of a unit-speed radial geodesic, that is,

$$\left\langle \frac{\partial}{\partial r}, \frac{\partial}{\partial r} \right\rangle = 1.$$

Moreover, we have

$$\begin{aligned} \frac{\partial}{\partial r} \left\langle \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta^i} \right\rangle &= \left\langle \nabla_{\frac{\partial}{\partial r}} \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta^i} \right\rangle + \left\langle \frac{\partial}{\partial r}, \nabla_{\frac{\partial}{\partial r}} \frac{\partial}{\partial \theta^i} \right\rangle \\ &= \left\langle \frac{\partial}{\partial r}, \nabla_{\frac{\partial}{\partial r}} \frac{\partial}{\partial \theta^i} \right\rangle \\ &= \left\langle \frac{\partial}{\partial r}, \nabla_{\frac{\partial}{\partial \theta^i}} \frac{\partial}{\partial r} \right\rangle \\ &= \frac{1}{2} \frac{\partial}{\partial \theta^i} \left\langle \frac{\partial}{\partial r}, \frac{\partial}{\partial r} \right\rangle \\ &= 0, \end{aligned}$$

hence $\left\langle \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta^i} \right\rangle$ is constant. However, if we let $r \to 0$, we have $\partial/\partial\theta^i \to 0$, therefore

$$\left\langle \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta^i} \right\rangle = 0.$$

$\square$

Corollary 3.20. *Under geodesic polar coordinate, the metric tensor has local expression*

$$g = \mathrm{d}r^2 + g_{ij}(r,\theta)\,\mathrm{d}\theta^i \otimes \mathrm{d}\theta^j,$$

where $[g_{ij}]_{i,j>0}$ *is positive definite.*

As an application, we prove that geodesics are locally length-minimizing as we promised.

Proposition 3.21. *Let* $\gamma : [0,1] \to M$ *be a geodesic contained in an open set* U*, where geodesic polar coordinate is defined on* U*. Let* $\tilde{\gamma}$ *be any curve contained in* U *with* $\tilde{\gamma}(0) = \gamma(0) = p$*,* $\tilde{\gamma}(1) = \gamma(1) = q$*. Then* $L[\gamma] \leq L[\tilde{\gamma}]$*.*

Proof. Let $q = \exp_p(v)$, φ is the geodesic polar coordinate. Then we have

$$\gamma(t) = \varphi(tr_0, \omega_0),\ \tilde{\gamma}(t) = \varphi(r(t), \omega(t))$$

such that $r(1) = r_0$, $\omega(t) \in \mathbb{S}^{n-1}$. Therefore

$$\begin{aligned} L[\gamma] &= \int_0^1 |\dot{\gamma}(t)|\,\mathrm{d}t \\ &= \int_0^1 r_0\,\mathrm{d}t = r_0, \\ L[\tilde{\gamma}] &= \int_0^1 (|\dot{r}|^2(t) + g_{ij}\dot{\theta}^i(t)\dot{\theta}^j(t))^{1/2} \\ &\geq \int_0^1 |\dot{r}(t)|\,\mathrm{d}t \\ &\geq \int_0^1 \dot{r}(t)\,\mathrm{d}t = r_0. \end{aligned}$$

□

Remark 3.22. The hypothesis of Proposition 3.21 can be weakened to $\exp_p$ is an immersion on U.

3.4 Hopf–Rinow Theorem

We now answer the problem whether length-minimizing curve always exists. The answer is Hopf–Rinow Theorem.

We adopt the metric geometry version of Hopf–Rinow Theorem from [1], using geometric approach to prove the theorem and keeping the differential tools minimal. We first define

Definition. A Riemannian manifold (M, g) is called *(geodesically) complete* if M is complete as a metric space.

Our theorem is

Theorem 3.23 (Hopf–Rinow–Cohn–Vossen)**.** *Let (M, g) be a Riemannian manifold, the following four assertions are equivalent:*

(1) M has the Heine–Borel property, i.e. every closed geodesic ball is compact;

(2) M is geodesically complete;

(3) Every geodesic $\gamma : [0, a) \to M$ can be extended to a continuous curve $\overline{\gamma} : [0, a] \to M$;

(4) There is a point $p \in M$ such that every length-minimizing geodesic $\gamma : [0, a) \to M$ with $\gamma(0) = p$ can be extended to a continuous curve $\overline{\gamma} : [0, a] \to M$.

We first establish several lemmas.

Lemma 3.24. *The length functional is lower semi-continuous in the following sense: let $\gamma_i, \gamma : [a, b] \to M$, if $\gamma_i \to \gamma$ pointwisely as $i \to \infty$, then*

$$L[\gamma] \leq \varliminf_{i\to\infty} L[\gamma_i].$$

Proof. Let Y be a partition of $[a, b]$ with $a = y_0 < \cdots < y_N = b$, denote

$$\Sigma(Y) = \sum_{i=1}^{N} d(\gamma_(y_{i-1}), \gamma_(y_i)).$$

Take $\varepsilon > 0$ and fix a partition Z such that $L[\gamma] - \Sigma(Z) < \varepsilon$. Now consider $\Sigma_j(Z)$ for curves γ_j corresponding to same partition Z. Choose j so large that the inequality $d(\gamma_j(z_i), \gamma(z_i)) < \varepsilon$ holds for all $z_i \in Z$. Then

$$L[\gamma] \leq \Sigma(Z) + \varepsilon \leq \Sigma_j(Z) + \varepsilon + (N + 1)\varepsilon \leq L[\gamma_j] + (N + 2)\varepsilon.$$

Since ε is arbitrary, the lemma holds. $\square$

Lemma 3.25. *Let γ_i be length-minimizing curves for $i = 1, 2, \cdots$, suppose $\gamma_i \to \gamma$ pointwisely, then γ also minimizes length.*

Proof. Let γ has end points p and q, then since $L[\gamma_i]$ equals to the distance between its end points, $L[\gamma_i] \to d(p, q)$. By the lower semi-continuity of length functional, we have

$$L[\gamma] \leq \varliminf_{i\to\infty} L[\gamma_i] = d(p, q) \leq L[\gamma]. \qquad \square$$

Lemma 3.26. *Let (M, g) be a compact Riemannian manifold (maybe with boundary), then any two points $p, q \in M$ can be joined by a length-minimizing curve.*

Proof. By the definition of distance, there exists a sequence of curves $\{\gamma_i\}$ with constant speed such that $\gamma_i(0) = p$, $\gamma_i(1) = q$ and $L[\gamma_i] \to d(p,q)$. Then let $L[\gamma_i] < A$ for all $i = 1, 2, \cdots$, we have $d(p,x) \leq L[\gamma_i|_{[0,a]}] < A$ for $x = \gamma_i(a)$, hence the family $\{\gamma_i\}$ is uniformly bounded. Moreover, fix $\varepsilon > 0$, let $\delta = \varepsilon/A$, we have

$$d(x,y) \leq L[\gamma_i|_{[a,b]}] \leq A(b-a) < \varepsilon$$

provided $x = \gamma_i(a)$, $y = \gamma_i(b)$, and $b - a < \delta$ for any $i = 1, 2, \cdots$, hence the family $\{\gamma_i\}$ is equicontinuous. Then the family verifies the conditions of Arzela–Ascoli Theorem, it converges (up to a subsequence) to a curve γ. Then by the lower semi-continuity of length functional, we have

$$L[\gamma] \leq \lim_{i\to\infty} L[\gamma_i] = d(p,q) \leq L[\gamma].$$

Thus γ is length-minimizing. $\square$

Proof of Theorem 3.23. Implications (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) are all easy, we prove (4) $\Longrightarrow$ (1).

Let $R = \sup\{\overline{B_r(p)}$ is a compact set$\}$, then $R > 0$ since manifolds are locally compact, $\overline{B_r(p)}$ is compact for r sufficiently small. We argue by contradiction. Suppose $R < +\infty$, that is, there exists noncompact geodesic balls. The argument is divided into two steps.

1. First we prove $B_R(p)$ is sequentially compact. Let $\{p_i\} \subset B_R(p)$, set $d(p, p_i) = r_i$. We may assume $r_i \to R$ as $i \to \infty$, otherwise $\{p_i\}$ is eventually contained in a smaller geodesic ball, and it has a convergent subsequence by the definition of R.

Now let $\gamma_i : [0, r_i] \to M$ be a length-minimizing curve joining p and p_i, whose existence is guaranteed by Lemma 3.26. Notice that γ_i's are parametrized by arc-length. We can choose a subsequence of $\{\gamma_i\}$ such that the restrictions of the curves to $[0, r_1]$ converge by Arzela–Ascoli Theorem. From this subsequence, we can choose a further subsequence such that the restrictions to $[0, r_2]$ converge, and so on. Then by Cantor diagonal procedure, we have a sequence $\{\gamma_{i_n}\}$ such that for $t \in [0, R)$, $\gamma_{i_n}(t)$ is well-defined for n sufficiently large and $\gamma_{i_n}(t) \to \gamma(t)$ as $n \to \infty$. Moreover, Arzela–Ascoli Theorem asserts γ is smooth for any restriction to $[0, r]$ provided $r < R$ by uniform convergence, hence γ is smooth.

Now by Lemma 3.25, γ is a length-minimizing curve, hence by the hypothesis (iv), γ can be extended to a continuous curve $\overline{\gamma} : [0, R] \to M$. Then let $q = \overline{\gamma}(R)$, fix $\varepsilon > 0$, take n sufficiently large such that $d(p_{i_n}, \gamma_{r_{i_n}}) < \varepsilon/2$ and $d(\gamma(r_{i_n}, q)) = R - r_{i_n} < \varepsilon/2$, we have

$$d(p_{i_n}, q) \leq d(p_{i_n}, \gamma(r_{i_n})) + d(\gamma(r_{i_n}), q) < \varepsilon.$$

Hence $p_{i_n} \to q$ as $n \to \infty$, we have $B_R(p)$ is sequentially compact.

2. Since $B_R(p)$ is sequentially compact, we have $\overline{B_R(p)}$ is compact. Now we show $\overline{B_{R+\varepsilon}(p)}$ is compact for some $\varepsilon > 0$. Since M is locally compact, for every $x \in \overline{B_R(0)}$ there is an $r(x) > 0$ such that $\overline{B_{r(x)}(x)}$ is compact. Then we can choose finite $x_i \in \overline{B_R(p)}$ such that $\{B_{r(x_i)}(x_i)\}$ covers $\overline{B_R(p)}$. The union of these geodesic balls is sequentially compact and contains the geodesic ball $B_{R+\varepsilon}(p)$ for $0 < \varepsilon < \min\{r(x_i)\}$. Hence $\overline{B_{R+\varepsilon}(p)}$ is compact, this contradicts the choice of R. □

We have two corollaries.

Corollary 3.27 (Hopf–Rinow Theorem)**.** *Let (M, g) be a Riemannian manifold, the following are equivalent:*

(1) M is geodesically complete;

(2) $\exp_p$ is defined on whole T_pM for any $p \in M$;

(3) $\exp_p$ is defined on whole T_pM for one $p \in M$.

Proof. (1) $\Longrightarrow$ (2): Let $S = \{r \in \mathbb{R}_{>0} : \gamma : [0, r) \text{ can be extended to } r\}$. We prove S is both open and closed, then the implication holds. Let $r \in S$, for any $q \in \partial B_r(p)$, define $r(q)$ as follows: let unit-speed geodesic $\gamma : [0, r] \to M$ joins p and q, then there exists a geodesic $\tilde{\gamma} : [r, r + r(q))$ such that $\tilde{\gamma}(r) = q$, $\dot{\tilde{\gamma}}(r) = \dot{\gamma}(r)$, that is, γ can be extended to $[0, r + r(q))$. Cover $\partial B_r(p)$ by finite many $B_{r(q_i)}(q_i)$, then for $0 < \varepsilon < \min\{r(q_i)\}$, $B_{r+\varepsilon}(p)$ is in the union of $B_r(p)$ and $B_{r(q_i)}(q_i)$, hence any geodesics $\sigma : [0, r + \varepsilon)$ can be extended to $\sigma : [0, r + \varepsilon] \to M$. This implies $(r - \varepsilon, r + \varepsilon) \subset S$, S is open.

Now assume $\{r_i\} \subset S$ converges to r. Let $\gamma : [0, r)$ be a geodesic, then define $\gamma(r) = \lim_{i \to \infty} \gamma(r_i)$, the limit exists since M is complete. Hence $r \in S$, S is closed.

(2) $\Longrightarrow$ (3) is trivial.

(3) $\Longrightarrow$ (1): If $\gamma : [0, a)$ is a length-minimizing geodesic with $\gamma(0) = p$, then $\gamma(t) = \exp_p(tv)$ for some v, and clearly it can be extended to a. □

Corollary 3.28. *If M is geodesically complete, then any two points can be joined by a length-minimizing geodesic.*

Proof. Let $p, q \in M$, $R > d(p, q)$. Then $q \in \overline{B_R(p)}$, the latter one is bounded and closed, hence by Theorem 3.23, it is compact. By Lemma 3.26, there is a length-minimizing geodesic joining p and q. □

Chapter 4

Curvature

4.1 Curvature Tensor and Curvaure Endomorphism

Riemann Curvature Tensor

We calculated in Proposition 3.14 that $\exp_{p*}|_0$ is identity, naturally we have the following problem:

Problem. Calculate $\exp_{p*}|_v : T_v(T_pM) \to T_{\exp_p(v)}M$.

Solution. To evaluate $\exp_{p*}|_v(\xi)$, we choose a line $v + s\xi$, and then

$$\exp_{p*}|_v(\xi) = \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \exp_p(v + s\xi).$$

Now we can introduce the one parameter family of geodesics

$$\gamma(t,s) = \exp_p(t(v + s\xi)),$$

and denote $\gamma(t) = \gamma(t,0)$. We calculate the variation vector field $J(t)$ of γ, and obtain the result by taking $t = 1$. Let $J_s(t) = \frac{\partial}{\partial s}\gamma(t,s)$, then $\dot{J}_s(t) = \nabla_{\dot{\gamma}_s(t)}\frac{\partial \gamma}{\partial s}$. Since $\nabla_{\dot{\gamma}_s(t)}\frac{\partial \gamma}{\partial t} = 0$, we have

$$\begin{aligned}
\ddot{J}_s(t) &= \nabla_{\dot{\gamma}_s(t)}\nabla_{\dot{\gamma}_s(t)}\frac{\partial \gamma}{\partial s} \\
&= \nabla_{\dot{\gamma}_s(t)}\nabla_{\frac{\partial \gamma}{\partial s}}\frac{\partial \gamma}{\partial t} \\
&= \nabla_{\frac{\partial \gamma}{\partial t}}\nabla_{\frac{\partial \gamma}{\partial s}}\frac{\partial \gamma}{\partial t} - \nabla_{\frac{\partial \gamma}{\partial s}}\nabla_{\frac{\partial \gamma}{\partial t}}\frac{\partial \gamma}{\partial t}.
\end{aligned}$$

Moreover, we have $[\partial_t, \partial_s] = 0$, then we denote

$$R\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial s}\right) = \nabla_{\frac{\partial}{\partial s}} \nabla_{\frac{\partial}{\partial t}} - \nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial s}} + \nabla_{[\frac{\partial}{\partial t}, \frac{\partial}{\partial s}]},$$

we obtain

$$\ddot{J}_s(t) + R\left(\frac{\partial \gamma}{\partial t}, \frac{\partial \gamma}{\partial s}\right) \frac{\partial \gamma}{\partial t} = 0.$$

Take $s = 0$, we have

$$\ddot{J}(t) + R(\dot{\gamma}(t), J(t))\dot{\gamma}(t) = 0. \tag{4.1}$$

We will show later that (4.1) is a system of ordinary differential equations, hence by solving the system with given initial value and taking $t = 1$, we obtain the answer of the problem. □

We make it into a definition.

Definition. The *Riemann curvature tensor* $R : \Gamma(TM) \times \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$ is defined by

$$R(X, Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z.$$

Remark 4.1. Many authors define the Riemann curvature tensor as the negative of above definition, e.g. in [7]. Please be careful with the sign of the tensor.

We need to explain the name "tensor", so we must show R is truly tensorial.

Lemma 4.2. *R is a tensor.*

Proof. $R(X, Y)Z$ is clearly tensorial in X and Y, we show that $R(X, Y)(fZ) = fR(X, Y)Z$ for $f \in C^\infty(M)$. We have

$$\begin{aligned}
\nabla_Y \nabla_X (fZ) &= \nabla_Y((Xf)Z + f\nabla_X Z) \\
&= (YXf)Z + (Xf)\nabla_Y Z + (Yf)\nabla_X Z + f\nabla_Y \nabla_X Z \\
-\nabla_X \nabla_Y (fZ) &= -(XYf)Z - (Yf)\nabla_X Z - (Xf)\nabla_Y Z - f\nabla_X \nabla_Y Z \\
\nabla_{[X,Y]}(fZ) &= ([X, Y]f)Z + f\nabla_{X,Y} Z,
\end{aligned}$$

add all three equalities and we reach the conclusion. □

Now we can look at Riemann curvature tensor locally. A rather complicated calculation shows:

Lemma 4.3. *Let $R = R_{ijk}{}^l \otimes \mathrm{d}x^i \otimes \mathrm{d}x^j \otimes \mathrm{d}x^k \otimes \partial_l$, then*

$$R_{ijk}{}^l = \partial_j \Gamma^l_{ik} - \partial_i \Gamma^l_{jk} + \Gamma^m_{ik} \Gamma^l_{jm} - \Gamma^m_{jk} \Gamma^l_{im}.$$

We also define a $(0,4)$-tensor by lowering the l index of $Rlijk$, that is:

Definition. $R(X,Y,Z,W) := \langle R(X,Y)Z, W\rangle$ is also called *Riemann curvature tensor*.

Example 4.4. Euclidean space $(\mathbb{R}^n, \delta)$ has $R \equiv 0$. Any metric admits zero curvature is called *flat*.

Curvature Endomorphism

Curvature also appears in another scene. Let us consider the second covariant differential of a tensor.

Definition. Let $T(\cdots)$ be a tensor, denote $(\nabla_{X,Y}T)(\cdots) := \nabla(\nabla T)(\cdots, Y, X)$.

Proposition 4.5. *For (r,s)-tensor T, we have*

$$\nabla_{X,Y}T = \nabla_X(\nabla_Y T) - \nabla_{\nabla_X Y}T. \tag{4.2}$$

Proof. Since covariant derivative commutes with contraction, we have

$$\begin{aligned}
\nabla_X(\nabla_Y T) &= \nabla_X(\mathrm{tr}_{1,s+2}\, Y \otimes \nabla T) \\
&= \mathrm{tr}_{1,s+2}(\nabla_X(Y \otimes \nabla T)) \\
&= \mathrm{tr}_{1,s+2}((\nabla_X Y) \otimes \nabla T + Y \otimes (\nabla_X \nabla T)) \\
&= \nabla_{\nabla_X Y}T + \nabla(\nabla T)(\cdots, Y, X) \\
&= \nabla_{\nabla_X Y}T + \nabla_{X,Y}T,
\end{aligned}$$

then the result follows. □

Definition. Define the *curvature endomorphism* $R(X,Y)$ on (r,s)-tensors by

$$R(X,Y)T = \nabla_Y \nabla_X T - \nabla_X \nabla_Y T + \nabla_{[X,Y]}T.$$

Remark 4.6. We need to show $R(X,Y)T$ is tensorial in T so that $R(X,Y)$ is a well-defined endomorphism. This is similar to Lemma 4.2.

Proposition 4.7. *For any (r,s)-tensor T, we have the following* Ricci identity*:*

$$\nabla_{Y,X}T - \nabla_{X,Y}T = R(X,Y)T.$$

Moreover, we have a explicit formula

$$\begin{aligned}
&(R(X,Y)T)(\omega_1, \cdots, \omega_r, X_1, \cdots, X_s) \\
={}& -\sum_{i=1}^{r} T(\omega_1, \cdots, R(X,Y)\omega_i, \cdots, \omega_r, X_1, \cdots, X_s) \\
&-\sum_{j=1}^{s} T(\omega_1, \cdots, \omega_r, X_1, \cdots, R(X,Y)X_j, \cdots, X_s)
\end{aligned} \tag{4.3}$$

Proof. Using equation (4.2), we have

$$\begin{aligned}\nabla_{Y,X}T - \nabla_{X,Y}T &= \nabla_Y\nabla_X T - \nabla_{\nabla_Y X}T - \nabla_X\nabla_Y T + \nabla_{\nabla_X Y}T \\ &= \nabla_Y\nabla_X T - \nabla_X\nabla_Y T + \nabla_{[X,Y]}T \\ &= R(X,Y)T,\end{aligned}$$

the second equality is torsion-freeness. Since $R(X,Y)$ clearly satisfies Leibniz Law and commutes with contraction, we can deduce the formula in the same way as covariant derivative. ☐

Ricci identity shows the curvature appears when we interchange the second covariant differential.

Properties of Curvature

Proposition 4.8. *Riemann curvature tensor has following symmetric properties: For* $X, Y, Z, W \in \Gamma(TM)$, *we have*

(1) $R(X,Y,Z,W) = -R(Y,X,Z,W) = -R(X,Y,W,Z)$;

(2) $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$ *(First Bianchi identity);*

(3) $R(X,Y,Z,W) = R(Z,W,X,Y)$;

(4) $(\nabla_X R)(Y,Z) + (\nabla_Y R)(Z,X) + (\nabla_Z R)(X,Y) = 0$ *(Second Bianchi identity).*

Proof. (1) The first equality is evident. We show the second equality. Consider the Hessian of $g(Z,W)$, then we have

$$\nabla^2 g(Z,W)(X,Y) = \langle \nabla^2_{Y,X}Z, W\rangle + \langle \nabla_X Z, \nabla_Y W\rangle + \langle \nabla_Y Z, \nabla_X W\rangle + \langle Z, \nabla^2_{Y,X}W\rangle.$$

Interchange X, Y, we have

$$\nabla^2 g(Z,W)(Y,X) = \langle \nabla^2_{X,Y}Z, W\rangle + \langle \nabla_Y Z, \nabla_X W\rangle + \langle \nabla_X Z, \nabla_Y W\rangle + \langle Z, \nabla^2_{X,Y}W\rangle.$$

These two equations must equal, hence

$$\langle \nabla^2_{Y,X}Z, W\rangle - \langle \nabla^2_{X,Y}Z, W\rangle = \langle Z, \nabla^2_{X,Y}W\rangle - \langle Z, \nabla^2_{Y,X}W\rangle,$$

this is equivalent to

$$R(X,Y,Z,W) = R(Y,X,W,Z).$$

(2) Since $R(X,Y)Z$ is tensorial, we can assume X, Y, Z are frames. Then all Lie bracket between X, Y, Z vanish, we have

$$\sum_{\text{cyc}} R(X,Y)Z = \nabla_Y\nabla_X Z - \nabla_X\nabla_Y Z + \nabla_Z\nabla_Y X - \nabla_Y\nabla_Z X$$

$$
\begin{aligned}
&+\nabla_X\nabla_Z Y-\nabla_Z\nabla_X Y\\
=&\nabla_Y[X,Z]+\nabla_Z[Y,X]+\nabla_X[Z,Y]\\
=&0.
\end{aligned}
$$

(3) By (1) and (2) we have

$$
\begin{aligned}
R(X,Y,Z,W)=&-R(Z,X,Y,W)-R(Y,Z,X,W)\\
=&R(Z,X,W,Y)+R(Y,Z,W,X)\\
=&-R(W,Z,X,Y)-R(X,W,Z,Y)\\
&-R(W,Y,Z,X)-R(Z,W,Y,X)\\
=&2R(Z,W,X,Y)+R(X,W,Y,Z)+R(W,Y,X,Z)\\
=&2R(Z,W,X,Y)-R(Y,X,W,Z)\\
=&2R(Z,W,X,Y)-R(X,Y,Z,W),
\end{aligned}
$$

which implies $2R(X,Y,Z,W)=2R(Z,W,X,Y)$.
(4) By the definition of covariant derivative, we have

$$
\begin{aligned}
(\nabla_X R)(Y,Z)=&[\nabla_X,R(Y,Z)]-R(\nabla_X Y,Z)-R(Y,\nabla_X Z)\\
=&[\nabla_X,\nabla_{[Y,Z]}]-[\nabla_X,[\nabla_Y,\nabla_Z]]-R(\nabla_X Y,Z)-R(Y,\nabla_X Z)\\
=&\nabla_{[X,[Y,Z]]}-[\nabla_X,[\nabla_Y,\nabla_Z]]-R(X,[Y,Z])\\
&-R(\nabla_X Y,Z)-R(Y,\nabla_X Z).
\end{aligned}
$$

Take summation cyclically, we see that terms involving ∇ vanish because of Jacobi identity. Moreover, since $\nabla_X Y-\nabla_Y X=[X,Y]$, the terms involving R also vanish. □

4.2 Sectional, Ricci and Scalar Curvature

We now define some special curvatures. Fix a Riemannian manifold (M,g).

Definition. Let $p\in M$, $\pi\subset T_pM$ be a 2-plane, $\pi=\mathrm{Span}\{u,v\}$. Then define the *sectional curvature* of π at p to be

$$
\mathrm{Sect}_p(\pi)=\frac{R_p(u,v,u,v)}{|u|^2|v|^2-\langle u,v\rangle^2}.
$$

Remark 4.9. One can show that sectional curvature does not depend on the choice of basis. For a proof, see [2, Proposition 3.1].

Proposition 4.10. *The sectional curvature determines the curvature tensor.*

Proof. Let R, R' have same sectional curvature, denote $\tilde{R} = R - R'$, then $\widetilde{\mathrm{Sect}} = 0$. We show that $\tilde{R} = 0$. First, we have

$$\begin{aligned}\tilde{R}(X,Y,X,W) &= \tilde{R}(X,Y-W,X,W) + \tilde{R}(X,W,X,W)\\ &= \tilde{R}(X,Y-W,X,W-Y) + \tilde{R}(X,Y-W,X,Y)\\ &= \tilde{R}(X,Y,X,Y) - \tilde{R}(X,W,X,Y)\\ &= \tilde{R}(X,Y,X,W),\end{aligned}$$

hence $\tilde{R}(X,Y,X,W) = 0$. Then we have

$$\begin{aligned}\tilde{R}(X,Y,Z,W) &= \tilde{R}(X,Y,Z,W-X) + \tilde{R}(X,Y,Z,X)\\ &= \tilde{R}(X-W,Y,Z,W-X) + \tilde{R}(W,Y,Z,W-X)\\ &= \tilde{R}(W,Y,Z,W) - \tilde{R}(W,Y,Z,X)\\ &= -\tilde{R}(W,Y,Z,X).\end{aligned}$$

By the same reasoning, we have

$$\begin{aligned}\tilde{R}(X,Y,Z,W) &= \tilde{R}(Z,W,X,Y)\\ &= -\tilde{R}(Y,W,X,Z)\\ &= -\tilde{R}(X,Z,Y,W).\end{aligned}$$

Passing the result to the $(1,3)$-tensor, by first Bianchi identity, we have

$$\tilde{R}(X,Z)Y + \tilde{R}(Y,Z)X + \tilde{R}(Z,X)Y = 0,$$

which implies $\tilde{R}(Y,Z)X = 0$. Then

$$\begin{aligned}0 &= \tilde{R}(Y,Z,X,W)\\ &= -\tilde{R}(W,Z,X,Y)\\ &= -\tilde{R}(X,Y,W,Z)\\ &= \tilde{R}(X,Y,Z,W).\end{aligned}$$

Thus we proved $\tilde{R} = 0$, and the result follows. □

We mention here a little observation.

Lemma 4.11. *If* Sect_p *is constant for all* 2*-planes in* T_pM*, say* K_p*, then we have*

$$R_p(X,Y,Z,W) = K_p(\langle X,Z\rangle\langle Y,W\rangle - \langle X,W\rangle\langle Y,Z\rangle).$$

Then we can prove the following Schur's Theorem.

Theorem 4.12 (Schur). *Let (M^n, g) be a Riemannian manifold with $n \geq 3$. If $\mathrm{Sect}_p(\pi)$ is independent from $\pi \subset T_pM$ for all $p \in M$, then M has constant sectional curvature.*

Proof. Since the tensor

$$R'(X,Y,Z,W) = \langle X,Z\rangle\langle Y,W\rangle - \langle X,W\rangle\langle Y,Z\rangle$$

evidently satisfies the Proposition 4.8, Proposition 4.10 shows $R = fR'$ for some $f \in C^\infty(M)$. We show that f is constant. Since $n \geq 3$, there exists three orthonormal vectors X, Y, Z. Take W arbitrary, then by second Bianchi identity, we have

$$(\nabla_X R)(Y,Z,W) + (\nabla_Y R)(Z,X,W) + (\nabla_Z R)(X,Y,W) = 0.$$

Since $\nabla_X R = \nabla_X(fR') = (Xf)R' + f(\nabla_X R')$, by taking summation cyclically we obtain

$$(Xf)R'(Y,Z)W + (Yf)R'(Z,X)W + (Zf)R'(X,Y)W = 0.$$

Since the sectional curvature is constant for any $\pi \subset T_pM$, the "sectional curvature" corresponding to R' is also constant, hence we can use Lemma 4.11 to obtain

$$\begin{aligned} 0 = &(Xf)K(\langle Y,W\rangle Z - \langle Z,W\rangle Y) \\ &+ (Yf)K(\langle Z,W\rangle X - \langle X,W\rangle Z) \\ &+ (Zf)K(\langle X,W\rangle Y - \langle Y,W\rangle X), \end{aligned}$$

which is equivalent to

$$\begin{aligned} 0 = &((Xf)\langle Y,W\rangle - (Yf)\langle X,W\rangle)Z \\ &+ ((Yf)\langle Z,W\rangle - (Zf)\langle Y,W\rangle)X \\ &+ ((Zf)\langle X,W\rangle - (Xf)\langle Z,W\rangle)Y. \end{aligned}$$

Since X, Y, Z are orthonormal, the coefficient of X, Y, Z must all equal to 0. Thus by taking $W = Y$, we obtain

$$Xf = (Xf)\langle Y,Y\rangle = (Yf)\langle X,Y\rangle = 0.$$

Since X is arbitrary, we have $f \equiv \text{const}$. This deduces M has constant sectional curvature. $\square$

Definition. Let $R_{ijk}{}^l$ be the local expression of Riemann curvature tensor, then we define the *Ricci curvature* by

$$\mathrm{Ric}_j{}^l = \mathrm{tr}_{13}\, R_{ijk}{}^l.$$

Equivalently, let $\{e_i\}$ be an orthonormal basis at p, then

$$\mathrm{Ric}_p(X) = \sum_{i=1}^{n} R(e_i, X)e_i.$$

Clearly, Ric is a self-adjoint linear transformation by Proposition 4.8 (3).

Definition. We define the *scalar curvature* by taking trace of Ric, or equivalently

$$\mathrm{Scal}(p) = \sum_{i=1}^{n} \langle \mathrm{Ric}_p(e_i), e_i \rangle$$

for an orthonormal basis $\{e_i\}$ at p.

Definition. A Riemannian manifold (M, g) is called an *Einstein manifold* if $\mathrm{Ric} = \lambda\delta$ for some $\lambda \in C^\infty(M)$.

We have another theorem of Schur on Einstein manifolds.

Theorem 4.13 (Schur). *Let M^n be an Einstein manifold with $n \geq 3$, then M has constant scalar curvature.*

First we need a lemma.

Lemma 4.14. *For any metric g, we have*

$$2\,\mathrm{tr}_{13} \nabla \mathrm{Ric} = \mathrm{d}\,\mathrm{Scal}\,.$$

Proof. We check the equation locally. The second Bianchi identity can be written as

$$R_{ijm}{}^{l}{}_{;k} + R_{kim}{}^{l}{}_{;j} + R_{jkm}{}^{l}{}_{;i} = 0,$$

or equivalently

$$R_{ijm}{}^{l}{}_{;k} - R_{ikm}{}^{l}{}_{;j} + R_{jkm}{}^{l}{}_{;i} = 0.$$

Multiply g^{im} we obtain

$$0 = \mathrm{Ric}_j{}^{l}{}_{;k} - \mathrm{Ric}_k{}^{l}{}_{;j} + R_{jk}{}^{il}{}_{;i}.$$

Let $l = j$, we obtain

$$\begin{aligned} 0 &= \mathrm{Ric}_j{}^{j}{}_{;k} - \mathrm{Ric}_k{}^{j}{}_{;j} - R_{kj}{}^{ij}{}_{;i} \\ &= \mathrm{Scal}_{;k} - \mathrm{Ric}_k{}^{j}{}_{;j} - \mathrm{Ric}_k{}^{i}{}_{;i} \\ &= \mathrm{Scal}_{;k} - 2\mathrm{Ric}_k{}^{i}{}_{;i}. \end{aligned}$$

Then the result follows. □

Proof of Theorem 4.13. Let $\mathrm{Ric} = \lambda\delta$, then by the lemma,

$$\begin{aligned} \mathrm{d}\,\mathrm{Scal} &= 2\,\mathrm{tr}_{1,3}\nabla\mathrm{Ric} \\ &= 2\,\mathrm{tr}_{1,3}\nabla(\lambda\delta) \\ &= 2\,\mathrm{tr}_{1,3}(\delta\otimes\mathrm{d}\lambda) \\ &= 2\,\mathrm{d}\lambda. \end{aligned}$$

However, we have

$$\begin{aligned} \mathrm{d}\,\mathrm{Scal} &= \mathrm{d}\,\mathrm{tr}_{1,2}\mathrm{Ric} \\ &= \mathrm{d}\lambda\,\mathrm{tr}_{1,2}\delta \\ &= n\,\mathrm{d}\lambda. \end{aligned}$$

This means $(n-2)\,\mathrm{d}\,\mathrm{Scal} = 0$, which implies $\mathrm{Scal} \equiv \mathrm{const}$ for $n \geq 3$. □

4.3 Jacobi Fields

At the beginning of Section 4.1, we introduced the one parameter family of geodesics $\gamma(t,s) = \exp_p(t(v+s\xi))$. It is a variation of curve $\gamma(t) = \gamma(t,0)$, and its variation vector field J satisfies (4.1). Motivated by this, we give the definition of Jacobi fields.

Definition. Let γ be a geodesic, a vector field J along γ is called a *Jacobi field*, if $\ddot{J}(t) + R(\dot\gamma(t), J(t))\dot\gamma(t) = 0$.

Let $\{e_i(t)\}$ be a parallel frame along γ, and $J(t) = f^i e_i$. Define $a^i_j(t) = R(\dot\gamma(t), e_i(t), \dot\gamma(t), e_j(t))$, then the equation of Jacobi field is equivalent to

$$\ddot{f}^i(t) + a^i_j(t) f^j(t) = 0,\ i = 1, \cdots, n.$$

Hence (4.1) is indeed a system of ODE. By ODE theory, given $f^i(0), \dot{f}^i(0)$ $i = 1, \cdots, n$, the $f^i(t)$'s are uniquely determined. Translate into geometric language, a Jacobi field J is determined by $J(0)$ and $\dot{J}(0)$.

We summarize above discussion.

Proposition 4.15. *Let $\mathscr{J}\gamma$ be the vector space of all Jacobi fields along γ, then*

$$\dim \mathscr{J}(\gamma) = 2n.$$

The proposition below shows only Jacobi fields that perpendicular to γ are interesting.

Proposition 4.16. *Let γ be a geodesic, J is a Jacobi field along γ. Then we have the decomposition*

$$J(t) = J^{\perp}(t) + (at+b)\dot{\gamma}(t),$$

where $J^{\perp}(t) \perp \dot{\gamma}(t)$. If a Jacobi field is perpendicular to γ, then we call it a normal Jacobi field.

Proof. We have

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\langle J(t), \dot{\gamma}(t)\rangle &= \langle \ddot{J}(t), \dot{\gamma}(t)\rangle \\ &= -\langle R(\dot{\gamma}(t), J(t))\dot{\gamma}(t), \dot{\gamma}(t)\rangle \\ &= 0.\end{aligned}$$

□

At the beginning of Section 4.1, we see that a one parameter family of geodesics gives rise to a Jacobi field. The converse is also true.

Proposition 4.17. *Let J be a Jacobi field, then J is the variation field of some one parameter family of geodesics.*

Proof. Given a Jacobi field J, it is determined by $J(0)$ and $\dot{J}(0)$. Let $\zeta(s)$ be the geodesic with initial tangent vector $J(0)$, and $T(s)$ be its tangent vector field. Let $W(s)$ be parallel along ζ and $W(0) = \dot{J}(0)$. Now set $\gamma(t,s) = \exp_{\zeta(s)}(t(T(s) + sW(s)))$. Then

$$\left[\frac{\partial}{\partial t}, \frac{\partial}{\partial s}\right] = \gamma_*[\partial_t, \partial_s] = 0,$$

hence we can interchange the partial derivative. Let U be the variation field of $\gamma(t,s)$, then U is a Jacobi field. We verify U and J has same initial value. We have

$$\begin{aligned}U(0) &= \left.\frac{\partial}{\partial s}\right|_{s=0} \gamma(0,s) \\ &= \left.\frac{\partial}{\partial s}\right|_{s=0} \exp_{\zeta(s)}(0) \\ &= \left.\frac{\partial}{\partial s}\right|_{s=0} \zeta(s) \\ &= \dot{\zeta}(0) = J(0),\end{aligned}$$

and

$$\dot{U}(0) = \left.\frac{\partial}{\partial t}\right|_{t=0} \left.\frac{\partial}{\partial s}\right|_{s=0} \exp_{\zeta(s)}(t(T(s) + sW(s)))$$

$$\begin{aligned}
&= \frac{\partial}{\partial s}\bigg|_{t=0} \frac{\partial}{\partial t}\bigg|_{s=0} \exp_{\zeta(s)}(t(T(s)+sW(s))) \\
&= \frac{\partial}{\partial s}\bigg|_{s=0} \exp_{\zeta_s *}|_0(T(s)+sW(s)) \\
&= \frac{\partial}{\partial s}\bigg|_{s=0} (T(s)+sW(s)) \\
&= W(0) = \dot{J}(0).
\end{aligned}$$

□

Now we can completely answer the calculation problem of differential of exponential map.

Proposition 4.18. *Let a Jacobi field J along $\gamma(t) = \exp_p(tv)$ satisfy $J(0) = 0$, $\dot{J}(0) = \xi$, then $J(t) = \exp_{p*}|_{tv}(t\xi)$.*

Proof. Let J be the variation vector field of $\gamma(t,s) = \exp_p(t(v+s\xi))$. Then

$$\begin{aligned}
J(t) &= \frac{\partial}{\partial s}\exp_p(t(v+s\xi)) \\
&= \exp_{p*}|_{tv}(t\xi).
\end{aligned}$$

Now consider the initial condition for J, we have

$$J(0) = \exp_{p*}|_0(0) = 0,$$

and

$$\begin{aligned}
\dot{J}(0) &= \frac{\partial}{\partial t}\bigg|_{t=0} \frac{\partial}{\partial s}\bigg|_{s=0} \exp_p(t(v+s\xi)) \\
&= \dot{J}(0) = \frac{\partial}{\partial s}\bigg|_{s=0} \frac{\partial}{\partial t}\bigg|_{t=0} \exp_p(t(v+s\xi)) \\
&= \frac{\partial}{\partial s}\bigg|_{s=0} \exp_{p*}|_0(v+s\xi) \\
&= \xi.
\end{aligned}$$

Then the conclusion follows by the uniqueness of Jacobi fields. □

Proposition 4.19 (Gauss Lemma)**.** $\langle \exp_{p*}|_v(\xi), \dot{\gamma}_v(1)\rangle = \langle \xi, v\rangle$.

Proof. Let J be a Jacobi field along γ_v with $J(0) = 0$, $\dot{J}(0) = \xi$. Then by Proposition 4.18, we have

$$\langle \exp_{p*}|_v(\xi), \dot{\gamma}_v(1)\rangle = \langle J(1), \dot{\gamma}_v(1)\rangle.$$

We differentiate above inner product, obtaining

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle J(t), \dot\gamma_v(t)\rangle = \langle \dot J(t), \dot\gamma_v(t), \rangle$$

and

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle \dot J(t), \dot\gamma_v(t)\rangle &= \langle \ddot J(t), \dot\gamma_v(t)\rangle \\ &= -\langle R(\dot\gamma_v, J)\dot\gamma_v, \dot\gamma_v\rangle \\ &= 0.\end{aligned}$$

Hence $\langle \dot J(t), \dot\gamma_v(t)\rangle$ is constant. Clearly $\langle \dot J(0), \dot\gamma_v(0)\rangle = \langle \xi, v\rangle$, then we have

$$\begin{aligned}\langle J(1), \dot\gamma_v(1)\rangle - \langle J(0), \dot\gamma_v(0)\rangle &= \int_0^1 \langle \xi, v\rangle \,\mathrm{d}t \\ &= \langle \xi, v\rangle.\end{aligned}$$

Notice that $\langle J(0), \dot\gamma_v(0)\rangle = 0$ and we obtain the result. □

Finally we consider the case where $\exp_{p*}$ degenerates.

Definition. Let γ be a geodesic starting at p, if $\exp_{p*}|_{\gamma(t_0)}$ is degenerate at $\gamma(t_0)$, then $\gamma(t_0)$ is called a *conjugate point* of p.

A simple criterion for conjugate points is

Proposition 4.20. $\exp_{p*}$ *degenerates at* $\gamma(t_0)$ *if and only if there is a nontrivial Jacobi field* J *with* $J(0) = J(t_0) = 0$.

Proof. $\exp_{p*}|_{\gamma(t_0)}(\xi) = 0$ if and only if Jacobi field $J(t) = \frac{\partial}{\partial s}\exp_p(t(v + s\xi))$ satisfies $J(0) = J(t_0) = 0$. □

Remark 4.21. This proposition shows conjugate is symmetric.

4.4 Index Form

Using the first variation of energy, we conclude that a length-minimizing curve must be a geodesic. However, the converse is generally not true. For instance, a geodesic on sphere starting from the north pole and crossing the south pole is not length-minimizing. We need to find a sufficient condition for a geodesic to be length-minimizing. Calculus hints to us that we should differentiate energy twice. This lead to the second variation formula.

Proposition 4.22 (Second variation formula)**.** *Let $\gamma : [0,a] \times (-\varepsilon, \varepsilon)$ be a variation, and $\gamma_0(t) = \gamma(t,0)$ is a geodesic, then*

$$E''(0) = \int_0^a |\dot{V}(t)|^2 - \langle R(\dot{\gamma}_0(t), V(t))\dot{\gamma}_0(t), V(t)\rangle \,\mathrm{d}t + \boxed{\langle \nabla_{V(t)} V(t), \dot{\gamma}_0(t)\rangle|_0^a}.$$

The boxed term is called boundary term*, and it vanishes when the variation is proper.*

Proof. We take the expression of $E'(s)$ in the first variation formula and differentiate, we have

$$\begin{aligned} E''(s) &= \int_0^a \frac{\partial}{\partial s}\left\langle \frac{\partial \gamma}{\partial t}, \nabla_{\frac{\partial\gamma}{\partial s}} \frac{\partial \gamma}{\partial t}\right\rangle \mathrm{d}t \\ &= \int_0^a \left\langle \nabla_{\frac{\partial\gamma}{\partial s}} \frac{\partial \gamma}{\partial t}, \nabla_{\frac{\partial\gamma}{\partial s}} \frac{\partial \gamma}{\partial t}\right\rangle + \left\langle \frac{\partial \gamma}{\partial t}, \nabla_{\frac{\partial\gamma}{\partial s}} \nabla_{\frac{\partial\gamma}{\partial s}} \frac{\partial \gamma}{\partial t}\right\rangle \mathrm{d}t, \end{aligned}$$

where

$$\left\langle \nabla_{\frac{\partial\gamma}{\partial s}} \frac{\partial \gamma}{\partial t}, \nabla_{\frac{\partial\gamma}{\partial s}} \frac{\partial \gamma}{\partial t}\right\rangle = \left\langle \nabla_{\frac{\partial\gamma}{\partial t}} \frac{\partial \gamma}{\partial s}, \nabla_{\frac{\partial\gamma}{\partial t}} \frac{\partial \gamma}{\partial s}\right\rangle = |\dot{V}(t)|^2$$

when $s = 0$, and

$$\nabla_{\frac{\partial\gamma}{\partial s}} \nabla_{\frac{\partial\gamma}{\partial s}} \frac{\partial \gamma}{\partial t} = \nabla_{\frac{\partial\gamma}{\partial s}} \nabla_{\frac{\partial\gamma}{\partial t}} \frac{\partial \gamma}{\partial s} = \nabla_{\frac{\partial\gamma}{\partial t}} \nabla_{\frac{\partial\gamma}{\partial t}} \frac{\partial \gamma}{\partial s} - R\left(\frac{\partial \gamma}{\partial s}, \frac{\partial \gamma}{\partial t}\right)\frac{\partial \gamma}{\partial s}.$$

Thus when $s = 0$, we have

$$E''(0) = \int_0^a |\dot{V}(t)|^2 - \langle R(\dot{\gamma}_0, V)\dot{\gamma}_0, V\rangle + \frac{\partial}{\partial t}\langle \dot{\gamma}_0, \nabla_V V\rangle - \langle \nabla_{\dot{\gamma}_0}\dot{\gamma}_0, \nabla_V V\rangle,$$

the last term is 0 since γ_0 is a geodesic, and we obtain the formula. □

Second variation formula gives rise to a symmetric bilinear form on the vector space of vector fields along a geodesic γ, which we will call it index form, to deduce whether a geodesic is locally length minimizing.

Definition. Let $\gamma : [0,a] \to M$ be a geodesic. The *index form* of γ is a bilinear form on $\Gamma(TM, \gamma)$ defined by

$$I(X,Y) = \int_0^a \langle \dot{X}, \dot{Y}\rangle - \langle R(\dot{\gamma}, X)\dot{\gamma}, Y\rangle \,\mathrm{d}t.$$

Clearly index form is symmetric.

Lemma 4.23. *Let $U \in \Gamma_0(TM,\gamma) = \{Y \in \Gamma(TM,\gamma) : \ Y(0) = Y(a) = 0\}$, then U is a Jacobi field if and only if $I(U,Y) = 0$ for any $Y \in \Gamma_0(TM,\gamma)$.*

Proof. First we observe that by integrate by parts, we have

$$I(U,Y) = -\int_0^a \langle \ddot{U} + R(\dot\gamma, U)\dot\gamma, Y\rangle.$$

If U is a Jacobi field, then $\ddot{U} + R(\dot\gamma, U)\dot\gamma = 0$, which implies $I(U,Y) = 0$. Conversely, if $I(U,Y) = 0$ for any $Y \in \Gamma_0(TM,\gamma)$, choose a smooth function f with $f(0) = f(a) = 0$ and positive elsewhere, let $Y = f(t)(\ddot{U} + R(\dot\gamma, U)\dot\gamma)$, then

$$0 = -f\int_0^a |\ddot{U} + R(\dot\gamma, U)\dot\gamma|^2 \,\mathrm{d}t,$$

this implies $\ddot{U} + R(\dot\gamma, U)\dot\gamma = 0$, that is, U is a Jacobi field. $\square$

We observe if a Jacobi field J does not vanish at the starting or ending point of the geodesic, the above integration by parts is changed into

$$\begin{aligned} I(Y,J) &= \langle Y, \dot{J}\rangle|_0^a - \int_0^a \langle \ddot{J} + R(\dot\gamma, J)\dot\gamma, Y\rangle \,\mathrm{d}t \\ &= \langle Y, \dot{J}\rangle|_0^a. \end{aligned}$$

The next proposition shows the positive definiteness of I is related to conjugate points.

Proposition 4.24. *Let $\gamma : [0,a] \to M$ be a geodesic, I be its index form.*

(1) If γ contains no conjugate points of $\gamma(0)$, then I is positive definite on $\Gamma_0(TM,\gamma)$.

(2) If $\gamma(a)$ is the only conjugate point of $\gamma(0)$, then I is positive semidefinite but not positive definite.

(3) If there is a $t_0 < a$ such that $\gamma(t_0)$ is conjugate to $\gamma(0)$, then there is a $U \in \Gamma_0(TM,\gamma)$ such that $I(U,U) < 0$.

Proof of Proposition 4.24 (1). Let $\gamma(0) = p$, $\tilde\gamma$ is the radial line in T_pM defined by $\tilde\gamma(t) = t\dot\gamma(0)$. Since γ contains no conjugate points of $\gamma(0)$, $\exp_p$ is nondegenerate on the whole $\tilde\gamma$. Hence there is a neighborhood U of $\tilde\gamma([0,a])$ such that $\exp_p : U \to M$ is an immersion. By Proposition 3.21 and the subsequent remark, γ is length-minimizing. Hence by Proposition 3.5, γ also minimizes energy. Let $\gamma(t,s) : [0,a] \times (-\varepsilon, \varepsilon) \to M$ be any proper variation, by taking ε small enough, we can assume the variation is contained in U. Then we have

$$E''(0) = \lim_{s\to 0} \frac{E(-s) + E(s) - 2E(0)}{s^2} \geq 0.$$

Since $E''(0)(V) = I(V,V)$ for any variation vector field, we have $I(V,V) \geq 0$ for any $V \in \Gamma_0(TM,\gamma)$. Now we must show $I(V,V) = 0$ implies $V = 0$. Let $I(V,V) = 0$, $X \in \Gamma_0(TM,\gamma)$ and $\delta > 0$, we have

$$0 \leq I(V+\delta X, V+\delta X) = I(V,V) + 2\delta I(V,X) + \delta^2 I(X,X),$$

this implies

$$2I(V,X) + \delta I(X,X) \geq 0.$$

Let $\delta \to 0$, we obtain $I(V,X) \geq 0$ for any $X \in \Gamma_0(TM,\gamma)$. Similarly, consider $I(V-\delta X, V-\delta X)$, we have $I(V,X) \leq 0$ for any $X \in \Gamma_0(TM,\gamma)$. This means $I(V,X) = 0$ for any $X \in \Gamma_0(TM,\gamma)$. Now by Lemma 4.23, V is a Jacobi field. But γ has no conjugate points of $\gamma(0)$, V must identically equal to 0. This proves I is positive definite. □

Before proving the rest of Theorem 4.24, we need the following *Index Lemma*.

Proposition 4.25 (Index Lemma). *Assume $\gamma : [0,a] \to M$ is a geodesic without conjugate points. Let $U \in \Gamma(TM,\gamma)$ with $U(0) = 0$, J be a Jacobi field such that $J(0) = 0$, $J(a) = U(a)$. Then $I(J,J) \leq I(U,U)$, the equality holds if and only if $U = J$.*

Proof. Since $U - J \in \Gamma_0(TM,\gamma)$, by Theorem 4.24 (1), we have $I(U-J,U-J) \geq 0$ with equality holds iff $U = J$. Then we have

$$0 \leq I(U-J,U-J) = I(U,U) - 2I(U,J) + I(J,J).$$

But $I(U,J) = \langle U, \dot{J}\rangle|_0^a = \langle J, \dot{J}\rangle|_0^a = I(J,J)$, thus

$$I(U,U) \geq I(J,J).$$ □

Proof of the rest of Proposition 4.24. (2) For any $0 < t_0 < a$, let $I_{[0,t_0]}$ denote the index form of $\gamma|_{[0,t_0]}$, i.e.,

$$I_{[0,t_0]}(X,Y) = \int_0^{t_0} \langle \dot{X}, \dot{Y}\rangle + \langle R(\dot{\gamma},X)\dot{\gamma}, Y\rangle \, dt.$$

Then for $X \in \Gamma_0(TM,\gamma|_{[0,t_0]})$, by (1) we have $I_{[0,t_0]}(X,X) \geq 0$. We now construct $\tau_{t_0}(U)$ for any $U \in \Gamma_0(TM,\gamma)$. Let $\{e_i(t)\}$ be a parallel frame along γ, $U = f^i(t)e_i(t)$. Then we define

$$\tau_{t_0}(U)(t) = f^i\left(\frac{a}{t_0}t\right) e_i\left(\frac{a}{t_0}t\right) \in \Gamma_0(TM,\gamma|_{[0,t_0]}).$$

Thus we have

$$I_{[0,t_0]}(\tau_{t_0}(U), \tau_{t_0}(U)) \geq 0,$$

let $t_0 \to a$ we obtain the conclusion. Moreover, since $\gamma(a)$ is conjugate to $\gamma(0)$, there exists a nontrivial Jacobi field J with $J(0) = J(a) = 0$, and $I(J,J) = \langle J, \dot{J}\rangle|_0^a = 0$. This shows I is positive semidefinite but not positive definite.
(3) Since $\gamma(0)$ is conjugate to $\gamma(t_0)$, there exists a Jacobi field $\tilde{J}_1$ along $\gamma|_{[0,t_0]}$ such that $\tilde{J}_1(0) = \tilde{J}_1(t_0) = 0$. Choose a smooth $m_1 : [0,a] \to \mathbb{R}$ with $\operatorname{supp} m_1 = [0,t_0]$, and set $J_1 = m_1\tilde{J}_1$. Let

$$V(t) = \begin{cases} J_1(t), & 0 \le t \le t_0, \\ 0, & t_0 \le t \le a, \end{cases}$$

then $V \in \Gamma(TM,\gamma)$ is smooth. Let $\delta > 0$ so small that $\gamma_{[t_0-\delta,t_0+\delta]}$ is contained in a normal neighborhood of $\gamma(t_0)$. Then there exists a Jacobi field $\tilde{J}_2$ along $\gamma|_{[t_0-\delta,t_0+\delta]}$ with $\tilde{J}_2(t_0-\delta) = J_1(t_0-\delta)$, $\tilde{J}_2(t_0+\delta) = 0$. Let $m_2 : [t_0-\delta, t_0+\delta]$ satisfy

$$\dot{J}_1 = \nabla_{\dot{\gamma}} m_2 \tilde{J}_2 = \frac{\mathrm{d}m_2}{\mathrm{d}t}\tilde{J}_2 + m_2\dot{\tilde{J}}_2$$

and

$$R(\dot{\gamma}(t_0-\delta), J_1(t_0-\delta))\dot{\gamma}(t_0-\delta) = m_2(t_0-\delta)R(\dot{\gamma}(t_0-\delta), \tilde{J}_2(t_0-\delta))\dot{\gamma}(t_0-\delta),$$

then by ODE theory, m_2 is the solution of a linear ODE of order 1, hence it exists and is smooth. Now set $J_2 = m_2\tilde{J}_2$, and

$$U(t) = \begin{cases} J_1(t), & 0 \le t \le t_0-\delta, \\ J_2(t), & t_0-\delta \le t \le t_0+\delta, \\ 0, & t_0+\delta \le t \le a, \end{cases}$$

then up to a third mollification, $U(t) \in \Gamma(TM,\gamma)$. Therefore by Index Lemma, we have

$$\begin{aligned} I(U,U) &= I_{[0,t_0-\delta]}(J_1,J_1) + I_{[t_0-\delta,t_0+\delta]}(J_2,J_2) + 0 \\ &< I_{[0,t_0-\delta]}(V,V) + I_{[t_0-\delta,t_0+\delta]}(V,V) \\ &= I_{[0,t_0]}(V,V) + I_{[t_0,a]}(V,V) \\ &= \langle J_1, \dot{J}_1\rangle|_0^{t_0} + 0 \\ &= 0. \end{aligned}$$

□

Translating Proposition 4.24 to the language of geometry, we have the following theorem.

Theorem 4.26 (Jacobi). *Let $\gamma : [0,a] \to M$ be a geodesic, then*

(1) *If γ has no conjugate points of $\gamma(0)$, then γ is length-minimizing under any small proper variation.*

(2) *If there is a $t_0 \in (0,a)$ such that $\gamma(t_0)$ is conjugate to $\gamma(0)$, then there is a proper variation γ_s such that $L[\gamma_s] < L[\gamma]$ for any s.*

4.5 Bochner Formula

We introduce a useful calculation tool. We will later apply it to distance function.

Theorem 4.27 (Bochner formula). *Let M be a Riemannian manifold, $f \in C^\infty(M)$. Then we have*

$$\Delta\frac{1}{2}|\nabla f|^2 = |\nabla^2 f|^2 + \langle\nabla\Delta f, \nabla f\rangle + \langle \mathrm{Ric}(\nabla f), \nabla f\rangle.$$

We first need a lemma.

Lemma 4.28. *Under geodesic coordinate around p, the Ricci identity (Proposition 4.7) of 1-forms is equivalent to*

$$f_{;kij} - f_{;kji} = -f_{;l}R_{ijk}{}^{l},$$

where $f_{;i}$ means $\partial_i f$.

Proof. On the one hand, since all Christoffel symbols vanish at p, we have

$$\begin{aligned}
\nabla_{\partial_i}\nabla_{\partial_j}\,\mathrm{d}f &= \nabla_{\partial_i}\nabla_{\partial_j} f_{;k}\,\mathrm{d}x^k \\
&= \nabla_{\partial_i}(\partial_j f_{;k}\,\mathrm{d}x^k + f_{;k}\nabla_{\partial_j}\mathrm{d}x^k) \\
&= \nabla_{\partial_i}(\partial_j f_{;k}\,\mathrm{d}x^k - f_{;k}\Gamma^k_{jl}\,\mathrm{d}x^l) \\
&= \partial_i\,\partial_j f_{;k}\,\mathrm{d}x^k + \partial_j f_{;k}\nabla_{\partial_i}\mathrm{d}x^k \\
&= f_{;kji} - \partial_j f_{;k}\Gamma^k_{il}\,\mathrm{d}x^l \\
&= f_{;kji}\,\mathrm{d}x^k,
\end{aligned}$$

then we have

$$\begin{aligned}
(R(\partial_i,\partial_j)\,\mathrm{d}f)(\partial_k) &= (\nabla_{\partial_j}\nabla_{\partial_i} - \nabla_{\partial_i}\nabla_{\partial_j} + \nabla_{[\partial_i,\partial_j]})\,\mathrm{d}f(\partial_k) \\
&= f_{;kij} - f_{;kji}.
\end{aligned}$$

On the other hand, we have

$$(R(\partial_i,\partial_j)\,\mathrm{d}f)(\partial_k) = -\,\mathrm{d}f(R(\partial_i,\partial_j)\,\partial_k)$$

$$
\begin{aligned}
&= -\,\mathrm{d}f(R_{ijk}{}^l\,\partial_l) \\
&= -f_{;m}\,\mathrm{d}x^m(R_{ijk}{}^l\,\partial_l) \\
&= -f_{;l}R_{ijk}{}^l,
\end{aligned}
$$

hence the equality holds. □

Proof of Bochner formula. Under geodesic normal coordinate we have the following calculation:

$$
\begin{aligned}
\Delta\frac{1}{2}|\nabla f|^2 &= \sum_{i,j}\left(\frac{1}{2}f_{;j}^2\right)_{;ii} \\
&= \sum_{i,j}(f_{;j}f_{;ji})_{;i} \\
&= \sum_{i,j}(f_{;ji}^2 + f_{;j}f_{;jii}) \\
&= \sum_{i,j}(f_{;ij}^2 + f_{;j}f_{;iji}) \\
&= \sum_{i,j}f_{;ij}^2 + \sum_{i,j}f_{;j}(f_{;iij} - f_{;k}R_{jii}{}^k) \\
&= \sum_{i,j}f_{;ij}^2 + \sum_{i,j}f_{;j}f_{;iij} + \sum_{i,j}f_{;j}f_{;k}R_{iji}{}^k \\
&= \sum_{i,j}f_{;ij}^2 + \sum_{i,j}f_{;j}f_{;iij} + \sum_{j}f_{;j}f_{;k}\mathrm{Ric}_j{}^k \\
&= |\nabla^2 f|^2 + \langle\nabla f, \nabla\Delta f\rangle + \langle\mathrm{Ric}(\nabla f), \nabla f\rangle.
\end{aligned}
$$

□

Chapter 5

Submanifolds

5.1 Second Fundamental Form

We first set up our stage. Let $f : \Sigma^k \to (M^n, g)$ be an immersion, equip Σ with pullback metric f^*g (still denoted by g for simplicity). There is a decomposition $\nabla_X Y = (\nabla_X Y)^\top + (\nabla_X Y)^\perp$ for $X, Y \in \Gamma(T\Sigma)$, and simple observation shows

Proposition 5.1. *Let ∇^Σ be the Levi–Civita connection on (Σ, g), then ∇^Σ is given by*

$$\nabla^\Sigma_X Y = (\nabla_X Y)^\top$$

for $X, Y \in \Gamma(T\Sigma)$.

Now we need the concept of normal bundle.

Definition. Let Σ be a submanifold of M, then for any $p \in M$ we have the decomposition

$$T_p M = T_p\Sigma \oplus N_p\Sigma,$$

where $N_p\Sigma$ is the orthogonal complement of $T_p\Sigma$. Then define the *normal bundle* of Σ to be

$$N\Sigma = \bigsqcup_{p \in M} N_p\Sigma.$$

It is similar to tangent bundle to make $N\Sigma$ into a vector bundle.

Thus we can define the second fundamental form of a submanifold.

Definition. Let Σ be a submanifold of M, then we define the *second fundamental form* of Σ to be

$$\begin{aligned} \mathrm{II} : \Gamma(T\Sigma) \times \Gamma(T\Sigma) &\to \Gamma(N\Sigma) \\ (X, Y) &\mapsto (\nabla_X Y)^\perp. \end{aligned}$$

Lemma 5.2. *The second fundamental form is symmetric.*

Proof. We have

$$\mathrm{II}(X,Y) - \mathrm{II}(Y,X) = (\nabla_X Y)^\perp - (\nabla_Y X)^\perp = ([X,Y])^\perp = 0. \qquad \square$$

Moreover, we can define the shape operator.

Definition. For a fixed $\xi \in \Gamma(N\Sigma)$, we define

$$\begin{aligned} S_\xi : \Gamma(T\Sigma) &\to \Gamma(T\Sigma) \\ X &\mapsto \nabla_X \xi - (\nabla_X \xi)^\perp. \end{aligned}$$

We have the following Weingarten formula.

Proposition 5.3. *For $\xi \in \Gamma(N\Sigma)$ and $X, Y \in \Gamma(T\Sigma)$, we have*

$$\langle S_\xi(X), Y\rangle = \langle \mathrm{II}(X,Y), -\xi\rangle.$$

In particular, shape operator is self-adjoint.

Proof. We have the calculation

$$\begin{aligned} \langle S_\xi(X), Y\rangle &= \langle \nabla_X \xi - (\nabla_X \xi)^\perp, Y\rangle \\ &= \langle \nabla_X \xi, Y\rangle - \langle (\nabla_X \xi)^\perp, Y\rangle \\ &= X\langle \xi, Y\rangle - \langle \xi, \nabla_X Y\rangle \\ &= \langle -\xi, \nabla_X Y\rangle \\ &= \langle -\xi, (\nabla_X Y)^\perp\rangle \\ &= \langle \mathrm{II}(X,Y), -\xi\rangle. \end{aligned} \qquad \square$$

For a two-sided hypersurface $\Sigma^{n-1} \subset M^n$, choose a unit normal vector field N, we can introduce the *second fundamental form tensor* h, which is defined by $h(X,Y) := \langle \mathrm{II}(X,Y), N\rangle$. In other words, $\mathrm{II}(X,Y) = h(X,Y)N$, and the Weingarten formula is $h(X,Y) = -\langle S(X), Y\rangle$.

5.2 Gauss–Codazzi Equations

In this section, our goal is to prove the following Gauss–Codazzi equations on the relation of curvature between a hypersurface and ambient manifold. After this, we give a brief discussion on principal curvatures.

In the following contents, we assume $\Sigma^{n-1} \subset M^{n-1}$, Σ is two-sided. We will use $\mathrm{D}_X Y$ to denote the Levi–Civita connection of M, and $\nabla_X Y$ for Σ. N will denote the unit normal vector field of Σ.

Theorem 5.4. *Under above settings, for $X, Y, Z, W \in \Gamma(T\Sigma)$ and $N \in \Gamma(N\Sigma)$, we have*

- *Gauss equation:*
$$R^\Sigma(X, Y, Z, W) = R^M(X, Y, Z, W) + h(X, Z)h(Y, W) - h(X, W)h(Y, Z);$$
- *Codazzi equation:*
$$R^M(X, Y, Z, N) = (\nabla_Y h)(X, Z) - (\nabla_X h)(Y, Z).$$

Proof. We first calculate $R^\Sigma(X, Y)Z$. We have

$$\begin{aligned}
R^\Sigma(X, Y)Z =& \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z \\
=& \mathrm{D}_Y \nabla_X Z - h(\nabla_X Z, Y)N - \mathrm{D}_X \nabla_Y Z + h(\nabla_Y Z, X)N \\
&+ D_{[X,Y]} Z - h([X, Y], Z)N \\
=& \mathrm{D}_Y \mathrm{D}_X Z - \mathrm{D}_Y h(X, Z)N - h(\nabla_X Z, Y)N \\
&- \mathrm{D}_X \mathrm{D}_Y Z + \mathrm{D}_X h(Y, Z)N + h(\nabla_Y Z, X)N \\
&+ \mathrm{D}_{[X,Y]} Z - h(\nabla_X Y - \nabla_Y X, Z)N \\
=& R^M(X, Y)Z - \mathrm{D}_Y h(X, Z)N + h(\nabla_Y X, Z) + h(X, \nabla_Y Z)N \\
&+ \mathrm{D}_X h(Y, Z)N - h(\nabla_X Y, Z)N - h(Y, \nabla_X Z)N \\
=& R^M(X, Y)Z \\
&- Yh(X, Z)N - h(X, Z)\mathrm{D}_Y N + h(\nabla_Y X, Z) + h(X, \nabla_Y Z)N \\
&+ Xh(Y, Z)N + h(Y, Z)\mathrm{D}_X N - h(\nabla_X Y, Z)N - h(Y, \nabla_X Z)N.
\end{aligned}$$

First we consider $\langle R^\Sigma(X, Y)Z, W\rangle$, by Weingarten formula we have

$$\begin{aligned}
\langle R^\Sigma(X, Y)Z, W\rangle &= \langle R^M(X, Y)Z, W\rangle + \langle h(Y, Z)\mathrm{D}_X N, W\rangle - \langle h(X, Z)\mathrm{D}_Y N, W\rangle \\
&= \langle R^M(X, Y)Z, W\rangle + h(X, Z)h(Y, W) - h(X, W)h(Y, Z).
\end{aligned}$$

Next we consider $\langle R^\Sigma(X, Y)Z, N\rangle$, notice that $\langle D_X N, N\rangle = \frac{1}{2} X\langle N, N\rangle = 0$, we have

$$\begin{aligned}
0 =& \langle R^M(X, Y)Z, N\rangle + Xh(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z) \\
&- Yh(X, Z) + h(\nabla_Y X, Z) + h(X, \nabla_Y Z) \\
=& \langle R^M(X, Y)Z, N\rangle + (\nabla_X h)(Y, Z) - (\nabla_Y h)(X, Z),
\end{aligned}$$

which is equivalent to Codazzi equation. □

We now turn to principal curvatures.

Definition. Let $p \in \Sigma^{n-1} \subset M^n$, the *principal curvatures* at p are the eigenvalues of shape operator S. Usually we denote κ_i, $i = 1, \cdots, n-1$ for principal curvatures.

Principal curvatures are extrinsic geometry quantities, but on 2-dimensional surfaces, their product happens to be intrinsic. This is the *Theorema Egregium*, a glorious achievement of classical differential geometry. At the era of Gauss, who discovered the theorem, Theorema Egregium needs tedious calculation to be proved. But using modern language, Theorema Egregium is almost self-evident. We now state and prove this theorem.

Theorem 5.5 (Theorema Egregium). *On a surface $\Sigma^2 \subset \mathbb{R}^3$, we define* Gaussian curvature $\kappa = \kappa_1\kappa_2$*, the product of principal curvatures. The Gaussian curvature is equal to the sectional curvature on a point, hence Gaussian curvature is intrinsic, i.e. only depends on metric.*

Proof. Since $S^i_j = -g^{ik}h_{kj}$, we have

$$\kappa = \kappa_1\kappa_2 = \det(S^i_j) = \frac{\det(h_{ij})}{\det(g_{ij})}.$$

However, by Gauss equation, we have

$$\begin{aligned}\mathrm{Sect}_p(T_p\Sigma) &= \frac{\langle R(\partial_i, \partial_j)\,\partial_i, \partial_j\rangle}{g_{ii}g_{jj} - g_{ij}^2} \\ &= \frac{1}{\det(g_{ij})}(0 + h_{ii}h_{jj} - h_{ij}^2) \\ &= \kappa.\end{aligned}$$

□

Finally we discuss totally umbilical hypersurfaces.

Definition. A point $p \in \Sigma^{n-1} \subset M^n$ is called *umbilical* if $h_p = cg_p$ for some $c \in \mathbb{R}$. Σ^{n-1} is called *totally umbilical* if $h = fg$ for some $f \in C^\infty(M)$.

Lemma 5.6. *A point is umbilical if and only if all the principal curvatures at this point are equal.*

Proof. This follows by $S^i_j = -g^{ik}h_{kj} = -fg^{ik}g_{kj} = -f\delta^i_j$. □

Proposition 5.7. *If Σ is a totally umbilical hypersurface in $(\mathbb{R}^n, \delta)$, i.e. $h = f\delta$, then f must be constant.*

Proof. By Codazzi equation, at $p \in \Sigma$, for $u, w, v \in T_p\Sigma$, we have

$$0 = \langle R(u,v)w, n\rangle = (\nabla h)(u,w,v) - (\nabla h)(v,w,u). \tag{5.1}$$

Since $h = f\delta$, we have

$$\nabla h = \nabla(f\delta) = \delta \otimes \mathrm{d}f + f\nabla\delta = \delta \otimes \mathrm{d}f.$$

Hence equation 5.1 is equivalent to

$$0 = \langle u, w\rangle \, \mathrm{d}f_p(v) - \langle v, w\rangle \, \mathrm{d}f|_p(u).$$

Choose v arbitrarily, $w \neq 0$ satisfying $\langle v, w\rangle = 0$ and $u = w$, we have $\mathrm{d}f|_p(v) = 0$. This shows $\mathrm{d}f|_p = 0$. Since p is arbitrary, $\mathrm{d}f = 0$ and this implies $f \equiv$ const. ☐

Remark 5.8. Some authors mistakenly think this proposition holds for general ambient manifold. We have a counterexample as follows. Let's consider the warped product $\mathbb{R} \times \mathbb{R}^2$ with metric $g = (\mathrm{d}x^1)^2 + F^2(x^1)((\mathrm{d}x^2)^2 + (\mathrm{d}x^3)^2)$, where F is a nonconstant function of x^1. If $\{\partial_i\}$ is coordinate vector field, then $N = \partial_1$ is orthonormal to $\Sigma = \mathbb{R}^2$. We have

$$\Gamma^1_{13} = \frac{1}{2} g^{11}(\partial_3 g_{11} + \partial_1 g_{13} - \partial_1 g_{13}) = \frac{\partial_3 F}{F} = \Gamma^2_{23},$$

and

$$\Gamma^2_{13} = \frac{1}{2} g^{22}(\partial_3 g_{12} + \partial_1 g_{23} - \partial_2 g_{13}) = 0.$$

Therefore we have

$$h(\partial_1, \partial_1) = -\langle \nabla_{\partial_1} \partial_3, \partial_1\rangle = -F^2\Gamma^1_{13} = -F\,\partial_3 F = h(\partial_2, \partial_2),$$

and

$$h(\partial_1, \partial_2) = -\langle \nabla_{\partial_1} \partial_3, \partial_2\rangle = -F^2\Gamma^2_{13} = 0.$$

Hence we have

$$h = -\frac{\partial_3 F}{F} g =: fg,$$

f is not constant as F is not constant.

If we assume a priori the classification theorem of constant sectional curvature spaces, we can have the following corollary.

Corollary 5.9. *A complete totally umbilical hypersurface Σ in $(\mathbb{R}^n, \delta)$ is either a hyperplane or a sphere.*

Proof. Let $p \in \Sigma$, $u, v \in T_p\Sigma$, then by Gauss equation, we have

$$\begin{aligned}
\mathrm{Sect}_p(\mathrm{Span}\{u, v\}) &= 0 + \frac{h(u,u)h(v,v) - h^2(u,v)}{\delta(u,u)\delta(v,v) - \delta^2(u,v)} \\
&= f^2 \frac{\delta(u,u)\delta(v,v) - \delta^2(u,v)}{\delta(u,u)\delta(v,v) - \delta^2(u,v)} \\
&= f^2 \\
&\equiv \mathrm{const} \geq 0.
\end{aligned}$$

When the constant is 0, Σ is a hyperplane; when the constant is positive, Σ is a sphere. □

5.3 Minimal Surfaces

We now give a brief introduction to minimal surfaces. We will use the word "area" to denote the volume of a submanifold.

Definition. Let $\Sigma^k \subset M^n$ be a submanifold (maybe with boundary), the *area functional* is defined as

$$A[\Sigma] = \int_\Sigma \mathrm{d}A_\Sigma,$$

where A_Σ is the volume element on Σ.

The guiding problem of this section is

Problem (Generalized Plateau problem). Given a fixed boundary submanifold $B^{k-1} \subset M^n$ ($k < n$), find the area-minimizer in all submanifolds Σ^k with $\partial\Sigma = B$.

We use variation to deduce a necessary condition for a submanifold to be area-minimizing. To describe the first variation formula of area, we introduce the notion of mean curvature.

Definition. Let $\Sigma^k \subset M$, define the *mean curvature vector* at a point $p \in \Sigma$ to be

$$\mathbf{H}_p = \sum_{i=1}^{k} \mathrm{II}_p(e_i, e_i),$$

where $\{e_i\}$ is an orthonormal basis of $T_p\Sigma$. When $k = n-1$, i.e. Σ is a hypersurface, we define *mean curvature* $H = \mathrm{tr}\, h$, the trace of second fundamental form tensor.

Proposition 5.10 (First variation of area). *Let M^n be a Riemannian manifold, Σ^k be a manifold with boundary with $k < n$, $f : \Sigma \times (-\varepsilon, \varepsilon) \to M$ be a family of immersions, denote $\Sigma_t = f(\Sigma, t)$. Suppose $f(p,t) = p$ for $p \in \partial\Sigma$ and $t \in (-\varepsilon, \varepsilon)$, the variation vector field $X := \frac{\partial}{\partial t}\Big|_{t=0} f \in \Gamma(TM, \Sigma_0)$. Then we have*

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} A[\Sigma_t] = -\int_{\Sigma_0} \langle \mathbf{H}, X \rangle \,\mathrm{d}A_{\Sigma_0}.$$

Proof. Denote $g_t = f_t^* g$ and $A_t = A_{\Sigma_t}$. Let

$$\mathscr{J}(p,t) = \frac{\sqrt{\det g_t}}{\sqrt{\det g_0}},$$

then we have $A_t(p) = \mathscr{J}(p,t) A_0(p)$. Now we can calculate (omitting p)

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \mathscr{J}(t) &= \frac{1}{\sqrt{\det g_0}} \cdot \frac{1}{2\sqrt{\det g_0}} \cdot \det g_0 \cdot g_0^{ij} \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} g_{ij} \\
&= \frac{1}{2} g_0^{ij} \left(\left\langle \nabla_{\partial_t} \frac{\partial f}{\partial x^i}, \frac{\partial f}{\partial x^j} \right\rangle + \left\langle \frac{\partial f}{\partial x^i}, \nabla_{\partial_t} \frac{\partial f}{\partial x^j} \right\rangle \right) \\
&= g_0^{ij} \left\langle \nabla_{\partial_t} \frac{\partial f}{\partial x^i}, \frac{\partial f}{\partial x^j} \right\rangle \\
&= g_0^{ij} \left\langle \nabla_{\partial_i} \frac{\partial f}{\partial t}, \frac{\partial f}{\partial x^j} \right\rangle \\
&= g_0^{ij} \left\langle \nabla_{\partial_i} X^\perp, \frac{\partial f}{\partial x^j} \right\rangle + g_0^{ij} \left\langle \nabla_{\partial_i} X^\top, \frac{\partial f}{\partial x^j} \right\rangle \\
&= -\left\langle g_0^{ij} \,\mathrm{II}\left(\frac{\partial f}{\partial x^i}, \frac{\partial f}{\partial x^j} \right), X^\perp \right\rangle + \operatorname{div} X^\top \\
&= -\langle \mathbf{H}, X^\perp \rangle + \operatorname{div} X^\top.
\end{aligned}$$

Notice that $-\langle \mathbf{H}, X^\perp \rangle = -\langle \mathbf{H}, X \rangle$, take integral we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} A[\Sigma_t] = -\int_{\Sigma_0} \langle \mathbf{H}, X \rangle \,\mathrm{d}A_{\Sigma_0} + \int_{\Sigma_0} \operatorname{div} X^\top \,\mathrm{d}A_{\Sigma_0}.$$

Since the variation fixes boundary, we have $X|_{\partial\Sigma_0} = 0$, then by divergence theorem, let N denote the outer unit normal vector field of $\partial\Sigma_0$, we have

$$\int_{\Sigma_0} \operatorname{div} X^\top \,\mathrm{d}A_{\Sigma_0} = \int_{\partial\Sigma_0} \langle X^\top, N \rangle \,\mathrm{d}A_{\partial\Sigma_0} = 0.$$

Hence we proved the first variation formula of area. $\square$

Thus we have a necessary condition for area-minimizing submanifold.

Corollary 5.11. *If Σ is a solution of generalized Plateau problem, then $\mathbf{H} = 0$.*

The corollary gives rise to the notion of minimal submanifolds.

Definition. Let Σ be a submanifold of M, if the mean curvature vector field $\mathbf{H} \equiv 0$, then we call Σ a *minimal submanifold*.

Remark 5.12. We do not demand a minimal submanifold is with boundary or actually minimzes area. But if the minimal submanifold is a graph, then it minimzes area.

Lemma 5.13. *Let $u : \Omega \subset \mathbb{R}^{n-1} \to \mathbb{R}$ be a smooth function, then its graph*

$$G[u] := \{(x^1, \cdots, x^{n-1}, u(x^1, \cdots, x^{n-1}))\}$$

is a minimal surface if and only if it satisfies the following minimal surface equation (MSE)*:*

$$\operatorname{div}_{\mathbb{R}^{n-1}} \left(\frac{\mathrm{D}u}{\sqrt{1 + |\mathrm{D}u|^2}} \right) = 0,$$

where D *is the Euclidean connection, and norm is Euclidean norm.*

Proof. Let

$$\begin{aligned} r : \Omega \subset \mathbb{R}^{n-1} &\to \mathbb{R}^n \\ (x^1, \cdots, x^{n-1}) &\mapsto (x^1, \cdots, x^{n-1}, u(x^1, \cdots, x^{n-1})), \end{aligned}$$

$r(\Omega) = G[u]$ has a unit normal vector field

$$N = \frac{(\mathrm{D}u, -1)}{\sqrt{1 + |\mathrm{D}u|^2}}.$$

Then we have

$$g_{ij} = \delta_{ij} + \frac{\partial u}{\partial x^i} \frac{\partial u}{\partial x^j},$$

or $g = I + \mathrm{D}u(\mathrm{D}u)^t$. Hence we have formally

$$\begin{aligned} (I + \mathrm{D}u(\mathrm{D}u)^t)^{-1} &= I - \mathrm{D}u(\mathrm{D}u)^t + (\mathrm{D}u(\mathrm{D}u)^t)^2 - (\mathrm{D}u(\mathrm{D}u)^t)^3 + \cdots \\ &= I - \mathrm{D}u(\mathrm{D}u)^t(1 - |\mathrm{D}u|^2 + |\mathrm{D}u|^4 + \cdots) \\ &= I - \frac{1}{1 + |\mathrm{D}u|^2} \mathrm{D}u(\mathrm{D}u)^t, \end{aligned}$$

and one can check we indeed obtain the inverse matrix. Moreover, we have second fundamental form tensor

$$h_{ij} = \langle r_{;ij}, N \rangle = -\frac{1}{\sqrt{1+|\mathrm{D}u|^2}} \cdot \frac{\partial^2 u}{\partial x^i \, \partial x^j}.$$

Thus by taking trace, we have

$$\begin{aligned}
H &= g^{ij} h_{ij} \\
&= -\sum_{i,j} \left(\delta_{ij} - \frac{1}{1+|\mathrm{D}u|^2} \frac{\partial u}{\partial x^i} \frac{\partial u}{\partial x^j} \right) \left(\frac{1}{\sqrt{1+|\mathrm{D}u|^2}} \cdot \frac{\partial^2 u}{\partial x^i \, \partial x^j} \right) \\
&= -\sum_{i,j} \frac{\partial_i(\partial_i u)\sqrt{1+|\mathrm{D}u|^2} - \partial_j u \, \partial_i u \, \partial_i(\partial_j u)(1+|\mathrm{D}u|^2)^{-1/2}}{1+|\mathrm{D}u|^2} \\
&= -\sum_{i,j} \frac{\partial}{\partial x^i} \left(\frac{\partial_i u}{\sqrt{1+|\mathrm{D}u|^2}} \right) \\
&= -\operatorname{div}_{\mathbb{R}^{n-1}} \left(\frac{\mathrm{D}u}{\sqrt{1+|\mathrm{D}u|^2}} \right),
\end{aligned}$$

where we used the fact

$$\begin{aligned}
\frac{\partial}{\partial x^i} \sqrt{1+|\mathrm{D}u|^2} &= \frac{\partial_i (1 + \sum_j (\partial_j u)^2)}{2\sqrt{1+|\mathrm{D}u|^2}} \\
&= \sum_j \frac{\partial_i(\partial_j u) \, \partial_j u}{\sqrt{1+|\mathrm{D}u|^2}}.
\end{aligned}$$

Hence we proved MSE. □

Remark 5.14. When $n = 3$, MSE was found by Lagrange in 1762.

Proposition 5.15. *A minimal graph in $\mathbb{R}^n$ is area-minimizing.*

Proof. We need to show if Σ_u is a minimal graph, Σ is a surface such that $\partial\Sigma_u = \partial\Sigma$, then $A[\Sigma_u] \leq A[\Sigma]$. Define a vector field $X \in \Gamma(T\mathbb{R}^n)$ such that

$$X = \frac{(-\mathrm{D}u, 1)}{\sqrt{1+|\mathrm{D}u|^2}}.$$

Since X does not depend on x^n, by MSE we have

$$\operatorname{div}_{\mathbb{R}^n} X = \operatorname{div}_{\mathbb{R}^{n-1}} \left(\frac{-\mathrm{D}u}{\sqrt{1+|\mathrm{D}u|^2}} \right) = 0.$$

Let Ω be the domain enclosed by Σ_u and Σ, N_{Σ_u} and N_Σ be the outward unit normal vector field of Σ_u and Σ respectively. Then by divergence theorem, we have

$$\begin{aligned} 0 &= \int_\Omega \operatorname{div}_{\mathbb{R}^n} X \,\mathrm{d}\,\mathrm{Vol}_\Omega \\ &= \int_{\Sigma_u} \langle X, N_{\Sigma_u} \rangle \,\mathrm{d}A_{\Sigma_u} + \int_\Sigma \langle X, N_\Sigma \rangle \,\mathrm{d}A_\Sigma. \end{aligned}$$

Since $N_{\Sigma_u} = \frac{(\mathrm{D}u,-1)}{\sqrt{1+|\mathrm{D}u|^2}}$, we have $\langle X, N_{\Sigma_u} \rangle = -1$, hence by Cauchy–Schwarz inequality,

$$A[\Sigma_u] = \int_\Sigma \langle X, N_\Sigma \rangle \,\mathrm{d}A_\Sigma \le \int_\Sigma |X||N_\Sigma| \,\mathrm{d}A_\Sigma = A[\Sigma],$$

where $|X| = |N_\Sigma| = 1$. $\square$

Finally we have a characterization of immersed minimal surfaces in Euclidean space.

Proposition 5.16. *Let $f : \Sigma^k \to \mathbb{R}^n$ ($k < n$) be an immersion, denote $\Delta f = (\Delta f^1, \cdots, \Delta f^n)$, then we have*

$$\Delta f = \mathbf{H}.$$

Proof. We use D to denote the Levi–Civita connection of $\mathbb{R}^n$ (i.e. directional derivative) and ∇ for Σ. First, we notice that this proposition is local, thus we can prove it on a sufficiently small coordinate neighborhood W on which f is an embedding. Moreover, notice that we have an identification

$$f_* Y = Yf = Yf^i \frac{\partial}{\partial x^i} = (Yf^1, \cdots, Yf^n) \in T\mathbb{R}^n,$$

thus we have $Y = Yf \in \Gamma(Tf(\Sigma))$. Now let $\{X_i\}$ be an orthonormal frame on W, we have

$$\begin{aligned} \Delta f &= \sum_i X_i(X_i f) - (\nabla_{X_i} X_i) f \\ &= \sum_i X_i(X_i) - \nabla_{X_i} X_i \\ &= \sum_i \mathrm{D}_{X_i} X_i - \nabla_{X_i} X_i \\ &= \sum_i \mathrm{II}(X_i, X_i) \\ &= \mathbf{H}. \end{aligned}$$

$\square$

Corollary 5.17. *An submanifold in Euclidean space is a minimal surface if and only if its every coordinate function is harmonic.*

Corollary 5.18. *There is no compact minimal submanifold without boundary in Euclidean space.*

Proof. Since the submanifold is minimal, all its coordinate functions are harmonic. But the submanifold is compact, the coordinate functions reach their maximum and minimum. Hence by Liouville's theorem, every coordinate functions is constant. This is impossible. □

Chapter 6

Curvature and Topology I

In this chapter, we will disucss some topological results derived by curvature restrictions. We will follow the order of Ricci curvature with positive lower bound, nonpositive sectional curvature and constant sectional curvature. The corresponding results are Bonnet–Myers Theorem, Cartan–Hadamard Theorem and some discussion of distance functions. More results on topology by curvature restrictions involves comparison theorem, which will be discussed later.

From this chapter, we will assume that all Riemannian manifolds are complete.

6.1 Bonnet–Myers Theorem

Bonnet–Myers Theorem asserts that a Riemannian manifold with positive lower bound for Ricci curvature is bounded, hence compact. Precisely, we have

Theorem 6.1 (Bonnet–Myers)**.** *Let* (M^n, g) *be a complete Riemannian manifold with* $\mathrm{Ric} \geq (n-1)Kg > 0$ *(here we lowered the index of Ricci tensor), then* $\mathrm{diam}(M) \geq \pi/\sqrt{K}$*. In particular,* M *is compact.*

Proof. We may scale the metric to make $K = 1$. Use contradiction argument, we may assume there exists $p, q \in M$ with $d(p,q) > \pi$. By completeness, there exists a unit speed geodesic $\gamma : [0,a] \to M$ such that $\gamma(0) = p$, $\gamma(a) = q$, so $d(p,q) = a > \pi$, γ is length-minimizing. Let $\{E_i(t)\}$ be a parallel orthonormal frame along γ with $E_n(t) = \dot{\gamma}(t)$, define

$$U_i(t) = \sin\left(\frac{\pi t}{a}\right).$$

Let I be the index form on γ, then we have

$$\begin{aligned}
\sum_{i=1}^{n-1} I(U_i, U_i) &= \sum_{i=1}^{n} \int_0^a \langle \dot{U}_i, \dot{U}_i \rangle - \langle R(\dot{\gamma}, U_i)\dot{\gamma}, U_i \rangle \, \mathrm{d}t \\
&= \sum_{i=1}^{n-1} \int_0^a -\langle \ddot{U}_i, U_i \rangle - \langle R(U_i, \dot{\gamma})U_i, \dot{\gamma} \rangle \, \mathrm{d}t \\
&= \sum_{i=1}^{n-1} \int_0^a \frac{\pi^2}{a^2} \sin^2\left(\frac{\pi t}{a}\right) - \sin^2\left(\frac{\pi t}{a}\right) \langle R(E_i, \dot{\gamma})E_i, \dot{\gamma} \rangle \, \mathrm{d}t \\
&= \int_0^a \left((n-1)\frac{\pi^2}{a^2} - \langle \mathrm{Ric}(\gamma), \gamma \rangle \right) \sin^2\left(\frac{\pi t}{a}\right) \\
&\leq \int_0^a (n-1)\left(\frac{\pi^2}{a^2} - 1\right) \sin^2\left(\frac{\pi t}{a}\right) \mathrm{d}t \\
&< 0,
\end{aligned}$$

the second equality is integration by parts. But γ is a length-minimizing geodesic, I must be positive definite, hence

$$\sum_{i=1}^{n-1} I(U_i, U_i) \geq 0,$$

this is a contradiction. □

Remark 6.2. (1) If we know a priori Cheng's maximal diameter theorem, the above proof can be regarded as a comparison between Jacobi fields on M and sphere. Thus this is a prototype of our proof to Rauch's comparison theorem.

(2) The condition cannot be weakened to $K = 0$, in fact, Sect > 0 is not enough. The surface $z = x^2 + y^2$ in $\mathbb{R}^3$ is an counterexample.

Corollary 6.3. *Under above settings, the universal covering $\pi : \tilde{M} \to M$ is compact. Moreover, $\pi_1(M)$ is finite.*

Proof. We can lift the metric of M to make π a Riemannian covering, then $\pi^* g$ has also a Ricci lower bound, and we can apply Bonnet–Myers Theorem. For the next claim, let $p \in M$, then $\pi^{-1}(p)$ is a discrete closed set in $\tilde{M}$, hence must be finite since $\tilde{M}$ is compact. Then the covering map is of finite sheet, $\pi_1(\tilde{M})$ has finite index in $\pi_1(M)$. But $\pi_1(\tilde{M})$ is trivial, hence $\pi_1(M)$ is finite. □

6.2 Cartan–Hadamard Theorem

Cartan–Hadamard Theorem gives the topological characterization of a Riemannian manifold with nonpositive sectional curvature.

Theorem 6.4 (Cartan–Hadamard)**.** *Let M^n be a complete Riemannian manifold with* Sect ≤ 0*, then for any $p \in M$ the exponential map $\exp_p : T_pM \to M$ is a universal covering.*

We decompose the proof into several lemmas.

Lemma 6.5 (Ambrose)**.** *Let $\varphi : (M, g) \to (N, h)$ be a local isometry. If (M, g) is complete, then φ is a Riemannian covering map.*

Proof. We first show that φ is surjective. Since φ is a local isometry, $\varphi(M)$ is open in N. Let $y \in N$ be a limit point of $\varphi(M)$, then there exists a $x \in M$ such that there is a geodesic $\sigma : [0, 1] \to N$ connecting $\varphi(x)$ and y. Since M is complete, this geodesic can be lifted to $\tilde{\sigma} : [0, 1] \to M$, and clearly $\varphi(\tilde{\gamma}(1)) = y$. Hence φ is surjective.

Now let $x \in M, x' \in N$ with $\varphi(x) = x'$. Choose δ so small that $\exp_{x'} : B'(\delta) \to B_{x'}(\delta)$ is a diffeomorphism, where $B'(\delta) = \{v \in T_{x'}N : |v| < \delta\}$. Since φ is a local isometry, $\varphi^{-1}(x')$ is discrete, say $\{x_i\}_{i \in I}$. For each $i \in I$, define

$$B^i(\delta) := \{v \in T_{x_i}M : |v| < \delta\},$$
$$B^i_\delta := \{y \in M : d(y, x_i) < \delta\}.$$

We now show the following three claims sequentially:

(1) $\varphi^{-1}(B_{x'}(\delta)) = \bigcup_{i \in I} B^i_\delta$;

(2) For each $i \in I$, $\varphi : B^i_\delta \to B_{x'}(\delta)$ is a diffeomorphism;

(3) If $i \neq j$, then B^i_δ and B^j_δ are disjoint.

For (1), $\bigcup_{i \in I} B^i_\delta \subset \varphi^{-1}(B_{x'}(\delta))$ is clear. Suppose $z \in \varphi^{-1}(B_{x'}(\delta))$, then there exists a unique geodesic $\gamma : [0, 1] \to N$ contained in $B_{x'}(\delta)$ such that $\gamma(0) = x'$, $\gamma(1) = \varphi(z)$ (combine Proposition 3.21 and Corollary 3.28). Since M is complete, we can lift γ to $\tilde{\gamma}$. Then by $\varphi(\tilde{\gamma}(0)) = x'$, we have some $\tilde{\gamma}(0) = x_i$ for some $i \in I$. Moreover, since $L[\tilde{\gamma}] = L[\gamma] < \delta$, we have $z \in B^i_\delta$. Hence $\varphi^{-1}(B_{x'}(\delta)) \subset \bigcup_{i \in I} B^i_\delta$. This is (1).

For (2), since M is complete, $\exp_{x_i} : B^i(\delta) \to B^i_\delta$ is well-defined. By naturality of exponential map (Proposition 3.16), we have the following commutative diagram:

$$\begin{tikzcd} B^i(\delta) \arrow[r, "d\varphi"] \arrow[d, "\exp_{x_i}"] & B'(\delta) \arrow[d, "\exp_{x'}"] \\ B^i_\delta \arrow[r, "\varphi"] & B_{x'}(\delta). \end{tikzcd}$$

Then

$$\varphi \circ \exp_{x_i} = \exp_{x'} \circ d\varphi$$

is a diffeomorphism. Since φ is a local isometry, $\exp_{x_i}$ is an immersion, but $\exp_{x_i}$ is surjective, hence $\exp_{x_i}$ is a diffeomorphism. Therefore $\varphi = \exp_{x'} \circ d\varphi \circ (\exp_{x_i})^{-1}$ is a diffeomorphism. This is (2).

For (3), let $z \in B^i_\delta \cap B^j_\delta$ for some i, j, then let $\gamma_i : [0,1] \to M$ and $\gamma_j : [0,1] \to M$ be the unique geodesics contained in B^i_δ and B^j_δ with $\gamma_i(0) = \gamma_j(0) = z$, $\gamma_i(1) = x_i$ and $\gamma_j(1) = x_j$. Let $\zeta : [0,1] \to N$ be the unique geodesic contained in $B_{x'}(\delta)$ with $\zeta(0) = \varphi(z)$, $\zeta(1) = x'$. Then by uniqueness, we have $\varphi \circ \gamma_i = \varphi \circ \gamma_j$. However, since φ is a local isometry, we have

$$\dot{\gamma}_i(0) = d\varphi^{-1}(\dot{\zeta}(0)) = \dot{\gamma}_j(0)$$

and

$$L[\gamma_i] = L[\zeta] = L[\gamma_j],$$

hence $\gamma_i(1) = \gamma_j(1)$, that is, $x_i = x_j$. This is (3). □

Lemma 6.6. *Let (M, g) be a Riemannian manifold, $p \in M$. If M contains no conjugate points of p, then $\exp_p : T_pM \to M$ is a covering map.*

Proof. Since p has no conjugate points, $\exp_p$ is nonsingular on T_pM, hence is an immersion. Then we can equip T_pM with pullback metric $\exp_p^* g$. If we can show $(T_pM, \exp_p^* g)$ is complete, then by Lemma 6.5, $\exp_p$ is a covering map. Let $\gamma : [0, a) \to T_pM$ be a length-minimizing geodesic, then since $\exp_p$ is a local isometry between $(T_pM, \exp_p^* g)$ and (M, g), $\exp_p \circ \gamma : [0, a) \to M$ must be a geodesic. By the definition of exponential map, we have $\exp_p \circ \gamma(t) = \exp_p(\dot{\gamma}(0)t)$, then it clearly can be extended to a. Thus by Theorem 3.23 (4), $(T_pM, \exp_p^* g)$ is complete. This proves the lemma. □

Lemma 6.7. *Let (M, g) be a complete Riemannian manifold with* $\mathrm{Sect} \leq 0$, *then for any $p \in M$, p contains no conjugate points.*

Proof. Only need to show for any geodesic $\gamma : [0, +\infty) \to M$ with $\gamma(0) = p$, the nontrivial Jacobi field J along γ with $J(0) = 0$ has no zeros besides 0. Consider the function $f(t) = \langle J(t), J(t)\rangle$, then we have

$$\dot{f}(t) = 2\langle J(t), \dot{J}(t)\rangle,$$

and

$$\begin{aligned}\ddot{f}(t) &= 2(\langle \dot{J}(t), \dot{J}(t)\rangle + \langle J(t), \ddot{J}(t)\rangle) \\ &= 2(\langle \dot{J}(t), \dot{J}(t)\rangle - \langle R(\dot{\gamma}, J)\dot{\gamma}, J\rangle(t) \\ &\geq \langle \dot{J}(t), \dot{J}(t)\rangle.\end{aligned}$$

Hence

$$\begin{aligned}\dot{f}(t) &\geq \int_0^t \langle \dot{J}(t), \dot{J}(t)\rangle \,\mathrm{d}t + \dot{f}(0) \\ &= \int_0^t \langle \dot{J}(t), \dot{J}(t)\rangle \,\mathrm{d}t + \langle \dot{J}(0), J(0)\rangle \\ &= \int_0^t \langle \dot{J}(t), \dot{J}(t)\rangle \,\mathrm{d}t \\ &> 0,\end{aligned}$$

unless $\dot{J}(t) \equiv 0$ for all $t \geq 0$, but this contradicts J being nontrivial. Hence by integrating $\dot{f}(t)$ we obtain $|J(t)|^2 > 0$ for $t > 0$, i.e. J has no zero other than 0. ☐

Now Cartan–Hadamard Theorem is the corollary of Lemma 6.6 and 6.7.

6.3 Cut Loci and Distance Functions

Distance functions control the topology of a Riemannian manifold via comparison theorems. But before we discuss comparison theorems, we first need to go through distance functions in detail.

We start this section by an example.

Example 6.8. Consider the cylinder $M = \mathbb{S}^1 \times \mathbb{R}$ with product metric. Then M is flat, and by Cartan–Hadamard Theorem, it has no conjugate points. Let $\gamma : [0, 2\pi] \to M$ be a generatrix circle, then γ is a geodesic. Denote $p = \gamma(0)$, $q = \gamma(\pi)$, then $\gamma|_{[0,\pi]}$ is a length-minimizing geodesic joining p and q. But it is not the only length-minimizing geodesic joining p and q, as $-\gamma|_{[\pi,2\pi]}$ is another one. Then q spoils the uniqueness of length-minimizing geodesic starts from p, without being conjugate to p. This inspires us to denote the notion of cut points.

Given $v \in T_pM$, $|v| = 1$, $\gamma_v : [0, +\infty) \to M$ be a geodesic with $\gamma_v(0) = p$, $\dot\gamma_v(0) = v$. Notice that $\gamma_v|_{[0,t_0]}$ is length-minimizing if $d(p, \gamma_v(t_0)) = t_0$.

Definition. Under above settings, let $t_0 = \sup\{t \in (0, +\infty) : d(p, \gamma_v(t)) = t\}$. If $t_0 < +\infty$, we call $\gamma_v(t_0)$ the *cut point* of p along γ_v. The *cut locus* of p is the set of all cut points of p, denoted by $\operatorname{Cut} p$.

By the definition and Jacobi Theorem (Theorem 4.26), the first conjugate point (if exists) must be cut point. But converse is generally not true, see Example 6.10. The following proposition characterizes cut points.

Proposition 6.9. *Let γ be a unit speed geodesic starts from p. If $\gamma(t_0)$ is the cut point of p along γ, then either $\gamma(t_0)$ is conjugate to p, or there exists two length-minimizing geodesics from p to $\gamma(t_0)$.*

Proof. Let the initial vector of γ be v. Choose a sequence $\{t_i\}$ decreasingly converges to t_0. Let σ_i be the unit speed length-minimizing geodesic joining p and t_i, with initial vector v_i. Define $b_i = d(p, \sigma_i(t_i)) = L[\sigma_i]$, then σ_i is defined on $[0, b_i]$ and $b_i \to t_0$. Hence up to a subsequence, we have $v_i \to y$ for some $y \in T_pM$ with $|y| = 1$. Now we have two cases.

(1) If $v \neq y$, then by some ODE theory, $\sigma_i \to \sigma(t) = \exp_p(ty)$ (up to a subsequence). By the lower semi-coninuity of length functional, we have
$$L[\sigma] \leq \lim_{i\to\infty} L[\sigma_i] \leq \lim_{i\to\infty} t_i = t_0.$$
On the other hand, $\gamma|_{[0,t_0]}$ is length-minimizing, we have
$$L[\sigma] \geq L[\gamma|_{0,t_0}] = t_0.$$
Hence $L[\sigma] = t_0$, σ is another length-minimizing geodesic joining p and $\gamma(t_0)$.

(2) If $v = y$, then
$$\exp_p(b_i v_i) = \gamma(t_i) = \exp_p(t_i v)$$
with both $b_i v_i \to t_0 v$ and $t_i v \to t_0 v$. This means $\exp_p$ is not injective in a neighborhood of $\gamma(t_0)$, hence $\exp_{p*}|_{\gamma(t_0)}$ degenerates, $\gamma(t_0)$ is conjugate to p.

□

Example 6.10. (1) Let $M = \mathbb{S}^n$. Then the south pole is conjugate to north pole, as well as the cut point of north pole.

(2) Let $M = \mathbb{RP}^n$. For simplicity we take $n = 2$. We regard $\mathbb{RP}^2$ as the upper hemisphere with antipodal points on equator identified. Then from north pole we move along a "great circle" to a point p on equator, the path is a length-minimizing geodesic. However, the path on the same "great circle" but on the other side is also a length-minimizing geodesic, so p is the cut point of north pole (along this geodesic). But by the naturality of exponential map, p is not conjugate to north pole, so this is an example of cut point not being conjugate point.

(3) Let $M = \mathbb{R}^2/\mathbb{Z}^2$ be the flat torus. Let $[0,1] \times [0,1]$ be the fundamental region, and denote $p = (0,0) = (0,1)$. Then $\gamma(t) = (t,0)$ is a geodesic. Notice that if $t_0 > 1/2$, $\gamma(t)$ is not the length-minimizing geodesic joining p and $(t_0,0)$, as $\sigma(t) = (1-t,0)$ is. By the same reason, we have the cut locus of p is $\{1/2\} \times [0,1/2] \cup [0,1/2] \times \{1/2\}$.

Let $S_p \subset T_pM$ be the unit sphere. For any $v \in S_p$, denote $\gamma_v : [0,+\infty) \to M$ be the geodesic with $\dot{\gamma}(0) = v$. We define a function

$$\tau : S_p \to (0,+\infty],\ \tau(v) = \begin{cases} +\infty, & \gamma_v \text{ contains no cut point,} \\ t_0, & \gamma_v(t_0) \text{ is the cut point of } p. \end{cases}$$

Proposition 6.11. *τ is continuous.*

Proof. I don't know! □

Chapter 7

Curvature and Transformation Groups

In this chapter, we will discuss the space forms which we defined by Lie groups in detail. We will show that they are constant curvature, and we will prove their uniqueness up to isometry. After this, we will state and prove Myers–Steenrod Theorem and its rigidity result.

7.1 Space Forms

Our goal of this section is to calculate the curvature of space forms. It's hard to calculate space forms directly using Lie group definition: this relies on the curvature formulae of Riemannian submersions. To overcome this difficulty, we recall the models of space forms, and use these models to calculate their curvature.

Models of Space Forms

Recall the space forms we introduced in Example 1.44:

- $\mathbb{R}^n$,
- $\mathbb{S}^n$: $(x^1)^2 + \cdots + (x^{n+1})^2 = 1$ in $\mathbb{R}^{n+1}$,
- $\mathbb{H}^n$: upper component of $(x^1)^2 + \cdots + (x^n)^2 - (x^{n+1})^2 = -1$ in $\mathbb{R}^{n,1}$.

We will adopt these three models and equip them with Riemannian metrics.

For $\mathbb{R}^n$, we have canonical (flat) metric $\delta = (\mathrm{d}x^1)^2 + \cdots + (\mathrm{d}x^n)^2$.

For $\mathbb{S}^n$, let $\iota : \mathbb{S}^n \to \mathbb{R}^{n+1}$ be the natural embedding, we have the canonical metric $g = \iota^*\delta$, where δ is the canonical metric of $\mathbb{R}^{n+1}$.

For $\mathbb{H}^n$, let $\iota : \mathbb{H}^n \to \mathbb{R}^{n,1}$ be the natural embedding. Denote $\ell = (\mathrm{d}x^1)^2 + \cdots + (\mathrm{d}x^n)^2 - (\mathrm{d}x^{n+1})^2$ be the Lorentz metric of $\mathbb{R}^{n,1}$, we have the canonical metric of $\mathbb{H}^n$ to be $g = \iota^*\ell$. We need to show that g is a Riemannian metric.

Lemma 7.1. *Let ℓ be the Lorentz metric of $\mathbb{R}^{n,1}$, $\iota : \mathbb{H}^n \to \mathbb{R}^{n,1}$ be the natural embedding. Then $\iota^*\ell$ is a Riemannian metric.*

Proof. Clearly we only need to show $\iota^*\ell$ is positive definite. Let $(x^1, \cdots, x^{n+1}) \in \mathbb{H}^n$, we consider its tangent vector $v = (v^1, \cdots, v^{n+1})$. This tangent vector satisfies equation

$$v^1x^1 + \cdots + v^nx^n - v^{n+1}x^{n+1} = 0.$$

We calculate the norm (in fact, we cannot call $\langle v, v\rangle$ norm now) of v, by Cauchy's inequality, we have

$$\begin{aligned}
\langle v, v\rangle &= (v^1)^2 + \cdots + (v^n)^2 - (v^{n+1})^2 \\
&= (v^1)^2 + \cdots + (v^n)^2 - \frac{1}{(x^{n+1})^2}(v^1x^1 + \cdots + v^nx^n)^2 \\
&\geq ((v^1)^2 + \cdots + (v^n)^2)\left(1 - \frac{(x^1)^2 + \cdots + (x^n)^2}{(x^{n+1})^2}\right) \\
&= \frac{1}{(x^{n+1})^2}((v^1)^2 + \cdots + (v^n)^2) \\
&\geq 0.
\end{aligned}$$

Clearly, when equality holds, we have $v = 0$. Thus $\iota^*\ell$ is a Riemannian metric. □

Curvature of Space Forms

Clearly, as $\mathbb{R}^n$ is equipped with flat metric, it has constant curvature 0. We write this into a proposition.

Proposition 7.2. *For all $p \in \mathbb{R}^n$, we have $\mathrm{Sect}_p\,\Pi = 0$ for any plane $\Pi \subset T_p\mathbb{R}^n$.*

To calculate the curvature of $\mathbb{S}^n$ and $\mathbb{H}^n$, we need to introduce the concept of totally geodesic submanifold.

Definition. Let M be a Riemannian manifold. A submanifold $\Sigma \subset M$ is called a *totally geodesic submanifold* if the second fundamental form vanishes identically. Clearly this is equivalent to every geodesic of M with initial vector in $T_p\Sigma$ is contained in Σ.

By the very definition and Gauss equation, we have

Lemma 7.3. *If Σ is a totally geodesic submanifold of M, then for $p \in \Sigma$ and plane $\Pi \subset T_p\Sigma$, the sectional curvatures with respective to Σ and M are equal.*

A nontrivial result is

Proposition 7.4. *Let (M, g) be a complete Riemannian manifold. Let $S \subset \operatorname{Isom}(M, g)$ be a subset, define $\operatorname{Fix} S$ be the fixed points of S. Then any component of $\operatorname{Fix} S$ is totally geodesic submanifold of M.*

Proof. Let $p \in \operatorname{Fix} S$, $V \subset T_pM$ be the Zariski tangent space, that is,

$$V := \{v \in T_pM : f_*|_p(v) = v, \ \forall f \in \operatorname{Fix} S\}.$$

V is well-defined since all f in $\operatorname{Fix} S$ fixes p. Let $v \in V$, then the initial data of geodesic $\gamma_v(t) = \exp_p(tv)$ are fixed by S. Since isometry preseves geodesics, we have $\gamma_v \subset \operatorname{Fix} S$. Thus the proposition is proved as soon as we have shown that $\operatorname{Fix} S$ is a submanifold.

Above argument shows $\exp_p$ is defined on the whole V. Let ε so small that $\exp_p : B(0, \varepsilon) \to B(p, \varepsilon)$ is a diffeomorphism. Then by Proposition 3.21, every geodesic starts from p in $B(p, \varepsilon)$ is length-minimizing. Let $q \in \operatorname{Fix} S \cap B(p, \varepsilon)$, then there exists a unique geodesic γ starts from p joining p and q. Since S fixes the initial and end points of γ, and S does not change γ's length, by the length-minimizing property of γ, we have γ fixed by S. Thus $\gamma \subset \operatorname{Fix} S$. This means $\exp_p$ is bijective on $V \cap B(0, \varepsilon)$, hence $\operatorname{Fix} S$ is a submanifold. The proposition is thus proved. $\square$

Remark 7.5. The first paragraph of above proof does not use the completeness assumption, thus if $\operatorname{Fix} S$ is a curve, above argument can show that $\operatorname{Fix} S$ is a geodesic. This can occur when M fails to be complete. Consider $\mathbb{R}^2 \backslash \{0\}$ equipped with pullback metric of $(\mathbb{R}^2, \delta)$ and f be the reflection with respective to the punctured x-axis. Then f is an isometry and fixes the punctured x-axis, which is a union of two geodesics.

Also we need Theorema Egregium in the following form.

Theorem 7.6 (Theorema Egregium)**.** *Let a M be a 2-dimensional Riemannian manifold (i.e. a Riemannian surface), $p \in M$. If the metric tensor has expression $\mathrm{d}s^2 = E\,\mathrm{d}u^2 + G\,\mathrm{d}v^2$ near p, then the sectional curvature at p is*

$$\operatorname{Sect}_p(T_pM) = -\frac{1}{\sqrt{EG}}\left(\frac{\partial}{\partial u}\left(\frac{\partial_u\sqrt{G}}{\sqrt{E}}\right) + \frac{\partial}{\partial v}\left(\frac{\partial_v\sqrt{E}}{\sqrt{G}}\right)\right)$$

We will not compute this equation here; For an elementary proof, see [9, Equation 2.131].

Now we calculate the curvature of $\mathbb{S}^n$ and $\mathbb{H}^n$.

For $\mathbb{S}^n$, we see that it's a closed subspace of complete metric space $\mathbb{R}^{n+1}$, hence is complete. Consider the orthogonal transformation

$$A = \operatorname{diag}(1, 1, 1, -1, \cdots, -1),$$

we have that A fixes $(x^1)^2 + (x^2)^2 + (x^3)^2 = 1$. Hence by Proposition 7.4, we only need to calculate the sectional curvature of $\mathbb{S}^2$ (this already shows $\mathbb{S}^n$ has constant curvature, since a Riemannian surface can only have one sectional curvature). Choose a coordinate

$$(u, v) \mapsto (\cos u \cos v, \cos u \sin v, \sin u),$$

it's known the metric tensor under this coordinate is

$$\mathrm{d}s^2 = \mathrm{d}u^2 + \cos^2 u \,\mathrm{d}v^2.$$

Using Theorema Egregium, we have

$$\begin{aligned} \operatorname{Sect}_p(T_p\mathbb{S}^2) &= -\frac{1}{\sqrt{\cos^2 u}}\left(\frac{\partial}{\partial u}\left(\frac{\partial_u \sqrt{\cos^2 u}}{\sqrt{1}}\right) + \frac{\partial}{\partial v}\left(\frac{\partial_v \sqrt{1}}{\sqrt{\cos^2 u}}\right)\right) \\ &= 1. \end{aligned}$$

Thus we obtain

Proposition 7.7. *$\mathbb{S}^n$ has constant sectional curvature* 1.

For $\mathbb{H}^n$, first consider the hyperbola

$$(x^1)^2 - (x^{n+1})^2 = -1,$$

we claim it is a geodesic. Let $\gamma(t) = (\cosh t, 0, \cdots, 0, \sinh t)$ $(t \geq 0)$ be the parametrization of the hyperbola, denote whose initial vector by

$$v = (1, 0, \cdots, 0).$$

Since $\mathbb{R}^{n,1}$ has flat metric, we have

$$\begin{aligned} \mathrm{D}_{\dot\gamma(t)}\dot\gamma(t) &= \frac{\mathrm{d}\dot\gamma(t)}{\mathrm{d}t} \\ &= (\cosh t, 0, \cdots, 0, \sinh t). \end{aligned}$$

The tangent space $T_{\gamma(t)}\mathbb{H}^n$ is defined by the equation

$$x^1 \cosh t = x^{n+1} \sinh t,$$

whose orthochronous complement is spanned by $(\cosh t, 0, \cdots, 0, \sinh t)$, which is $\mathrm{D}_{\dot\gamma(t)}\dot\gamma(t)$. Hence $\nabla_{\dot\gamma(t)}\dot\gamma(t) = 0$, γ is a geodesic. Let $w \in T_0\mathbb{H}^n$, we can rotate w to v, and the rotation can be extended to a orthochronous transformation of $\mathbb{R}^{n,1}$ since it's restricted in $\mathbb{R}^n \times \{0\}$. This orthochronous transformation carries γ to a geodesic with initial vector w, and can be defined on the whole $\mathbb{R}_{\geq 0}$. Thus by Hopf–Rinow Theorem, $\mathbb{H}^n$ is complete.

After proving $\mathbb{H}^n$ is complete, we can calculate its curvature using totally geodesic submanifold. The orthochronous transformation

$$A = \operatorname{diag}(1, 1, -1, \cdots, -1, 1)$$

fixes $(x^1)^2 + (x^2)^2 - (x^{n+1})^2 = -1$. Consider the coordinate

$$(u, v) \mapsto (\cosh u \cos v, \cosh u \sin v, \sinh u),$$

one can calculate that the expression of metric tensor under this coordinate is

$$\mathrm{d}s^2 = \mathrm{d}u^2 + \cosh^2 u \, \mathrm{d}v^2.$$

Using Theorema Egregium, we have

$$\begin{aligned}
\operatorname{Sect}_p(T_p\mathbb{H}^2) &= -\frac{1}{\sqrt{\cosh^2 u}} \left(\frac{\partial}{\partial u} \left(\frac{\partial_u \sqrt{\cosh^2 u}}{\sqrt{1}} \right) + \frac{\partial}{\partial v} \left(\frac{\partial_v \sqrt{1}}{\sqrt{\cosh^2 u}} \right) \right) \\
&= -1.
\end{aligned}$$

Thus we obtain

Proposition 7.8. $\mathbb{H}^n$ *has sectional curvature* -1.

7.2 Uniqueness of Space Forms

In this section we prove that constant sectional curvature spaces are unique in some sense. We have some comments for this.

First, let (M, g) be a Riemannian manifold with constant sectional curvature K, we can scale g by a positive constant C to make (M, Cg) into a space with constant sectional curvature $0, 1, -1$, depending on the sign of K. Therefore we only need to consider spaces with sectional curvature $0, 1, -1$.

Second, we cannot have uniqueness unless we have some topological restriction. Let's consider any Riemannian covering with total space to be a space form, then the base space has constant curvature. This enlights us that we may consider universal covering of constant curvature spaces.

Third, the uniqueness is up to an isometry. For instance, $\mathrm{O}(n+1)/\mathrm{O}(n)$ as a homogeneous space may be considered different from $\mathbb{S}^n$ ontologically, but they are isometric.

Taking all above into consideration, we have the following theorem.

Theorem 7.9. *Let M^n be a Riemannian manifold with constant sectional curvature of $0, 1, -1$, then the universal covering of M must be isometric to space forms $\mathbb{R}^n$, $\mathbb{S}^n$ or $\mathbb{H}^n$.*

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