# A Concise Note on Differential Geometry

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# Preface

# Part I

# Smooth Manifolds

# Part II

# Riemannian Geometry

# Chapter 1

# Metric Structure

In this chapter we introduce the metric structure on a Riemannian manifold, i.e. using the metric tensor to make the manifold into a metric space. We will prove that the topology induced by metric coincides with the topology carried by the manifold. Then we will study the length-minimizing problem: which curve minimize the distance between two points? The answer is two-sided: we will prove that length-minimizing curves are geodesics, and geodesics are *locally* length-minimizing.

## 1.1 Metric Structure on Riemannian Manifolds

Let $(M, g)$ be a Riemannian manifold, $\gamma : [0, 1] \to M$ be a regular curve (i.e. $\gamma$ is an immersion). We define the length functional of regular curves.

**Definition.** The *length functional* $L[\cdot]$ on the set of regular curves is defined by

$$L[\gamma] = \int_0^1 \sqrt{g(\dot{\gamma}(t), \dot{\gamma}(t))}\,\mathrm{d}t.$$

It is a well-known result from calculus that any regular curve can be reparametrized by arc-length. From now on the word "curve" means a regular curve.

Now we define the distance between two points.

**Definition.** Let $p, q \in M$ be two points on $M$, then we define their *distance* by

$$d(p, q) = \inf_{\gamma \in C_{p,q}} L[\gamma],$$

where $C_{p,q}$ is the set of all regular curves joining $p$ and $q$.

**Proposition 1.1.** *The distance function $d : M \times M \to \mathbb{R}$ has the following properties:*

*(1)* $d(p,q) \geq 0$, *and* $d(p,q) = 0 \iff p = q$;

*(2)* $d(p,q) = d(q,p)$;

*(3)* $d(p,r) \leq d(p,q) + d(q,r)$.

*Thus the distance function makes $M$ into a metric space.*

*Proof.* Only need to show $d(p,q) = 0 \iff p = q$, all else are trivial. We assume $p \neq q$, need to show $d(p,q) > 0$. Let $\gamma : [0,1] \to M$ be any curve joining $p$ and $q$. Choose a local chart $(U, \varphi)$ such that $\varphi(U) = B_r(0)$, $q \neq U$. By Jordan–Brouwer Separation Theorem, $\gamma$ must intersect $\partial U$ at $s := \gamma(c)$. Then we have

$$L[\gamma] \geq L[\gamma|_{[0,c]}] = \int_0^c \sqrt{g_{ij}\dot{x}^i(\gamma(t))\dot{x}^j(\gamma(t))}\, \mathrm{d}t.$$

Regarding $g : \overline{U} \times \mathbb{S}^{n-1} \to \mathbb{R}$, $g$ is a continuous function on a compact set, thus it attains its minimum $g(x)(v,v) \geq m$, and $m > 0$ since $v \in \mathbb{S}^{n-1} \neq 0$. Thus we have

$$L[\gamma] \geq L[\gamma|_{[0,c]}] \geq m \int_0^c |\dot{x}(\gamma(t))|\, \mathrm{d}t \geq mr > 0. \tag{1.1}$$

$mr$ does not depend on $\gamma$, hence $d(p,q) \geq mr > 0$. □

However, the metric space topology is nothing but the original topology carried by the manifold.

**Proposition 1.2.** *The metric space topology on $M$ coincides with the manifold topology.*

We first need a lemma.

**Lemma 1.3.** *The distance function to $p$ defined by $r(q) = d(p,q)$ is continuous with respective to the manifold topology.*

*Proof.* Since manifolds satisfy the second countable axiom, the Sequence Lemma holds. Then it's equivalent to show for any $q_i \to q$ in manifold topology, we have $r(q_i) \to r(q)$. Without loss of generality we can assume $\{q_i\} \subset U$ and $(U, \varphi)$ is a local chart such that $\varphi(q) = 0$, $\varphi(U) = B_r(0)$. Let $\delta$ be the Euclidean metric on $B_r(0)$, then by regarding $g$ as a continuous function on $\overline{U} \times \mathbb{S}^{n-1}$ again, we have $g \leq M\delta$ for some $M > 0$. By assumption, $q_i \to q$ in manifold topology implies $L_\delta[\psi_i] \to 0$, where $\psi_i(t) = t\varphi(q_i)$, the radial line joining $\varphi(q)$ and $\varphi(q_i)$ in $\varphi(U) = B_r(0)$. Let $\varphi^{-1}\psi_i = \gamma_i$, then we have

$$L_g[\gamma_i] = \int_0^1 \sqrt{g(\dot{\gamma}_i(t), \dot{\gamma}_i(t))}\, \mathrm{d}t \leq M \int_0^1 \sqrt{\delta(\dot{\psi}_i(t), \dot{\psi}_i(t))}\, \mathrm{d}t \leq M L_\delta[\psi_i].$$

Since $r$ is Lipschitz, i.e. $|r(q) - r(s)| \le d(q, s)$, we have

$$d(q_i, q) \le L_g[\gamma_i] \le ML_\delta[\psi_i] \to 0,$$

hence $r$ is continuous. $\square$

*Proof of Proposition 1.2.* Since distance function is continuous, metric balls are open in manifold topology. Now we prove the converse.

Let $U$ be open with respective to manifold topology. Let $p \in U$, $V$ be a neighborhood of $p$ so small that $\varphi(V) = B_r(0)$ for some $r > 0$. The estimate (1.1) shows if $q \notin V$ then $d(p, q) \ge mr$ for some fixed $m > 0$, then by taking contrapositive statement, we have $q \in V$ if $d(p, q) < mr$. Therefore $B_p(mr) \subset U$, then $U$ is open with respective to metric space topology. $\square$

## 1.2 Length-Minimizing Curves

This section and the next is guided by the following problem:

*Problem.* Let $p, q$ be two distinct points on a Riemannian manifold $(M, g)$, then what is the curve $\gamma$ satisfying $L[\gamma] = d(p, q)$?

This section we will show if the length-minimizing curve exists, then it is a *geodesic*. Next section we will show if $p, q$ are sufficiently closed, *the* geodesic joining $p$ and $q$ is length-minimizing. Finally, the existence of length-minimizing curves is related to Hopf-Rinow Theorem, which we will discuss at the last section of the chapter.

To find the length-minimizing curve, we "gather" curves with same initial and end points, which is called a variation.

**Definition.** Let $\gamma_0 : [0, a] \to M$ be a curve, a *variation* of $\gamma_0$ is a differentiable map $\gamma : [0, a] \times (-\varepsilon, \varepsilon) \to M$ such that $\gamma(t, 0) = \gamma_0(t)$. If $\gamma(0, s) = \gamma_0(0)$ and $\gamma(a, s) = \gamma_0(a)$ for any $s \in (-\varepsilon, \varepsilon)$, then we call the variation a *proper variation*. We call $\frac{\partial}{\partial s}\Big|_{s=0} \gamma(s, t) =: V(t)$ the *variation vector field*.

Now we introduce the energy functional, which is easier to calculate.

**Definition.** The *energy functional* on the set of curves is defined by

$$E[\gamma] = \int_0^a \frac{1}{2} |\dot{\gamma}(t)|^2 \, \mathrm{d}t,$$

where $\gamma : [0, a] \to M$ is a regular curve.

We will prove that a curve is energy-minimizing if and only if it is length-minimizing.

**Lemma 1.4.** *For a curve $\gamma : [0, a] \to M$, we have*

$$L^2[\gamma] \leq 2aE[\gamma],$$

*with equality holds if and only if $|\dot{\gamma}(t)| = \text{const.}$*

*Proof.* This is Cauchy–Schwarz inequality. □

**Proposition 1.5.** *If $\gamma$ is length-minimizing, then it is energy-minimizing.*

*Proof.* Let $\tilde{\gamma}$ be another curve, then we have

$$2aE[\gamma] = L^2[\gamma] \leq L^2[\tilde{\gamma}] \leq 2aE[\tilde{\gamma}].$$ □

Our aim is to prove the converse.

**Proposition 1.6.** *If $\gamma$ is an energy-minimizing curve, then it is length-minimizing.*

To prove this, we need to differentiate the variation.

**Proposition 1.7** (First variation formula)**.** *Let $\gamma(t, s)$ be a variation, define its energy $E(s) = \int_0^a \frac{1}{2} \left| \frac{\partial}{\partial t}\gamma(t, s) \right|^2 \mathrm{d}t$, then we have*

$$E'(0) = \boxed{\langle V, \dot{\gamma} \rangle |_0^a} - \int_0^a \langle V(t), \nabla_{\dot{\gamma}_0(t)} \dot{\gamma}_0(t) \rangle.$$

*The boxed term is called* boundary term*, and it vanishes when the variation is proper.*

*Proof.* This is a calculation. We have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}E(s) &= \int_0^a \left\langle \frac{\partial\gamma}{\partial t}, \nabla_{\frac{\partial}{\partial s}} \frac{\partial\gamma}{\partial t} \right\rangle \mathrm{d}t \\
&= \int_0^a \left\langle \frac{\partial\gamma}{\partial t}, \nabla_{\frac{\partial}{\partial t}} \frac{\partial\gamma}{\partial s} \right\rangle \mathrm{d}t \\
&= \int_0^a \frac{\partial}{\partial t} \left\langle \frac{\partial\gamma}{\partial t}, \frac{\partial\gamma}{\partial s} \right\rangle - \left\langle \frac{\partial\gamma}{\partial s}, \nabla_{\frac{\partial}{\partial t}} \frac{\partial\gamma}{\partial t} \right\rangle \mathrm{d}t.
\end{aligned}$$

Take $s = 0$, we obtain

$$\begin{aligned}
E'(0) &= \int_0^a \frac{\partial}{\partial t} \langle V(t), \dot{\gamma}_0(t) \rangle - \langle V(t), \nabla_{\dot{\gamma}_0(t)} \dot{\gamma}_0(t) \rangle \,\mathrm{d}t \\
&= \langle V, \dot{\gamma}_0 \rangle |_0^a - \int_0^a \langle V(t), \nabla_{\dot{\gamma}_0(t)} \dot{\gamma}_0(t) \rangle \,\mathrm{d}t.
\end{aligned}$$ □

Now we can derive the definition of a geodesic.

**Definition.** A curve $\gamma : [0, a]$ is called a *geodesic* if $\nabla_{\dot\gamma(t)}\dot\gamma(t) = 0$ for $t \in [0, a]$.

*Remark* 1.8. Geodesics are constant speed, this can be shown by $\frac{\mathrm{d}}{\mathrm{d}t}|\dot\gamma(t)|^2 = 2\langle\dot\gamma(t), \nabla_{\dot\gamma(t)}\dot\gamma(t)\rangle = 0$.

**Corollary 1.9.** *$\gamma$ is a critical value for all proper variation if and only if $\nabla_{\dot\gamma(t)}\dot\gamma(t) = 0$, that is, $\gamma$ is a geodesic.*

We can give the proof of Proposition 1.6 now.

*Proof of Proposition 1.6.* Let $\gamma : [0, a] \to M$ be a curve such that for any $\tilde\gamma : [0, a] \to M$ with $\tilde\gamma(0) = \gamma(0)$, $\tilde\gamma(1) = \gamma(1)$, the inequality $E[\gamma] \le E[\tilde\gamma]$ holds, we show that $L[\gamma] \le L[\tilde\gamma]$. Let $\gamma(t, s)$ be any proper variation with $\gamma(t, 0) = \gamma(t)$, then $\gamma$ is a critical point of $E(s)$. Hence by Corollary 1.9, $\gamma$ is a geodesic. Now we can reparametrize $\tilde\gamma$ into arc-length, obtaining $\hat{\tilde\gamma}$. Therefore

$$L^2[\gamma] = 2aE[\gamma] \le 2aE\left[\hat{\tilde\gamma}\right] = L^2\left[\hat{\tilde\gamma}\right] = L^2[\tilde\gamma],$$

which implies $L[\gamma] \le L[\tilde\gamma]$. $\square$

Combining all results above, we have

**Proposition 1.10.** *If a curve is length-minimizing, then it is a geodesic.*

## 1.3 Geodesics and Exponential Maps

To prove the local length-minimizing property of geodesic, we need to introduce the exponential map.

We first need to investigate the equation that determine a geodesic.

**Proposition 1.11.** *Given $p \in M$ and $v \in T_pM$, there exists a unique geodesic $\gamma$ (whose domain may not be maximal) such that $\gamma(0) = p$, $\dot\gamma(0) = v$.*

*Proof.* Let $(U, \varphi)$ be a local chart containing $p$, compose $\varphi$ with $\gamma$ we obtain coordinate curves $x^i$'s. Then the geodesic equation is equivalent to

$$\ddot x^k(t) + \Gamma^k_{ij}(\gamma(t))\dot x^i(t)\dot x^j(t) = 0,\ k = 1, \cdots, n.$$

This is a system of second order ordinary differential equations, by the unique existence theorem of ODE, the solution is completely determined by $x^i$'s and $\dot x^i$'s, that is, $p$ and $v$. $\square$

Since the solution of an ODE relies continuously on its initial value, we have the following proposition.

**Proposition 1.12.** *For any $p \in M$, there exists a neighborhood $V$ of $p$, such that there exists $\delta > 0$, $\varepsilon > 0$ and a differentiable map $\gamma : (-\delta, \delta) \times \mathscr{U} \to M$, where $\mathscr{U} = \{(q, v) \in TM : q \in V, v \in T_q M, |v| < \varepsilon\}$, such that $\gamma(t; q, v)$ is a geodesic with $\gamma(0) = q$, $\dot{\gamma}(0) = v$.*

A proof can be found in [4, Chapter 3, Lemma 1].

Observe that $\gamma(\lambda t; p, v) = \gamma(t; p, \lambda v)$. Denote $\gamma(t; p, v)$ by $\gamma_v(t)$, then above observation can be written as $\gamma_{\lambda v}(t) = \gamma_v(\lambda t)$. Therefore, we can shorten the initial vector to lengthen the domain of geodesic.

**Definition.** Let $U \subset T_p M$ be a neighborhood of origin, such that for any $v \in U$, $\gamma_v(1)$ is defined (existence is guaranteed by Proposition 1.12). We define the *exponential map* at $p$ by

$$\begin{aligned} \exp_p : U &\to M \\ v &\mapsto \gamma_v(1). \end{aligned}$$

*Remark* 1.13. We can scale the initial vector and obtain

$$\exp_p(v) = \gamma_v(1) = \gamma_{v/|v|}(|v|).$$

This means the action of exponential map on $v$ is to move forward the distance $|v|$ along the geodesic with initial direction $v/|v|$.

**Proposition 1.14.** $\exp_{p*}|_0 : T_0(T_p M) \to T_p M$ *is identity (we identify $T_0(T_p M)$ with $T_p M$).*

*Proof.* We have

$$\exp_{p*}|_0(v) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (tv) = v. \qquad \square$$

**Corollary 1.15.** *There exists a ball $B_\varepsilon(0) \subset T_p M$ such that $\exp_p : B_\varepsilon(0) \to M$ is a diffeomorphism onto its image.*

*Proof.* Since $\exp_{p*}|_0$ is identity, it is nondegenerate, the corollary follows from Inverse Function Theorem. $\square$

**Example 1.16.** (1) We know that the geodesics on $S^n$ are great circles, hence $\exp_p$ is defined on the whole $T_p M$. But $\exp_p$ is not injective, since

$$\exp_p(0) = \exp(2\pi v) = p$$

for unit vector $v$ in $T_p M$.

(2) Let $M = \mathbb{S}^1 \times \mathbb{R}$ be the cylinder. We know from elementary differential geometry that the geodesics on cylinder are directrix circles, helices and generatrix lines. Then in local charts $(e^{2\pi it}, s) \mapsto (t, s)$, we know $\exp_p$ is not injective in the direction $(1, 0)$, and injective in other directions.

We postpone the discussion on whether the exponential map can be defined on the whole tangent space, the answer is Hopf–Rinow Theorem, which will be discussed in next section.

Now we prove that geodesics are locally length-minimizing. For this, we introduce some local charts. Given a Riemannian manifold $(M, g)$ and $p \in M$, let $\exp_p : B_\varepsilon(0) \to \exp(B_\varepsilon(0)) = B_\varepsilon(p)$ be a diffeomorphism.

**Definition.** We define *geodesic normal coordinate* as follows: Let $\{e_i\}$ be an orthonormal basis of Euclidean space $(T_pM, \delta)$, $\{\alpha^i\}$ be its dual basis. Then we define the coordinate by

$$q \in B_\varepsilon(p) \mapsto (\alpha^1(\exp_p^{-1}(q)), \cdots, \alpha^n(\exp_p^{-1}(q))).$$

**Proposition 1.17.** *Under geodesic normal coordinate, we have*

$$g_{ij}(p) = \delta_{ij},\ \Gamma_{ij}^k(p) = 0.$$

*Proof.* Since $\exp_p$ is a diffeomorphism, we have $\frac{\partial}{\partial x^i}\Big|_p = \exp_{p*}|_0(e_i) = e_i$, hence $g_{ij} = \delta(e_i, e_j) = \delta_{ij}$. Moreover, let $x(t) = ty$ for $y \in T_pM - \{0\}$, then $x(t)$ is the coordinate of some geodesic in $B_\varepsilon(p)$, thus it satisfies the equation

$$\ddot{x}^k(t) + \Gamma_{ij}^k(x(t))\dot{x}^i(t)\dot{x}^j(t) = 0.$$

Since $\dot{x}^i = y^i \neq 0$, $\ddot{x}^k = 0$, we have $\Gamma_{ij}^k(ty) = 0$. Let $y \to 0$ we obtain the conclusion. $\square$

Next we introduce the geodesic polar coordinate.

**Definition.** We define *geodesic polar coordinate* as follows: Let $(r, \theta^1, \cdots, \theta^{n-1})$ be a polar coordinate on Euclidean space $(T_pM, \delta)$, then we defined the coordinate by

$$q \in B_\varepsilon(p) - \{p\} \mapsto (r(\exp_p^{-1}(q)), \theta^1(\exp_p^{-1}(q)), \cdots, \theta^{n-1}(\exp_p^{-1}(q))).$$

**Proposition 1.18.** *Under geodesic polar coordinate, we have*

$$\left\langle \frac{\partial}{\partial r}, \frac{\partial}{\partial r} \right\rangle = 1,\ \left\langle \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta^i} \right\rangle = 0.$$

*Proof.* To make things clear, we write the inverse of geodesic polar coordinate as

$$F : (r, \omega) \mapsto \exp_p(r\omega)$$

for $r \in (0, \varepsilon)$, $\omega \in \mathbb{S}^{n-1}$. Then we use $\partial_0, \partial_1, \cdots, \partial_{n-1}$ to denote the tangent vectors in $(0, \varepsilon) \times \mathbb{S}^{n-1}$, we have

$$\begin{aligned} \frac{\partial}{\partial r} &= F_*(\partial_0), \\ \frac{\partial}{\partial \theta^i} &= F_*(\partial_i),\ i = 1, \cdots, n-1. \end{aligned}$$

First we know that $\partial_0$ is the tangent vector of radial line $r\omega$, hence $\partial/\partial r$ is the tangent vector of a unit-speed radial geodesic, that is,

$$\left\langle \frac{\partial}{\partial r}, \frac{\partial}{\partial r} \right\rangle = 1.$$

Moreover, we have

$$\begin{aligned} \frac{\partial}{\partial r} \left\langle \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta^i} \right\rangle &= \left\langle \nabla_{\frac{\partial}{\partial r}} \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta^i} \right\rangle + \left\langle \frac{\partial}{\partial r}, \nabla_{\frac{\partial}{\partial r}} \frac{\partial}{\partial \theta^i} \right\rangle \\ &= \left\langle \frac{\partial}{\partial r}, \nabla_{\frac{\partial}{\partial r}} \frac{\partial}{\partial \theta^i} \right\rangle \\ &= \left\langle \frac{\partial}{\partial r}, \nabla_{\frac{\partial}{\partial \theta^i}} \frac{\partial}{\partial r} \right\rangle \\ &= \frac{1}{2} \frac{\partial}{\partial \theta^i} \left\langle \frac{\partial}{\partial r}, \frac{\partial}{\partial r} \right\rangle \\ &= 0, \end{aligned}$$

hence $\left\langle \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta^i} \right\rangle$ is constant. However, if we let $r \to 0$, we have $\partial/\partial\theta^i \to 0$, therefore

$$\left\langle \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta^i} \right\rangle = 0.$$

□

**Corollary 1.19.** *Under geodesic polar coordinate, the metric tensor has local expression*

$$g = \mathrm{d}r^2 + g_{ij}(r, \theta)\, \mathrm{d}\theta^i \otimes \mathrm{d}\theta^j,$$

*where* $[g_{ij}]_{i,j>0}$ *is positive definite.*

As an application, we prove that geodesics are locally length-minimizing as we promised.

**Proposition 1.20.** *Let $\gamma : [0,1] \to M$ be a geodesic contained in an open set $U$, where geodesic polar coordinate is defined on $U$. Let $\tilde{\gamma}$ be any curve contained in $U$ with $\tilde{\gamma}(0) = \gamma(0) = p$, $\tilde{\gamma}(1) = \gamma(1) = q$. Then $L[\gamma] \leq L[\tilde{\gamma}]$.*

*Proof.* Let $q = \exp_p(v)$, $\varphi$ is the geodesic polar coordinate. Then we have

$$\gamma(t) = \varphi(tr_0, \omega_0),\ \tilde{\gamma}(t) = \varphi(r(t), \omega(t))$$

such that $r(1) = r_0$, $\omega(t) \in \mathbb{S}^{n-1}$. Therefore

$$\begin{aligned} L[\gamma] &= \int_0^1 |\dot{\gamma}(t)|\,\mathrm{d}t \\ &= \int_0^1 |v|\,\mathrm{d}t = r_0, \\ L[\tilde{\gamma}] &= \int_0^1 (|\dot{r}^2(t) + g_{ij}\dot{\theta}^i(t)\dot{\theta}^j(t))^{1/2} \\ &\geq \int_0^1 |\dot{r}(t)|\,\mathrm{d}t \\ &\geq \int_0^1 \dot{r}(t)\,\mathrm{d}t = r_0. \end{aligned}$$

□

*Remark* 1.21. The hypothesis of Proposition 1.20 can be weakened to $\exp_p$ is an immersion on $U$.

## 1.4 Hopf–Rinow Theorem

We now answer the problem whether length-minimizing curve always exists. The answer is Hopf–Rinow Theorem.

We adopt the metric geometry version of Hopf–Rinow Theorem from [1], using geometric approach to prove the theorem and keeping the differential tools minimal. We first define

**Definition.** A Riemannian manifold $(M, g)$ is called *(geodesically) complete* if $M$ is complete as a metric space.

Our theorem is

**Theorem 1.22** (Hopf–Rinow–Cohn–Vossen)**.** *Let $(M, g)$ be a Riemannian manifold, the following four assertions are equivalent:*

*(1) $M$ has the Heine–Borel property, i.e. every closed geodesic ball is compact;*

*(2) $M$ is geodesically complete;*

*(3) Every geodesic $\gamma : [0, a) \to M$ can be extended to a continuous curve $\overline{\gamma} : [0, a] \to M$;*

*(4) There is a point $p \in M$ such that every length-minimizing geodesic $\gamma : [0, a) \to M$ with $\gamma(0) = p$ can be extended to a continuous curve $\overline{\gamma} : [0, a] \to M$.*

We first establish several lemmas.

**Lemma 1.23.** *The length functional is lower semi-continuous in the following sense: let $\gamma_i, \gamma : [a, b] \to M$, if $\gamma_i \to \gamma$ pointwisely as $i \to \infty$, then*

$$L[\gamma] \leq \varliminf_{i\to\infty} L[\gamma_i].$$

*Proof.* Let $Y$ be a partition of $[a, b]$ with $a = y_0 < \cdots < y_N = b$, denote

$$\Sigma(Y) = \sum_{i=1}^{N} d(\gamma_(y_{i-1}), \gamma_(y_i)).$$

Take $\varepsilon > 0$ and fix a partition $Z$ such that $L[\gamma] - \Sigma(Z) < \varepsilon$. Now consider $\Sigma_j(Z)$ for curves $\gamma_j$ corresponding to same partition $Z$. Choose $j$ so large that the inequality $d(\gamma_j(z_i), \gamma(z_i)) < \varepsilon$ holds for all $z_i \in Z$. Then

$$L[\gamma] \leq \Sigma(Z) + \varepsilon \leq \Sigma_j(Z) + \varepsilon + (N + 1)\varepsilon \leq L[\gamma_j] + (N + 2)\varepsilon.$$

Since $\varepsilon$ is arbitrary, the lemma holds. □

**Lemma 1.24.** *Let $\gamma_i$ be length-minimizing curves for $i = 1, 2, \cdots$, suppose $\gamma_i \to \gamma$ pointwisely, then $\gamma$ also minimizes length.*

*Proof.* Let $\gamma$ has end points $p$ and $q$, then since $L[\gamma_i]$ equals to the distance between its end points, $L[\gamma_i] \to d(p, q)$. By the lower semi-continuity of length functional, we have

$$L[\gamma] \leq \varliminf_{i\to\infty} L[\gamma_i] = d(p, q) \leq L[\gamma].$$ □

**Lemma 1.25.** *Let $(M, g)$ be a compact Riemannian manifold (maybe with boundary), then any two points $p, q \in M$ can be joined by a length-minimizing curve.*

*Proof.* By the definition of distance, there exists a sequence of curves $\{\gamma_i\}$ with constant speed such that $\gamma_i(0) = p$, $\gamma_i(1) = q$ and $L[\gamma_i] \to d(p,q)$. Then let $L[\gamma_i] < A$ for all $i = 1, 2, \cdots$, we have $d(p,x) \le L[\gamma_i|_{[0,a]}] < A$ for $x = \gamma_i(a)$, hence the family $\{\gamma_i\}$ is uniformly bounded. Moreover, fix $\varepsilon > 0$, let $\delta = \varepsilon/A$, we have

$$d(x,y) \le L[\gamma_i|_{[a,b]}] \le A(b-a) < \varepsilon$$

provided $x = \gamma_i(a)$, $y = \gamma_i(b)$, and $b - a < \delta$ for any $i = 1, 2, \cdots$, hence the family $\{\gamma_i\}$ is equicontinuous. Then the family verifies the conditions of Arzela–Ascoli Theorem, it converges (up to a subsequence) to a curve $\gamma$. Then by the lower semi-continuity of length functional, we have

$$L[\gamma] \le \lim_{i\to\infty} L[\gamma_i] = d(p,q) \le L[\gamma].$$

Thus $\gamma$ is length-minimizing. □

*Proof of Theorem 1.22.* Implications (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) are all easy, we prove (4) $\Longrightarrow$ (1).

Let $R = \sup\{\overline{B_r(p)}$ is a compact set$\}$, then $R > 0$ since manifolds are locally compact, $\overline{B_r(p)}$ is compact for $r$ sufficiently small. We argue by contradiction. Suppose $R < +\infty$, that is, there exists noncompact geodesic balls. The argument is divided into two steps.

1. First we prove $B_R(p)$ is sequentially compact. Let $\{p_i\} \subset B_R(p)$, set $d(p,p_i) = r_i$. We may assume $r_i \to R$ as $i \to \infty$, otherwise $\{p_i\}$ is eventually contained in a smaller geodesic ball, and it has a convergent subsequence by the definition of $R$.

Now let $\gamma_i : [0, r_i] \to M$ be a length-minimizing curve joining $p$ and $p_i$, whose existence is guaranteed by Lemma 1.25. Notice that $\gamma_i$'s are parametrized by arc-length. We can choose a subsequence of $\{\gamma_i\}$ such that the restrictions of the curves to $[0, r_1]$ converge by Arzela–Ascoli Theorem. From this subsequence, we can choose a further subsequence such that the restrictions to $[0, r_2]$ converge, and so on. Then by Cantor diagonal procedure, we have a sequence $\{\gamma_{i_n}\}$ such that for $t \in [0, R)$, $\gamma_{i_n}(t)$ is well-defined for $n$ sufficiently large and $\gamma_{i_n}(t) \to \gamma(t)$ as $n \to \infty$. Moreover, Arzela–Ascoli Theorem asserts $\gamma$ is smooth for any restriction to $[0, r]$ provided $r < R$ by uniform convergence, hence $\gamma$ is smooth.

Now by Lemma 1.24, $\gamma$ is a length-minimizing curve, hence by the hypothesis (iv), $\gamma$ can be extended to a continuous curve $\overline{\gamma} : [0, R] \to M$. Then let $q = \overline{\gamma}(R)$, fix $\varepsilon > 0$, take $n$ sufficiently large such that $d(p_{i_n}, \gamma_{r_{i_n}}) < \varepsilon/2$ and $d(\gamma(r_{i_n}, q)) = R - r_{i_n} < \varepsilon/2$, we have

$$d(p_{i_n}, q) \le d(p_{i_n}, \gamma(r_{i_n})) + d(\gamma(r_{i_n}), q) < \varepsilon.$$

Hence $p_{i_n} \to q$ as $n \to \infty$, we have $B_R(p)$ is sequentially compact.

2. Since $B_R(p)$ is sequentially compact, we have $\overline{B_R(p)}$ is compact. Now we show $\overline{B_{R+\varepsilon}(p)}$ is compact for some $\varepsilon > 0$. Since $M$ is locally compact, for every $x \in \overline{B_R(0)}$ there is an $r(x) > 0$ such that $\overline{B_{r(x)}(x)}$ is compact. Then we can choose finite $x_i \in \overline{B_R(p)}$ such that $\{B_{r(x_i)}(x_i)\}$ covers $\overline{B_R(p)}$. The union of these geodesic balls is sequentially compact and contains the geodesic ball $B_{R+\varepsilon}(p)$ for $0 < \varepsilon < \min\{r(x_i)\}$. Hence $\overline{B_{R+\varepsilon}(p)}$ is compact, this contradicts the choice of $R$. □

We have two corollaries.

**Corollary 1.26** (Hopf–Rinow Theorem)**.** *Let $(M, g)$ be a Riemannian manifold, the following are equivalent:*

*(1) $M$ is geodesically complete;*

*(2) $\exp_p$ is defined on whole $T_pM$ for any $p \in M$;*

*(3) $\exp_p$ is defined on whole $T_pM$ for one $p \in M$.*

*Proof.* (1) $\implies$ (2): Let $S = \{r \in \mathbb{R}_{>0} : \gamma : [0, r)$ can be extended to $r\}$. We prove $S$ is both open and closed, then the implication holds. Let $r \in S$, for any $q \in \partial B_r(p)$, define $r(q)$ as follows: let unit-speed geodesic $\gamma : [0, r] \to M$ joins $p$ and $q$, then there exists a geodesic $\tilde{\gamma} : [r, r + r(q))$ such that $\tilde{\gamma}(r) = q$, $\dot{\tilde{\gamma}}(r) = \dot{\gamma}(r)$, that is, $\gamma$ can be extended to $[0, r + r(q))$. Cover $\partial B_r(p)$ by finite many $B_{r(q_i)}(q_i)$, then for $0 < \varepsilon < \min\{r(q_i)\}$, $B_{r+\varepsilon}(p)$ is in the union of $B_r(p)$ and $B_{r(q_i)}(q_i)$, hence any geodesics $\sigma : [0, r + \varepsilon)$ can be extended to $\sigma : [0, r + \varepsilon] \to M$. This implies $(r - \varepsilon, r + \varepsilon) \subset S$, $S$ is open.

Now assume $\{r_i\} \subset S$ converges to $r$. Let $\gamma : [0, r)$ be a geodesic, then define $\gamma(r) = \lim_{i\to\infty} \gamma(r_i)$, the limit exists since $M$ is complete. Hence $r \in S$, $S$ is closed.

(2) $\implies$ (3) is trivial.

(3) $\implies$ (1): If $\gamma : [0, a)$ is a length-minimizing geodesic with $\gamma(0) = p$, then $\gamma(t) = \exp_p(tv)$ for some $v$, and clearly it can be extended to $a$. □

**Corollary 1.27.** *If $M$ is geodesically complete, then any two points can be joined by a length-minimizing geodesic.*

*Proof.* Let $p, q \in M$, $R > d(p, q)$. Then $q \in \overline{B_R(p)}$, the latter one is bounded and closed, hence by Theorem 1.22, it is compact. By Lemma 1.25, there is a length-minimizing geodesic joining $p$ and $q$. □

# Chapter 2

# Curvature

## 2.1 Curvature Tensor and Curvaure Endomorphism

### Riemann Curvature Tensor

We calculated in Proposition 1.14 that $\exp_{p*}|_0$ is identity, naturally we have the following problem:

*Problem.* Calculate $\exp_{p*}|_v : T_v(T_pM) \to T_{\exp_p(v)}M$.

*Solution.* To evaluate $\exp_{p*}|_v(\xi)$, we choose a line $v + s\xi$, and then

$$\exp_{p*}|_v(\xi) = \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \exp_p(v + s\xi).$$

Now we can introduce the one parameter family of geodesics

$$\gamma(t,s) = \exp_p(t(v + s\xi)),$$

and denote $\gamma(t) = \gamma(t,0)$. We calculate the variation vector field $J(t)$ of $\gamma$, and obtain the result by taking $t = 1$. Let $J_s(t) = \frac{\partial}{\partial s}\gamma(t,s)$, then $\dot{J}_s(t) = \nabla_{\dot{\gamma}_s(t)}\frac{\partial \gamma}{\partial s}$. Since $\nabla_{\dot{\gamma}_s(t)}\frac{\partial \gamma}{\partial t} = 0$, we have

$$\begin{aligned}
\ddot{J}_s(t) &= \nabla_{\dot{\gamma}_s(t)}\nabla_{\dot{\gamma}_s(t)}\frac{\partial \gamma}{\partial s} \\
&= \nabla_{\dot{\gamma}_s(t)}\nabla_{\frac{\partial \gamma}{\partial s}}\frac{\partial \gamma}{\partial t} \\
&= \nabla_{\frac{\partial \gamma}{\partial t}}\nabla_{\frac{\partial \gamma}{\partial s}}\frac{\partial \gamma}{\partial t} - \nabla_{\frac{\partial \gamma}{\partial s}}\nabla_{\frac{\partial \gamma}{\partial t}}\frac{\partial \gamma}{\partial t}.
\end{aligned}$$

Moreover, we have $[\partial_t, \partial_s] = 0$, then we denote

$$R\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial s}\right) = \nabla_{\frac{\partial}{\partial s}} \nabla_{\frac{\partial}{\partial t}} - \nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial s}} + \nabla_{[\frac{\partial}{\partial t}, \frac{\partial}{\partial s}]},$$

we obtain

$$\ddot{J}_s(t) + R\left(\frac{\partial \gamma}{\partial t}, \frac{\partial \gamma}{\partial s}\right)\frac{\partial \gamma}{\partial t} = 0.$$

Take $s = 0$, we have

$$\ddot{J}(t) + R(\dot{\gamma}(t), J(t))\dot{\gamma}(t) = 0. \tag{2.1}$$

We will show later that (2.1) is a system of ordinary differential equations, hence by solving the system with given initial value and taking $t = 1$, we obtain the answer of the problem. □

We make it into a definition.

**Definition.** The *Riemann curvature tensor* $R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ is defined by

$$R(X, Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z.$$

*Remark* 2.1. Many authors define the Riemann curvature tensor as the negative of above definition, e.g. in [3]. Please be careful with the sign of the tensor.

We need to explain the name "tensor", so we must show $R$ is truly tensorial.

**Lemma 2.2.** *$R$ is a tensor.*

*Proof.* $R(X, Y)Z$ is clearly tensorial in $X$ and $Y$, we show that $R(X, Y)(fZ) = fR(X, Y)Z$ for $f \in C^\infty(M)$. We have

$$\begin{aligned}
\nabla_Y \nabla_X (fZ) &= \nabla_Y ((Xf)Z + f\nabla_X Z) \\
&= (YXf)Z + (Xf)\nabla_Y Z + (Yf)\nabla_X Z + f\nabla_Y \nabla_X Z \\
-\nabla_X \nabla_Y (fZ) &= -(XYf)Z - (Yf)\nabla_X Z - (Xf)\nabla_Y Z - f\nabla_X \nabla_Y Z \\
\nabla_{[X,Y]}(fZ) &= ([X, Y]f)Z + f\nabla_{X,Y} Z,
\end{aligned}$$

add all three equalities and we reach the conclusion. □

Now we can look at Riemann curvature tensor locally. A rather complicated calculation shows:

**Lemma 2.3.** *Let* $R = R_{ijk}{}^l \otimes \mathrm{d}x^i \otimes \mathrm{d}x^j \otimes \mathrm{d}x^k \otimes \partial_l$, *then*

$$R_{ijk}{}^l = \partial_j \Gamma^l_{ik} - \partial_i \Gamma^l_{jk} + \Gamma^m_{ik}\Gamma^l_{jm} - \Gamma^m_{jk}\Gamma^l_{im}.$$

We also define a $(0, 4)$-tensor by lowering the $l$ index of $Rlijk$, that is:

**Definition.** $R(X, Y, Z, W) := \langle R(X, Y)Z, W\rangle$ is also called *Riemann curvature tensor*.

**Example 2.4.** Euclidean space $(\mathbb{R}^n, \delta)$ has $R \equiv 0$. Any metric admits zero curvature is called *flat*.

## Curvature Endomorphism

Curvature also appears in another scene. Let us consider the second covariant differential of a tensor.

**Definition.** Let $T(\cdots)$ be a tensor, denote $(\nabla_{X,Y}T)(\cdots) := \nabla(\nabla T)(\cdots, Y, X)$.

**Proposition 2.5.** *For* $(r, s)$*-tensor* $T$*, we have*

$$\nabla_{X,Y}T = \nabla_X(\nabla_Y T) - \nabla_{\nabla_X Y}T. \tag{2.2}$$

*Proof.* Since covariant derivative commutes with contraction, we have

$$\begin{aligned}
\nabla_X(\nabla_Y T) &= \nabla_X(\mathrm{tr}_{1,s+2}\, Y \otimes \nabla T) \\
&= \mathrm{tr}_{1,s+2}(\nabla_X(Y \otimes \nabla T)) \\
&= \mathrm{tr}_{1,s+2}((\nabla_X Y) \otimes \nabla T + Y \otimes (\nabla_X \nabla T)) \\
&= \nabla_{\nabla_X Y}T + \nabla(\nabla T)(\cdots, Y, X) \\
&= \nabla_{\nabla_X Y}T + \nabla_{X,Y}T,
\end{aligned}$$

then the result follows. □

**Definition.** Define the *curvature endomorphism* $R(X, Y)$ on $(r, s)$-tensors by

$$R(X, Y)T = \nabla_Y \nabla_X T - \nabla_X \nabla_Y T + \nabla_{[X,Y]}T.$$

*Remark* 2.6. We need to show $R(X, Y)T$ is tensorial in $T$ so that $R(X, Y)$ is a well-defined endomorphism. This is similar to Lemma 2.2.

**Proposition 2.7.** *For any* $(r, s)$*-tensor* $T$*, we have the following* Ricci identity*:*

$$\nabla_{Y,X}T - \nabla_{X,Y}T = R(X, Y)T.$$

*Moreover, we have a explicit formula*

$$\begin{aligned}
&(R(X, Y)T)(\omega_1, \cdots, \omega_r, X_1, \cdots, X_s) \\
= &-\sum_{i=1}^{r} T(\omega_1, \cdots, R(X, Y)\omega_i, \cdots, \omega_r, X_1, \cdots, X_s) \\
&-\sum_{j=1}^{s} T(\omega_1, \cdots, \omega_r, X_1, \cdots, R(X, Y)X_j, \cdots, X_s)
\end{aligned} \tag{2.3}$$

*Proof.* Using equation (2.2), we have

$$\begin{aligned}\nabla_{Y,X}T - \nabla_{X,Y}T &= \nabla_Y\nabla_X T - \nabla_{\nabla_Y X}T - \nabla_X\nabla_Y T + \nabla_{\nabla_X Y}T \\ &= \nabla_Y\nabla_X T - \nabla_X\nabla_Y T + \nabla_{[X,Y]}T \\ &= R(X,Y)T,\end{aligned}$$

the second equality is torsion-freeness. Since $R(X,Y)$ clearly satisfies Leibniz Law and commutes with contraction, we can deduce the formula in the same way as covariant derivative. □

Ricci identity shows the curvature appears when we interchange the second covariant differential.

## Properties of Curvature

**Proposition 2.8.** *Riemann curvature tensor has following symmetric properties: For $X, Y, Z, W \in \mathfrak{X}(M)$, we have*

*(1)* $R(X,Y,Z,W) = -R(Y,X,Z,W) = -R(X,Y,W,Z)$;

*(2)* $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$ *(First Bianchi identity);*

*(3)* $R(X,Y,Z,W) = R(Z,W,X,Y)$;

*(4)* $(\nabla_X R)(Y,Z) + (\nabla_Y R)(Z,X) + (\nabla_Z R)(X,Y) = 0$ *(Second Bianchi identity).*

*Proof.* (1) The first equality is evident. We show the second equality. Consider the Hessian of $g(Z,W)$, then we have

$$\nabla^2 g(Z,W)(X,Y) = \langle\nabla^2_{Y,X}Z, W\rangle + \langle\nabla_X Z, \nabla_Y W\rangle + \langle\nabla_Y Z, \nabla_X W\rangle + \langle Z, \nabla^2_{Y,X}W\rangle.$$

Interchange $X, Y$, we have

$$\nabla^2 g(Z,W)(Y,X) = \langle\nabla^2_{X,Y}Z, W\rangle + \langle\nabla_Y Z, \nabla_X W\rangle + \langle\nabla_X Z, \nabla_Y W\rangle + \langle Z, \nabla^2_{X,Y}W\rangle.$$

These two equations must equal, hence

$$\langle\nabla^2_{Y,X}Z, W\rangle - \langle\nabla^2_{X,Y}Z, W\rangle = \langle Z, \nabla^2_{X,Y}W\rangle - \langle Z, \nabla^2_{Y,X}W\rangle,$$

this is equivalent to

$$R(X,Y,Z,W) = R(Y,X,W,Z).$$

(2) Since $R(X,Y)Z$ is tensorial, we can assume $X,Y,Z$ are frames. Then all Lie bracket between $X,Y,Z$ vanish, we have

$$\begin{aligned}\sum_{\text{cyc}} R(X,Y)Z =&\nabla_Y\nabla_X Z - \nabla_X\nabla_Y Z + \nabla_Z\nabla_Y X - \nabla_Y\nabla_Z X\\ &+ \nabla_X\nabla_Z Y - \nabla_Z\nabla_X Y\\ =&\nabla_Y[X,Z] + \nabla_Z[Y,X] + \nabla_X[Z,Y]\\ =&0.\end{aligned}$$

(3) By (1) and (2) we have

$$\begin{aligned}R(X,Y,Z,W) =&-R(Z,X,Y,W) - R(Y,Z,X,W)\\ =&R(Z,X,W,Y) + R(Y,Z,W,X)\\ =&-R(W,Z,X,Y) - R(X,W,Z,Y)\\ &-R(W,Y,Z,X) - R(Z,W,Y,X)\\ =&2R(Z,W,X,Y) + R(X,W,Y,Z) + R(W,Y,X,Z)\\ =&2R(Z,W,X,Y) - R(Y,X,W,Z)\\ =&2R(Z,W,X,Y) - R(X,Y,Z,W),\end{aligned}$$

which implies $2R(X,Y,Z,W) = 2R(Z,W,X,Y)$.
(4) By the definition of covariant derivative, we have

$$\begin{aligned}(\nabla_X R)(Y,Z) =&[\nabla_X, R(Y,Z)] - R(\nabla_X Y, Z) - R(Y, \nabla_X Z)\\ =&[\nabla_X, \nabla_{[Y,Z]}] - [\nabla_X, [\nabla_Y, \nabla_Z]] - R(\nabla_X Y, Z) - R(Y, \nabla_X Z)\\ =&\nabla_{[X,[Y,Z]]} - [\nabla_X, [\nabla_Y, \nabla_Z]] - R(X, [Y,Z])\\ &- R(\nabla_X Y, Z) - R(Y, \nabla_X Z).\end{aligned}$$

Take summation cyclically, we see that terms involving $\nabla$ vanish because of Jacobi identity. Moreover, since $\nabla_X Y - \nabla_Y X = [X,Y]$, the terms involving $R$ also vanish. □

## 2.2 Sectional, Ricci and Scalar Curvature

We now define some special curvatures. Fix a Riemannian manifold $(M,g)$.

**Definition.** Let $p \in M$, $\pi \subset T_pM$ be a 2-plane, $\pi = \text{Span}\{u,v\}$. Then define the *sectional curvature* of $\pi$ at $p$ to be

$$\text{Sect}_p(\pi) = \frac{R_p(u,v,u,v)}{|u|^2|v|^2 - \langle u,v\rangle^2}.$$

*Remark* 2.9. One can show that sectional curvature does not depend on the choice of basis. For a proof, see [2, Proposition 3.1].

**Proposition 2.10.** *The sectional curvature determines the curvature tensor.*

*Proof.* Let $R, R'$ have same sectional curvature, denote $\tilde{R} = R - R'$, then $\widetilde{\mathrm{Sect}} = 0$. We show that $\tilde{R} = 0$. First, we have

$$\begin{aligned} \tilde{R}(X,Y,X,W) &= \tilde{R}(X,Y-W,X,W) + \tilde{R}(X,W,X,W) \\ &= \tilde{R}(X,Y-W,X,W-Y) + \tilde{R}(X,Y-W,X,Y) \\ &= \tilde{R}(X,Y,X,Y) - \tilde{R}(X,W,X,Y) \\ &= \tilde{R}(X,Y,X,W), \end{aligned}$$

hence $\tilde{R}(X,Y,X,W) = 0$. Then we have

$$\begin{aligned} \tilde{R}(X,Y,Z,W) &= \tilde{R}(X,Y,Z,W-X) + \tilde{R}(X,Y,Z,X) \\ &= \tilde{R}(X-W,Y,Z,W-X) + \tilde{R}(W,Y,Z,W-X) \\ &= \tilde{R}(W,Y,Z,W) - \tilde{R}(W,Y,Z,X) \\ &= -\tilde{R}(W,Y,Z,X). \end{aligned}$$

By the same reasoning, we have

$$\begin{aligned} \tilde{R}(X,Y,Z,W) &= \tilde{R}(Z,W,X,Y) \\ &= -\tilde{R}(Y,W,X,Z) \\ &= -\tilde{R}(X,Z,Y,W). \end{aligned}$$

Passing the result to the $(1,3)$-tensor, by first Bianchi identity, we have

$$\tilde{R}(X,Z)Y + \tilde{R}(Y,Z)X + \tilde{R}(Z,X)Y = 0,$$

which implies $\tilde{R}(Y,Z)X = 0$. Then

$$\begin{aligned} 0 &= \tilde{R}(Y,Z,X,W) \\ &= -\tilde{R}(W,Z,X,Y) \\ &= -\tilde{R}(X,Y,W,Z) \\ &= \tilde{R}(X,Y,Z,W). \end{aligned}$$

Thus we proved $\tilde{R} = 0$, and the result follows. □

We mention here a little observation.

**Lemma 2.11.** *If $\mathrm{Sect}_p$ is constant for all 2-planes in $T_pM$, say $K_p$, then we have*

$$R_p(X,Y,Z,W) = K_p(\langle X,Z\rangle\langle Y,W\rangle - \langle X,W\rangle\langle Y,Z\rangle).$$

Then we can prove the following Schur's Theorem.

**Theorem 2.12** (Schur)**.** *Let $(M^n, g)$ be a Riemannian manifold with $n \geq 3$. If $\mathrm{Sect}_p(\pi)$ is independent from $\pi \subset T_pM$ for all $p \in M$, then $M$ has constant sectional curvature.*

*Proof.* Since the tensor

$$R'(X,Y,Z,W) = \langle X,Z\rangle\langle Y,W\rangle - \langle X,W\rangle\langle Y,Z\rangle$$

evidently satisfies the Proposition 2.8, Proposition 2.10 shows $R = fR'$ for some $f \in C^\infty(M)$. We show that $f$ is constant. Since $n \geq 3$, there exists three orthonormal vectors $X, Y, Z$. Take $W$ arbitrary, then by second Bianchi identity, we have

$$(\nabla_X R)(Y,Z,W) + (\nabla_Y R)(Z,X,W) + (\nabla_Z R)(X,Y,W) = 0.$$

Since $\nabla_X R = \nabla_X(fR') = (Xf)R' + f(\nabla_X R')$, by taking summation cyclically we obtain

$$(Xf)R'(Y,Z)W + (Yf)R'(Z,X)W + (Zf)R'(X,Y)W = 0.$$

Since the sectional curvature is constant for any $\pi \subset T_pM$, the "sectional curvature" corresponding to $R'$ is also constant, hence we can use Lemma 2.11 to obtain

$$\begin{aligned} 0 =& (Xf)K(\langle Y,W\rangle Z - \langle Z,W\rangle Y) \\ &+ (Yf)K(\langle Z,W\rangle X - \langle X,W\rangle Z) \\ &+ (Zf)K(\langle X,W\rangle Y - \langle Y,W\rangle X), \end{aligned}$$

which is equivalent to

$$\begin{aligned} 0 =& ((Xf)\langle Y,W\rangle - (Yf)\langle X,W\rangle)Z \\ &+ ((Yf)\langle Z,W\rangle - (Zf)\langle Y,W\rangle)X \\ &+ ((Zf)\langle X,W\rangle - (Xf)\langle Z,W\rangle)Y. \end{aligned}$$

Since $X, Y, Z$ are orthonormal, the coefficient of $X, Y, Z$ must all equal to 0. Thus by taking $W = Y$, we obtain

$$Xf = (Xf)\langle Y,Y\rangle = (Yf)\langle X,Y\rangle = 0.$$

Since $X$ is arbitrary, we have $f \equiv \text{const}$. This deduces $M$ has constant sectional curvature. □

**Definition.** Let $R_{ijk}{}^{l}$ be the local expression of Riemann curvature tensor, then we define the *Ricci curvature* by

$$\mathrm{Ric}_j{}^l = \mathrm{tr}_{13} R_{ijk}{}^l.$$

Equivalently, let $\{e_i\}$ be an orthonormal basis at $p$, then

$$\mathrm{Ric}_p(X) = \sum_{i=1}^{n} R(e_i, X)e_i.$$

Clearly, Ric is a self-adjoint linear transformation by Proposition 2.8 (3).

**Definition.** We define the *scalar curvature* by taking trace of Ric, or equivalently

$$\mathrm{Scal}(p) = \sum_{i=1}^{n} \langle \mathrm{Ric}_p(e_i), e_i \rangle$$

for an orthonormal basis $\{e_i\}$ at $p$.

**Definition.** A Riemannian manifold $(M, g)$ is called an *Einstein manifold* if $\mathrm{Ric} = \lambda\delta$ for some $\lambda \in C^\infty(M)$.

We have another theorem of Schur on Einstein manifolds.

**Theorem 2.13** (Schur). *Let $M^n$ be an Einstein manifold with $n \geq 3$, then $M$ has constant scalar curvature.*

First we need a lemma.

**Lemma 2.14.** *For any metric $g$, we have*

$$2\,\mathrm{tr}_{13} \nabla \mathrm{Ric} = \mathrm{d}\,\mathrm{Scal}\,.$$

*Proof.* We check the equation locally. The second Bianchi identity can be written as

$$R_{ijm}{}^l{}_{;k} + R_{kim}{}^l{}_{;j} + R_{jkm}{}^l{}_{;i} = 0,$$

or equivalently

$$R_{ijm}{}^l{}_{;k} - R_{ikm}{}^l{}_{;j} + R_{jkm}{}^l{}_{;i} = 0.$$

Multiply $g^{im}$ we obtain

$$0 = \mathrm{Ric}_j{}^l{}_{;k} - \mathrm{Ric}_k{}^l{}_{;j} + R_{jk}{}^{il}{}_{;i}.$$

Let $l = j$, we obtain

$$\begin{aligned} 0 &= \mathrm{Ric}_j{}^j{}_{;k} - \mathrm{Ric}_k{}^j{}_{;j} - R_{kj}{}^{ij}{}_{;i} \\ &= \mathrm{Scal}_{;k} - \mathrm{Ric}_k{}^j{}_{;j} - \mathrm{Ric}_k{}^i{}_{;i} \\ &= \mathrm{Scal}_{;k} - 2\mathrm{Ric}_k{}^i{}_{;i}. \end{aligned}$$

Then the result follows. □

*Proof of Theorem 2.13.* Let $\mathrm{Ric} = \lambda\delta$, then by the lemma,

$$\begin{aligned} \mathrm{d}\,\mathrm{Scal} &= 2\,\mathrm{tr}_{1,3}\,\nabla\,\mathrm{Ric} \\ &= 2\,\mathrm{tr}_{1,3}\,\nabla(\lambda\delta) \\ &= 2\,\mathrm{tr}_{1,3}(\delta \otimes \mathrm{d}\lambda) \\ &= 2\,\mathrm{d}\lambda. \end{aligned}$$

However, we have

$$\begin{aligned} \mathrm{d}\,\mathrm{Scal} &= \mathrm{d}\,\mathrm{tr}_{1,2}\,\mathrm{Ric} \\ &= \mathrm{d}\lambda\,\mathrm{tr}_{1,2}\,\delta \\ &= n\,\mathrm{d}\lambda. \end{aligned}$$

This means $(n-2)\,\mathrm{d}\,\mathrm{Scal} = 0$, which implies $\mathrm{Scal} \equiv \text{const}$ for $n \geq 3$. □

## 2.3 Jacobi Fields

At the beginning of Section 2.1, we introduced the one parameter family of geodesics $\gamma(t,s) = \exp_p(t(v + s\xi))$. It is a variation of curve $\gamma(t) = \gamma(t,0)$, and its variation vector field $J$ satisfies (2.1). Motivated by this, we give the definition of Jacobi fields.

**Definition.** Let $\gamma$ be a geodesic, a vector field $J$ along $\gamma$ is called a *Jacobi field*, if $\ddot{J}(t) + R(\dot\gamma(t), J(t))\dot\gamma(t) = 0$.

Let $\{e_i(t)\}$ be a parallel frame along $\gamma$, and $J(t) = f^i e_i$. Define $a^i_j(t) = R(\dot\gamma(t), e_i(t), \dot\gamma(t), e_j(t))$, then the equation of Jacobi field is equivalent to

$$\ddot{f}^i(t) + a^i_j(t) f^j(t) = 0,\ i = 1, \cdots, n.$$

Hence (2.1) is indeed a system of ODE. By ODE theory, given $f^i(0), \dot{f}^i(0)$ $i = 1, \cdots, n$, the $f^i(t)$'s are uniquely determined. Translate into geometric language, a Jacobi field $J$ is determined by $J(0)$ and $\dot{J}(0)$.

We summarize above discussion.

**Proposition 2.15.** *Let $\mathscr{J}\gamma$ be the vector space of all Jacobi fields along $\gamma$, then*

$$\dim \mathscr{J}(\gamma) = 2n.$$

The proposition below shows only Jacobi fields that perpendicular to $\gamma$ are interesting.

**Proposition 2.16.** *Let $\gamma$ be a geodesic, $J$ is a Jacobi field along $\gamma$. Then we have the decomposition*

$$J(t) = J^{\perp}(t) + (at+b)\dot{\gamma}(t),$$

*where $J^{\perp}(t) \perp \dot{\gamma}(t)$. If a Jacobi field is perpendicular to $\gamma$, then we call it a* normal Jacobi field.

*Proof.* We have

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\langle J(t), \dot{\gamma}(t)\rangle &= \langle \ddot{J}(t), \dot{\gamma}(t)\rangle \\ &= -\langle R(\dot{\gamma}(t), J(t))\dot{\gamma}(t), \dot{\gamma}(t)\rangle \\ &= 0.\end{aligned}$$

□

At the beginning of Section 2.1, we see that a one parameter family of geodesics gives rise to a Jacobi field. The converse is also true.

**Proposition 2.17.** *Let $J$ be a Jacobi field, then $J$ is the variation field of some one parameter family of geodesics.*

*Proof.* Given a Jacobi field $J$, it is determined by $J(0)$ and $\dot{J}(0)$. Let $\zeta(s)$ be the geodesic with initial tangent vector $J(0)$, and $T(s)$ be its tangent vector field. Let $W(s)$ be parallel along $\zeta$ and $W(0) = \dot{J}(0)$. Now set $\gamma(t,s) = \exp_{\zeta(s)}(t(T(s) + sW(s)))$. Then

$$\left[\frac{\partial}{\partial t}, \frac{\partial}{\partial s}\right] = \gamma_*[\partial_t, \partial_s] = 0,$$

hence we can interchange the partial derivative. Let $U$ be the variation field of $\gamma(t,s)$, then $U$ is a Jacobi field. We verify $U$ and $J$ has same initial value. We have

$$\begin{aligned}U(0) &= \left.\frac{\partial}{\partial s}\right|_{s=0} \gamma(0,s) \\ &= \left.\frac{\partial}{\partial s}\right|_{s=0} \exp_{\zeta(s)}(0) \\ &= \left.\frac{\partial}{\partial s}\right|_{s=0} \zeta(s) \\ &= \dot{\zeta}(0) = J(0),\end{aligned}$$

and

$$\begin{aligned}\dot{U}(0) &= \frac{\partial}{\partial t}\bigg|_{t=0}\frac{\partial}{\partial s}\bigg|_{s=0}\exp_{\zeta(s)}(t(T(s)+sW(s)))\\ &= \frac{\partial}{\partial s}\bigg|_{t=0}\frac{\partial}{\partial t}\bigg|_{s=0}\exp_{\zeta(s)}(t(T(s)+sW(s)))\\ &= \frac{\partial}{\partial s}\bigg|_{s=0}\exp_{\zeta s*}|_0(T(s)+sW(s))\\ &= \frac{\partial}{\partial s}\bigg|_{s=0}(T(s)+sW(s))\\ &= W(0)=\dot{J}(0).\end{aligned}$$

□

Now we can completely answer the calculation problem of differential of exponential map.

**Proposition 2.18.** *Let a Jacobi field $J$ along $\gamma(t)=\exp_p(tv)$ satisfy $J(0)=0$, $\dot{J}(0)=\xi$, then $J(t)=\exp_{p*}|_{tv}(t\xi)$.*

*Proof.* Let $J$ be the variation vector field of $\gamma(t,s)=\exp_p(t(v+s\xi))$. Then

$$\begin{aligned}J(t) &= \frac{\partial}{\partial s}\exp_p(t(v+s\xi))\\ &= \exp_{p*}|_{tv}(t\xi).\end{aligned}$$

Now consider the initial condition for $J$, we have

$$J(0)=\exp_{p*}|_0(0)=0,$$

and

$$\begin{aligned}\dot{J}(0) &= \frac{\partial}{\partial t}\bigg|_{t=0}\frac{\partial}{\partial s}\bigg|_{s=0}\exp_p(t(v+s\xi))\\ &= \dot{J}(0)=\frac{\partial}{\partial s}\bigg|_{s=0}\frac{\partial}{\partial t}\bigg|_{t=0}\exp_p(t(v+s\xi))\\ &= \frac{\partial}{\partial s}\bigg|_{s=0}\exp_{p*}|_0(v+s\xi)\\ &= \xi.\end{aligned}$$

Then the conclusion follows by the uniqueness of Jacobi fields. □

**Proposition 2.19** (Gauss Lemma). $\langle\exp_{p*}|_v(\xi),\dot{\gamma}_v(1)\rangle=\langle\xi,v\rangle$.

*Proof.* Let $J$ be a Jacobi field along $\gamma_v$ with $J(0) = 0$, $\dot{J}(0) = \xi$. Then by Proposition 2.18, we have

$$\langle \exp_{p*}|_v(\xi), \dot{\gamma}_v(1)\rangle = \langle J(1), \dot{\gamma}_v(1)\rangle.$$

We differentiate above inner product, obtaining

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle J(t), \dot{\gamma}_v(t)\rangle = \langle \dot{J}(t), \dot{\gamma}_v(t), \rangle$$

and

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle \dot{J}(t), \dot{\gamma}_v(t)\rangle &= \langle \ddot{J}(t), \dot{\gamma}_v(t)\rangle \\ &= -\langle R(\dot{\gamma}_v, J)\dot{\gamma}_v, \dot{\gamma}_v\rangle \\ &= 0.\end{aligned}$$

Hence $\langle \dot{J}(t), \dot{\gamma}_v(t)\rangle$ is constant. Clearly $\langle \dot{J}(0), \dot{\gamma}_v(0)\rangle = \langle \xi, v\rangle$, then we have

$$\begin{aligned}\langle J(1), \dot{\gamma}_v(1)\rangle - \langle J(0), \dot{\gamma}_v(0)\rangle &= \int_0^1 \langle \xi, v\rangle \,\mathrm{d}t \\ &= \langle \xi, v\rangle.\end{aligned}$$

Notice that $\langle J(0), \dot{\gamma}_v(0)\rangle = 0$ and we obtain the result. □

Finally we consider the case where $\exp_{p*}$ degenerates.

**Definition.** Let $\gamma$ be a geodesic starting at $p$, if $\exp_{p*}|_{\gamma(t_0)}$ is degenerate at $\gamma(t_0)$, then $\gamma(t_0)$ is called a *conjugate point* of $p$.

A simple criterion for conjugate points is

**Proposition 2.20.** $\exp_{p*}$ *degenerates at* $\gamma(t_0)$ *if and only if there is a nontrivial Jacobi field* $J$ *with* $J(0) = J(t_0) = 0$.

*Proof.* $\exp_{p*}|_{\gamma(t_0)}(\xi) = 0$ if and only if Jacobi field $J(t) = \frac{\partial}{\partial s}\exp_p(t(v + s\xi))$ satisfies $J(0) = J(t_0) = 0$. □

*Remark* 2.21. This proposition shows conjugate is symmetric.

## 2.4 Cartan–Hadamard Theorem

In this section we prove the celebrated Cartan–Hadamard Theorem.

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