A Concise Note on Differential Geometry

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Preface

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Chapter 1

Metric Structure

In this chapter we introduce the metric structure on a Riemannian manifold, i.e. using the metric tensor to make the manifold into a metric space. We will prove that the topology induced by metric coincides with the topology carried by the manifold. Then we will study the length-minimizing problem: which curve minimize the distance between two points? The answer is two-sided: we will prove that length-minimizing curves are geodesics, and geodesics are *locally* length-minimizing.

1.1 Metric Structure on Riemannian Manifolds

Let (M, g) be a Riemannian manifold, $\gamma : [0, 1] \to M$ be a regular curve (i.e. γ is an immersion). We define the length functional of regular curves.

Definition. The *length functional* $L[\cdot]$ on the set of regular curves is defined by

$$L[\gamma] = \int_0^1 \sqrt{g(\dot{\gamma}(t), \dot{\gamma}(t))} \, \mathrm{d}t.$$

It is a well-known result from calculus that any regular curve can be reparametrized by arc-length. From now on the word "curve" means a regular curve.

Now we define the distance between two points.

Definition. Let $p, q \in M$ be two points on M, then we define their *distance* by

$$d(p, q) = \inf_{\gamma \in C_{p,q}} L[\gamma],$$

where $C_{p,q}$ is the set of all regular curves joining p and q.

Proposition 1.1. *The distance function $d : M \times M \to \mathbb{R}$ has the following properties:*

(1) $d(p,q) \geq 0$, *and* $d(p,q) = 0 \iff p = q$;

(2) $d(p,q) = d(q,p)$;

(3) $d(p,r) \leq d(p,q) + d(q,r)$.

Thus the distance function makes M into a metric space.

Proof. Only need to show $d(p,q) = 0 \iff p = q$, all else are trivial. We assume $p \neq q$, need to show $d(p,q) > 0$. Let $\gamma : [0,1] \to M$ be any curve joining p and q. Choose a local chart (U, φ) such that $\varphi(U) = B_r(0)$, $q \neq U$. By Jordan–Brouwer Separation Theorem, γ must intersect ∂U at $s := \gamma(c)$. Then we have

$$L[\gamma] \geq L[\gamma|_{[0,c]}] = \int_0^c \sqrt{g_{ij}\dot{x}^i(\gamma(t))\dot{x}^j(\gamma(t))}\,\mathrm{d}t.$$

Regarding $g : \overline{U} \times \mathbb{S}^{n-1} \to \mathbb{R}$, g is a continuous function on a compact set, thus it attains its minimum $g(x)(v,v) \geq m$, and $m > 0$ since $v \in \mathbb{S}^{n-1} \neq 0$. Thus we have

$$L[\gamma] \geq L[\gamma|_{[0,c]}] \geq m \int_0^c |\dot{x}(\gamma(t))|\,\mathrm{d}t \geq mr > 0. \tag{1.1}$$

mr does not depend on γ, hence $d(p,q) \geq mr > 0$. □

However, the metric space topology is nothing but the original topology carried by the manifold.

Proposition 1.2. *The metric space topology on M coincides with the manifold topology.*

We first need a lemma.

Lemma 1.3. *The distance function to p defined by $r(q) = d(p,q)$ is continuous with respective to the manifold topology.*

Proof. Since manifolds satisfy the second countable axiom, the Sequence Lemma holds. Then it's equivalent to show for any $q_i \to q$ in manifold topology, we have $r(q_i) \to r(q)$. Without loss of generality we can assume $\{q_i\} \subset U$ and (U, φ) is a local chart such that $\varphi(q) = 0$, $\varphi(U) = B_r(0)$. Let δ be the Euclidean metric on $B_r(0)$, then by regarding g as a continuous function on $\overline{U} \times \mathbb{S}^{n-1}$ again, we have $g \leq M\delta$ for some $M > 0$. By assumption, $q_i \to q$ in manifold topology implies $L_\delta[\psi_i] \to 0$, where $\psi_i(t) = t\varphi(q_i)$, the radial line joining $\varphi(q)$ and $\varphi(q_i)$ in $\varphi(U) = B_r(0)$. Let $\varphi^{-1}\psi_i = \gamma_i$, then we have

$$L_g[\gamma_i] = \int_0^1 \sqrt{g(\dot{\gamma}_i(t), \dot{\gamma}_i(t))}\,\mathrm{d}t \leq M \int_0^1 \sqrt{\delta(\dot{\psi}_i(t), \dot{\psi}_i(t))}\,\mathrm{d}t \leq M L_\delta[\psi_i].$$

Since r is Lipschitz, i.e. $|r(q) - r(s)| \leq d(q,s)$, we have

$$d(q_i, q) \leq L_g[\gamma_i] \leq ML_\delta[\psi_i] \to 0,$$

hence r is continuous. $\square$

Proof of Proposition 1.2. Since distance function is continuous, metric balls are open in manifold topology. Now we prove the converse.

Let U be open with respective to manifold topology. Let $p \in U$, V be a neighborhood of p so small that $\varphi(V) = B_r(0)$ for some $r > 0$. The estimate (1.1) shows if $q \notin V$ then $d(p,q) \geq mr$ for some fixed $m > 0$, then by taking contrapositive statement, we have $q \in V$ if $d(p,q) < mr$. Therefore $B_p(mr) \subset U$, then U is open with respective to metric space topology. $\square$

1.2 Length-Minimizing Curves

This section and the next is guided by the following problem:

Problem. Let p, q be two distinct points on a Riemannian manifold (M, g), then what is the curve γ satisfying $L[\gamma] = d(p,q)$?

This section we will show if the length-minimizing curve exists, then it is a *geodesic*. Next section we will show if p, q are sufficiently closed, *the* geodesic joining p and q is length-minimizing. Finally, the existence of length-minimizing curves is related to Hopf-Rinow Theorem, which we will discuss at the last section of the chapter.

To find the length-minimizing curve, we "gather" curves with same initial and end points, which is called a variation.

Definition. Let $\gamma_0 : [0,a] \to M$ be a curve, a *variation* of γ_0 is a differentiable map $\gamma : [0,a] \times (-\varepsilon, \varepsilon) \to M$ such that $\gamma(t,0) = \gamma_0(t)$. If $\gamma(0,s) = \gamma_0(0)$ and $\gamma(a,s) = \gamma_0(a)$ for any $s \in (-\varepsilon, \varepsilon)$, then we call the variation a *proper variation*. We call $\left.\frac{\partial}{\partial s}\right|_{s=0} \gamma(s,t) =: V(t)$ the *variation vector field*.

Now we introduce the energy functional, which is easier to calculate.

Definition. The *energy functional* on the set of curves is defined by

$$E[\gamma] = \int_0^a \frac{1}{2} |\dot{\gamma}(t)|^2 \, \mathrm{d}t,$$

where $\gamma : [0,a] \to M$ is a regular curve.

We will prove that a curve is energy-minimizing if and only if it is length-minimizing.

Lemma 1.4. *For a curve $\gamma : [0, a] \to M$, we have*

$$L^2[\gamma] \leq 2aE[\gamma],$$

with equality holds if and only if $|\dot{\gamma}(t)| = \text{const}$.

Proof. This is Cauchy–Schwarz inequality. □

Proposition 1.5. *If γ is length-minimizing, then it is energy-minimizing.*

Proof. Let $\tilde{\gamma}$ be another curve, then we have

$$2aE[\gamma] = L^2[\gamma] \leq L^2[\tilde{\gamma}] \leq 2aE[\tilde{\gamma}]. \qquad \square$$

Our aim is to prove the converse.

Proposition 1.6. *If γ is an energy-minimizing curve, then it is length-minimizing.*

To prove this, we need to differentiate the variation.

Proposition 1.7 (First variation formula)**.** *Let $\gamma(t, s)$ be a variation, define its energy* $E(s) = \int_0^a \frac{1}{2} \left| \frac{\partial}{\partial t} \gamma(t, s) \right|^2 \mathrm{d}t$, *then we have*

$$E'(0) = \boxed{\langle V, \dot{\gamma} \rangle |_0^a} - \int_0^a \langle V(t), \nabla_{\dot{\gamma}_0(t)} \dot{\gamma}_0(t) \rangle.$$

The boxed term is called boundary term, *and it vanishes when the variation is proper.*

Proof. This is a calculation. We have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s} E(s) &= \int_0^a \left\langle \frac{\partial \gamma}{\partial t}, \nabla_{\frac{\partial}{\partial s}} \frac{\partial \gamma}{\partial t} \right\rangle \mathrm{d}t \\
&= \int_0^a \left\langle \frac{\partial \gamma}{\partial t}, \nabla_{\frac{\partial}{\partial t}} \frac{\partial \gamma}{\partial s} \right\rangle \mathrm{d}t \\
&= \int_0^a \frac{\partial}{\partial t} \left\langle \frac{\partial \gamma}{\partial t}, \frac{\partial \gamma}{\partial s} \right\rangle - \left\langle \frac{\partial \gamma}{\partial s}, \nabla_{\frac{\partial}{\partial t}} \frac{\partial \gamma}{\partial t} \right\rangle \mathrm{d}t.
\end{aligned}$$

Take $s = 0$, we obtain

$$\begin{aligned}
E'(0) &= \int_0^a \frac{\partial}{\partial t} \langle V(t), \dot{\gamma}_0(t) \rangle - \langle V(t), \nabla_{\dot{\gamma}_0(t)} \dot{\gamma}_0(t) \rangle \, \mathrm{d}t \\
&= \langle V, \dot{\gamma}_0 \rangle |_0^a - \int_0^a \langle V(t), \nabla_{\dot{\gamma}_0(t)} \dot{\gamma}_0(t) \rangle \, \mathrm{d}t. \qquad \square
\end{aligned}$$

Now we can derive the definition of a geodesic.

Definition. A curve $\gamma : [0, a]$ is called a *geodesic* if $\nabla_{\dot{\gamma}(t)}\dot{\gamma}(t) = 0$ for $t \in [0, a]$.

Remark 1.8. Geodesics are constant speed, this can be shown by $\frac{\mathrm{d}}{\mathrm{d}t}|\dot{\gamma}(t)|^2 = 2\langle\dot{\gamma}(t), \nabla_{\dot{\gamma}(t)}\dot{\gamma}(t)\rangle = 0$.

Corollary 1.9. *γ is a critical value for all proper variation if and only if $\nabla_{\dot{\gamma}(t)}\dot{\gamma}(t) = 0$, that is, γ is a geodesic.*

We can give the proof of Proposition 1.6 now.

Proof of Proposition 1.6. Let $\gamma : [0, a] \to M$ be a curve such that for any $\tilde{\gamma} : [0, a] \to M$ with $\tilde{\gamma}(0) = \gamma(0)$, $\tilde{\gamma}(1) = \gamma(1)$, the inequality $E[\gamma] \leq E[\tilde{\gamma}]$ holds, we show that $L[\gamma] \leq L[\tilde{\gamma}]$. Let $\gamma(t, s)$ be any proper variation with $\gamma(t, 0) = \gamma(t)$, then γ is a critical point of $E(s)$. Hence by Corollary 1.9, γ is a geodesic. Now we can reparametrize $\tilde{\gamma}$ into arc-length, obtaining $\hat{\tilde{\gamma}}$. Therefore

$$L^2[\gamma] = 2aE[\gamma] \leq 2aE\left[\hat{\tilde{\gamma}}\right] = L^2\left[\hat{\tilde{\gamma}}\right] = L^2[\tilde{\gamma}],$$

which implies $L[\gamma] \leq L[\tilde{\gamma}]$. □

Combining all results above, we have

Proposition 1.10. *If a curve is length-minimizing, then it is a geodesic.*

1.3 Geodesics and Exponential Maps

To prove the local length-minimizing property of geodesic, we need to introduce the exponential map.

We first need to investigate the equation that determine a geodesic.

Proposition 1.11. *Given $p \in M$ and $v \in T_pM$, there exists a unique geodesic γ (whose domain may not be maximal) such that $\gamma(0) = p$, $\dot{\gamma}(0) = v$.*

Proof. Let (U, φ) be a local chart containing p, compose φ with γ we obtain coordinate curves x^i's. Then the geodesic equation is equivalent to

$$\ddot{x}^k(t) + \Gamma^k_{ij}(\gamma(t))\dot{x}^i(t)\dot{x}^j(t) = 0,\ k = 1, \cdots, n.$$

This is a system of second order ordinary differential equations, by the unique existence theorem of ODE, the solution is completely determined by x^i's and $\dot{x}^i$'s, that is, p and v. □

Since the solution of an ODE relies continuously on its initial value, we have the following proposition.

Proposition 1.12. *For any $p \in M$, there exists a neighborhood V of p, such that there exists $\delta > 0$, $\varepsilon > 0$ and a differentiable map $\gamma : (-\delta, \delta) \times \mathscr{U} \to M$, where $\mathscr{U} = \{(q, v) \in TM : q \in V, v \in T_qM, |v| < \varepsilon\}$, such that $\gamma(t; q, v)$ is a geodesic with $\gamma(0) = q$, $\dot{\gamma}(0) = v$.*

A proof can be found in [1, Chapter 3, Lemma 1].

Observe that $\gamma(\lambda t; p, v) = \gamma(t; p, \lambda v)$. Denote $\gamma(t; p, v)$ by $\gamma_v(t)$, then above observation can be written as $\gamma_{\lambda v}(t) = \gamma_v(\lambda t)$. Therefore, we can shorten the initial vector to lengthen the domain of geodesic.

Definition. Let $U \subset T_pM$ be a neighborhood of origin, such that for any $v \in U$, $\gamma_v(1)$ is defined (existence is guaranteed by Proposition 1.12). We define the *exponential map* at p by

$$\begin{aligned} \exp_p : U &\to M \\ v &\mapsto \gamma_v(1). \end{aligned}$$

Remark 1.13. We can scale the initial vector and obtain

$$\exp_p(v) = \gamma_v(1) = \gamma_{v/|v|}(|v|).$$

This means the action of exponential map on v is to move forward the distance $|v|$ along the geodesic with initial direction $v/|v|$.

Proposition 1.14. $\exp_{p*}|_0 : T_0(T_pM) \to T_pM$ *is identity (we identify $T_0(T_pM)$ with T_pM).*

Proof. We have

$$\exp_{p*}|_0(v) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}(tv) = v. \qquad \square$$

Corollary 1.15. *There exists a ball $B_\varepsilon(0) \subset T_pM$ such that* $\exp_p : B_\varepsilon(0) \to M$ *is a diffeomorphism onto its image.*

Proof. Since $\exp_{p*}|_0$ is identity, it is nondegenerate, the corollary follows from Inverse Function Theorem. $\square$

Example 1.16. (1) We know that the geodesics on S^n are great circles, hence $\exp_p$ is defined on the whole T_pM. But $\exp_p$ is not injective, since

$$\exp_p(0) = \exp(2\pi v) = p$$

for unit vector v in T_pM.

(2) Let $M = \mathbb{S}^1 \times \mathbb{R}$ be the cylinder. We know from elementary differential geometry that the geodesics on cylinder are directrix circles, helices and generatrix lines. Then in local charts $(e^{2\pi i t}, s) \mapsto (t, s)$, we know $\exp_p$ is not injective in the direction $(1, 0)$, and injective in other directions.

We postpone the discussion on whether the exponential map can be defined on the whole tangent space, the answer is Hopf–Rinow Theorem, which will be discussed in next section.

Now we prove that geodesics are locally length-minimizing. For this, we introduce some local charts. Given a Riemannian manifold (M, g) and $p \in M$, let $\exp_p : B_\varepsilon(0) \to \exp(B_\varepsilon(0)) = B_\varepsilon(p)$ be a diffeomorphism.

Definition. We define *geodesic normal coordinate* as follows: Let $\{e_i\}$ be an orthonormal basis of Euclidean space (T_pM, δ), $\{\alpha^i\}$ be its dual basis. Then we define the coordinate by

$$q \in B_\varepsilon(p) \mapsto (\alpha^1(\exp_p^{-1}(q)), \cdots, \alpha^n(\exp_p^{-1}(q))).$$

Proposition 1.17. *Under geodesic normal coordinate, we have*

$$g_{ij}(p) = \delta_{ij}, \ \Gamma_{ij}^k(p) = 0.$$

Proof. Since $\exp_p$ is a diffeomorphism, we have $\frac{\partial}{\partial x^i}\Big|_p = \exp_{p*}|_0(e_i) = e_i$, hence $g_{ij} = \delta(e_i, e_j) = \delta_{ij}$. Moreover, let $x(t) = ty$ for $y \in T_pM - \{0\}$, then $x(t)$ is the coordinate of some geodesic in $B_\varepsilon(p)$, thus it satisfies the equation

$$\ddot{x}^k(t) + \Gamma_{ij}^k(x(t))\dot{x}^i(t)\dot{x}^j(t) = 0.$$

Since $\dot{x}^i = y^i \neq 0$, $\ddot{x}^k = 0$, we have $\Gamma_{ij}^k(ty) = 0$. Let $y \to 0$ we obtain the conclusion. $\square$

Next we introduce the geodesic polar coordinate.

Definition. We define *geodesic polar coordinate* as follows: Let $(r, \theta^1, \cdots, \theta^{n-1})$ be a polar coordinate on Euclidean space (T_pM, δ), then we defined the coordinate by

$$q \in B_\varepsilon(p) - \{p\} \mapsto (r(\exp_p^{-1}(q)), \theta^1(\exp_p^{-1}(q)), \cdots, \theta^{n-1}(\exp_p^{-1}(q))).$$

Proposition 1.18. *Under geodesic polar coordinate, we have*

$$\left\langle \frac{\partial}{\partial r}, \frac{\partial}{\partial r} \right\rangle = 1, \ \left\langle \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta^i} \right\rangle = 0.$$

Proof. To make things clear, we write the inverse of geodesic polar coordinate as

$$F : (r, \omega) \mapsto \exp_p(r\omega)$$

for $r \in (0, \varepsilon)$, $\omega \in \mathbb{S}^{n-1}$. Then we use $\partial_0, \partial_1, \cdots, \partial_{n-1}$ to denote the tangent vectors in $(0, \varepsilon) \times \mathbb{S}^{n-1}$, we have

$$\begin{aligned}
\frac{\partial}{\partial r} &= F_*(\partial_0), \\
\frac{\partial}{\partial \theta^i} &= F_*(\partial_i),\ i = 1, \cdots, n-1.
\end{aligned}$$

First we know that ∂_0 is the tangent vector of radial line $r\omega$, hence $\partial/\partial r$ is the tangent vector of a unit-speed radial geodesic, that is,

$$\left\langle \frac{\partial}{\partial r}, \frac{\partial}{\partial r} \right\rangle = 1.$$

Moreover, we have

$$\begin{aligned}
\frac{\partial}{\partial r}\left\langle \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta^i} \right\rangle &= \left\langle \nabla_{\frac{\partial}{\partial r}} \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta^i} \right\rangle + \left\langle \frac{\partial}{\partial r}, \nabla_{\frac{\partial}{\partial r}} \frac{\partial}{\partial \theta^i} \right\rangle \\
&= \left\langle \frac{\partial}{\partial r}, \nabla_{\frac{\partial}{\partial r}} \frac{\partial}{\partial \theta^i} \right\rangle \\
&= \left\langle \frac{\partial}{\partial r}, \nabla_{\frac{\partial}{\partial \theta^i}} \frac{\partial}{\partial r} \right\rangle \\
&= \frac{1}{2} \frac{\partial}{\partial \theta^i} \left\langle \frac{\partial}{\partial r}, \frac{\partial}{\partial r} \right\rangle \\
&= 0,
\end{aligned}$$

hence $\left\langle \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta^i} \right\rangle$ is constant. However, if we let $r \to 0$, we have $\partial/\partial\theta^i \to 0$, therefore

$$\left\langle \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta^i} \right\rangle = 0.$$

□

Corollary 1.19. *Under geodesic polar coordinate, the metric tensor has local expression*

$$g = \mathrm{d}r^2 + g_{ij}(r, \theta)\, \mathrm{d}\theta^i \otimes \mathrm{d}\theta^j,$$

where $[g_{ij}]_{i,j>0}$ *is positive definite.*

As an application, we prove that geodesics are locally length-minimizing as we promised.

Proposition 1.20. *Let $\gamma : [0,1] \to M$ be a geodesic contained in an open set U, where geodesic polar coordinate is defined on U. Let $\tilde{\gamma}$ be any curve contained in U with $\tilde{\gamma}(0) = \gamma(0) = p$, $\tilde{\gamma}(1) = \gamma(1) = q$. Then $L[\gamma] \le L[\tilde{\gamma}]$.*

Proof. Let $q = \exp_p(v)$, φ is the geodesic polar coordinate. Then we have

$$\gamma(t) = \varphi(tr_0, \omega_0),\ \tilde{\gamma}(t) = \varphi(r(t), \omega(t))$$

such that $r(1) = r_0$, $\omega(t) \in \mathbb{S}^{n-1}$. Therefore

$$\begin{aligned} L[\gamma] &= \int_0^1 |\dot{\gamma}(t)|\, \mathrm{d}t \\ &= \int_0^1 |v|\, \mathrm{d}t = r_0, \\ L[\tilde{\gamma}] &= \int_0^1 (|\dot{r}^2(t) + g_{ij}\dot{\theta}^i(t)\dot{\theta}^j(t))^{1/2} \\ &\ge \int_0^1 |\dot{r}(t)|\, \mathrm{d}t \\ &\ge \int_0^1 \dot{r}(t)\, \mathrm{d}t = r_0. \end{aligned}$$

□

Remark 1.21. The hypothesis of Proposition 1.20 can be weakened to $\exp_p$ is an immersion on U.

Bibliography

[1] Hung-Hsi Wu, Chunli Shen, and Yanlin Yu. *Introduction to Riemannian Geometry*. Higher Education Press, 2014.