Notes on
Advanced Methods in Geometric Analysis

Last compile: 4 March 2025

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Contents

Section 1. Maximum Principle

In this section, we discuss the maximum principle for linear operator, weak and strong solutions.

First we give the definition of ellipticity. Assume $\Omega \subset \mathbb{R}^n$ is a bounded domain, and

$$Lu = a^{ij}(x)\nabla_i\nabla_j u + b^i(x)\nabla_i u + c(x)u,$$

the (x) in the coefficients indicates they only depend on x. Moreover, we assume $a^{ij} = a^{ji}$. Let $\lambda(x)$ and $\Lambda(x)$ be the smallest and greatest eigenvalue of a^{ij} respectively.

Definition 1.1. We say the operator L is

(1) *elliptic* if $\lambda(x) > 0$;
(2) *strictly elliptic* if $\lambda(x) \geq \lambda_0$ for some $\lambda_0 > 0$;
(3) *uniformly elliptic* if L is elliptic and $\lambda_1 \leq \frac{\Lambda(x)}{\lambda(x)} \leq \lambda_2$ for some $\lambda_1, \lambda_2 > 0$.

In the following context, we shall assume that $a^{ij}(x)$, $b^i(x)$ and $c(x)$ are continuous on $\bar{\Omega}$, which implies L is uniformly elliptic.

Now we can discuss the weak maximum principle.

Theorem 1.2. *Assume $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$, $Lu \geq 0$, L is uniformly elliptic and $c \equiv 0$. Then*

$$\sup_{\Omega} u = \sup_{\partial\Omega} u.$$

Proof. Let's first assume that $Lu > 0$. If $x_0 \in \Omega^\circ$ is the maximal point of u, then we have

$$\nabla u(x_0) = 0,\ \nabla_i\nabla_j u(x_0) \leq 0.$$

This implies $a^{ij}\nabla_i\nabla_j u(x_0) \leq 0$, which can imply that $Lu(x_0) \leq 0$, contradiction!

To handle $Lu \geq 0$, we consider an auxilary function

$$\tilde{u}(x) = u(x) + \varepsilon e^{Ax^1},$$

where x^1 is the first component of $\mathbf{x}$, and ε, A need to be determined. We compute the operation of L on $\tilde{u}$:

$$\begin{aligned}
L\tilde{u}(x) &= Lu(x) + \varepsilon L(e^{Ax^1}) \\
&= Lu(x) + \varepsilon(a^{11}(x)\nabla_1\nabla_1(e^{Ax^1}) + b^1(x)\nabla_1(e^{Ax^1})) \\
&= Lu(x) + \varepsilon(A^2 e^{Ax^1} a^{11}(x) + b^1(x)Ae^{Ax^1}) \\
&= Lu(x) + \varepsilon A e^{Ax^1}(Aa^{11}(x) + b^1(x)),
\end{aligned}$$

here $a^{11}(x) \geq c_0 > 0$ since a^{ij} is positive definite, $b^1(x) \geq -c_1$. So choose A so large that $Ac_0 - c_1 > 0$, we have

$$L\tilde{u}(x) \geq \varepsilon A(c_0 A - c_1) > 0,$$

then by the first paragraph of this proof, we have $\sup_{\Omega}\tilde{u}(x) = \sup_{\partial\Omega}\tilde{u}(x)$. Let $\varepsilon \to 0$, we obtain $\sup_{\Omega} u(x) = \sup_{\partial\Omega} u(x)$. •

Remark 1.3. If the condition is $Lu \leq 0$, then the conclusion is changed to $\inf_{\Omega} u = \inf_{\partial\Omega} u$.

Corollary 1.4. *Assume $\Omega \subset \mathbb{R}^n$ is a bounded domain, $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$, L is (uniformly) elliptic. If $Lu \geq 0$, $c \leq 0$, then*

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} \max\{u, 0\}.$$

Proof. Consider $\Omega^+ \subset \Omega$ on which $u > 0$, then

$$L_0 u = a^{ij}\nabla_i\nabla_j u + b^i\nabla_i u \geq -cu(x) \geq 0.$$

Apply weak maximum principle (Theorem 1.2) then we obtain the conclusion. •

Remark 1.5. If $Lu \leq 0$ and $c \leq 0$, then $\inf_{\Omega} u \geq \inf_{\partial\Omega}\min\{u, 0\}$; If $Lu = 0$, then $\sup_{\Omega}|u| = \sup_{\partial\Omega}|u|$.

An application of weak maximum principle is uniqueness of solution, or comparison principle.

Proposition 1.6. *Let L be elliptic on Ω, $c \le 0$, $u, v \in C^2(\Omega) \cap C^0(\bar{\Omega})$. Then*

(1) *If $Lu = Lv$ on Ω and $u = v$ on $\partial\Omega$, then $u \equiv v$ on Ω;*
(2) *If $Lu \ge Lv$ on Ω, $u \le v$ on $\partial\Omega$, then $u \le v$ on Ω.*

These two statements are one-line corollaries of Theorem 1.2 and Corollary 1.4.

Next we discuss the strong maximum principle. Simply speaking, strong maximum principle asserts that if u attains its maximum at an interior point, then $u \equiv \text{const}$. To develop the strong maximum principle, we first need the Hopf lemma.

Lemma 1.7. *Let L be uniformly elliptic, $Lu \ge 0$, $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ and $x_0 \in \partial\Omega$. We assume:*

(1) *either $c = 0$ or $c \le 0$, $u(x_0) \ge 0$;*
(2) *at x_0, we can find $B(y, R) \subset \Omega$ such that $x_0 \in \partial B(y, R)$;*
(3) *for all $x \in \Omega$, $u(x) < u(x_0)$,*

then $\frac{\partial u}{\partial \mathbf{n}}(x_0) > 0$, where $\mathbf{n}$ is the outer normal vector.

Proof. Let $0 < r < R$, consider the annulus

$$A(r, R) = B(y, R) - B(y, r).$$

Let $v(x) = e^{-A\rho^2} - e^{-AR^2}$, where $\rho = |x - y| > r$, then

$$v(x) = \begin{cases} \ge 0 \text{ on } A(r, R), \\ \quad 0 \text{ on } \partial B(y, R). \end{cases}$$

Direct calculation gives

$$\begin{aligned} Lv(x) &= e^{-A\rho^2}(4A^2 a^{ij}(x_i - y_i)(x_j - y_j) - 2A(a^{ii} + b^i(x_i - y_i))) + cv(x) \\ &\ge e^{-A\rho^2}(4A^2\lambda(x)\rho^2 - 2A(a^{ii} + |\mathbf{b}|\rho) + c), \end{aligned}$$

where $\mathbf{b} = (b^1, \cdots, b^n)$. By uniformly ellipticity, a^{ii}/λ, $|\mathbf{b}|/\lambda$ and c/λ are bounded, so we can choose A so large that $Lv \ge 0$ on the annulus $A(r, R)$. Since $u(x) - u(x_0) < 0$ on $\partial B(y, r)$, there is a constant ε such that $u(x) - u(x_0) + \varepsilon v(x) \le 0$ on $\partial B(y, r)$. This inequality also holds on $\partial B(y, R)$ since on which $v = 0$. Hence we have $L(u(x) - u(x_0) + \varepsilon v(x)) \ge -cu(x_0) \ge 0$ in $A(r, R)$, and $u(x) - u(x_0) + \varepsilon v(x) \le 0$ on $\partial A(r, R)$. Then by Corollary 1.4, this implies $u(x) - u(x_0) + \varepsilon v(x) \le 0$ on whole $A(r, R)$. Taking the normal derivative at x_0, we obtain

$$\frac{\partial u}{\partial \mathbf{n}}(x_0) \ge -\varepsilon \frac{\partial v}{\partial \mathbf{n}}(x_0) = -\varepsilon v'(R) > 0$$

as required. •

Now we can state the strong maximum principle.

Theorem 1.8. *Let L be uniformly elliptic, $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$, $Lu \ge 0$. Assume either $c = 0$ or $c \le 0$ and c/λ is bounded, then if u attains maximum at $x_0 \in \Omega$, then $u \equiv \text{const}$.*

Proof. Suppose u is not constant and attains maximum in an interior point. If $x_0 \in \Omega$ is the maximum point, then we can find an interior ball $B(y, R) \subset \Omega$ such that $x_0 \in B(y, R)$ and $u(x) < u(x_0)$ in Ω. Then by Hopf lemma (Lemma 1.7), $\frac{\partial u}{\partial \mathbf{n}}(x_0) > 0$, contradict to $\nabla u(x_0) = 0$. •

Remark 1.9. The condition $Lu \ge 0$ often comes from some curvature condition.

An important application of maximum principle is C^0-estimate.

Theorem 1.10. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $Lu = f$, L elliptic, $c \leq 0$ and $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$. Then there exists a constant C depending on Ω and the coefficients of L, such that*

$$\sup_{\Omega} |u| \leq \sup_{\partial\Omega} |u| + C \sup_{\Omega} |f|,$$

where $C = C(\mathrm{diam}(\Omega), L)$.

Proof. Without loss of generality we assume Ω lies in $\{\mathbf{x}: 0 < x^1 < d\}$ for some $d > 0$. Let

$$v(x) = \sup_{\partial\Omega} |u| + (e^{\alpha d} - e^{\alpha x^1}) \sup_{\Omega} |f|,$$

where $\alpha > 0$. Thus for $\lambda\alpha^2 + b^1\alpha \geq 1$, we have

$$\begin{aligned} Lv &= -(a^{11}\alpha^2 + b^1\alpha)e^{\alpha x^1} \sup_{\Omega} |f| + cv(x) \\ &\leq -(\lambda\alpha^2 + b^1\alpha)e^{\alpha x^1} \sup_{\Omega} |f| \\ &\leq -\sup_{\Omega} |f| \\ &\leq f. \end{aligned}$$

Moreover, $v \geq \sup_{\partial\Omega} |u|$, hence we have

$$\begin{cases} Lv \leq f \text{ in } \Omega \\ \quad v \geq u \text{ on } \partial\Omega, \end{cases}$$

then by comparison principle (Proposition 1.6), we obtain $v \geq u$ in Ω. Hence for $C \geq \sup_{\Omega}(e^{\alpha d} - e^{\alpha x^1})$, which only depends on $\mathrm{diam}(\Omega)$, we have $\sup_{\Omega} |u| \leq \sup_{\partial\Omega} |u| + C \sup_{\Omega} |f|$. •

We also consider elliptic operator of divergence from

$$Lu = \nabla_i (a^{ij} \nabla_j u), \tag{1.1}$$

ellipticity is defined as before, depending on the positivity of a^{ij}.

Definition 1.11. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, L defined as (1.1). We say $Lu = 0$ $(\geq 0, \leq 0)$ in the weak sense if for any $v \in C_0^1(\Omega)$ the equality

$$\int_{\Omega} a^{ij} \nabla_j u \nabla_i v = 0 \ (\leq 0, \geq 0)$$

holds.

Now we discuss the weak maximum principle for weak solutions.

Theorem 1.12. *Assume $a^{ij}(x) \in C^1(\Omega)$ and $u \in C^2(\Omega)$. If $Lu \geq 0$ in the weak sense, then*

$$\sup_{\Omega} u = \sup_{\partial\Omega} u.$$

Proof. Assume $\sup_{\Omega} u > \sup_{\Omega u} = u_0$. Let $x_0 \in \Omega$ such that $u(x_0) > u_0$. Choose $\Omega^1 \subset\subset \Omega$ and c such that $u(x) - u(x_0) - c > 0$ on Ω^1 and $u(x) - u(x_0) - c = 0$ on $\partial\Omega^1$. Let $\varphi(x) = u(x) - u(x_0) - c$ and

$$v(x) = \begin{cases} \varphi(x) \text{ in } \Omega^1, \\ \quad 0 \text{ on } \Omega \setminus \Omega^1. \end{cases}$$

By assumption $Lu \geq 0$, we have

$$\begin{aligned} 0 &\geq \int_\Omega a^{ij}\nabla_i u \nabla_j v \\ &\geq \int_\Omega \lambda |\nabla v|^2. \end{aligned}$$

Hence $\nabla v = 0$, $c = \text{const}$ on Ω^1. But $v = 0$ on $\partial\Omega^1$, contradicts to $v = \varphi > 0$ on Ω^1. •

We weaken the assumption of C^∞-ness of coefficent and u. There are two cases. The first one is

$$u \in W^{1,2}(\Omega) = \{f \in L^2(\Omega)\colon \int |\nabla f|^2 < +\infty\}$$

and assume

$$Lu = \nabla_i(a^{ij}\nabla_j u + b^i u) + c^i \nabla_i u + du.$$

Assume L has bounded coefficient, that is,

$$\sum |a^{ij}|^2 \leq \Lambda$$

and

$$\frac{1}{\lambda^2}\left(\sum(|b^i|^2 + |c^i|^2)\right) + \frac{1}{\lambda}|d|^2 \leq A.$$

We say u satisfies $Lu \geq 0$ in the weak sense if for all $v \in C_0^1(\Omega)$, we have

$$\int (a^{ij}\nabla_j u + b^1 u)\nabla_i v - (c^i \nabla_i u + du)v \leq 0.$$

Theorem 1.13. *Under above assumption, $Lu \geq 0$ implies*

$$\sup_\Omega u \leq \sup_{\partial\Omega} u^+.$$

Proof. See [GT01, Theorem 8.1]. •

Remark 1.14. If $Lu = g + \nabla_i f^i$ has weak solution $u \in W^{1,2}(\Omega)$, $g \in L^{q/2}(\Omega)$ and $f^i \in L^q(\Omega)$ provided $q > n$, then from the Moser iteration, we can obtain L^∞-bound of u.

The second case is $u \in W^{2,n}_{\text{loc}}(\Omega) \cap C^0(\bar{\Omega})$. Consider the equation $Lu = f$ for $Lu = a^{ij}\nabla_i\nabla_j u + b^i \nabla_i u + cu$, such u is called a *strong solution*. Then we have the Alexander–Backerman–Pucci maximum principle.

Theorem 1.15. *Let $Lu \geq f$, L elliptic, $D = \det a^{ij}$ and $D^* = D^{1/n}$. Assume $\frac{|b|}{D^*}, \frac{|f|}{D^*} \in L^n(\Omega)$, $c \leq 0$ in Ω, $u \in W^{2,n}_{\text{loc}}(\Omega) \cap C^0(\bar{\Omega})$, then*

$$\sup_\Omega u \leq \sup_{\partial\Omega} u^+ + \tilde{C}\|f\|_{L^n}.$$

Proof. See [GT01, Theorem 9.1]. •

Remark 1.16. The ABP maximum principle has a geometric proof. It often applies to non-divergence type of elliptic PDEs like Monge–Amperé equations.

Now we discuss some practical useful principle on manifolds.

(1) Let (M, g) be a compact Riemannian manifold, $u \in C^2(M)$. If x_0 is the maximum point of u, then $\nabla u(x_0) = 0$ and $e^\varphi \Delta e^{-\varphi} u(x_0) < 0$. Hence we can estimate $\tilde{\Delta} u(x)$ from below to get

$$\tilde{\Delta} u(x) \geq f(x, u, \nabla u).$$

(2) Let (M, g) be a noncompact complete Riemannian manifold. Then the Omori–Yau maximum principle tells that for $u^* = \sup u$, there exists a sequence of points $\{x_k\}$ such that

$$u(x_k) > u^* - \frac{1}{k}.\ |\nabla u|(x_k) < \frac{1}{k}.\ \Delta u(x_k) < \frac{1}{k}.$$

Bibliography

[GT01] David Gilbarg and Neil S. Trudinger. *Elliptic partial differential equations of second order*. Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.

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