Notes on
Advanced Methods in Geometric Analysis

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Contents

Lecture 1. Maximum Principle

In this lecture, we discuss the maximum principle for linear operator, weak and strong solutions.

First we give the definition of ellipticity. Assume $\Omega \subset \mathbb{R}^n$ is a bounded domain, and

$$Lu = a^{ij}(x)\nabla_i\nabla_j u + b^i(x)\nabla_i u + c(x)u,$$

the (x) in the coefficients indicates they only depend on x. Moreover, we assume $a^{ij} = a^{ji}$. Let $\lambda(x)$ and $\Lambda(x)$ be the smallest and greatest eigenvalue of a^{ij} respectively.

Definition 1.1. We say the operator L is

(1) *elliptic* if $\lambda(x) > 0$;
(2) *strictly elliptic* if $\lambda(x) \geq \lambda_0$ for some $\lambda_0 > 0$;
(3) *uniformly elliptic* if L is elliptic and $\lambda_1 \leq \frac{\Lambda(x)}{\lambda(x)} \leq \lambda_2$ for some $\lambda_1, \lambda_2 > 0$.

In the following context, we shall assume that $a^{ij}(x)$, $b^i(x)$ and $c(x)$ are continuous on $\bar{\Omega}$, which implies L is uniformly elliptic.

Now we can discuss the weak maximum principle.

Theorem 1.2. *Assume $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$, $Lu \geq 0$, L is uniformly elliptic and $c \equiv 0$. Then*

$$\sup_{\Omega} u = \sup_{\partial\Omega} u.$$

Proof. Let's first assume that $Lu > 0$. If $x_0 \in \Omega^\circ$ is the maximal point of u, then we have

$$\nabla u(x_0) = 0,\ \nabla_i\nabla_j u(x_0) \leq 0.$$

This implies $a^{ij}\nabla_i\nabla_j u(x_0) \leq 0$, which can imply that $Lu(x_0) \leq 0$, contradiction!

To handle $Lu \geq 0$, we consider an auxilary function

$$\tilde{u}(x) = u(x) + \varepsilon e^{Ax^1},$$

where x^1 is the first component of $\mathbf{x}$, and ε, A need to be determined. We compute the operation of L on $\tilde{u}$:

$$\begin{aligned}
L\tilde{u}(x) &= Lu(x) + \varepsilon L(e^{Ax^1}) \\
&= Lu(x) + \varepsilon(a^{11}(x)\nabla_1\nabla_1(e^{Ax^1}) + b^1(x)\nabla_1(e^{Ax^1})) \\
&= Lu(x) + \varepsilon(A^2e^{Ax^1}a^{11}(x) + b^1(x)Ae^{Ax^1}) \\
&= Lu(x) + \varepsilon Ae^{Ax^1}(Aa^{11}(x) + b^1(x)),
\end{aligned}$$

here $a^{11}(x) \geq c_0 > 0$ since a^{ij} is positive definite, $b^1(x) \geq -c_1$. So choose A so large that $Ac_0 - c_1 > 0$, we have

$$L\tilde{u}(x) \geq \varepsilon A(c_0A - c_1) > 0,$$

then by the first paragraph of this proof, we have $\sup_{\Omega}\tilde{u}(x) = \sup_{\partial\Omega}\tilde{u}(x)$. Let $\varepsilon \to 0$, we obtain $\sup_{\Omega} u(x) = \sup_{\partial\Omega} u(x)$. •

Remark 1.3. If the condition is $Lu \leq 0$, then the conclusion is changed to $\inf_{\Omega} u = \inf_{\partial\Omega} u$.

Corollary 1.4. *Assume $\Omega \subset \mathbb{R}^n$ is a bounded domain, $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$, L is (uniformly) elliptic. If $Lu \geq 0$, $c \leq 0$, then*

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} \max\{u, 0\}.$$

Proof. Consider $\Omega^+ \subset \Omega$ on which $u > 0$, then

$$L_0 u = a^{ij}\nabla_i\nabla_j u + b^i\nabla_i u \geq -cu(x) \geq 0.$$

Apply weak maximum principle (Theorem 1.2) then we obtain the conclusion. •

Remark 1.5. If $Lu \leq 0$ and $c \leq 0$, then $\inf_{\Omega} u \geq \inf_{\partial\Omega}\min\{u, 0\}$; If $Lu = 0$, then $\sup_{\Omega}|u| = \sup_{\partial\Omega}|u|$.

An application of weak maximum principle is uniqueness of solution, or comparison principle.

Proposition 1.6. *Let L be elliptic on Ω, $c \le 0$, $u, v \in C^2(\Omega) \cap C^0(\bar{\Omega})$. Then*

(1) *If $Lu = Lv$ on Ω and $u = v$ on $\partial\Omega$, then $u \equiv v$ on Ω;*
(2) *If $Lu \ge Lv$ on Ω, $u \le v$ on $\partial\Omega$, then $u \le v$ on Ω.*

These two statements are one-line corollaries of Theorem 1.2 and Corollary 1.4.

Next we discuss the strong maximum principle. Simply speaking, strong maximum principle asserts that if u attains its maximum at an interior point, then $u \equiv \text{const}$. To develop the strong maximum principle, we first need the Hopf lemma.

Lemma 1.7. *Let L be uniformly elliptic, $Lu \ge 0$, $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ and $x_0 \in \partial\Omega$. We assume:*

(1) *either $c = 0$ or $c \le 0$, $u(x_0) \ge 0$;*
(2) *at x_0, we can find $B(y, R) \subset \Omega$ such that $x_0 \in \partial B(y, R)$;*
(3) *for all $x \in \Omega$, $u(x) < u(x_0)$,*

then $\frac{\partial u}{\partial \mathbf{n}}(x_0) > 0$, where $\mathbf{n}$ is the outer normal vector.

Proof. Let $0 < r < R$, consider the annulus

$$A(r, R) = B(y, R) - B(y, r).$$

Let $v(x) = e^{-A\rho^2} - e^{-AR^2}$, where $\rho = |x - y| > r$, then

$$v(x) = \begin{cases} \ge 0 \text{ on } A(r, R), \\ \quad 0 \text{ on } \partial B(y, R). \end{cases}$$

Direct calculation gives

$$\begin{aligned} Lv(x) &= e^{-A\rho^2}(4A^2 a^{ij}(x_i - y_i)(x_j - y_j) - 2A(a^{ii} + b^i(x_i - y_i))) + cv(x) \\ &\ge e^{-A\rho^2}(4A^2\lambda(x)\rho^2 - 2A(a^{ii} + |\mathbf{b}|\rho) + c), \end{aligned}$$

where $\mathbf{b} = (b^1, \cdots, b^n)$. By uniformly ellipticity, a^{ii}/λ, $|\mathbf{b}|/\lambda$ and c/λ are bounded, so we can choose A so large that $Lv \ge 0$ on the annulus $A(r, R)$. Since $u(x) - u(x_0) < 0$ on $\partial B(y, r)$, there is a constant ε such that $u(x) - u(x_0) + \varepsilon v(x) \le 0$ on $\partial B(y, r)$. This inequality also holds on $\partial B(y, R)$ since on which $v = 0$. Hence we have $L(u(x) - u(x_0) + \varepsilon v(x)) \ge -cu(x_0) \ge 0$ in $A(r, R)$, and $u(x) - u(x_0) + \varepsilon v(x) \le 0$ on $\partial A(r, R)$. Then by Corollary 1.4, this implies $u(x) - u(x_0) + \varepsilon v(x) \le 0$ on whole $A(r, R)$. Taking the normal derivative at x_0, we obtain

$$\frac{\partial u}{\partial \mathbf{n}}(x_0) \ge -\varepsilon \frac{\partial v}{\partial \mathbf{n}}(x_0) = -\varepsilon v'(R) > 0$$

as required. •

Now we can state the strong maximum principle.

Theorem 1.8. *Let L be uniformly elliptic, $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$, $Lu \ge 0$. Assume either $c = 0$ or $c \le 0$ and c/λ is bounded, then if u attains maximum at $x_0 \in \Omega$, then $u \equiv \text{const}$.*

Proof. Suppose u is not constant and attains maximum in an interior point. If $x_0 \in \Omega$ is the maximum point, then we can find an interior ball $B(y, R) \subset \Omega$ such that $x_0 \in B(y, R)$ and $u(x) < u(x_0)$ in Ω. Then by Hopf lemma (Lemma 1.7), $\frac{\partial u}{\partial \mathbf{n}}(x_0) > 0$, contradict to $\nabla u(x_0) = 0$. •

Remark 1.9. The condition $Lu \ge 0$ often comes from some curvature condition.

An important application of maximum principle is C^0-estimate.

Theorem 1.10. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $Lu = f$, L elliptic, $c \le 0$ and $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$. Then there exists a constant C depending on Ω and the coefficients of L, such that*

$$\sup_{\Omega} |u| \le \sup_{\partial\Omega} |u| + C \sup_{\Omega} |f|,$$

where $C = C(\mathrm{diam}(\Omega), L)$.

Proof. Without loss of generality we assume Ω lies in $\{\mathbf{x}\colon 0 < x^1 < d\}$ for some $d > 0$. Let

$$v(x) = \sup_{\partial\Omega} |u| + (e^{\alpha d} - e^{\alpha x^1}) \sup_{\Omega} |f|,$$

where $\alpha > 0$. Thus for $\lambda\alpha^2 + b^1\alpha \ge 1$, we have

$$\begin{aligned} Lv &= -(a^{11}\alpha^2 + b^1\alpha)e^{\alpha x^1} \sup_{\Omega} |f| + cv(x) \\ &\le -(\lambda\alpha^2 + b^1\alpha)e^{\alpha x^1} \sup_{\Omega} |f| \\ &\le -\sup_{\Omega} |f| \\ &\le f. \end{aligned}$$

Moreover, $v \ge \sup_{\partial\Omega} |u|$, hence we have

$$\begin{cases} Lv \le f \text{ in } \Omega \\ \quad v \ge u \text{ on } \partial\Omega, \end{cases}$$

then by comparison principle (Proposition 1.6), we obtain $v \ge u$ in Ω. Hence for $C \ge \sup_{\Omega}(e^{\alpha d} - e^{\alpha x^1})$, which only depends on $\mathrm{diam}(\Omega)$, we have $\sup_{\Omega} |u| \le \sup_{\partial\Omega} |u| + C \sup_{\Omega} |f|$. •

We also consider elliptic operator of divergence from

$$Lu = \nabla_i(a^{ij}\nabla_j u), \tag{1.1}$$

ellipticity is defined as before, depending on the positivity of a^{ij}.

Definition 1.11. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, L defined as (1.1). We say $Lu = 0$ ($\ge 0, \le 0$) in the weak sense if for any $v \in C_0^1(\Omega)$ the equality

$$\int_{\Omega} a^{ij}\nabla_j u \nabla_i v = 0 \ (\le 0, \ge 0)$$

holds.

Now we discuss the weak maximum principle for weak solutions.

Theorem 1.12. *Assume $a^{ij}(x) \in C^1(\Omega)$ and $u \in C^2(\Omega)$. If $Lu \ge 0$ in the weak sense, then*

$$\sup_{\Omega} u = \sup_{\partial\Omega} u.$$

Proof. Assume $\sup_{\Omega} u > \sup_{\Omega u} = u_0$. Let $x_0 \in \Omega$ such that $u(x_0) > u_0$. Choose $\Omega^1 \subset\subset \Omega$ and c such that $u(x) - u(x_0) - c > 0$ on Ω^1 and $u(x) - u(x_0) - c = 0$ on $\partial\Omega^1$. Let $\varphi(x) = u(x) - u(x_0) - c$ and

$$v(x) = \begin{cases} \varphi(x) \text{ in } \Omega^1, \\ \quad 0 \text{ on } \Omega \setminus \Omega^1. \end{cases}$$

By assumption $Lu \geq 0$, we have

$$0 \geq \int_\Omega a^{ij}\nabla_i u \nabla_j v$$
$$\geq \int_\Omega \lambda |\nabla v|^2.$$

Hence $\nabla v = 0$, $c = \text{const}$ on Ω^1. But $v = 0$ on $\partial\Omega^1$, contradicts to $v = \varphi > 0$ on Ω^1. •

We weaken the assumption of C^∞-ness of coefficent and u. There are two cases. The first one is

$$u \in W^{1,2}(\Omega) = \{f \in L^2(\Omega) \colon \int |\nabla f|^2 < +\infty\}$$

and assume

$$Lu = \nabla_i(a^{ij}\nabla_j u + b^i u) + c^i \nabla_i u + du.$$

Assume L has bounded coefficient, that is,

$$\sum |a^{ij}|^2 \leq \Lambda$$

and

$$\frac{1}{\lambda^2}\left(\sum (|b^i|^2 + |c^i|^2)\right) + \frac{1}{\lambda}|d|^2 \leq A.$$

We say u satisfies $Lu \geq 0$ in the weak sense if for all $v \in C_0^1(\Omega)$, we have

$$\int (a^{ij}\nabla_j u + b^1 u)\nabla_i v - (c^i \nabla_i u + du)v \leq 0.$$

Theorem 1.13. *Under above assumption, $Lu \geq 0$ implies*

$$\sup_\Omega u \leq \sup_{\partial\Omega} u^+.$$

Proof. See [GT01, Theorem 8.1]. •

Remark 1.14. If $Lu = g + \nabla_i f^i$ has weak solution $u \in W^{1,2}(\Omega)$, $g \in L^{q/2}(\Omega)$ and $f^i \in L^q(\Omega)$ provided $q > n$, then from the Moser iteration, we can obtain L^∞-bound of u.

The second case is $u \in W^{2,n}_{\text{loc}}(\Omega) \cap C^0(\bar{\Omega})$. Consider the equation $Lu = f$ for $Lu = a^{ij}\nabla_i\nabla_j u + b^i \nabla_i u + cu$, such u is called a *strong solution*. Then we have the Alexander–Backerman–Pucci maximum principle.

Theorem 1.15. *Let $Lu \geq f$, L elliptic, $D = \det a^{ij}$ and $D^* = D^{1/n}$. Assume $\frac{|b|}{D^*}, \frac{|f|}{D^*} \in L^n(\Omega)$, $c \leq 0$ in Ω, $u \in W^{2,n}_{\text{loc}}(\Omega) \cap C^0(\bar{\Omega})$, then*

$$\sup_\Omega u \leq \sup_{\partial\Omega} u^+ + \tilde{C}\|f\|_{L^n}.$$

Proof. See [GT01, Theorem 9.1]. •

Remark 1.16. The ABP maximum principle has a geometric proof. It often applies to non-divergence type of elliptic PDEs like Monge–Amperé equations.

Now we discuss some practical useful principle on manifolds.

(1) Let (M, g) be a compact Riemannian manifold, $u \in C^2(M)$. If x_0 is the maximum point of u, then $\nabla u(x_0) = 0$ and $e^\varphi \Delta e^{-\varphi} u(x_0) < 0$. Hence we can estimate $\tilde{\Delta} u(x)$ from below to get

$$\tilde{\Delta} u(x) \geq f(x, u, \nabla u).$$

(2) Let (M, g) be a noncompact complete Riemannian manifold. Then the Omori–Yau maximum principle tells that for $u^* = \sup u$, there exists a sequence of points $\{x_k\}$ such that

$$u(x_k) > u^* - \frac{1}{k},\ |\nabla u|(x_k) < \frac{1}{k},\ \Delta u(x_k) < \frac{1}{k}.$$

Lecture 2. Existence of Solutions

In this lecture, we discuss inverse function theorem on Banach spaces, Fredholm alternative on Hilbert spaces and Sobolev embedding. These utils enable us to discuss the existence of solutions.

First we define the Frechet derivative. Let B_1, B_2 be two Banach spaces, $U_1 \subset B_1, U_2 \subset B_2$ are open subsets.

Definition 2.1. A map $F : B_1 \to B_2$ is called *Frechet differentiable* at $u \in U_1$ if there exists a bounded linear mapping $L : B_1 \to B_2$ such that

$$\frac{\|F(u+h) - F(u) - Lh\|_{B_2}}{\|h\|_{B_1}} \to 0 \text{ as } h \to 0 \text{ in } B_1.$$

The linear mapping $L =: \delta_u F$ is called the *Frechet derivative* of F at u. We say F is a C^1-map (continuously Frechet differentiable) at u if the map $v \to \delta_v F$ is continuous at u.

Now we state the inverse function theorem and implicit function theorem.

Theorem 2.2. *Let $\Phi : U \to V$ be a C^1-map. Assume $\Phi(0) = 0$, $\delta_u \Phi(0)$ is an isomorphism, then there exists an open neighborhood U_0 of 0 and $V_0 \subset B_2$ such that for all $v \in V_0$, $\Phi(u) = v$ is solvable for u.*

Theorem 2.3. *Let B_1, B_2 be Banach spaces, $W \subset B_1 \times X$ be an open subset. On open subset $V \subset B_2$, $\Phi : W \to V$ is C^1, $\Phi(u_0, t_0) = 0$, $u_0 \in B_1$, $t_0 \in X$. Assume the bounded linear operator $\delta_u \Phi(u_0, t_0)$ is invertible, then there exists a small neighborhood V_0 of t_0 such that $\Phi(u, t) = 0$ is solvable for $u = u(t)$.*

Remark 2.4. Apply to PDE, consider $F(u) = v$, F is C^1 between Banach spaces (e.g. $C^{0,\alpha}, W^{k,p}, L^p$) if the linearized operator $\delta_u F(u_0)$ is invertible.

Example 2.5. For an elliptic operator F, if δF is uniformly elliptic, implicit function theorem implies $F(u) = v$ is locally solvable for "weak solution" in $W^{k,p}$ sense. Next, use the elliptic estimate (Schauder estimate or L^p-estimate) to boost up regularity to get regular enough solution.

Proof. Take $U = V$, assume $dF(0) = \mathrm{id}$ (otherwise, consider $\tilde{F} = (dF^{-1})(0)F$). Let $F(u) = u + \eta(u)$, then

$$F(u) = v \iff u = v - \eta(u).$$

Let $G(u) = v - \eta(u)$, then the equation is equivalent to $G(u) = u$. Note $dF(u) = Iu + d\eta(u)$, hence $dF(0) = I + d\eta(0)$, so $d\eta(0) = 0$. Also $d\eta$ is a bounded linear operator, hence there exists a $\delta > 0$, such that for $\|u\| < \delta$, we have

$$\|d\eta(u) - d\eta(0)\| < \frac{1}{2}.$$

Thus for u_1, u_2 with $\|u_i\| < \delta$, we have

$$\|\eta(u_1) - \eta(u_2)\| \le \frac{1}{2}\|u_1 - u_2\|.$$

Now let $u_1 = v$, $u_2 = G(u_1)$,…,$u_n = G(u_{n-1})$,… We thus have

$$\begin{aligned}\|u_{n+1} - u_{m+1}\| &= \|G(u_n) - G(u_m)\| \\ &\le \frac{1}{2}\|u_n - u_m\| \\ &\le \left(\frac{1}{2}\right)^{n-m-1} \|u_2 - u_1\|,\end{aligned}$$

hence $\{u_n\}$ is a Cauchy sequence. Therefore $\lim_{n\to\infty} u_n = u_\infty$ exists in Banach space, and $G(u_n) = u_{n+1}$ implies $G(u_\infty) = u_\infty$. •

Next we discuss Arzela–Ascoli theorem.

Theorem 2.6. *Let X be a compact metric space, $\{f_k(x)\}$ be a sequence of functions such that*

(1) $f_k(x)$ is uniformly bounded, i.e. there exists M such that $\forall x \in X$, $\forall k \in \mathbb{N}$, we have $|f_k(x)| < M$;

(2) $f_k(x)$ is uniformly equicontinuous, i.e. for any $\varepsilon > 0$, there exists $\delta > 0$, such that for $\|x-y\| < \delta$, $|f_k(x) - f_k(y)| < \varepsilon$ independent from x and k.

Then there exists a subsequence $\{f_{k_j}\}$ such that $f_{k_j} \rightrightarrows f$ in some compact subset of X.

Remark 2.7. For PDE, $\{f_k\} \subset W^{k,p}, L^p$ or $C^{0,\alpha}$. Then (1) requires checking the boundedness of Sobolev, Hölder norms, and (2) requires checking the boundedness of weak derivatives.

Now we discuss Fredholm alternative. For simplicity, we only discuss Hilbert space version.

Definition 2.8. Let V_1, V_2 be normed linear spaces, $T : V_1 \to V_2$ is called a *compact operator* if for any bounded sequence $\{u_k\} \subset V_1$, there exists a subsequence $\{u_{k_j}\}$ such that $\{Tu_{k_j}\}$ converges in V_2.

Assume H is a Hilbert space, $T : H \to H$ is a compact operator, then one can check that $T^* : H \to H$ is also a compact operator.

Theorem 2.9. *Assumption as above, we have*

(1) The spectral of T is discrete, i.e. $\Lambda = \{\lambda_i \in \mathbb{R} | \lambda_i \neq 0, Tx = \lambda_i x\}$ is discrete. Moreover, $\ker(\lambda_i I - T)$ and $\ker(\lambda_i I - T^*)$ are of finite dimensional.

(2) If $\lambda \neq 0$, $\lambda \in \Lambda$, then

$$(\lambda I - T)x = y \quad \text{and} \quad (\lambda I - T^*)x = y$$

have unique solutions x for each y and $(\lambda I - T)^{-1}, (\lambda I - T^*)^{-1}$ exist and are bounded.

(3) For $\lambda \in \Lambda$, $(\lambda I - T)x = y$ is solvable if and only if $y \perp \ker(\lambda I - T^*)$, $(\lambda I - T^*)x = y$ is solvable if and only if $y \perp \ker(\lambda I - T)$.

Remark 2.10. We often use (Hodge) Laplacian Δ on a Riemannian manifold and $\Delta_{\bar{\partial}}$ on a complex manifold.

Application: Prescribed Gaussian curvature equation on a compact Riemannian surface.

Assume (Σ, ds_0^2) is a compact Riemannian surface, K_0 is the Gaussian curvature of ds_0^2. Let $ds^2 = e^{2f} ds_0^2$ be the hyperbolic metric $K = -1$. Then f satisfies

$$\Delta f + e^{2f} = -K_0, \tag{2.2}$$

where Δ is the Hodge Laplacian. We rewrite (2.2) as $\Delta f + e^{2f} - 1 = -K_0 - 1$. Let $F(f) = \Delta f + e^{2f} - 1$, then $\delta F(\varphi) = (\Delta + 2)\varphi$. Since all eigenvalues of $\Delta > 0$, $\Delta + 2$ is invertible. We apply iteration on

$$(\Delta + 2)f = (-K_0 - 1) + (1 + 2f - e^{2f}). \tag{2.3}$$

Set

$$\begin{aligned}
&f_1 = (\Delta + 2)^{-1}(-K_0 - 1) \\
&f_2 = f_1 + (\Delta + 2)^{-1}(1 + 2f_1 - e^{2f_1}) \\
&\cdots \\
&f_n = f_{n-1} + (\Delta + 2)^{-1}(1 + 2f_{n-1} - e^{2f_{n-1}}) \\
&\cdots
\end{aligned}$$

We can check when $\| - K_0 - 1\| < \varepsilon$ ($\varepsilon = 1/2$ will do) in some Sobolev norm, (2.3) has unique solution f for $\|f\| < \delta$. •

Definition 2.11. A bounded linear operator $T : H \to H$ is called *Fredholm* if

(1) Both $\ker(T), \mathrm{coker}(T)$ are of finite dimensional.

(2) $\mathrm{im}(T)$ is closed.

(3) $\mathrm{Index}(T) = \dim\ker(T) - \dim\mathrm{coker}(T)$.

Now we review some aspects about Sobolev spaces. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, define *Sobolev space*

$$W^{k,p}(\Omega) := \left\{ f \in L^p(\Omega) \,\middle|\, \int_\Omega |\nabla^i f|^p < +\infty,\ i = 1, \cdots, k \right\} \subset L^p(\Omega).$$

We define *Sobolev norm*

$$\|f\|^p_{W^{k,p}(\Omega)} := \sum_{|\alpha|\le k} \int_\Omega |\nabla^\alpha f|^p.$$

Define $W^{k,p}_0(\Omega)$ to be the completion of $C^k_0(\Omega)$ with respective to $W^{k,p}$-norm, whose elements should be understood as Cauchy sequences.

Now we state the Sobolev embedding theorem.

Theorem 2.12. *There are embeddings*

$$W^{k,p}_0(\Omega) \hookrightarrow \begin{cases} W^{l,q}_0(\Omega) \text{ if } 1 \le q \le p, 0 \le l \le k, l - \dfrac{n}{q} < k - \dfrac{n}{p} < l; \\ L^{\frac{np}{n-p}}(\Omega) \text{ if } k - \dfrac{n}{p} < 0; \\ C^m(\bar{\Omega}) \text{ if } 0 \le m \le k - \dfrac{n}{p}; \\ C^{m,\alpha}(\bar{\Omega}) \text{ if } 0 < m + \alpha < k - \dfrac{n}{p}. \end{cases}$$

Moreover, all embeddings are compact.

The following special cases are useful.

(1) When $k = 1, p = 2$, we have *Sobolev inequality* $W^{1,2}_0(\Omega) \hookrightarrow L^{\frac{2n}{n-2}}(\Omega)$ for $n \ge 3$, or equivalently

$$\|f\|_{\frac{2n}{n-2}} \le C(\Omega)\|\nabla f\|_2.$$

(2) We have *Morrey's inequality*

$$W^{1,p}_0(\Omega) \hookrightarrow \begin{cases} L^{\frac{np}{n-p}}(\Omega),\ n > p; \\ C^0(\Omega),\ p > n; \\ C^{0,\alpha},\ 0 < \alpha < 1 - \dfrac{n}{p}. \end{cases}$$

(3) For $f \in W^{k,p}_0(\Omega)$ with $p > n$, we have $f \in C^{k-1}(\bar{\Omega})$.

(4) For $f \in W^{k,p}_0(\Omega)$ with $pk > n$, we have $f \in C^p(\bar{\Omega})$.

Basic Question. In the embedding picture, why $k - \frac{n}{p}$ appears?

Philosophy. Meaningful inequalities in geometry must be scaling invariant.

Answer. $k - \frac{n}{p}$ comes from scaling argument.

(1) If $u \in C^{0,\alpha}(\Omega)$, then $|u(x) - u(y)| \le C|x - y|^\alpha$ (Hölder). Without loss of generality we can assume $\Omega = B_1(0)$, $y = 0$. Let $u_\lambda(x) = u(\lambda x)$, then

$$|u_\lambda(x) - u_\lambda(y)| = |u(\lambda x) - u(\lambda y)| \le C|\lambda|^\alpha |x|^\alpha,$$

hence

$$\frac{|u_\lambda(x) - u_\lambda(y)|}{|\lambda|^\alpha} \le C|x|^\alpha.$$

That is, Hölder function is of $C^{0,\alpha}$-regular.

(2) Assume $u \in L^2(B_1(0))$. Let $u_\lambda(x) = u(\lambda x)$, then

$$\begin{aligned} \left(\int_{B_\lambda(0)} u^2_\lambda\right)^{1/2} &= \left(\int_{B_1(0)} u^2(\lambda x)\frac{d(\lambda x)}{\lambda^n}\right)^{1/2} \\ &= \lambda^{-n/2}\|u_\lambda\|_{L^2(B_\lambda(0))}. \end{aligned}$$

That is, as domain scales λ yields to the L^2-function scales like $\lambda^{-n/2}$. Similarly, if $u \in L^p(B_1(0))$, u is regular in the sense of $C^{-n/p}$.

(3) Assume $\nabla u \in L^2(B_1(0))$. Let $u_\lambda(x) = u(\lambda x)$, then

$$\begin{aligned}\left(\int_{B_\lambda(0)} |\nabla u_\lambda|^2\right)^{1/2} &= \left(\int_{B_1(0)} \frac{\lambda^2}{\lambda^n} |\nabla u(\lambda x)|^2 d(\lambda x)\right)^{1/2} \\ &= \lambda^{1-n/2} \|\nabla u\|_{L^2(B_\lambda(0))}.\end{aligned}$$

Similarly, if $u \in W^{k,p}(B_1(0))$, then u scales like $\lambda^{k-n/p}$.

Summary. The three types of functions have regularity:

(1) Hölder function is $C^{0,\alpha}$-regular;
(2) L^p-function is $C^{-n/p}$-regular;
(3) $W^{k,p}$-function is $C^{k-n/p}$-regular.

Now it's times to understand the Sobolev embedding.

$$W_0^{1,p}(\Omega) \hookrightarrow \begin{cases} L^{\frac{np}{n-p}}(\Omega), \ p < n; \\ C^{0,\alpha}, \ 0 < \alpha < 1 - \dfrac{n}{p}. \end{cases}$$

We have two situations:

(1) $p > n$, $u \in W_0^{1,p}(\Omega)$, then u is $C^{1-n/p}$-regular at most;
(2) $p < n$, $1 - \frac{n}{p} < 0$, then we expect $u \in L^q(\Omega)$ for some q. Translate to inequality, we have

$$\|u\|_{L^q(\Omega)} \leq C \|\nabla u\|_{L^p(\Omega)}.$$

By our "philosophy", the regularity on both sides of the inequality must equal. The left side is $C^{-n/q}$-regular, and the right side is $C^{1-n/p}$-regular. Thus

$$-\frac{n}{q} = 1 - \frac{n}{p},$$

we obtain $q = \frac{np}{n-p}$.

Bibliography

[GT01] David Gilbarg and Neil S. Trudinger. *Elliptic partial differential equations of second order*. Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.

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