2024 BICMR Summer School on Differential Geometry

Riemannian Geometry

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Contents

Chapter 1

Differentiable Manifolds

In this lecture, we review some basic notions of differentiable manifolds.

1.1 Differentiable Manifolds and Maps

Definition. Let M^n be a Hausdorff space with countable topological basis. If there exists an open cover $\{U_\alpha\}$ of M, and homeomorphisms $\varphi_\alpha : U_\alpha \to \varphi_\alpha(U_\alpha)$ onto its image $\varphi_\alpha(U_\alpha) \subset \mathbb{R}^n$ open, such that

(1) $M = \bigcup_\alpha U_\alpha$,

(2) if $U_\alpha \cap U_\beta \neq \varnothing$, then $\varphi_\alpha^{-1} \circ \varphi_\beta : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$ is differentiable (we mean C^∞ here),

then M is called an n**-dimensional differentiable manifold**.

Moreover, we call $(U_\alpha, \varphi_\alpha)$ a **local chart**, $\{(U_\alpha, \varphi_\alpha)\}$ an **atlas**, and we say the atlas induces a **differentiable structure** on M.

Remark 1.1. We often assume the atlas is *maximal*, that is, there is no more local chart being compatible with the atlas.

Example 1.2. We illustrate some examples of differentiable manifolds.

(1) $\mathbb{R}^n$ itself is a differentiable manifolds, with single local chart $(\mathbb{R}^n, \mathrm{id})$.

(2) $\mathbb{S}^n := \{x \in \mathbb{R}^{n+1} | \ (x^1)^2 + \cdots + (x^{n+1})^2 = 1\}$. We use stereographic projection as local chart. Define the stereographic projection from north pole

$$\varphi_N : \mathbb{S}^n \backslash \{(0, \cdots, 0, 1)\} \to \mathbb{R}^n$$

$$x \mapsto \left(\frac{x^1}{1+x^{n+1}}, \cdots, \frac{x^n}{1+x^{n+1}} \right)$$

Similarly define φ_S to be stereographic projection from the south pole. Then we have

$$\begin{aligned} \varphi_S \circ \varphi_N^{-1} : \mathbb{R}\backslash\{0\} &\to \mathbb{R}\backslash\{0\} \\ (y^1, \cdots, y^n) &\mapsto \left(\frac{y^1}{\sum_i (y^i)^2}, \cdots, \frac{y^n}{\sum_i (y^i)^2} \right) \end{aligned}$$

is clearly differentiable.

(3) Let $M_1^{n_1}, M_2^{n_2}$ be differentiable manifolds, then $M_1 \times M_2$ has the **product manifold** structure. To be precise, let M_1, M_2 have atlas $\{U_\alpha, \varphi_\alpha\}, \{(V_\beta, \psi_\beta)\}$, then $M_1 \times M_2$ has atlas $(U_\alpha \times V_\beta, \varphi_\alpha \times \psi_\beta)$. In particular, we have

- (flat) n-torus $\mathbb{T}^n = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$ (n times);
- cylinder $\mathbb{S}^1 \times \mathbb{R}$ (or generally $\mathbb{S}^k \times \mathbb{R}^{n-k}$).

(4) Real projective space $\mathbb{RP}^n$. Let equivalence relation $\sim$ on $\mathbb{R}^{n+1}\backslash\{0\}$ be $x \sim y \iff x = \lambda y,\ \lambda \neq 0$. Then define $\mathbb{RP}^n = (\mathbb{R}^{n+1}\backslash\{0\})/\sim$. We now define the differentiable structure on $\mathbb{RP}^n$. Let $U_i = \{x \in \mathbb{RP}^n :\ x = [x^1, \cdots, x^{n+1}],\ x^i \neq 0\}$, and

$$\begin{aligned} \varphi_i : U_i &\to \mathbb{R}^n \\ x &\mapsto \left(\frac{x^1}{x^i}, \cdots, \frac{x^{i-1}}{x^i}, \frac{x^{i+1}}{x^i}, \cdots, \frac{x^{n+1}}{x^i} \right) \end{aligned}$$

We check $\varphi_j \circ \varphi_i^{-1}$ on $U_i \cap U_j$. We may assume $i < j$, then

$$\begin{aligned} \varphi_j \circ \varphi_i^{-1}(y^1, \cdots, y^n) &= \varphi_j \left([y^1, \cdots, y^{i-1}, 1, y^{i+1}, \cdots, y^n] \right) \\ &= \left(\frac{y^1}{y^j}, \cdots, \frac{y^{i-1}}{y^j}, \frac{1}{y^j}, \frac{y^{i+1}}{y^j}, \cdots, \frac{y^n}{y^j} \right) \end{aligned}$$

is differentiable.

We now give the definition of differentiable maps.

Definition. A map $f : M \to N$ is **differentiable** at $p \in M$ if there exists local chart (U, φ) of p and (V, ψ) of $f(p)$, such that $\psi \circ f \circ \varphi^{-1}$ is differentiable at $\varphi(p)$.

Remark 1.3. (1) If $\tilde{\varphi}, \tilde{\psi}$ are another chart at p and $f(p)$, then we have

$$\tilde{\psi} \circ f \circ \tilde{\varphi}^{-1} = (\tilde{\psi} \circ \psi^{-1}) \circ (\psi \circ f \circ \varphi^{-1}) \circ (\varphi \circ \tilde{\varphi})$$

is still differentiable at p by the compatiblility of charts, so differentiable maps are well-defined.

(2) When $N=\mathbb{R}$, f is also called a **differentiable function**.

Notation 1.4. We use $C^\infty(M,N)$ to denote the $\mathbb{R}$-vector space of differentiable maps between M and N, $C^\infty(M)$ to denote the $\mathbb{R}$-algebra of differentiable functions on M. We use $C_p^\infty(M)$ to denote the $\mathbb{R}$-algebra of germs of differentiable functions at p. We often use $\gamma : I\subset\mathbb{R}\to M$ to denote a **differentiable curve** on M.

1.2 Tangent Spaces and Tangent Maps

Definition. Let $\gamma : I\to M$ be a curve, $\gamma(0)=p$. We define the **tangent vector along γ at** p as a mapping $\dot\gamma(0) : C_p^\infty(M)\to\mathbb{R}$, $\dot\gamma(0)f=\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}(f\circ\gamma)(t)$. Then we define the **tangent space at** p

$$T_pM := \{\dot\gamma(0) \mid \gamma : I\to M \text{ differentiable}, \ \gamma(0)=o=p\}.$$

Proposition 1.5. *We have the* ***Leibniz rule*** $\dot\gamma(0)(fg)=(\dot\gamma(0)g)f(p)+(\dot\gamma(0)f)g(p)$. *So a tangent vector is a* derivative *on* $C_p^\infty(M)$.

We now calculate the local representation of a tangent vector. Fix a chart $\varphi=(x^1,\cdots,x^n)$, we have

$$\begin{aligned}\dot\gamma(0)f &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0}(f\circ\gamma)(t)\\ &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0}(f\circ\varphi^{-1})\circ(\varphi\circ\gamma)(t)\\ &= \sum_{i=1}^n \frac{\partial}{\partial x^i}\bigg|_{\varphi(p)}(f\circ\varphi^{-1})\,\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0}x^i(\gamma(t)) \quad \text{(Chain rule)}\end{aligned} \tag{1.1}$$

Using equation (1.1), we can describe T_pM as a vector space.

Proposition 1.6. *T_pM is a real vector space of dimension n. Moreover, given a local chart $\varphi=(x^1,\cdots,x^n)$, we have*

$$T_pM=\mathrm{Span}\left\{\frac{\partial}{\partial x^i}\bigg|_p\right\}$$

where $\partial/\partial x^i|_p$ is the tangent vector of $\sigma_i(t)=\varphi^{-1}(\varphi(p)+te_i)$, $e_i=(0,\cdots,1,\cdots,0)$ with only i-th component being 1. Thus we have

$$\frac{\partial}{\partial x^i}\bigg|_p f=\frac{\partial}{\partial x^i}(f\circ\varphi^{-1})\bigg|_\varphi(p)$$

Proof. Clearly T_pM has natural vector space structure. Thus by the definition of $\frac{\partial}{\partial x^i}\Big|_p$'s, $\mathrm{Span}\left\{\frac{\partial}{\partial x^i}\Big|_p\right\} \subset T_pM$. For the converse inclusion, let $v \in T_pM$, then there is a curve $\gamma: I \to M$ with $\dot\gamma(0) = v$. Then by (1.1), $\dot\gamma(0)$ is a linear combination of $\frac{\partial}{\partial x^i}\Big|_p$'s, hence $T_pM \subset \mathrm{Span}\left\{\frac{\partial}{\partial x^i}\Big|_p\right\}$. □

Definition (Tangent maps)**.** Let $f : M \to N$ be a differentiable map, we define $f_{*p} : T_pM \to T_{f(p)}M$ as

$$f_{*p}(v)(g) = v(g \circ f)$$

for any $g \in C^\infty_{f(p)}N$. In particular, if $N = \mathbb{R}$, given $v \in T_pM$, let $\dot\gamma(0) = v$, then $f_{*p}(v) = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}(f\circ\gamma)(t)$.

Again we can look at the local representation of f_{*p}. Let $\varphi = (x^1, \cdots, x^n)$, $\psi = (y^1, \cdots, y^m)$ be local charts containing p and $f(p)$. Let $v = \sum_{i=1}^n v^i \frac{\partial}{\partial x^i}\Big|_p = \dot\sigma(0)$, then $\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}(\varphi\circ\sigma)(t) = (v^1, \cdots, v^n)$. Thus we have

$$\begin{aligned} f_{*p}(v)(g) &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0}(g\circ f\circ\sigma)(t) \\ &= \sum_{i,j}\frac{\partial}{\partial y^j}(g\circ\psi^{-1})\bigg|_{\psi\circ f(p)}\frac{\partial}{\partial x^i}(\psi\circ f\circ\varphi^{-1})^j\bigg|_{\varphi(p)}\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0}(\varphi\circ\sigma)^i(t) \\ &= \sum_{i,j} v^i\frac{\partial}{\partial x^i}(\psi\circ f\circ\varphi^{-1})^j\bigg|_{\varphi(p)}\frac{\partial}{\partial y^j}\bigg|_{f(p)} g \end{aligned}$$

In particular, we have

$$f_{*p}\left(\frac{\partial}{\partial x^i}\bigg|_p\right) = \sum_j \frac{\partial}{\partial x^i}(\psi\circ f\circ\varphi^{-1})^j\bigg|_{\varphi(p)}\frac{\partial}{\partial y^j}\bigg|_{f(p)}$$

We can easy to verity the following chain rule:

Proposition 1.7. *Let $f : M \to N$, $g : N \to P$ be differentiable maps, then we have $(g \circ f)_{*p} = g_{*f(p)} \circ f_{*p}$.*

Definition (Diffeomorphism)**.** A map $f : M \to N$ is called a **diffeomorphism** if f is bijective, and f, f^{-1} are both differentiable.

Proposition 1.8. *If $f : M \to N$ is a diffeomorphism, then $f_{*p} : T_pM \to T_{f(p)}N$ is an isomorphism.*

This proposition can be easily proved by chain rule.

Remark 1.9. (1) The Proposition 1.8 shows that dimension of a manifold is well-defined in the category (differentiable manifolds, differentiable maps).

(2) Since we can do calculus locally on manifolds, the *Inverse function theorem* is valid on differentiable manifolds. That is, if $f_{*p} : T_pM \to T_{f(p)}N$ is an isomorphism, then $f : M \to N$ is a local diffeomorphism at p.

1.3 Tangent Bundles and Vector Fields

Definition (Tangent bundle)**.** Assume differentiable manifold M^n has atlas $\{U_\alpha, \varphi_\alpha\}$, define

$$TM := \bigsqcup_{p\in M} T_pM$$
$$\pi : TM \to M,\ (p,v) \mapsto p$$

We give an atlas of TM to make it into a $2n$-dimensional differentiable manifold. Let

$$\Phi_\alpha : \bigsqcup_{p\in U_\alpha} T_pM \to \mathbb{R}^{2n}$$
$$(p,v) \mapsto (\varphi_\alpha(p), (v^1, \cdots, v^n))$$

where $v = \sum_{i=1}^n v^i \left.\frac{\partial}{\partial x^i}\right|_p$. Let's check

$$\Phi_\beta \circ \Phi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \times \mathbb{R}^n \to \varphi_\beta(U_\alpha \cap U_\beta) \times \mathbb{R}^n$$
$$(x^1, \cdots, x^n, v^1, \cdots, v^n) \mapsto \left(\varphi_\beta \circ \varphi_\alpha^{-1}(x), \sum_{i=1}^n \frac{\partial(\varphi_\beta \circ \varphi_\alpha^{-1})^1}{\partial x^i} v^i, \cdots, \sum_{i=1}^n \frac{\partial(\varphi_\beta \circ \varphi_\alpha^{-1})^n}{\partial x^i} v^i\right)$$

Clearly it is differentiable, then $\{(\pi^{-1}(U_\alpha), \Phi_\alpha)\}$ induces a differentiable structure on TM.

We call T_pM a **fiber** over p, and $\pi : TM \to M$ the projection.

Definition (Vector field)**.** A **vector field** is a differentiable map $X : M \to TM$ such that $X(p) \in T_pM$.

Notation 1.10. We use $\mathfrak{X}(M)$ to denote the collection of vector fields on M.

Proposition 1.11. *$X \in \mathfrak{X}(M^n)$ if and only if in any local chart (U, φ), we have $X(p) = \sum_{i=1}^n X^i(p) \left.\frac{\partial}{\partial x^i}\right|_p$ for $X^i \in C^\infty(U)$, $i = 1, 2, \cdots, n$.*

This proposition is equivalent to X can be a mapping $C^\infty(M) \to C^\infty(M)$ defined by $Xf(p) = X(p)f$.

Definition (Lie bracket)**.** For $X,Y \in \mathfrak{X}(M)$, define $[X,Y] = XY - YX$, then $[X,Y] \in \mathfrak{X}(M)$.

Remark 1.12. We explain the definition more explicitly. If we act two vector fields on the product of two functions, we have

$$\begin{aligned}(XY)_p(fg) &= X_p(Y(fg)) = X_p(gYf + fYg)\\ &= \boxed{X_pg \cdot Y_pf + X_pf \cdot Y_pg} + g(p)X_pYf + f(p)X_pYg\end{aligned}$$

The boxed thing is bad, it spoils Leibniz rule. But if we substract $YX_p(fg)$, the boxed thing is cancelled. So $XY - YX \in \mathfrak{X}(M)$.

Proposition 1.13. *On some local chart, we have* $\left[\left.\frac{\partial}{\partial x^i}\right|_p, \left.\frac{\partial}{\partial x^j}\right|_p\right] = 0.$

Proof. This is equivalent to mixed partial derivative is commutative for smooth functions in $\mathbb{R}^n$. $\square$

Chapter 2

Metric and Connection

In this lecture we introduce the Riemannian metric on a differentiable manifold, and the connection compatible with metric, i.e., Levi–Civita connection. Moreover, we introduce the covariant derivative of a vector field along a curve, and parallel transport of vectors along a curve.

Notation 2.1. From now on we adopt *Einstein summation convention*: any index appear twice as both upper index and lower index means taking summation respective to the index. For example, a vector field on a local chart can be expressed as

$$X = X^i \frac{\partial}{\partial x^i} = \sum_{i=1}^{n} X^i \frac{\partial}{\partial x^i}$$

2.1 Riemannian Metric

Definition. Let M^n be a differentiable manifold. A **metric** (or Riemannian metric) on M is a smooth assignment on each T_pM, $\forall p \in M$, a symmetric positive definite bilinear form g_p, that is for $X, Y \in T_pM$ we have

1. $g_p(X,Y) = g_p(Y,X)$;
2. $g_p(X,X) \geq 0$, $g_p(X,X) = 0 \iff X = 0$.

"Smooth" means in any local chart (U, φ) we have

$$g_{ij}(p) = g\left(\left.\frac{\partial}{\partial x^i}\right|_p, \left.\frac{\partial}{\partial x^j}\right|_p \right)$$

is smooth about p for any indices i,j. Then g is a symmetric positive definite $(0,2)$-tensor

$$g = g_{ij}\,\mathrm{d}x^i \otimes \mathrm{d}x^j$$

Proposition 2.2. *Any differentiable manifold M admits a Riemannian metric.*

Proof. We use partition of unity. Let $\{U_\alpha, x^i_\alpha\}$ be a locally finite atlas of M, $\{\phi_\alpha\}$ be a partition of unity subordinate to $\{U_\alpha\}$, i.e., $\operatorname{supp}\phi_\alpha \subset\subset U_\alpha$ and $\sum_\alpha \phi_\alpha = 1$. On the local chart (U_α, x^i_α), let $g_\alpha = \sum_{i=1}^n \mathrm{d}x^i_\alpha \otimes \mathrm{d}x^i_\alpha$. Set

$$g = \sum_\alpha \phi_\alpha g_\alpha,$$

we can check g is indeed a Riemannian metric on M. □

Remark 2.3. For an n-form $\omega \in \bigwedge^n M$ with $\operatorname{supp}\omega$ compact, we can define its integral as

$$\int_M \omega = \sum_\alpha \int_M \phi_\alpha \omega$$

One can check the definition is independent from the choice of partition of unity.

Example 2.4. (1) $\mathbb{R}^n$ has the Euclidean metric $g = \sum_{i=1}^n \mathrm{d}x^i \otimes \mathrm{d}x^i$, $g_{ij} = \delta_{ij}$.

(2) Let $f : M \to (N,h)$ be an immersion, then we define $f^*h(X,Y)|_p = h(f_{*p}X, f_{*p}Y)$, f^*h is the induced metric from h by immersion.

(3) Let $f : (M,g) \to (N,h)$ be an immersion, if $g = f^*h$, then f is called a **local isometry**. If $f : M \to N$ is a diffeomorphism, then f is called an **isometry**.

(4) The standard metric on $\mathbb{S}^n$ is the induced metric by the embedding $i : \mathbb{S}^n \hookrightarrow (\mathbb{R}^{n+1}, \delta_{ij})$.

(5) If $(M_1, g_1), (M_2, g_2)$ are Riemannian manifolds, then their product manifold has product metric $g_1 \times g_2$. In particular, equip $\mathbb{S}^1$ with standard metric g, $(\mathbb{T}^n, g^n) = (\mathbb{S}^1 \times \cdots \times \mathbb{S}^1, g \times \cdots \times g)$ is the flat torus (we will explain the word "flat" later).

(6) Let $f : M \to (N,g)$ be a covering map, then f^*g is a Riemannian metric on M, called Riemannian covering map. In particular, let $\operatorname{Isom}(M) := \{f : M \to M \mid f \text{ is isometry}\}$ denote the isometry group of M, $\Gamma \subset \operatorname{Isom}(M)$ be a subgroup, then M/Γ is a manifold, and $f : M \to M/\Gamma$ is a Riemannian covering map. Examples of this manner are

- $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$;
- $\mathbb{RP}^n = \mathbb{S}^n/\{\mathrm{id}, A\}$, where A is the antipodal map.

2.2 Metric Structure

Definition. Let $\gamma : [0,1] \to M$ be a curve, define its **length** to be

$$\begin{aligned} L(\gamma) &:= \int_0^1 |\dot\gamma(t)| \,\mathrm{d}t \\ &= \int_0^1 \sqrt{g(\gamma(t))(\dot\gamma(t),\dot\gamma(t))} \,\mathrm{d}t \end{aligned}$$

Let $p,q \in M$, define their **distance** to be

$$d(p,q) = \inf_{\gamma \in C_{p,q}} L(\gamma)$$

where $C_{p,q}$ denotes the collection of all smooth curve joining p and q.

Proposition 2.5. *The distance function $d : M \times M \to \mathbb{R}$ has the following properties:*

(1) $d(p,q) \geq 0$*, and* $d(p,q) = 0 \iff p = q$*;*

(2) $d(p,q) = d(q,p)$*;*

(3) $d(p,r) \leq d(r,q) + d(p,q)$*.*

Thus the distance function makes M into a metric space.

Proof. Only need to show $d(p,q) = 0 \iff p = q$, all else are trivial. We assume $p \neq q$, need to show $d(p,q) > 0$. Let $\gamma : [0,1] \to M$ be any curve joining p and q. Choose a local chart (U,φ) such that $\varphi(U) = B_r(0)$, $q \notin U$. By Jordan–Brouwer Separation Theorem, γ must intersect ∂U at $s := \gamma(c)$. Then we have

$$L(\gamma) \geq L(\gamma|_{[0,c]}) = \int_0^c \sqrt{g_{ij}\dot x^i(\gamma(t))\dot x^j(\gamma(t))} \,\mathrm{d}t$$

Regarding $g : \overline{U} \times \mathbb{S}^{n-1} \to \mathbb{R}$, g is a continuous function on a compact set, thus it attains its minimum $g(x)(v,v) \geq m$, and $m > 0$ since $v \in \mathbb{S}^{n-1} \neq 0$. Thus we have

$$L(\gamma|_{[0,c]}) \geq m \int_0^c |\dot x(\gamma(t))| \,\mathrm{d}t \geq mr > 0$$

mr does not depend on γ, hence $d(p,q) \geq mr > 0$. □

Proposition 2.6. *(M,d) with metric topology coincides with its original topology.*

For a proof, we refer to John Lee's *Introduction to Smooth Manifolds*, 2nd ed., Theorem 13.29.

2.3 Levi–Civita Connection

Definition. (Affine connection) Let M be a differentiable manifold, if the operator $\nabla :$ $\mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$, denoting $\nabla_X Y$, satisfies

(1) $\nabla_{(fX+gY)} Z = f\nabla_X Z + g\nabla_Y Z$ for $f, g \in C^\infty(M)$,

(2) $\nabla_X(Y+Z) = \nabla_X Y + \nabla_X Z$,

(3) $\nabla_X(fY) = (Xf)Y + f\nabla_X Y$,

then ∇ is called an **affine connection** on M.

Definition. An affine connection ∇ on Riemannian manifold (M,g) is called **Levi–Civita connection** if it satisfies

(LC1) $Xg(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$,

(LC2) $\nabla_X Y - \nabla_Y X - [X,Y] = 0$.

Proposition 2.7. *On a Riemannian manifold (M,g) there exists a unique Levi–Civita connection.*

Proof. We have the Koszul formula:

$$\begin{aligned} 2g(\nabla_X Y, Z) =& Xg(Y,Z) + Yg(X,Z) - Zg(X,Y) \\ &+ g([X,Y],Z) - g([X,Z],Y) - g([Y,Z],X) \end{aligned}$$

The formula shows Levi–Civita connection is unique, and can be used as the definition of Levi–Civita connection. □

We check Levi–Civita connection locally. First we introduce the Christoffel symbols:

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma_{ij}^k \frac{\partial}{\partial x^k}$$

Then (LC1) is equivalent to

$$\frac{\partial g_{ij}}{\partial x^k} = \Gamma_{ki}^l g_{lj} + \Gamma_{kj}^l g_{li}$$

(LC2) is equivalent to

$$\Gamma_{ij}^k = \Gamma_{ji}^k$$

Using (LC1) and (LC2), we obtain

Proposition 2.8. *We have the expression of Γ_{ij}^k:*

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl} \left(\frac{\partial g_{il}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l}. \right)$$

Let vector fields $X = X^i \frac{\partial}{\partial x^i}, Y = Y^i \frac{\partial}{\partial x^i}$, then we have

$$\nabla_X Y = X^i \left(\frac{\partial Y^k}{\partial x^i} + Y^j \Gamma_{ij}^k \right) \frac{\partial}{\partial x^k} \tag{2.1}$$

This shows that $\nabla_X Y(p)$ depends only on $X(p)$ and the value of Y along the curve $\gamma(t)$ with $\gamma(0) = p, \dot{\gamma}(0) = X(p)$. Using this, we can introduce the covariant derivative of vector fields along curves.

2.4 Covariant Derivative

Definition. Let $\gamma : [0,1] \to M$ be a curve, Y is a vector field along γ. Then define

$$\frac{\nabla}{\mathrm{d}t} Y := \nabla_{\dot{\gamma}(t)} Y$$

We look at covariant derivative locally. Choose a local chart (U, φ), let $Y = Y^i \frac{\partial}{\partial x^i}$, $\dot{\gamma}(t) = \dot{\gamma}^i(t) \frac{\partial}{\partial x^i}$, then by equation (2.1), we have

$$\begin{aligned} \frac{\nabla}{\mathrm{d}t} Y &= \dot{\gamma}^i(t) \left(\frac{\partial Y^k(t)}{\partial x^i} + Y^j(t) \Gamma_{ij}^k(\gamma(t)) \right) \frac{\partial}{\partial x^k} \\ &= \left(\dot{Y}^k(t) + Y^j(t) \dot{\gamma}^i(t) \Gamma_{ij}^k(\gamma(t)) \right) \frac{\partial}{\partial x^k} \quad \text{(chain rule)} \end{aligned} \tag{2.2}$$

Definition (Parallel transport)**.** Let $\gamma : [0,1] \to M$ be a curve, Y is a vector field along γ. If $\nabla Y / \mathrm{d}t = 0$, then we call Y is **parallel** along γ.

Proposition 2.9. *Given a curve $\gamma : [0,1] \to M$ and an initial vector $Y_{\gamma(0)} \in T_{\gamma(0)} M$, then there exists a unique parallel vector field along γ with initial vector $Y_{\gamma(0)}$.*

Proof. Parallel transport satisfies (2.2), and it is a second order ordinary differential equation. By the unique existence theorem of solution of ODEs, the proposition is proved. □

Definition. Let $\gamma : [0,1] \to M$ be a curve, $\gamma(0) = p, \gamma(1) = q$, we define a mapping $P_\gamma : T_p M \to T_q M$ as follows: Let $Y_0 \in T_p M$, there exists a unique parallel vector field Y along γ with $Y(0) = Y_0$, then we define $P_\gamma(Y_0) = Y(1)$. Clearly P_γ is linear.

Proposition 2.10. *P_γ is an isometry, hence an isomorphism.*

Proof. Using notation above, let $X_0, Y_0 \in T_p M$, X, Y are parallel vector field along γ with $X(0) = X_0, Y(0) = Y_0$. Then we have

$$\frac{\mathrm{d}}{\mathrm{d}t} g(X,Y) = g(\nabla_{\dot{\gamma}(t)} X, Y) + g(X, \nabla_{\dot{\gamma}(t)} Y) = 0,$$

since X,Y are parallel. Thus $g(X,Y)$ is constant, we have $g(X_0,Y_0) = g(P_\gamma(X_0),P_\gamma(Y_0))$. ☐

The last proposition reveals the meaning of the word "connection", it means ∇ "connects" different tangent spaces.

Proposition 2.11. *Let $\gamma : [0,1] \to M$ be a curve with $\gamma(0) = p$, X,Y be vector fields with $X(p) = \dot\gamma(0)$. Then $\nabla_X Y(p) = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} P_\gamma^{-1}(Y(\gamma(t)))$.*

Proof. Let $\{e_1(t),\cdots,e_n(t)\}$ be a parallel frame along γ, then $\nabla_{\dot\gamma(t)} e_i(t) = 0$, in particular, $\nabla_{\dot\gamma(0)} e_i(0) = 0$. Let $Y(\gamma(t)) = Y^i(\gamma(t))e_i(t)$, thus

$$\begin{aligned}
\nabla_X Y(p) &= \nabla_{\dot\gamma(0)} Y^i(\gamma(0))e_i(0) \\
&= \dot\gamma(0)Y^i(\gamma(0))e_i(0) + Y^i(0)\nabla_{\dot\gamma(0)} e_i(0) \\
&= \left(\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} Y^i(\gamma(t)) \right) e_i
\end{aligned}$$

On the other hand, parallel transport gives

$$P_\gamma^{-1}(Y(\gamma(t))) = Y^i(\gamma(t))e_i(0),$$

hence

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} P_\gamma^{-1}(Y(\gamma(t))) &= \left(\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} Y^i(\gamma(t)) \right) e_i \\
&= \nabla_X Y(p)
\end{aligned}$$ ☐

Remark 2.12. The symbol $\frac{\nabla}{\mathrm{d}t}$ is rarely used in literature, so we will simply use $\nabla_{\dot\gamma(t)} X$ to denote covariant derivative.

Chapter 3

Geodesics and Curvature

In this lecture, we first introduce the concepts of geodesics and exponential maps. By differentiating exponential map, we can introduce the concept of curvature and Jacobi fields. Using exponential map, we can also introduce geodesic normal coordinate and geodesic polar coordinate. As an application, we will use geodesic polar coordinate to show geodesics are locally length-minimizing. Finally, we will introduce the notion of conjugate points.

3.1 Geodesics and Exponential Maps

Definition. A curve $\gamma : [0,1] \to M$ is called a **geodesic** if $\nabla_{\dot\gamma(t)}\dot\gamma(t) = 0$.

Remark 3.1. Geodesics are constant speed, i.e., $|\dot\gamma(t)| \equiv \text{const}$. This can be shown by $\frac{\mathrm{d}}{\mathrm{d}t}\langle\dot\gamma(t),\dot\gamma(t)\rangle = 2\langle\nabla_{\dot\gamma(t)}\dot\gamma(t),\dot\gamma(t)\rangle = 0$.

In a local chart (U,φ), let $\varphi\circ\gamma(t) = (x^1(t),\cdots,x^n(t))$, then $\dot\gamma(t) = \dot x^i(t)\left.\frac{\partial}{\partial x^i}\right|_{\gamma(t)}$. Thus the geodesic equation is

$$\nabla_{\dot\gamma(t)}\dot\gamma(t) = 0 \iff \ddot x^k(t) + \Gamma^k_{ij}(\gamma(t))\dot x^i(t)\dot x^j(t) = 0,\ k = 1,\cdots,n,$$

with $\gamma(0) = p, \dot\gamma(0) = v \in T_pM$.

Since the solution of an ODE relies continuously on initial value, we have the following proposition.

Proposition 3.2. *For any $p \in M$, there exists a neighborhood V of p, such that there exists $\delta > 0, \varepsilon > 0$ and a differentiable map $\gamma : (-\delta,\delta)\times\mathscr{U} \to M$, where $\mathscr{U} = \{(q,v) \in TV \mid q \in V,\ v \in T_pM,\ |v| < \varepsilon\}$, such that $\gamma(t;q,v)$ is a geodesic with $\gamma(0) = q, \dot\gamma(0) = v$.*

The idea of the proof is the ODE theorem we mentioned above, but the proof is not simply using only ODE theory. A proof of an equivalent proposition can be found in Wu Hung–Hsi, et. al.'s *Introduction to Riemannian Geometry* Chapter 3, Lemma 1.

Observe that $\gamma(\lambda t; p, v) = \gamma(t; p, \lambda v)$. Denote $\gamma(t; p, v) = \gamma_v(t)$, then above observation can be written as $\gamma_{\lambda v}(t) = \gamma_v(\lambda t)$. Therefore, we can shorten the initial vector to lengthen the domain of geodesic.

Definition (Exponential map)**.** Let $U \subset T_pM$ be a neighborhood of origin, such that for any $v \in U$, $\gamma_v(1)$ is defined (such neighborhood exists by Proposition 3.2). We define the **exponential map** at p to be

$$\begin{aligned} \exp_p : U &\to M \\ v &\mapsto \gamma_v(1) \end{aligned}$$

Remark 3.3. We scale the initial vector and can obtain

$$\exp_p(v) = \gamma_v(1) = \gamma_{v/|v|}(|v|)$$

This means the exponential map act on v is moving forward distance $|v|$ along the geodesic with initial direction $v/|v|$.

Proposition 3.4. $\exp_{p*}|_0 : T_0(T_pM) \to T_pM$ *is identity (we identify* $T_0(T_pM)$ *with* T_pM*).*

Proof. We have

$$\exp_{p*}|_0(v) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \exp_p(tv) = v. \qquad \square$$

Corollary 3.5. *There is a ball* $B_\varepsilon(0) \subset T_pM$ *such that* $\exp_p : B_\varepsilon(0) \to M$ *is a diffeomorphism onto its image.*

Proof. Since $\exp_{p*}|_0$ is identity, it is nondegenerate, the corollary follows by Inverse Function Theorem. $\square$

Example 3.6.

(1) We know that the geodesics on $\mathbb{S}^n$ are great circles, hence $\exp_p$ is defined on the whole T_pM. But $\exp_p$ is not injective, since $\exp_p(0) = \exp_p(2\pi v) = p$ for unit vector v in T_pM.

(2) Let $M = \mathbb{S}^1 \times \mathbb{R}$ be the cylinder. We know from elementary differential geometry that the geodesics on cylinder are directrix circles, helices and generatrix lines. Then in local chart $(e^{2\pi i t}, s) \mapsto (t, s)$, we know the $\exp_p$ is not injective in the direction $(1,0)$, and injective in other directions.

Definition. If $\exp_p$ can be defined on whole T_pM for any $p \in M$, we say M is **geodesically complete**.

We have the following important theorem.

Theorem 3.7 (Hopf–Rinow)**.** *Let M be a Riemannian manifold, the following are equivalent:*

(1) M is geodesically complete.

(2) $\exp_p : T_pM \to M$ *is well-defined for some $p \in M$.*

(3) The Heine–Borel property *holds, that is, any closed bounded set is compact on M.*

(4) M is complete as a metric space, that is, any Cauchy sequence converges.

We will not prove Hopf–Rinow theorem here. For a proof, one can refer to Peter Petersen's *Riemannian Geometry*, 3rd ed., Theorem 5.7.1.

3.2 Curvature

We know $\exp_{p*}|_0$ is identity, and we want to ask:

Question. What is $\exp_{p*}|_v : T_v(T_pM) \to T_{\exp_p(v)}M$?

To calculate $\exp_{p*}|_v(\xi)$, we choose a line $v + s\xi$, and then

$$\exp_{p*}|_v(\xi) = \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \exp_p(v + s\xi)$$

Now we can introduce a family of geodesics $\gamma(t,s) = \gamma_s(t) = \exp_p(t(v+s\xi))$, and denote $\gamma(t) = \gamma(t,0)$. Let $J_s(t) = \frac{\partial}{\partial s}\gamma(t,s)$, then $\dot{J}_s(t) = \nabla_{\dot{\gamma}_s(t)}\frac{\partial\gamma}{\partial s}$. Since $\nabla_{\dot{\gamma}_s(t)}\frac{\partial\gamma}{\partial t} = 0$, we have

$$\begin{aligned}
\ddot{J}_s(t) &= \nabla_{\dot{\gamma}_s(t)}\nabla_{\dot{\gamma}_s(t)}\frac{\partial\gamma}{\partial s} \\
&= \nabla_{\dot{\gamma}_s(t)}\nabla_{\frac{\partial\gamma}{\partial s}}\frac{\partial\gamma}{\partial t} \quad \text{(torsion-freeness)} \\
&= \nabla_{\frac{\partial\gamma}{\partial t}}\nabla_{\frac{\partial\gamma}{\partial s}}\frac{\partial\gamma}{\partial t} - \nabla_{\frac{\partial\gamma}{\partial s}}\nabla_{\frac{\partial\gamma}{\partial t}}\frac{\partial\gamma}{\partial t}.
\end{aligned}$$

Denote

$$R\left(\frac{\partial\gamma}{\partial t}, \frac{\partial\gamma}{\partial s}\right) = \nabla_{\frac{\partial\gamma}{\partial s}}\nabla_{\frac{\partial\gamma}{\partial t}} - \nabla_{\frac{\partial\gamma}{\partial t}}\nabla_{\frac{\partial\gamma}{\partial s}} + \nabla_{\left[\frac{\partial\gamma}{\partial t}, \frac{\partial\gamma}{\partial s}\right]},$$

then we have

$$\frac{\partial^2}{\partial t^2}J_s(t) + R\left(\frac{\partial\gamma}{\partial t}, \frac{\partial\gamma}{\partial s}\right)\frac{\partial\gamma}{\partial t} = 0.$$

Let $s = 0$, we have

$$\ddot{J}(t) + R(\dot{\gamma}(t), J(t))\dot{\gamma}(t) = 0.$$

We make it into a definition.

Definition (Riemann curvature tensor)**.** Let $R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ defined by

$$R(X,Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z.$$

We also look at Riemann curvature tensor locally. Tedious calculation will show Riemann curvature tensor is truly tensorial. Using a more simple notation $\partial_i = \partial/\partial x^i$, we have

$$\begin{aligned} R(\partial_i, \partial_j)\, \partial_k &= \nabla_{\partial_j} \nabla_{\partial_i} \partial_k - \nabla_{\partial_i} \nabla_{\partial_j} \partial_k \\ &= \nabla_{\partial_j} (\Gamma^l_{ik}\, \partial_l) - \nabla_{\partial_i} (\Gamma^l_{jk}\, \partial_l), \end{aligned}$$

and

$$\begin{aligned} R^l_{ijk} &= (\partial_j \Gamma^l_{ik} - \partial_i \Gamma^l_{jk} + \Gamma^m_{ik} \Gamma^l_{jm} - \Gamma^m_{jk} \Gamma^l_{im})\, \partial_l \\ &= \partial^2 g + \partial g * \partial g \end{aligned}$$

We also define $R_{ijkl} = R^m_{ijk} g_{ml}$, or $R(X,Y,Z,W) = g(R(X,Y)Z,W)$.

Example 3.8. $(\mathbb{R}, \delta)$ has $R \equiv 0$. Any metric admits zero curvature is call **flat**.

Proposition 3.9. *Riemann curvature tensor has following symmetric properties: For $X,Y,Z,W \in \mathfrak{X}(M)$, we have*

(1) $R(X,Y,Z,W) = -R(Y,X,Z,W) = -R(X,Y,W,Z) = R(Z,W,X,Y)$*;*

(2) $R(X,Y,Z,W) + R(Y,Z,X,W) + R(Z,X,Y,W) = 0$ *(First Bianchi Identity).*

Proof. Tedious calculation. □

Definition (Sectional curvature)**.** Let $p \in M$, $\pi \subset T_pM$ be a 2-plane, $\pi = \text{Span}\{X,Y\}$. Then define the sectional curvature at p of π as

$$K_p(\pi) := \frac{R(X,Y,X,Y)}{|X|^2|Y|^2 - \langle X,Y \rangle^2}.$$

Remark 3.10. One can show sectional curvature does not depend on the choice of basis. A proof can be found in Manfredo do Carmo's *Riemannian Geometry*, Proposition 3.1.

Proposition 3.11. *Let M be a Riemannian manifold. $R(X,Y,Z,W)$ as a $(0,4)$-tensor is determined by all $K_p(\pi)$.*

For a proof, see Wu Hung–Hsi et. al., *Introduction to Riemannian Geometry*, Chapter 2 Lemma 2.

We mention one little observation. If all sectional curvature at p is constant K_p, then

$$R_p(X,Y,Z,W) = K_p(\langle X,Z \rangle \langle Y,W \rangle - \langle X,W \rangle \langle Y,Z \rangle)$$

For a theorem about constant sectoinal curvature, we mention here Schur's Theorem.

Theorem 3.12 (Schur)**.** *Let (M^n, g) be a Riemannian manifold, $n \geq 3$. If $K_p(\pi)$ is independent of $\pi \subset T_pM$ for any $p \in M$, then M has constant sectional curvature.*

We define another two important curvature.

Definition (Ricci curvature)**.** Let (M^n, g) be a Riemannian manifold. Define **Ricci curvature tensor** $\mathrm{Ric} : \mathfrak{X}(M) \to \mathfrak{X}(M)$ as

$$\mathrm{Ric}_p(X) = \sum_{i=1}^{n} R_p(e_i, X)e_i,$$

where $\{e_i\}$ is an orthonormal frame around p. It's easy to check the definition is independent from the choice of orthonormal frame, and Ric is self-adjoint.

Definition (Scalar curvature)**.** Let (M^n, g) be a Riemannian manifold. We define **Scalar curvature** $\mathrm{Scal} \in C^\infty(M)$ as

$$\mathrm{Scal}(p) = \sum_{i=1}^{n} \langle \mathrm{Ric}_p(e_i), e_i(p) \rangle$$

for an orthonormal frame $\{e_i\}$ around p.

Definition. Let (M, g) be a Riemannian manifold, if $\mathrm{Ric} = \lambda(p)g$ for a $\lambda \in C^\infty(M)$, we call M an **Einstein manifold**.

We also mention here

Theorem 3.13 (Schur)**.** *Let M^n be an Einstein manifold with $n \geq 3$, then M has constant scalar curvature.*

3.3 Jacobi Fields

Definition. Let γ be a geodesic, a vector field J along γ is called a **Jacobi field** if $\ddot{J} + R(\dot{\gamma}, J)\dot{\gamma} = 0$.

Let $\{e_i(t)\}$ be a parallel orthonormal frame along γ, $J(t) = \sum_i f_i(t)e_i(t)$. Define $a_{ij}(t) = \langle R(\dot{\gamma}(t), e_i(t))\dot{\gamma}(t), e_j(t) \rangle$, then the qeuation for Jacobi field is equivalent to

$$\ddot{f}_i(t) + \sum_j a_i j(t) f_j(t) = 0,\ i = 1, 2, \cdots, n.$$

By ODE theory, given $f_i(0), \dot{f}_i(0)$, $i = 1, 2, \cdots, n$, the $f_i(t)$'s are uniquely determined. Translating into language of Jacobi field, we have a Jacobi field is uniquely determined by $J(0)$ and $\dot{J}(0)$.

Notation 3.14. Let γ be a geodesic on a Riemannian manifold M, the vector space of Jacobi fields along γ is denoted by $\mathscr{J}(\gamma)$.

Above discussion can be summarized as the following proposition.

Proposition 3.15. *Let M^n be a Riemannian manifold, γ be a geodesic, then* $\dim \mathscr{J}(\gamma) = 2n$.

The next proposition shows only the Jacobi field at tangential direction are interesting.

Proposition 3.16. *Let J be a Jacobi field along γ, we have the decomposition $J(t) = J^{\perp}(t) + (at+b)\dot{\gamma}(t)$, where $J^{\perp}(t) \perp \dot{\gamma}(t)$. If a Jacobi field is perpendicular to γ, we call it a* ***normal Jacobi field***

Proof. We have

$$\begin{aligned}
\frac{\mathrm{d}^2}{\mathrm{d}t^2}\langle J(t), \dot{\gamma}(t)\rangle &= \langle \ddot{J}(t), \dot{\gamma}(t)\rangle \\
&= -\langle R(\dot{\gamma}, J)\dot{\gamma}, \dot{\gamma}\rangle \\
&= 0.
\end{aligned}$$

□

We have the following Gauss Lemma.

Proposition 3.17 (Gauss Lemma)**.** $\langle \exp_{p*}|_v(\xi), \dot{\gamma}_v(1)\rangle = \langle \xi, v\rangle$.

Proof. Let one-parameter geodesic family $\gamma(t,s) = \exp_p(t(v+s\xi))$, then using the calculation in the beginning of section 3.2, we have

$$\begin{aligned}
J(t) = \left.\frac{\partial}{\partial s}\right|_{s=0} \gamma(t,s) &= \exp_{p*}|_{tv}(t\xi) \\
&= t\exp_{p*}|_{tv}(\xi).
\end{aligned}$$

Moreover, we have $\langle J(t), \dot{\gamma}_v(t)\rangle = at+b$ (Proposition 3.16),

$$\begin{aligned}
a = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \langle J(t), \dot{\gamma}_v(t)\rangle &= \langle \dot{J}(0), \dot{\gamma}_v(0)\rangle + \langle J(0), \nabla_v \dot{\gamma}_v(t)\rangle \\
&= \langle \dot{J}(0), v\rangle = \left\langle \left.\left(\exp_{p*}|_{tv}(\xi) + t\frac{\mathrm{d}}{\mathrm{d}t}\exp_{p*}|_{tv}(\xi)\right)\right|_{t=0}, v\right\rangle \\
&= \langle \exp_{p*}|_0(\xi), v\rangle = \langle \xi, v\rangle,
\end{aligned}$$

and $b = \langle J(0), v\rangle = 0$. But we have

$$t\langle \exp_{p*}|_{tv}(\xi), \dot{\gamma}_v(t)\rangle = \langle J(t), \dot{\gamma}_v(t)\rangle = \langle \xi, v\rangle t.$$

Let $t = 1$ we obtain the conclusion. □

Example 3.18. We calculate the Jacobi field on Riemannian manifolds admit a constant sectional curvature. By scaling the metric, we can assume $K=0,1,-1$. The corresponding simply connected complete Riemannian manifolds are called **space forms**, which are $\mathbb{R}^n,\mathbb{S}^n,\mathbb{H}^n$. The Jacobi field equation is

$$\begin{cases} \ddot{J}(t)+KJ(t)=0, \\ J(0)=0, \dot{J}(0)=\xi. \end{cases}$$

Let $\xi(t)$ be the parallel transport along a geodesic with $\xi(0)=\xi$, then solving the equation by eigenvalue method, we obtain

$$J(t)=\begin{cases} t\xi(t), & K=0, \\ \sin(t)\xi(t), & K=1, \\ \sinh(t)\xi(t), & K=-1. \end{cases}$$

3.4 Some Local Charts

In this section, we adopt the traditional terminology "coordinate" to mean chart.

First we introduce the geodesic normal coordinate. Given a Riemannian manifold (M,g) and $p\in M$, there exists an $\varepsilon>0$ such that $\exp_p : B_\varepsilon(0)\to\exp_p(B_\varepsilon(0)) =: B_\varepsilon(p)$ is an diffeomorphism. Let $\{e_i\}$ be an orthonormal basis of Euclidean space (T_pM,g_p), $\{\alpha^i\}$ be the dual basis of $\{e_i\}$, then we construct the **geodesic normal coordinate**

$$q\in B_\varepsilon(p)\mapsto(\alpha^1(\exp_p^{-1}(q)),\cdots,\alpha^n(\exp_p^{-1}(q))).$$

Proposition 3.19. *Under geodesic normal coordinate, we have*

$$g_{ij}(p)=\delta_{ij},\ \Gamma_{ij}^k(p)=0$$

Proof. Since $\exp_p$ is a diffeomorphism, we have $\left.\frac{\partial}{\partial x^i}\right|_p=\exp_{p*}|_0(e_i)=e_i$, hence $g_{ij}=g_p(e_i,e_j)=\delta_{ij}$. Moreover, let $x(t)=ty$ for $y\in T_pM\backslash\{0\}$, then $x(t)$ is the coordinate of some geodesic in $B_\varepsilon(p)$, thus it satisfies the equation

$$\ddot{x}^k(t)+\Gamma_{ij}^k(x(t))\dot{x}^i(t)\dot{x}^j(t)=0.$$

Since $\ddot{x}^k(t)=0$, $\dot{x}^i(t)=y^i\neq 0$, we must have $\Gamma_{ij}^k(ty)=0$. Let $y\to 0$ and we obtain the conclusion. □

Next we introduce the geodesic polar coordinate. Let $(r,\theta^1,\cdots,\theta^{n-1})$ be the polar coordinate on Euclidean space (T_pM,g_p), and we define the **geodesic polar coordinate** by

$$q\in B_\varepsilon(p)\backslash\{p\}\mapsto(r(\exp_p^{-1}(q)),\theta^1(\exp_p^{-1}(q)),\cdots,\theta^{n-1}(\exp_p^{-1}(q))).$$

Proposition 3.20. *Under geodesic polar coordinate, we have*

$$g\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) = 1,\ g\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta^i}\right) = 0$$

Proof. To make things clear, we write the inverse of geodesic polar coordinate as

$$F : (r, \omega) \mapsto \exp_p(r\omega)$$

for $r \in (0, +\infty), \omega \in \mathbb{S}^{n-1}$. Then we use $\partial_r, \partial_{\theta^1}, \cdots, \partial_{\theta^{n-1}}$ to denote the tangent vectors in $(0, +\infty) \times \mathbb{S}^{n-1}$, we have

$$\begin{aligned}\frac{\partial}{\partial r} &= F_*(\partial_r)\\ \frac{\partial}{\partial \theta^i} &= F_*(\partial_{\theta^i}),\ i = 1, \cdots, n.\end{aligned}$$

First we know ∂_r is the tangent vector of direction $r\omega$, hence $\partial/\partial r$ is the tangent vector of a unit-speed radial geodesic, that is

$$g\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) = 1.$$

Moreover, we have

$$\begin{aligned}\frac{\partial}{\partial r} g\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta^i}\right) &= g\left(\nabla_{\frac{\partial}{\partial r}} \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta^i}\right) + g\left(\frac{\partial}{\partial r}, \nabla_{\frac{\partial}{\partial r}} \frac{\partial}{\partial \theta^i}\right)\\ &= g\left(\frac{\partial}{\partial r}, \nabla_{\frac{\partial}{\partial r}} \frac{\partial}{\partial \theta^i}\right)\\ &= g\left(\frac{\partial}{\partial r}, \nabla_{\frac{\partial}{\partial \theta^i}} \frac{\partial}{\partial r}\right) \quad \text{(torsion-freeness)}\\ &= \frac{1}{2}\frac{\partial}{\partial \theta^i} g\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right)\\ &= 0,\end{aligned}$$

hence $g\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta^i}\right)$ is constant. But if we let $r \to 0$, we have $\partial/\partial\theta^i \to 0$, therefore

$$g\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta^i}\right) = 0$$

□

Corollary 3.21. *Under geodesic polar coordinate, the metric tensor has local expression*

$$g = \mathrm{d}r^2 + g_{ij}(r, \theta)\,\mathrm{d}\theta^i \otimes \mathrm{d}\theta^j$$

As an application, we prove geodesics are locally length-minimizing.

Proposition 3.22. *Let $\gamma : [0,1] \to M$ be a geodesic in $B_\varepsilon(p)$, $\tilde{\gamma} : [0,1] \to M$ be any curve in $B_\varepsilon(p)$ with $\gamma(0) = \tilde{\gamma}(0) = p$, $\gamma(1) = \tilde{\gamma}(1) = q$. Then $L(\gamma) \le L(\tilde{\gamma})$.*

Proof. Let $q = \exp_p(v)$, φ be the geodesic polar coordinate, then we have

$$\gamma(t) = (tr_0, \omega_0),\ \tilde{\gamma}(t) = (r(t), \omega(t))$$

such that $\omega_0, \omega(t) \in \mathbb{S}^{n-1}$, $r(1) = r_0$. Therefore

$$\begin{aligned}
L(\gamma) &= \int_0^1 g_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))\,\mathrm{d}t \\
&= \int_0^1 |v|\,\mathrm{d}t = r_0 \\
L(\tilde{\gamma}) &= \int_0^1 (|\dot{r}(t)|^2 + g_{ij}\dot{\theta}^i(t)\dot{\theta}^j(t))^{1/2}\,\mathrm{d}t \\
&\ge \int_0^1 |\dot{r}(t)|\,\mathrm{d}t \\
&\ge \int_0^1 \dot{r}(t) = r_0
\end{aligned}$$

□

3.5 Conjugate Points

Definition (Conjugate points)**.** If $\exp_{p*}|_{\dot{\gamma}_v(t_0)}$ is degenerate at $\dot{\gamma}_v(t_0)$, then we call $\gamma_v(t_0)$ a **conjugate point** of p along γ.

Proposition 3.23. *We have $\exp_p$ is degenerate at $\gamma(t_0)$ if and only if there is a Jacobi field J not identically equal to 0 such that $J(0) = J(t_0) = 0$.*

Proof. $\exp_{p*}|_{\dot{\gamma}_v(t_0)}(\xi) = 0$ if and only if $J(t) = \left.\frac{\partial}{\partial s}\right|_{s=0} \exp_p(t(v + s\xi))$ satisfies $J(0) = J(t) = 0$. □

Remark 3.24. This proposition shows conjugate is symmetric.

Chapter 4

Variation Formula and Index Form

In this lecture we introduce the variation of energy. The variation formulae are closely related to minimizing property of geodesics. As an application of second variation formula, we introduce the Bonnet–Myers Theorem. Regarding the second variation formula as a quadric form, we have the notion of index form. We will explain the relation between index form and conjugate points. Finally, we will briefly mention the Morse Index Theorem.

Let $\gamma : [0,a] \to M$ be a curve, we define two functional

$$L(\gamma) = \int_0^a |\dot{\gamma}(t)| \, \mathrm{d}t$$
$$E(\gamma) = \int_0^a \frac{1}{2} |\dot{\gamma}(t)|^2 \, \mathrm{d}t.$$

Then by Cauchy–Schwarz inequality, we have

$$L(\gamma)^2 \leq 2aE(\gamma),$$

with equality holds if and only if $|\dot{\gamma}(t)| = \text{const}$.

Proposition 4.1. *If γ is a length-minimizing geodesic, then γ is energy-minimizing.*

Proof. Let $\tilde{\gamma}$ be another curve, then

$$2aE(\gamma) = L^2(\gamma) \leq L^2(\tilde{\gamma}) \leq 2aE(\tilde{\gamma})$$ □

Our aim is to prove the converse.

Proposition 4.2. *If γ is energy-minimizing, then γ is a length-minimizing geodesic.*

4.1 First Variation Formula

Definition (Variation)**.** Let $\gamma_0 : [0,a] \to M$ be a curve, a **variation** of γ_0 is a differentiable map $\gamma : [0,a] \times (-\varepsilon, \varepsilon) \to M$ such that $\gamma(t,0) = \gamma_0(t)$. If $\gamma(0,s) = \gamma_0(0)$, $\gamma(a,s) = \gamma_0(a)$ for any $s \in (-\varepsilon, \varepsilon)$, then we call it a **proper variation**. We call $\frac{\partial}{\partial s}\Big|_{s=0} \gamma(t,s) =: V(t)$ the **variation vector field**.

Proposition 4.3 (First variation formula)**.** *Let $\gamma(t,s)$ be a variation, its energy $E(s) = \int_0^a \frac{1}{2}\left|\frac{\partial}{\partial t}\gamma(t,s)\right|^2 \mathrm{d}t$, then we have*

$$E'(0) = \langle V, \dot{\gamma} \rangle|_0^a - \int_0^a \left\langle V(t), \nabla_{\dot{\gamma}_0(t)} \dot{\gamma}_0(t) \right\rangle \mathrm{d}t.$$

Proof. We calculate

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s} E(s) &= \int_0^a \left\langle \frac{\partial \gamma}{\partial t}, \nabla_{\frac{\partial}{\partial s}} \frac{\partial \gamma}{\partial t} \right\rangle \mathrm{d}t \\
&= \int_0^a \left\langle \frac{\partial \gamma}{\partial t}, \nabla_{\frac{\partial}{\partial t}} \frac{\partial \gamma}{\partial s} \right\rangle \mathrm{d}t & \text{(LC2)} \\
&= \int_0^a \left(\frac{\partial}{\partial t} \left\langle \frac{\partial \gamma}{\partial t}, \frac{\partial \gamma}{\partial s} \right\rangle - \left\langle \frac{\partial \gamma}{\partial s}, \nabla_{\frac{\partial}{\partial t}} \frac{\partial \gamma}{\partial t} \right\rangle \right) \mathrm{d}t. & \text{(LC1)}
\end{aligned}$$

Take $s = 0$, we obtain

$$\begin{aligned}
E'(0) &= \int_0^a \left(\frac{\partial}{\partial t} \langle V(t), \dot{\gamma}_0(t) \rangle - \langle V(t), \nabla_{\dot{\gamma}_0(t)} \dot{\gamma}_0(t) \rangle \right) \mathrm{d}t \\
&= \langle V, \dot{\gamma} \rangle|_0^a - \int_0^a \left\langle V(t), \nabla_{\dot{\gamma}_0(t)} \dot{\gamma}_0(t) \right\rangle \mathrm{d}t.
\end{aligned}$$

□

Corollary 4.4. *$E'(0) = 0$ for all proper variation if and only if $\nabla_{\dot{\gamma}_0(t)} \dot{\gamma}_0(t) = 0$, that is, γ_0 is a geodesic.*

Now we can give a proof of energy-minimizing curves are length-minimizing geodesics.

Proof of Proposition 4.2. Let $\gamma : [0,a] \to M$ be a curve such that for any $\tilde{\gamma} : [0,1] \to M$ with $\gamma(0) = \tilde{\gamma}(0)$, $\gamma(1) = \tilde{\gamma}(1)$, the inequality $E(\gamma) \leq E(\tilde{\gamma})$ holds, we show that $L(\gamma) \leq L(\tilde{\gamma})$. Let $\gamma(t,s)$ be any variation with $\gamma(t,0) = \gamma$, then γ is a critical point of $E(s)$. Hence by Corollary 4.4, γ is a geodesic. Then we can reparameterize $\tilde{\gamma}$ into arc-length, obtaining $\hat{\tilde{\gamma}}$. Therefore

$$L^2(\gamma) = 2aE(\gamma) \leq 2aE\left(\hat{\tilde{\gamma}}\right) = L^2\left(\hat{\tilde{\gamma}}\right) = L^2(\tilde{\gamma})$$

□

4.2 Second Variation Formula

Since we concentrate on critical points of variation of energy, we define second variation formula only for geodesics.

Proposition 4.5 (Second variation formula)**.** *Let $\gamma_0 : [0,a] \to M$ be a geodesic, then*

$$E''(0) = \int_0^a \left(|\dot{V}(t)|^2 - \langle R(\dot{\gamma}_0(t), V(t))\dot{\gamma}_0(t), V(t)\rangle\right) \mathrm{d}t + \boxed{\langle \nabla_{V(t)} V(t), \dot{\gamma}_0(t)\rangle|_0^a}.$$

The boxed term is called ***boundary term****, and it vanishes when the variation is proper.*

Proof. We take the expression of $E'(s)$ from the proof of first variation formula and differentiate

$$\begin{aligned} E''(s) &= \int_0^a \frac{\partial}{\partial s}\left\langle \frac{\partial \gamma}{\partial t}, \nabla_{\frac{\partial \gamma}{\partial s}} \frac{\partial \gamma}{\partial t}\right\rangle \mathrm{d}t \\ &= \int_0^a \left(\left\langle \nabla_{\frac{\partial \gamma}{\partial s}} \frac{\partial \gamma}{\partial t}, \nabla_{\frac{\partial \gamma}{\partial s}} \frac{\partial \gamma}{\partial t}\right\rangle + \left\langle \frac{\partial \gamma}{\partial t}, \nabla_{\frac{\partial \gamma}{\partial s}} \nabla_{\frac{\partial \gamma}{\partial s}} \frac{\partial \gamma}{\partial t}\right\rangle\right) \mathrm{d}t, \end{aligned}$$

where

$$\left\langle \nabla_{\frac{\partial \gamma}{\partial s}} \frac{\partial \gamma}{\partial t}, \nabla_{\frac{\partial \gamma}{\partial s}} \frac{\partial \gamma}{\partial t}\right\rangle = \left\langle \nabla_{\frac{\partial \gamma}{\partial t}} \frac{\partial \gamma}{\partial s}, \nabla_{\frac{\partial \gamma}{\partial t}} \frac{\partial \gamma}{\partial s}\right\rangle = |\dot{V}(t)|^2$$

and

$$\nabla_{\frac{\partial \gamma}{\partial s}} \nabla_{\frac{\partial \gamma}{\partial s}} \frac{\partial \gamma}{\partial t} = \nabla_{\frac{\partial \gamma}{\partial s}} \nabla_{\frac{\partial \gamma}{\partial t}} \frac{\partial \gamma}{\partial s} = \nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial \gamma}{\partial s}} \frac{\partial \gamma}{\partial s} - R\left(\frac{\partial \gamma}{\partial s}, \frac{\partial \gamma}{\partial t}\right)\frac{\partial \gamma}{\partial s}$$

Thus

$$E''(s) = \int_0^a \left(|\dot{V}(t)|^2 - \langle R(\dot{\gamma}, V)\dot{\gamma}, V\rangle + \frac{\partial}{\partial t}\left\langle \frac{\partial \gamma}{\partial t}, \nabla_{\frac{\partial \gamma}{\partial s}} \frac{\partial \gamma}{\partial s}\right\rangle - \left\langle \nabla_{\frac{\partial \gamma}{\partial t}} \frac{\partial \gamma}{\partial t}, \nabla_{\frac{\partial \gamma}{\partial s}}, \frac{\partial \gamma}{\partial s}\right\rangle\right) \mathrm{d}t,$$

the last term is 0, hence second variation formula holds by taking $s = 0$. □

We now use second variation formula to prove the famous Bonnet–Myers Theorem.

Theorem 4.6 (Bonnet–Myers)**.** *Let (M^n, g) be a complete Riemannian manifold with* $\mathrm{Ric} \geq (n-1)Kg > 0$*, then* $\mathrm{diam}(M,g) \leq \pi/\sqrt{K}$*. In particular, M is compact.*

Proof. We can scale the metric and assume $K = 1$. We prove the theorem by contradiction.

Assume there exists $p, q \in M$ joined by length-minimizing geodesic $\gamma : [0,1] \to M$, with $d(p,q) = L(\gamma) > \pi$. Since γ is length-minimizing, $E''(0)(V,V) \geq 0$ for any proper variation vector field V. Let $\{e_1(t), \cdots, e_{n-1}(t), \dot{\gamma}(t)/|\dot{\gamma}(t)|\}$ be a parallel orthonormal frame along γ, and set $V_i(t) = \sin(\pi t)e_i(t)$, $i = 1, \cdots, n-1$. Then $V_i(0) = V_i(1) = 0$ for each $i = 1, \cdots, n-1$, V_i's are proper variation vector field. Thus we have

$$\begin{aligned}
E''(0)(V_i,V_i) &= \int_0^1 (|\dot{V}_i(t)|^2 - \langle R(\dot{\gamma}(t),V(t))\dot{\gamma}(t),V(t)\rangle)\,\mathrm{d}t \\
&= \int_0^1 (-\langle V_i,\ddot{V}_i\rangle - \langle R(\dot{\gamma}(t),V(t))\dot{\gamma}(t),V(t)\rangle)\,\mathrm{d}t \quad \text{(integration by parts)} \\
&= \int_0^1 (\pi^2\sin^2(\pi t) - \sin^2(\pi t)L^2(\gamma)K(e_n,e_i))\,\mathrm{d}t
\end{aligned}$$

Take summation we have

$$\begin{aligned}
\sum_{i=1}^{n-1} E''(0)(V_i,V_i) &= \int_0^1 \sin^2(\pi t)((n-1)\pi^2 - L^2(\gamma)\langle \mathrm{Ric}(e_n),e_n\rangle)\,\mathrm{d}t \\
&\leq \int_0^1 (n-1)\sin^2(\pi t)(\pi^2 - L^2(\gamma))\,\mathrm{d}t \\
&< 0
\end{aligned}$$

Then there must exist a V_i such that $E''(0)(V_i,V_i) < 0$, contradiction! Hence we proved $\mathrm{diam}(M) \leq \pi$. Moreover, by Hopf–Rinow theorem, M is bounded implies M is compact (M is automatically closed as a topological space). $\square$

Involving some theory of covering spaces, we have the following corollary.

Corollary 4.7. *The universal covering $\tilde{M} \to M$ is compact. Moreover, $\pi_1(M)$ is finite.*

Proof. We can lift g to $\tilde{M}$ to make $\pi : \tilde{M} \to M$ a Riemannian covering, then π^*g also admits a Ricci curvature bounded below. By Bonnet–Myers theorem, $\tilde{M}$ is compact. For the next claim, let $p \in M$, then $\pi^{-1}(p)$ is a discret closed set in $\tilde{M}$, hence must be finite. Then the covering map is of finite sheet, $\pi_1(\tilde{M})$ has finite index in $\pi_1(M)$. But $\pi_1(\tilde{M})$ is trivial, $\pi_1(M)$ must be finite. $\square$

Remark 4.8. (1) We cannot weaken the condition to $K = 0$, in fact, even $\mathrm{Sect} > 0$ is not enough. The surface $z = x^2 + y^2$ in $\mathbb{R}^3$ is an counterexample.

(2) If (M^n, g) satisfies $\mathrm{Ric} \geq (n-1)g$ and $\mathrm{diam}(M,g) = \pi$, then M must isometric to $\mathbb{S}^n$. This is Cheng's Maximal Diameter Theorem.

4.3 Index Form

Definition (Index form)**.** Let $\gamma : [0,a] \to M$ be a geodesic. The **index form** of γ is a biliear form on $\mathscr{V} := \{\text{vector fields along } \gamma\}$ defined by

$$I(X,Y) = \int_0^a (\langle \dot{X},\dot{Y}\rangle - \langle R(\dot{\gamma},X)\dot{\gamma},Y\rangle)\,\mathrm{d}t$$

Lemma 4.9. *Let $U \in \mathscr{V}_0 := \{Y \in \mathscr{V} \mid Y(0) = Y(a) = 0\}$, then U is a Jacobi field if and only if $I(U,Y) = 0$ for any $Y \in \mathscr{V}_0$.*

Proof. First we observe

$$I(U,Y) = -\int_0^a \langle \ddot{V} + R(\dot{\gamma}, V)\dot{\gamma}, Y\rangle \, \mathrm{d}t$$

If V is a Jacobi field, then $\ddot{V} + R(\dot{\gamma}, V)\dot{\gamma} = 0$, which implies $I(U,Y) = 0$. Conversely, if $I(U,Y) = 0$ for any $Y \in \mathscr{V}_0$, we choose $Y = \ddot{V} + R(\dot{\gamma}, V)\dot{\gamma}$, then

$$0 = -\int_0^a |\ddot{V} + R(\dot{\gamma}, V)\dot{\gamma}|^2 \, \mathrm{d}t,$$

this implies $\ddot{V} + R(\dot{\gamma}, V)\dot{\gamma} = 0$, that is, V is a Jacobi field. □

We observe if a Jacobi field J is not in $\mathscr{V}_0$, the above integration by parts is changed into

$$\begin{aligned} I(Y,J) &= \langle Y, \dot{J}\rangle|_0^a - \int_0^a \langle \ddot{V} + R(\dot{\gamma}, V)\dot{\gamma}, Y\rangle \\ &= \langle Y, \dot{J}\rangle|_0^a \end{aligned} \tag{4.1}$$

This will be useful in some calculation.

Next theorem shows the positive definiteness of I is related to conjugate points.

Theorem 4.10. *Let $\gamma[0,a] \to M$ be a geodesic, I be its index form.*

(1) If γ has no conjugate points of $\gamma(0)$, then I is positive definite on $\mathscr{V}_0$.

(2) If $\gamma(a)$ is the only conjugate point of $\gamma(0)$, then I is positive semidefinite but not positive definite.

(3) If there is a $t_0 < a$ such that $\gamma(t_0)$ is conjugate to $\gamma(0)$, then there is a $U \in \mathscr{V}$ such that $I(U,U) < 0$.

Proof of Theorem 4.10 (1). [1] Let $\gamma(0) = p$, $\tilde{\gamma}$ is the radial line in T_pM defined by $\tilde{\gamma}(t) = \dot{\gamma}(0)t$. Since γ has no conjugate points of $\gamma(0)$, the exponential map $\exp_p$ is not degenerate on the whole $\tilde{\gamma}$. Hence there is a neighborhood U of $\tilde{\gamma}([0,a])$ such that $\exp_p : U \to M$ is an immersion. Now by carefully modifying the proof of Proposition 3.22, we have γ is the length-minimizing curve in $\exp_p(U)$. Hence by Proposition 4.1, γ also minimizes energy. Let $\gamma(t,s) : [0,a] \times (-\varepsilon, \varepsilon) \to M$ be any proper variation of γ, by taking ε small enough we can assume every γ_s is in $\exp_p(U)$. Then we have

$$E''(0) = \lim_{s\to 0} \frac{E(-s) + E(s) - 2E(0)}{s^2} \geq 0.$$

[1] This proof is not the proof provided on class.

Since $E''(0)(V,V) = I(V,V)$ for any variation vector field V, we have $I(V,V) \geq 0$ for all $V \in \mathscr{V}_0$. Now we must show that $I(V,V) = 0$ implies $V = 0$. Let $I(V,V) = 0$, $X \in \mathscr{V}_0$ and $\delta > 0$, we have

$$0 \leq I(V+\delta X, V+\delta X) = I(V,V) + 2\delta I(V,X) + \delta^2 I(X,X),$$

this implies

$$2I(V,X) + \delta I(X,X) \geq 0,$$

let $\delta \to 0$, we obtain $I(V,X) \geq 0$ for all $X \in \mathscr{V}_0$. Similarly, consider $I(V-\delta X, V-\delta X)$, we obtain $I(V,X) \leq 0$ for all $X \in \mathscr{V}_0$. This means $I(V,X) = 0$ for all $X \in \mathscr{V}_0$. By Proposition 4.9, V is a Jacobi field. But γ has no conjugate points of $\gamma(0)$, V must identically equal to 0. This proves I being positive definite. □

Before proving the rest of Theorem 4.10, we need the *Index Lemma*.

Proposition 4.11 (Index Lemma). *Assume γ is a geodesic without conjugate points. Let $U \in \mathscr{V}$ with $U(0) = 0$, J be a Jacobi field such that $J(0) = 0$, $J(a) = U(a)$, then $I(J,J) \leq I(U,U)$. The equality holds if and only if $U = J$.*

Proof. Since $U - J \in \mathscr{V}_0$, by Theorem 4.10 (1), we have $I(U-J, U-J) \geq 0$ with equality holds if and only if $U = J$. Then

$$I(U-J, U-J) = I(U,U) - 2I(U,J) + I(J,J)$$

But $I(U,J) = \langle U, \dot{J} \rangle |_0^a = \langle J, \dot{J} \rangle |_0^a = I(J,J)$, thus

$$I(U,U) \geq I(J,J)$$ □

Proof of the rest of Theorem 4.10. (2) For any $0 < t_0 < a$, let $I_{[0,t_0]}$ denote

$$I_{[0,t_0]}(X,Y) = \int_0^{t_0} (\langle \dot{X}, \dot{Y} \rangle + \langle R(\dot{\gamma}, X)\dot{\gamma}, Y \rangle)\, \mathrm{d}t.$$

Then for any $X \in \mathscr{V}_0|_{[0,t_0]}$, by (1) we have $I_{[0,t_0]}(X,X) \geq 0$. We now construct $\tau_{t_0}(U)$ for any $U \in \mathscr{V}_0$. Let $\{e_i(t)\}$ be a parallel frame, $U(t) = \sum_{i=1}^n f_i(t) e_i(t)$. Then we define

$$\tau_{t_0}(U)(t) = \sum_{i=1}^n f\left(\frac{a}{t_0} t\right) e_i\left(\frac{a}{t_0} t\right) \in \mathscr{V}_0|_{[0,t_0]}.$$

Thus we have

$$I_{[0,t_0]}(\tau_{t_0}(U), \tau_{t_0}(U)) \geq 0,$$

let $t_0 \to a$ then we obtain the conclusion. Moreover, since $\gamma(a)$ is conjugate to $\gamma(0)$, there is a Jacobi field J with $J(0) = J(a) = 0$, $J \not\equiv 0$, and $I(J,J) = \langle J, \dot{J} \rangle |_0^a = 0$. This shows I is positive semidefinite but not positive define.

(3) Since $\gamma(t_0)$ is conjugate to $\gamma(0)$, there exists a Jacobi field J_1 along $\gamma|_{[0,t_0]}$ such that $J_1(0) = J_1(t_0) = 0$. Let

$$V(t) = \begin{cases} J_1(t), & t \in [0,t_0], \\ 0, & t \in [t_0,a]. \end{cases}$$

Let $\delta > 0$ so small that $\gamma|_{[t_0-\delta,t_0+\delta]}$ is contained in a normal neighborhood of $\gamma(t_0)$, then there exists a Jacobi field J_2 along $\gamma|_{[t_0-\delta,t_0+\delta]}$ with $J_2(t_0-\delta) = J_1(t_0-\delta)$, $J_2(t_0+\delta) = 0$. Define

$$U(t) = \begin{cases} J_1(t), & t \in [0,t_0-\delta], \\ J_2(t), & t \in [t_0-\delta,t_0+\delta], \\ 0, & t \in [t_0+\delta,a]. \end{cases}$$

Then we have $I(V,V) = I_{[0,t_0]}(J_1,J_1) = 0$, and[2]

$$\begin{aligned} I(U,U) &= I_{[0,t_0-\delta]}(J_1,J_1) + I_{[t_0-\delta,t_0+\delta]}(J_2,J_2) \\ &< I_{[0,t_0-\delta]}(V,V) + I_{[t_0-\delta,t_0+\delta]}(V,V) \quad \text{(Index Lemma)} \\ &= I(V,V) \\ &= 0 \end{aligned}$$

□

Translating Theorem 4.10 into language of geometry, we have the following Jacobi Theorem.

Theorem 4.12 (Jacobi). *Let $\gamma : [0,a] \to M$ be a geodesic, then*

(1) If γ has no conjugate points of $\gamma(0)$, then γ is length-minimizing under any small proper variation.

(2) If there is a $t_0 \in (0,a)$ such that $\gamma(t_0)$ is conjugate to $\gamma(0)$, then there is a proper variation $\gamma_s(t)$ such that $L(\gamma_s) < L(\gamma)$ for all s.

Finally we mention something about *Morse index*. Define Morse index

$$\operatorname{ind}(\gamma) = \max\{\dim S | \ S \leq \mathscr{V}_0, \ I \text{ is negative definite on } S\}$$

The famous *Morse Index Theorem* claims

$$\operatorname{ind}(\gamma) = \#\{\text{conjugate points (counting multiplicity)}\} < +\infty$$

For more information, one can refer to John Milnor's *Morse Theory*.

[2] Actually we allow piecewise smooth curve in variation and index form.

Chapter 5

Comparison Theorems

In this lecture, we first state and prove Rauch comparison theorem. Then we discuss cut points and distance function, as preparation for Hessian and Laplace comparison theorem. Using Rauch comparison theorem as a model, we then state and prove Hessian, Laplace and Bishop–Gromov relative comparison theorems. Finally, as an application, we prove Cheng's maximal radius theorem.

5.1 Rauch Comparison Theorem

We first state and prove the Rauch comparison theorem.

Theorem 5.1 (Rauch comparison)**.** *Let $M^n, \tilde{M}^n$ be Riemannian manifolds, $\gamma : [0,a] \to M$, $\tilde{\gamma} : [0,a] \to \tilde{M}$ be unit speed geodesics, and $J, \tilde{J}$ be Jacobi fields along $\gamma, \tilde{\gamma}$ respectively, such that $J(0) = \tilde{J}(0) = 0$, $\langle \dot{J}(0), \dot{\gamma}(0) \rangle = \left\langle \dot{\tilde{J}}(0), \dot{\tilde{\gamma}}(0) \right\rangle$, $|\dot{J}(0)| = \left|\dot{\tilde{J}}(0)\right|$. Assume that*

(1) γ has no conjugate points of $\gamma(0)$ along γ;

(2) $K_\gamma(\dot{\gamma}, v) \geq K_{\tilde{\gamma}}\left(\dot{\tilde{\gamma}}, \tilde{v}\right)$ with $|v| = |\tilde{v}| = 1$.

Then $|J(t)| \leq |\tilde{J}(t)|$ for all $t \in [0,a]$.

Remark 5.2. Before giving the proof of the theorem, we notice that when $\dim M = \dim \tilde{M} = 2$, the theorem reduces to the Liouville–Sturm comparison theorem in ODE theory.

Proof of Theorem 5.1. Decompose $J, \tilde{J}$ into $J^\perp(t) + (at+b)\dot{\gamma}(t)$, $\tilde{J}^\perp(t) + (\tilde{a}t + \tilde{b})\dot{\tilde{\gamma}}(t)$ respectively. Since $J(0) = \tilde{J}(0) = 0$, we have $b = \tilde{b} = 0$; Moreover, since $\langle \dot{J}(0), \dot{\gamma}(0) \rangle = \left\langle \dot{\tilde{J}}(0), \dot{\tilde{\gamma}}(0) \right\rangle$, we have $a = \tilde{a}$. Since $|J^\perp|^2 = |J|^2 - (at)^2$, $|\tilde{J}^\perp|^2 = |\tilde{J}|^2 - (\tilde{a}t)^2$, we can just compare $J^\perp$ and $\tilde{J}^\perp$. So we assume without loss of generality that $J, \tilde{J}$ are normal Jacobi fields, that is, $\dot{J}(0) \perp \dot{\gamma}(0)$, $\dot{\tilde{J}}(0) \perp \dot{\tilde{\gamma}}(0)$.

Let $f(t) = |J(t)|^2$, $\tilde{f}(t) = |\tilde{J}(t)|^2$. By l'Hospital's rule, we have

$$\lim_{\varepsilon \to 0} \frac{\tilde{f}(\varepsilon)}{f(\varepsilon)} = 1,$$

hence only need to show $\tilde{f}/f$ is monotonically increasing, which is equivalent to

$$\frac{\dot{\tilde{f}}}{\tilde{f}} \geq \frac{\dot{f}}{f} \iff \frac{\left\langle \dot{\tilde{J}}, \tilde{J} \right\rangle}{\left|\tilde{J}\right|^2} \geq \frac{\langle \dot{J}, J \rangle}{|J|^2}.$$

We can check the last inequality pointwisely, so we can scale the Jacobi fields to make $|\tilde{J}| = |J|$. Then we write the inequality into a lemma. □

Lemma 5.3. *If $J, \tilde{J}$ are Jacobi fields along $\gamma, \tilde{\gamma}$ respectively, with $J(0) = \tilde{J}(0) = 0$, $\dot{J}(0) \perp \dot{\gamma}(0)$, $\dot{\tilde{J}}(0) \perp \dot{\tilde{\gamma}}(0)$, and $|J(c)| = |\tilde{J}(c)|$. Moreover, the conditions (1) and (2) in Theorem 5.1 hold, then $\left\langle \dot{\tilde{J}}(c), \tilde{J}(c) \right\rangle \geq \langle \dot{J}(c), J(c) \rangle$.*

Proof. First we notice that

$$\langle \dot{J}(c), J(c) \rangle = I_{[0,c]}(J, J),\ \left\langle \dot{\tilde{J}}(c), \tilde{J}(c) \right\rangle = I_{[0,c]}\left(\tilde{J}, \tilde{J}\right).$$

From now on we will simply use I as $I_{[0,c]}$. (Here we abuse the notation, using same I to denote index form on different curves. We will write the curve in subscript in case of ambiguous if necessary.) Let $\{e_1(t), \cdots, e_{n-1}(t), \dot{\gamma}(t)\}$, $\left\{\tilde{e}_1(t), \cdots, \tilde{e}_{n-1}(t), \dot{\tilde{\gamma}}(t)\right\}$ be parallel orthonormal frames along $\gamma, \tilde{\gamma}$ respectively, such that $J(c) = \alpha e_1(c)$, $\tilde{J}(c) = \alpha \tilde{e}_1(c)$. Let

$$J(t) = \sum_{i=1}^{n-1} h_i(t) e_i(t),\ \tilde{J}(t) = \sum_{i=1}^{n-1} \tilde{h}_i(t) \tilde{e}_i(t),$$

then $h_i(c) = \tilde{h}_i(c) = 0$, $i = 2, \cdots, n-1$. Define $U(t) = \sum_{i=1}^{n-1} \tilde{h}_i(t) e_i(t)$ along γ, then $U(c) = J(c)$, $U(0) = J(0) = 0$, $|U(t)| = |\tilde{J}(T)|$. Thus by Index Lemma, $I_\gamma(J, J) \leq I_\gamma(U, U)$. However, we have

$$\begin{aligned}
I_\gamma(U, U) &= \int_0^c \left(|\dot{U}|^2 - \langle R(\dot{\gamma}, U)\dot{\gamma}, U \rangle\right) \mathrm{d}t \\
&= \int_0^c \left(\left|\dot{\tilde{J}}\right|^2 - |U|^2 K(\dot{\gamma}, U/|U|)\right) \mathrm{d}t \\
&\leq \int_0^c \left(\left|\dot{\tilde{J}}\right|^2 - \left|\tilde{J}\right| K\left(\dot{\tilde{\gamma}}, \tilde{J}/|\tilde{J}|\right)\right) \mathrm{d}t
\end{aligned}$$

$$= I_{\tilde{\gamma}}(\tilde{J},\tilde{J})$$

Thus we have $\left\langle \dot{\tilde{J}}(c), \tilde{J}(c) \right\rangle \geq \langle \dot{J}(c), J(c) \rangle$. □

Corollary 5.4. *If Riemannian manifold* (M^n, g) *satisfies* Sect ≤ 0, *then* $|J(t)| \geq t$, *that is,* $\left|\exp_{p*}|_v(w)\right| \geq |w|$.

Proof. Compare (M^n, g) with Euclidean space $(\mathbb{R}^n, \delta)$. □

The corollary implies the following important theorem.

Theorem 5.5. *Let M be a complete Riemannian manifold with* Sect ≤ 0, *$p \in M$ be any point. Then M has no conjugate points of p.*

Theorem 5.5 is known as the first part of the celebrated Cartan–Hadamard Theorem. We state the rest part of the theorem as follows.

Theorem 5.6 (Cartan–Hadamard)**.** *Let M^n be a complete Riemannian manifold with* Sect ≤ 0, *then the universal cover of M is diffeomorphic to $\mathbb{R}^n$.*

Part of the Proof. Without loss of generality, we assume M is simply connected. Then we only need to show M is diffeomorphic to $\mathbb{R}^n$. Since M is complete, $\exp_p : T_pM \to M$ is surjective, and by Theorem 5.5, the $\exp_p$ is nondegenerate. Thus $\exp_p$ is a local isometry between $(T_pM, \exp_p^* g) \to (M, g)$. We show that $\exp_p^* g$ is complete. Let $\gamma_v(t) : [0, +\infty) \to T_pM, t \mapsto vt$ for any unit vector $v \in T_pM$, then $\exp_p(\gamma)$ is a geodesic in (M, g). By the definition of $\exp_p^* g$, $\gamma_v(t)$ is a geodesic in $(T_pM, \exp_p^* g)$, hence the exponential map $\exp_0 : T_0(T_pM) \to T_pM$ is well-defined at $0 \in T_pM$. Hence by Hopf–Rinow Theorem, $\exp_p^* g$ is complete. Now the proof reduces to the following proposition. □

Proposition 5.7. *Let $f : M \to N$ be local isometry between Riemannian manifolds, with M complete. Then N is complete and f is a covering map.*

We stop here, and refer to Peter Peterson's *Riemannian Geometry*, 3rd ed., Lemma 5.6.4.

We mention something more. Cartan–Hadamard Theorem can be used to prove the uniqueness of space forms, that is, simply connected complete constant sectional curvature Riemannian manifolds are unique up to an isometry. This can be found in Wu Hung–Hsi et. al.'s *Introduction to Riemannian Geometry*, Chapter 5.

5.2 Cut Points and Distance Function

From this section, we assume the Riemannian manifold (M, g) appear in the context is complete.

We start this section by an example.

Example 5.8. Consider the cylinder $M = \mathbb{S}^1 \times \mathbb{R}$. M is flat, that is, has constant sectional curvature 0, hence has no conjugate points (Cartan–Hadamard). Let $\gamma : [0, 2\pi] \to M$ be a generatrix circle, then γ is a geodesic. Denote $p = \gamma(0)$, $q = \gamma(\pi)$. Then $\gamma|_{[0,\pi]}$ is a length-minimizing geodesic joining p and q. But it is not the only length-minimizing geodesic joining p and q, since $-\gamma|_{[\pi,2\pi]}$ is another one. Then q spoils the uniqueness of length-minimizing geodesic starts from p, without being conjugate to p. This inspires us to define the notion of *cut points*.

Given $v \in T_pM$, $|v| = 1$, $\gamma_v : [0, +\infty) \to M$ be a geodesic with $\gamma(0) = p$, $\dot{\gamma}_v(0) = v$. Notice that $\gamma_v|_{[0,t_0]}$ is length-minimizing if $d(p, \gamma_v(t_0)) = t_0$. If γ_v contains conjugate points, then γ_v is in general not locally length-minimizing.

Definition. Under above settings, let $t_0 = \sup\{t \in (0, +\infty) |\, d(p, \gamma_v(t)) = t\}$. If $t_0 < +\infty$, we call $\gamma_v(t_0)$ the **cut point** of p along γ_v. Define $\mathrm{Cut}(p)$ to be the set of all cut points of p, called the **cut locus** of p.

By definition and Jacobi's theorem, the first conjugate point (if exists) must be the cut point. But converse is in general not true (See Example 5.10.) The following proposition characterizes cut points.

Proposition 5.9. *Let γ be a unit speed geodesic. If $\gamma(t_0)$ is the cut point of p along γ, then either $\gamma(t_0)$ is conjugate to p along γ, or there exists two length-minimizing geodesics from p to $\gamma(t_0)$.*

Proof. Let the initial vector of γ be v. Choose a sequence $\{t_i\}$ decreasingly converges to t_0. Let σ_i be the unit speed length-minimizing geodesic joining p and $\gamma(t_i)$, with initial vector v_i. Define $b_i = d(p, \gamma(t_i)) = L(\sigma_i)$, then σ_i is defined on $[0, b_i]$. Hence up to a subsequence, we have $b_i v_i \to t_0 y$, where $|y| = 1$. If $v \neq y$, then $\sigma_i \to \sigma(t) = \exp_p(ty)$ (up to a subsequence), thus σ, γ are two length-minimizing geodesics joining p and $\gamma(t_0)$. If $v = y$, then $\exp_p(b_i v_i) = \gamma(t_i) = \exp_(t_i v)$, $\exp_p$ is not one-to-one near $t_0 y$. Hence $\exp_{p*}|_{t_0 y}$ is degenerate, at $\exp_p(t_0 y) = \gamma(t_0)$, that is, $\gamma(t_0)$ is conjugate to p. □

Example 5.10. (1) Let $M = \mathbb{S}^n$. Then the south pole is conjugate to north pole, as well as the cut point of north pole.

(2) Let $M = \mathbb{RP}^n$. For simplicity we take $n = 2$. We regard $\mathbb{RP}^2$ as the upper hemisphere with identifying equator's antipodal points (This is actually CW decomposition). Then from the north pole we move along a "great circle" to a point q on the equator, the path is a length-minimizing geodesic. However, the path on the same "great circle" but on the other side is also a length-minimizing geodesic, so p is the cut point of north pole (along this geodesic). But by the naturality of exponential map, q is not conjugate to north pole, so this is an example of cut point not being conjugate point.

(3) Let $M = \mathbb{R}^2/\mathbb{Z}^2$ be the flat torus. Let $[0,1]\times[0,1]$ be the fundamental region, and denote $p = (0,0) = (0,1)$. Then $\gamma(t) = (t,0)$ is a geodesic. Notice that if $t_0 > 1/2$, $\gamma(t)$ is not the length-minimizing geodesic joining p and $(t_0,0)$, since $\sigma(t) = (1-t,0)$ is the length-minimizing geodesic joining p and $(t_0,0)$. By the same reason, we have the cut locus of p is $\{1/2\}\times[0,1/2]\cup[0,1/2]\times\{1/2\}$.

Let $S_p \subset T_pM$ be the unit sphere. For any $v \in S_p$, denote $\gamma_v : [0,+\infty) \to M$ be the geodesic with $\dot{\gamma}(0) = v$. We define a function

$$\tau : S_p \to (0,+\infty],\ \tau(v) = \begin{cases} +\infty, & \gamma_v \text{ contains no cut point,} \\ t_0, & \gamma_v(t_0) \text{ is the cut point of } p. \end{cases}$$

We have the following properties of τ and cut locus.

Proposition 5.11. *Let τ be defined as above. We have*

(1) τ is a continuous function.

(2) $\mathrm{Cut}(p)$ is closed in M.

(3) $\mathrm{Cut}(p)$ is of zero-measure.

The proof is complicated, so we refer to Wu Hung–Hsi et. al.'s *Introduction to Riemannian Geometry*, Chapter 10.

Proposition 5.12. *Denote $\Sigma(p) = \{tv|\ v \in S_p,\ t \in [0,\tau(v))\}$, then*

$$\exp_p : \Sigma(p) \to \exp_p(\Sigma(p))$$

is a diffeomorphism, and $M = \exp_p(\Sigma(0)) \sqcup \mathrm{Cut}(p)$.

Proof. Since there is no cut points of p in $\Sigma(p)$ by the definition of τ, there is no conjugate points of p in $\Sigma(p)$. Hence $\exp_{p*}$ is nondegenerate in $\Sigma(p)$, $\exp_p$ is an immersion. We need to show $\exp_p$ is one-to-one. For if not, let $\exp_p(t_1v_1) = \exp_p(t_2v_2) =: q$ for $v_1 \neq v_2$. Let $\gamma_1(t) = \exp_p(tv_1)$, $\gamma_2(t) = \exp_p(tv_2)$. Since $M\backslash\mathrm{Cut}(p)$ is open, there is a neighborhood of q contained in $\Sigma(p)$, hence there is an $\varepsilon > 0$ such that $\gamma_2(t_2+\varepsilon) =: r \in \Sigma(p)$. Then $\gamma_1 \cup \gamma_2|_{[t_2,t_2+\varepsilon]}$ realizes the length-minimizing path from p to r, hence by the first variation formula, it is a geodesic. But $\gamma_1 \cup \gamma_2|_{[t_2,t_2+\varepsilon]}$ is not smooth at q (otherwise $-\dot{\gamma}_1(t_1) = -\dot{\gamma}_2(t_2)$ will lead to the same geodesic), contradicting to a geodesic must be smooth. Thus we proved the first claim.

The second claim is clear. $\square$

Now we define the injective radius of a point and a manifold.

Definition. Let M be a Riemannian manifold, $p \in M$, we define the **injective radius** by

$$\mathrm{inj}(p) := \sup\{r \in (0,+\infty) \mid B_r(p) \subset \Sigma(p)\}$$
$$\mathrm{inj}(M) := \inf_{p \in M} \mathrm{inj}(p)$$

A simple observation leads to the following property.

Proposition 5.13. *If M is compact, then* $\mathrm{inj}(M) > 0$.

However, if the manifold is not compact, the injective radius may be 0.

Example 5.14. Let's consider the surface of revolution by rotating $y = 1/x$ $(x > 0)$ around x-axis, then the point $(x, 1/x, 0)$ has injective radius π/x, which converges to 0 as $x \to +\infty$. Hence the surface has injective radius 0.

We now discuss some differential properties of distance function.

Let $p \in M$, denote $d_p : M \to \mathbb{R}$, $d_p(q) = d(p,q)$. By triangle inequality, d_p is Lipschitz continuous. But we have more to say.

Proposition 5.15. *Let $p \in M$, we have $d_p \in C^\infty(M \backslash (\{p\} \cup \mathrm{Cut}(p)))$. Moreover, $|\nabla d_p| = 1$ at where d_p is smooth.*

Proof. We just need to calculate ∇d_p in $M \backslash (\{p\} \cup \mathrm{Cut}(p))$. Let $q \in M \backslash (\{p\} \cup \mathrm{Cut}(p))$, and $\exp_p(lv) = q$, $|v| = 1$. Then there exists a neighborhood $U \subset T_pM$ of $\{tv \in T_pM \mid t \in [0,l]\}$ such that $\exp_p(U) \subset M \backslash \mathrm{Cut}(p)$. Now for $z \in U$, we have $d(\exp_p(z), p) = |z|$. Thus by taking the geodesic polar coordinate (U, φ), we have $d_p(\varphi^{-1}(r,\theta)) = r$. Therefore we have

$$\nabla d_p = g^{rr}\frac{\partial d_p}{\partial r}\frac{\partial}{\partial r} + g^{ij}\frac{\partial d_p}{\partial \theta^i}\frac{\partial}{\partial \theta^j} = \frac{\partial}{\partial r}.$$

(Recall $g = \mathrm{d}r^2 + g_{ij}\,\mathrm{d}\theta^i \otimes \mathrm{d}\theta^j$ under polar coordinate.) Thus d_p is smooth at $q \in M \backslash (\{p\} \cup \mathrm{Cut}(p))$, and $|\nabla d_p| = 1$. $\square$

Now we compute the Hessian of d_p for some situation.

Proposition 5.16. *We have $\nabla^2 d_p(\nabla d_p, X) = 0$ for any $X \in T_qM$, $q \in M \backslash (\{p\} \cup \mathrm{Cut}(p))$.*

Proof. Let's just calculate

$$\begin{aligned}
\nabla^2 d_p(\nabla d_p, X) &= X(\nabla d_p)(d_p) - (\nabla_X(\nabla d_p))(d_p) \\
&= X\langle \nabla d_p, \nabla d_p \rangle - \langle \nabla_X(\nabla d_p), \nabla d_p \rangle \\
&= X\langle \nabla d_p, \nabla d_p \rangle - \frac{1}{2}X\langle \nabla d_p, \nabla d_p \rangle \\
&= 0. \quad (\text{Since } |\nabla d_p| = 1)
\end{aligned}$$

$\square$

Proposition 5.17. *Let J be a Jacobi field along geodesic γ such that $J(0)=0$, $J\perp\dot\gamma$. Then*

$$\nabla^2 d_p|_{\gamma(t)}(J(t),J(t)) = \langle \dot J(t), J(t)\rangle \quad (t<\tau(\dot\gamma(0)))$$

Proof. First we notice that $\nabla d_p = \dot\gamma$ (One can show this by using geodesic polar coordinate as in Proposition 5.15). Then let's calculate

$$\begin{aligned}\nabla^2 d_p|_{\gamma(t)}(J(t),J(t)) &= J\langle J,\nabla d_p\rangle - \langle \nabla_J J, \nabla d_p\rangle\\ &= \langle J, \nabla_J(\nabla d_p)\rangle\\ &= \langle J, \nabla_{\nabla d_p} J\rangle \quad \text{(torsion-freeness)}\\ &= \langle J, \dot J\rangle. \end{aligned}$$

□

Now we are well-prepared to march towards Hessian comparison theorem and Laplace comparison theorem.

5.3 Hessian and Laplace Comparison Theorems

We first state and prove Hessian comparison theorem.

Theorem 5.18 (Hessian comparison)**.** *Let $\gamma:[0,l]\to M$, $\tilde\gamma:[0,l]\to M$ be geodesics without cut points of $\gamma(0)$ and $\tilde\gamma(0)$ respectively. Assume $K_{\gamma(t)}(\dot\gamma(t),v)\geq K_{\tilde\gamma(t)}\left(\dot{\tilde\gamma}(t),\tilde v\right)$ for $|v|=|\tilde v|=1$. Then $\nabla^2 d_p(X,X)\leq \tilde\nabla^2\tilde d_{\tilde p}\left(\tilde X,\tilde X\right)$, for all $X\in T_{\gamma(t)}M$, $\tilde X\in T_{\tilde\gamma(t)}M$, with $|X|=|\tilde X|$, $\langle X,\dot\gamma\rangle = \left\langle \tilde X,\dot{\tilde\gamma}\right\rangle$.*

Proof. By Proposition 5.16, we can assume without loss of generality that $X\perp\dot\gamma$, $\tilde X\perp\dot{\tilde\gamma}$. Let $J,\tilde J$ be normal Jacobi fields on $\gamma,\tilde\gamma$ such that $J(0)=0, J(t)=X$ and $\tilde J=0,\tilde J(t)=\tilde X$. Then by Proposition 5.17, we have

$$\nabla^2 d_p(X,X) = \langle \dot J(t),J(t)\rangle,\ \tilde\nabla^2\tilde d_{\tilde p}\left(\tilde X,\tilde X\right) = \left\langle \dot{\tilde J}(t),\tilde J(t)\right\rangle.$$

Now $J,\tilde J$ satisfy the assumptions of Lemma 5.3, hence we have

$$\nabla^2 d_p(X,X) = \langle \dot J(t),J(t)\rangle \leq \left\langle \dot{\tilde J}(t),\tilde J(t)\right\rangle = \tilde\nabla^2\tilde d_{\tilde p}\left(\tilde X,\tilde X\right).$$

□

Example 5.19. We compute $\nabla^2 d_p$ on constant sectional curvature manifolds. As usual we scale the metric to make $K=0,1,-1$. Then the results are

$$\nabla^2 d_p = \begin{cases} \frac{1}{d_p}(g-\mathrm{d}(d_p)\otimes\mathrm{d}(d_p)), & K=0,\\ \cot d_p(g-\mathrm{d}(d_p)\otimes\mathrm{d}(d_p)), & K=1,\\ \coth d_p(g-\mathrm{d}(d_p)\otimes\mathrm{d}(d_p)), & K=-1,\end{cases}$$

where g is the metric tensor, and $\mathrm{d}(d_p)$ is the exterior differential of d_p.

Now we state the Laplace comparison theorem.

Theorem 5.20. *Let (M^n, g) be a Riemannian manifold, M_K^n be a Riemannian manifold with constant sectional curvature K. Let $\gamma : [0,l] \to M$, $\gamma_K : [0,l] \to M$ be geodesics without cut points. Assume* Ric $\geq (n-1)Kg$, *then* $\Delta d_p|_{\gamma(l)} \leq \Delta_k d_k|_{\gamma_K(l)}$.

Remark 5.21. Before giving the proof, we give some remark on the theorem. The theorem has a relatively weaker assumption that only Ricci curvature is bounded below, so we cannot copy the proof of Rauch comparison theorem as we did in Hessian comparison theorem. Moreover, the conclusion is also weakened (only compare to space forms) correspondingly. However, we can modify the proof of Bonnet–Myers theorem to obtain a proof of Laplace comparison theorem.

Proof. Let $\{e_1, \cdots, e_{n-1}, e_n = \dot{\gamma}(l)\}$, $\{e_1^K, \cdots, e_{n-1}^K, e_n^K = \dot{\gamma}_K(l)\}$ be orthonormal bases of $T_{\gamma(l)}M$, $T_{\gamma_K(l)}M_K$ respectively. Let J_i, J_i^K be Jacobi fields such that $J_i(0) = 0, J_i(l) = e_i$ and $J_i^K(0) = 0, J_i^K(l) = e_i^K$, for $i = 1, \cdots, n-1$. Then by Proposition 5.16 and Proposition 5.17, we have

$$\begin{aligned}
\Delta d_p|_{\gamma(l)} &= \sum_{i=1}^{n-1} \nabla^2 d_p|_{\gamma(l)}(J_i, J_i) \\
&= \sum_{i=1}^{n-1} \langle \dot{J}, J \rangle \\
&= \sum_{i=1}^{n-1} I(J_i, J_i),
\end{aligned}$$

and similarly $\Delta_K d_K = \sum_{i=1}^{n-1} I(J_i^K, J_i^K)$. We need to show that

$$\sum_{i=1}^{n-1} I(J_i, J_i) \leq \sum_{i=1}^{n-1} I(J_i^K, J_i^K).$$

Let $\{e_i(t)\}$, $\{e_i^K(t)\}$ be parallel, then

$$J_i^K(t) = \sum_{j=1}^{n-1} h_{ij}(t) e_j^K(t), \ i = 1, \cdots, n-1,$$

where $h_{ij}(t) = f_K(t)\delta_{ij}$, $\ddot{f}_K + K f_K = 0$. Define $U_i = \sum_{j=1}^{n-1} h_{ij}(t) e_j(t)$ along γ in M, $i = 1, \cdots, n-1$. Then $J_i(0) = U_i(0) = 0$, $J_i(l) = U_i(l)$. By Index Lemma, we have

$$\sum_{i=1}^{n-1} I(J_i, J_i) \leq \sum_{i=1}^{n-1} (U_i, U_i).$$

Moreover, we have

$$I(U_i,U_i) = \int_0^l \left(|\dot{U}_i|^2 - \langle R(\dot\gamma,U_i)\dot\gamma,U_i\rangle\right)\mathrm{d}t,$$
$$I(J_i^K,J_i^K) = \int_0^l \left(|\dot{J}_i^K|^2 - \left\langle R(\dot\gamma_K,J_i^K)\dot\gamma_K,J_i^K\right\rangle\right)\mathrm{d}t,$$

and $|\dot{U}_i|^2 = |\dot{J}_i^K|^2$ for $i=1,\cdots,n-1$. Now we have

$$\begin{aligned}\sum_{i=1}^{n-1}\langle R(\dot\gamma,U_i)\dot\gamma,U_i\rangle &= f_K^2(t)\langle R(\dot\gamma,e_i)\dot\gamma,e_i\rangle\\ &= f_K^2(t)\langle \mathrm{Ric}(\dot\gamma),\dot\gamma\rangle\\ &\geq f_K^2(t)(n-1)\\ &= \sum_{i=1}^{n-1}\left\langle R(\dot\gamma_K,J_i^K)\dot\gamma_K,J_i^K\right\rangle.\end{aligned}$$

This implies the conclusion. □

Remark 5.22. (1) There is a conclusion that if q lies in the geodesic sphere $B_r(p) = \{d_p = r\}$, then $\Delta d_p(q) = H_{B_r(p)}(q)$, where $H_{B_r(p)}(q)$ is the mean curvature of $B_r(p)$ at q. So Laplace comparison theorem can be also regarded as a comparison theorem of mean curvature.

(2) We notice that the equality holds in Laplace comparison theorem if and only if $K(\dot\gamma,v) = K$ for any $|v| = 1$, $v \perp \dot\gamma$.

(3) The case of $K=0$ is $\Delta d_p \leq \frac{n-1}{d_p}$. This inequality also holds in the sence of distribution, that is, for any $\varphi \in C_c^\infty(M)$, $\varphi \geq 0$, we have

$$\int_M d_p\Delta\varphi \leq \int_M \frac{n-1}{d_p}\cdot\varphi.$$

For a proof, we refer to Schoen, Yau's *Lectures on Differential Geometry*, Proposition 1.1 in Chapter 1.

5.4 Volume Comparison Theorem

In this section we assume all the manifolds appear are orientable.

We first introduce the Riemann volume form.

Definition. Let (M^n, g) be a Riemannian manifold, define an n-form $\mathrm{d}\,\mathrm{Vol}_g$ as

$$\mathrm{d}\,\mathrm{Vol}_g(e_1, \cdots, e_n) = 1$$

for an orthonormal basis $\{e_i\}$ in T_pM for any $p \in M$. $\mathrm{d}\,\mathrm{Vol}_g$ is call **Riemann volume form**.

Since if $(e_1', \cdots, e_n') = (e_1, \cdots, e_n)A$, for any n-form ω we have

$$\omega(e_1', \cdots, e_n') = (\det A)\omega(e_1, \cdots, e_n),$$

the Riemann volume form is well-defined.

Locally we have

$$\mathrm{d}\,\mathrm{Vol}_g = \sqrt{\det(g_{ij})}\,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n.$$

Definition. Let $\Omega \subset M$, the **volume** of Ω is defined as

$$\mathrm{Vol}(\Omega) := \int_\Omega \mathrm{d}\,\mathrm{Vol}_g\,.$$

If M is compact, we can also define

$$\mathrm{Vol}(M) := \int_M \mathrm{d}\,\mathrm{Vol}_g\,.$$

Integration on manifold is defined by partition of unity, this is almost impossible to compute. However, we have a local chart $(\Sigma(p), \exp_p^{-1})$ with $M \backslash \Sigma(p)$ is of zero-measure. So we have

$$\int_M \mathrm{d}\,\mathrm{Vol}_g = \int_{\exp_p(\Sigma(p))} \mathrm{d}\,\mathrm{Vol}_g = \int_{\Sigma(p)} \exp_p^*(\mathrm{d}\,\mathrm{Vol}_g).$$

We now calculate Riemann volume form in geodesic polar coordinate. Let J_i be Jacobi fields with $J_i(0) = 0$, $J_i(r, \theta) = \partial_{\theta^i}$ for $i = 1, \cdots, n$. Then

$$\mathrm{d}\,\mathrm{Vol}_g = \sqrt{\det(g_{ij})}\,\mathrm{d}r \wedge \mathrm{d}\theta^1 \wedge \cdots \wedge \mathrm{d}\theta^{n-1},$$

and denote $\mathscr{J} = \sqrt{\det(g_{ij})}$.

Proposition 5.23. *We have* $\frac{\partial}{\partial r} \log \mathscr{J} = \Delta d_p$.

Proof. We calculate

$$\frac{\partial}{\partial r} \log \mathscr{J} = \frac{1}{2} \frac{1}{\det(g_{ij})} \det(g_{ij}) g^{ij} (\langle \dot{J}_i, J_j \rangle + \langle J_i, \dot{J}_j \rangle)$$

$$
\begin{aligned}
&= g^{ij}\langle \dot{J}_i, J_j\rangle \\
&= \operatorname{tr}_g \nabla^2 d_p \\
&= \Delta d_p
\end{aligned}
$$

The third equality uses both Proposition 5.16 and Proposition 5.17. □

Thus we have the following Bishop's theorem.

Theorem 5.24 (Bishop)**.** *Let (M^n, g) be a Riemannian manifold with $\mathrm{Ric} \geq (n-1)Kg$. Let $\gamma : [0,l] \to M$ be a geodesic without cut points. Then $\frac{\mathscr{J}}{\mathscr{J}_K}(r)$ is nonincreasing with respective to r. ($\mathscr{J}$, $\mathscr{J}_K$ is defined as above.)*

Proof. We have

$$
\begin{aligned}
\frac{\partial}{\partial r}\log\frac{\mathscr{J}}{\mathscr{J}_K} &= \frac{\partial}{\partial r}\log\mathscr{J} - \frac{\partial}{\partial r}\log\mathscr{J}_K \\
&= \Delta d_p - \Delta_K d_K \\
&\leq 0
\end{aligned}
$$

by Laplace comparison theorem. □

Now we can state and prove Bishop–Gromov comparison theorem, which is also known as volume comparison theorem.

Theorem 5.25 (Bishop–Gromov comparison)**.** *Let (M^n, g) be a Riemannian manifold with $\mathrm{Ric} \geq (n-1)Kg$. Define geodesic annuli $A_{s,r}(p) := \{\exp_p(tv) \mid s < t < \min r, t_v,\ v \in S_p\}$ and annuli $A^K_{r,s}$ in constant sectional curvature space M_K with radii $r < s$. Then we have*

$$
\frac{\mathrm{Vol}(A_{r_3,r_4}(p))}{\mathrm{Vol}_K(A^K_{r_3,r_4})} \leq \frac{\mathrm{Vol}(A_{r_1,r_2}(p))}{\mathrm{Vol}_K(A^K_{r_1,r_2})}
$$

provided $r_1 < \min\{r_2, r_3\} < \max\{r_2, r_3\} < r_4$.

Corollary 5.26. *(1) Let $r_1 = r_3 = 0$ in Theorem 5.25, we have*

$$
\frac{\mathrm{Vol}(B_{r_2}(p))}{\mathrm{Vol}_K(B^K_{r_2})} \leq \frac{\mathrm{Vol}(B_{r_1}(p))}{\mathrm{Vol}_K(B^K_{r_1})}
$$

provided $r_1 < r_2$.

(2) Moreover, let $r_1 \to 0$, we have $\mathrm{Vol}(B_r(p)) \leq \mathrm{Vol}_K(B^K_r)$.

Before proving the theorem, we need a lemma from calculus.

Lemma 5.27. *If $f(t),g(t)>0$, f/g is nonincreasing, then*

$$\frac{\int_s^r f(t)\,\mathrm{d}t}{\int_s^r g(t)\,\mathrm{d}t}$$

is nonincreasing with respective to r,s.

Proof. We show the function is nonincreasing with respective to one variable, the other is similar. Let $s<r_1<r_2$, we need to show $\int_s^{r_1} f\int_s^{r_2} g\geq \int_s^{r_2} f\int_s^{r_1} g$. We have

$$\begin{aligned}\int_s^{r_1} f\int_s^{r_2} g-\int_s^{r_2} f\int_s^{r_1} g&=\int_s^{r_1} f\left(\int_s^{r_1} g+\int_{r_1}^{r_2} g\right)-\left(\int_s^{r_1} f+\int_{r_1}^{r_2} f\right)\int_s^{r_1} g\\&=\int_s^{r_1} f\int_{r_1}^{r_2} g-\int_{r_1}^{r_2} f\int_s^{r_1} g\end{aligned}$$

By Intermediate Value Theorem, there exists $s\leq t_1\leq r_1\leq t_2\leq r_2$ such that

$$\frac{\int_s^{r_1} f}{\int_s^{r_1} g}=\frac{f(t_1)}{g(t_1)}\geq\frac{f(t_2)}{g(t_2)}=\frac{\int_{r_1}^{r_2} f}{\int_{r_1}^{r_2} g}.$$

Hence the lemma is proved. □

Proof of Theorem 5.25. We just need to show $\mathrm{Vol}(A_{s,r}(p))/\mathrm{Vol}(A^K_{s,r})$ is nonincreasing with respective to r,s. Using geodesic polar coordinate, denote

$$\chi(r,\theta)=\begin{cases}1, & r<t_\theta,\\ 0, & r\geq t_\theta,\end{cases}$$

and

$$\chi_K(t)=\begin{cases}1, & K\leq 0,\\ \begin{cases}1, & t\leq \pi/\sqrt{K},\\ 0, & t>\pi/\sqrt{K},\end{cases} & K>0.\end{cases}$$

Then by Bonnet–Myers theorem, $t_\theta<\pi/\sqrt{K}$ if $K>0$, then $\frac{\chi}{\chi_K}(t,\theta)$ is nonincreasing with respective to t. Now we have

$$\begin{aligned}\frac{\mathrm{Vol}(A_{s,r}(p))}{\mathrm{Vol}_K(A^K_{s,r})}&=\frac{\int_{S_p}\int_r^s \chi(r,\theta)\mathscr{J}(r,\theta)\,\mathrm{d}r\wedge\mathrm{d}\theta^1\wedge\cdots\wedge\mathrm{d}\theta^{n-1}}{\int_r^s \chi_K(t)\mathscr{J}_K(t)\,\mathrm{d}t}\\&=\frac{1}{\mathrm{Vol}\,\mathbb{S}^{n-1}}\int_{S_p}\frac{\int_r^s \chi(t,\theta)\mathscr{J}(t,\theta)\,\mathrm{d}t}{\int_r^s \chi_K(t)\mathscr{J}_K(t)\,\mathrm{d}t}\,\mathrm{d}\theta\end{aligned}$$

Using Theorem 5.24, we have $\frac{\chi\mathscr{J}}{\chi_K\mathscr{J}_K}$ is nonincreasing (with respective to t). Then the theorem follows by above lemma. □

As a corollary, we have another theorem of Bishop.

Theorem 5.28 (Bishop). *Let Riemannian manifold (M^n,g) satisfy* $\mathrm{Ric} \geq (n-1)Kg > 0$*, then* $\mathrm{Vol}(M) \leq \mathrm{Vol}(\mathbb{S}^n(1/\sqrt{K}))$. *The equality holds if and only if M is isometric to $\mathbb{S}^n(1/\sqrt{K})$.*

Proof. The inequality holds from Bonnet–Myers theorem and volume comparison theorem. The equality holds if and only if the equality holds in Laplace comparison theorem, that is, M is isometric to $\mathbb{S}^n(1/\sqrt{K})$. □

The important application of volume comparison theorem is Cheng's maximal radius theorem.

Theorem 5.29 (Cheng). *If (M^n,g) is a Riemannian manifold with* $\mathrm{Ric} \geq (n-1)Kg$*, and* $\mathrm{diam}(M) = \pi/\sqrt{K}$*, then M is isometric to $\mathbb{S}^n(1/\sqrt{K})$.*

Proof. We scale the metric to let $K = 1$. Since M is compact by Bonnet–Myers theorem, there exists $p,q \in M$ such that $d(p,q) = \pi$. Consider $B_r(p)$ and $B_{\pi-r}(q)$, by triangle inequality, we have $B_r(p) \cap B_{\pi-r}(q) = \varnothing$. Hence we have

$$\begin{aligned}
\mathrm{Vol}(M) &\geq \mathrm{Vol}(B_r(p)) + \mathrm{Vol}(B_{\pi-r}(q)) \\
&= \frac{\mathrm{Vol}(B_r(p))}{\mathrm{Vol}_1(B_r^1)} \cdot \mathrm{Vol}_1(B_r^1) + \frac{\mathrm{Vol}(B_{\pi-r}(q))}{\mathrm{Vol}_1(B_{\pi-r}^1)} \cdot \mathrm{Vol}_1(B_{\pi-r}^1) \\
&\geq \frac{\mathrm{Vol}(B_\pi(p))}{\mathrm{Vol}_1(B_\pi^1)} \cdot \mathrm{Vol}_1(B_r^1) + \frac{\mathrm{Vol}(B_\pi(q))}{\mathrm{Vol}_1(B_\pi^1)} \cdot \mathrm{Vol}_1(B_{\pi-r}^1) \\
&= \frac{\mathrm{Vol}(M)}{\mathrm{Vol}(\mathbb{S}^n)}(\mathrm{Vol}_1(B_r^1) + \mathrm{Vol}_1(B_{\pi-r}^1)) \\
&= \mathrm{Vol}(M),
\end{aligned}$$

where the second inequality is Bishop–Gromov comparison theorem, and last equality is $\mathrm{Vol}_1(B_r^1) + \mathrm{Vol}_1(B_{\pi-r}^1) = \mathrm{Vol}(\mathbb{S}^n)$ on $\mathbb{S}^n$. Hence the equality in Bishop–Gromov comparison theorem holds, that is, the equality in Laplace comparison theorem holds. Then M must be isometric to $\mathbb{S}^n$. □

As for noncompact manifolds, volume comparison theorem has following corollary.

Corollary 5.30. *Let M be a complete noncompact Riemannian manifold with* $\mathrm{Ric} \geq 0$*, then* $\mathrm{Vol}(B_r(p)) \leq \omega_n r^n = \mathrm{Vol}_0(B_r^0)$.

The corollary gives an upper bound of the growth of the volume of geodesic ball. Moreover, the following Calabi–Yau theorem (not the one in complex geometry) gives a lower bound.

Theorem 5.31 (Calabi–Yau). *Let M be a complete noncompact Riemannian manifold with* $\mathrm{Ric} \geq 0$*, then* $\mathrm{Vol}(B_r(p)) \geq C(n)\mathrm{Vol}(B_1(p))r$.

For a proof, we refer to Schoen, Yau's *Lectures on Differential Geometry*, Theorem 4.1 in Chapter 1.

Chapter 6

Bochner Formula and Application

We state and prove the useful Bochner formula, and apply it on distance function. We will give another proof of Laplace comparison theorem using Bochner formula, and prove Cheeger–Gromoll splitting theorem.

6.1 Bochner Formula

Theorem 6.1 (Bochner formula)**.** *Let M be a Riemannian manifold, $f \in C^\infty(M)$. Then we have*

$$\Delta\frac{1}{2}|\nabla f|^2 = |\nabla^2 f|^2 + \langle \nabla\Delta f, \nabla f\rangle + \langle \mathrm{Ric}(\nabla f), \nabla f\rangle.$$

We first need a lemma.

Lemma 6.2. *Under geodesic normal coordinate around p, the Ricci identity (Proposition A.7) of 1-form is equivalent to*

$$f_{;kij} - f_{;kji} = -f_{;l}R^l_{ijk},$$

where $f_{;i}$ means $\partial_i f$.

Proof. First we notice that

$$\begin{aligned}
\nabla_{\partial_i}\nabla_{\partial_j}\,\mathrm{d}f &= \nabla_{\partial_i}(\partial_j f_{;k}\,\mathrm{d}x^k + f_{;k}\,\mathrm{d}x^k) \\
&= \partial_i\partial_j f_{;k}\,\mathrm{d}x^k - \nabla_{\partial_i}\Gamma^k_{jl}\,\mathrm{d}x^l \\
&= f_{;kji}\,\mathrm{d}x^k
\end{aligned}$$

since all $\Gamma^k_{jl} = 0$ at p. Then we have

$$\begin{aligned}(R(\partial_i,\partial_j)\,\mathrm{d}f)(\partial_k) &= (\nabla_{\partial_j}\nabla_{\partial_i} - \nabla_{\partial_i}\nabla_{\partial_j} + \nabla_{[\partial_i,\partial_j]})\,\mathrm{d}f(\partial_k)\\ &= (f_{;kij} - f_{;kji}).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}(R(\partial_i,\partial_j)\,\mathrm{d}f)(\partial_k) &= -\,\mathrm{d}f(R(\partial_i,\partial_j)\,\partial_k)\\ &= -\,\mathrm{d}f(R^l_{ijk}\,\partial_l)\\ &= -f_{;m}\,\mathrm{d}x^m(R^l_{ijk}\,\partial_l)\\ &= -f_{;l}R^l_{ijk},\end{aligned}$$

hence the equality holds. □

Proof of Bochner formula. Under geodesic normal coordinate we have the following calculation

$$\begin{aligned}\Delta\frac{1}{2}|\nabla f|^2 &= \sum_{i,j}\left(\frac{1}{2}f_{;j}^2\right)_{;ii}\\ &= \sum_{i,j}(f_{;j}f_{;ji})_{;i}\\ &= \sum_{i,j}(f_{;ji}^2 + f_{;j}f_{;jii})\\ &= \sum_{i,j}(f_{;ij}^2 + f_{;j}f_{;iji}) \quad \text{(Hessian is interchangeable)}\\ &= \sum_{i,j}f_{;ij}^2 + \sum_{i,j}f_{;j}(f_{;iij} - f_{;k}R^k_{jii}) \quad \text{(Ricci identity)}\\ &= \sum_{i,j}f_{;ij}^2 + \sum_{i,j}f_{;j}f_{;iij} + \sum_{i,j}f_{;k}R^k_{iji}\\ &= \sum_{i,j}f_{;ij}^2 + \sum_{i,j}f_{;iij}f_{;j} + \sum_{j}f_{;j}f_{;k}\,\mathrm{Ric}^k_j\\ &= |\nabla f|^2 + \langle\Delta\nabla f,\nabla f\rangle + \langle\mathrm{Ric}(\nabla f),\nabla f\rangle.\end{aligned}$$

□

As a first application, we use Bochner formula to provide an alternative proof of Bishop's theorem (Theorem 5.24).

Alternative proof of Theorem 5.24. We apply Bochner formula on distance function d_p. Notice that $|\nabla d_p| = 1$, thus we have

$$0 = |\nabla^2 d_p|^2 + \langle\nabla\Delta d_p,\nabla d_p\rangle + \langle\mathrm{Ric}(\nabla d_p),\nabla d_p\rangle.$$

Now we use the geodesic polar coordinate. First let L by the corresponding linear transformation of $\nabla^2 d_p$, by Proposition 5.16 and Proposition 5.17, we know that L has $n-1$ nonzero eigenvalue, say $\lambda_1, \cdots, \lambda_{n-1}$. Then by Cauchy–Schwartz inequality, we have

$$\begin{aligned}
|\nabla^2 d_p|^2 = \operatorname{tr}(L \circ L) &= \sum_{k=1}^{n-1} \lambda_k^2 \\
&\geq \frac{1}{n-1} \left(\sum_{k=1}^{n-1} \lambda_k \right)^2 = \frac{1}{n-1} (\operatorname{tr} L)^2 \\
&= \frac{1}{n-1} (\Delta d_p)^2.
\end{aligned}$$

Next, we notice that $\nabla d_p = \partial_r$, hence

$$\langle \nabla \Delta d_p, \nabla d_p \rangle = \langle \partial_r (\Delta d_p) \partial_r, \partial_r \rangle = \frac{\partial (\Delta d_p)}{\partial r}.$$

Moreover, by the assumption, we have

$$\langle \operatorname{Ric}(\nabla d_p), \nabla d_p \rangle \geq (n-1) K \langle \nabla d_p, \nabla d_p \rangle = (n-1) K.$$

Add three equalities together, and since $\Delta d_p = \partial_r (\log \mathscr{J})$, we have

$$\frac{\partial^2}{\partial r^2} (\log \mathscr{J}) + \frac{1}{n-1} \left(\frac{\partial}{\partial r} \log \mathscr{J} \right) + (n-1) K \leq 0. \tag{6.1}$$

Let $\Phi = \mathscr{J}^{1/(n-1)}$, then the equation (6.1) is equivalent to

$$\ddot{\Phi} + K \Phi \leq 0.$$

However, let $\Phi_K = \mathscr{J}_K^{1/(n-1)}$, we have $\ddot{\Phi}_K + K \Phi_K = 0$, and $\Phi_K(0) = \Phi(0) = 0, \Phi_K'(0) = \Phi'(0) = 1$. Then by Sturm–Liouville comparison theorem, we have Φ / Φ_K is nonincreasing. This derives $\mathscr{J} / \mathscr{J}_K$ is nonincreasing. $\square$

Remark 6.3. From the original proof of Bishop's theorem, we know it is equivalent to Laplace comparison theorem. Hence this argument also gives a new proof of Laplace comparison theorem.

6.2 Spliting Theorem

Theorem 6.4 (Cheeger–Gromoll)**.** *Let M be a complete Riemannian manifold with* $\operatorname{Ric} \geq 0$*. Assume M admits a line, that is, there is a unit speed geodesic $\gamma : \mathbb{R} \to M$ such that $d(\gamma(t), \gamma(s)) = |t-s|$. Then M is isometric to $(\mathbb{R} \times N, g = \mathrm{d}t^2 + g_N)$.*

Corollary 6.5. *Under the same assumption, M is isometric to $\mathbb{R}^k \times N^{n-k}$, where N^{n-k} does not contain any line.*

Before we start the proof, we first state some results from theory of partial differential equations.

Theorem 6.6 (Maximum principle)**.** *If $f \in H^2(M)$ satisfies $\Delta f \leq 0$ in weak sense, and f attains its local minimum, then f is constant.*

Theorem 6.7 (Weyl's lemma)**.** *If $f \in H^2(M)$ satisfies $\Delta f = 0$, then f is smooth and harmonic.*

Proof of Theorem 6.4. The proof uses the *Busemann function*. We first define and discuss some properties of Busemann function.

Fix $t \in \mathbb{R}$, for $x \in M$, define $b_t^+(x) = d(x, \gamma(t)) - t$. Then by triangle inequality, if $t_1 > t_2$, then

$$b_{t_1}^+(x) - b_{t_2}^+(x) = d(x, \gamma(t_1)) - d(x, \gamma(t_2)) - (t_1 - t_2) \leq 0,$$

hence b_t^+ is nonincreasing with respective to t. Moreover, we have $b_t^+(x) \geq -d(x, \gamma(0))$ by triangle inequality, hence b_t^+ is bounded below. Therefore, for any fixed x the limit $\lim_{t \to +\infty} b_t^+(x)$ exists, and we defined it as $B^+(x)$. Similarly we define $b_t^-(x) = d(x, \gamma(-t)) - t$, and thus $B^-(x) = \lim_{t \to +\infty} b_t^-(x)$ is well-defined.

Now we discuss the properties of B^+ and B^-. We first have

$$B^+(x) + B^-(x) = \lim_{t \to +\infty} (d(x, \gamma(t)) + d(x, \gamma(-t)) - 2t) \geq 0,$$

and the equality holds when $x \in \gamma(\mathbb{R})$. Moreover, by Laplace comparison theorme we have

$$\Delta b_t^+ \leq \frac{n-1}{b_t^+ + t},$$

hence for any test function $\varphi \in C_c^\infty(M)$, $\varphi \geq 0$ we have

$$\begin{aligned}\int_M b_t^+ \Delta \varphi = \int_M \varphi \Delta b_t^+ &= \int_M \varphi \Delta d(x, \gamma(t)) \\ &\leq \int_M \varphi \cdot \frac{n-1}{b_t^+ + t}.\end{aligned}$$

By Lebesgue Dominated Convergence Theorem, let $t \to +\infty$ we have

$$\int_M \varphi \Delta B^+ = \int_M B^+ \Delta \varphi \leq 0,$$

then $\Delta B^+ \leq 0$ in weak sense. Similarly $\Delta B^- \leq 0$ in weak sense. Hence $\Delta(B^+ + B^-) \leq 0$, $B^+ + B^- \geq 0$, and $(B^+ + B^-)(\gamma(t_0)) = 0$. Then $B^+ + B^-$ satisfies the assumption of maximum principle, this implies $B^+ + B^- \equiv 0$ on M. Thus $\Delta B^+ = \Delta B^- = 0$, by Weyl's lemma B^+ and B^- are smooth and harmonic. Moreover, since $|\nabla d(x,\gamma(t))| = 1$ holds almost everywhere, we have $|B^+| = 1$. Using Bochner formula, we have

$$0 = \Delta\frac{1}{2}|\nabla B^+|^2 = |\nabla^2 B^+|^2 + \langle \nabla\Delta B^+, \nabla B^+\rangle + \langle \mathrm{Ric}(\nabla B^+), \nabla B^+\rangle \geq |\nabla^2 B^+|^2,$$

hence $\nabla^2 B^+ = 0$.

Now we come to the geometric part. Let $s \in \mathbb{R}$ be a regular value of B^+, define $N = \{x \in M |\ B^+(x) = s\}$, then N is a regular submanifold of codimension 1 by Implicit Function Theorem. Now we introduce the map

$$\begin{aligned} f : \mathbb{R} \times N &\to M \\ (t,p) &\mapsto \exp_{B^+(p)}(t\nabla B^+(p)). \end{aligned}$$

Since $\nabla^2 B^+ = 0$, ∇B^+ is parallel, so we can give $f^{-1}(x)$ of $x \in M$ by taking the integral curve of ∇f passing x and trace back to N, ODE theory shows f^{-1} is smooth. Therefore f is a diffeomorphism. We need to show f induces a product metric on $\mathbb{R} \times N$. Let (U,φ) be a local chart of N, with coordinate $(x^1,\cdots,x^{n-1})$, define the Fermi coordinate $(x^1,\cdots,x^{n-1},t)$ by

$$(x^1,\cdots,x^{n-1},t) \mapsto \exp_{\varphi^{-1}(x^1,\cdots,x^{n-1})}(t\nabla B^+(\varphi^{-1}(x^1,\cdots,x^{n-1}))).$$

Then we have

$$g\left(\frac{\partial}{\partial t},\frac{\partial}{\partial t}\right) = |\nabla B^+|^2 = 1,$$

and

$$g\left(\frac{\partial}{\partial t},\frac{\partial}{\partial x^i}\right) = g\left(\nabla B^+,\frac{\partial}{\partial x^i}\right) = 0$$

for $i = 1,\cdots,n-1$. We need to show $g_{ij} = g(\partial_i,\partial_j)$ is independent from t, that is,

$$\frac{\partial}{\partial t} g\left(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\right) = 0.$$

However, since ∇B^+ is parallel, we have

$$\begin{aligned} \frac{\partial}{\partial t} g\left(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\right) &= g\left(\nabla_{\frac{\partial}{\partial t}}\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\right) + g\left(\nabla_{\frac{\partial}{\partial t}}\frac{\partial}{\partial x^j},\frac{\partial}{\partial x^i}\right) \\ &= g\left(\nabla_{\frac{\partial}{\partial x^i}}(\nabla B^+),\frac{\partial}{\partial x^j}\right) + g\left(\nabla_{\frac{\partial}{\partial x^j}}(\nabla B^+),\frac{\partial}{\partial x^i}\right) \\ &= 0. \end{aligned}$$

This finishes our proof. $\square$

Chapter 7

Heintze–Karcher Comparison Theorem

Now we finished our tour in basic classes, and enter the topic classes. We first go through geometry of submanifolds rapidly, and then provide the Heintze–Karcher comparison theorem. After this, we give two applications: Alexandorv's theorem on constan mean curvature (CMC) hypersurface and Levy–Gromov isoperimetric inequality.

7.1 Geometry of Submanifolds

Let $f : \Sigma^k \hookrightarrow (M^n, g)$ be an immersion, equip Σ with pullback metric f^*g (still denoted by g for simplicity). There is a decomposition $\nabla_X Y = (\nabla_X Y)^\top + (\nabla_X Y)^\perp$ for $X, Y \in \mathfrak{X}(\Sigma)$, and simple observation shows

Proposition 7.1. *Let ∇^Σ be the Levi–Civita connection on Σ, then ∇^Σ is given by*

$$\nabla^\Sigma_X Y = (\nabla_X Y)^\top$$

for $X, Y \in \mathfrak{X}(\Sigma)$.

Now we need the conception of normal bundle.

Definition. Let Σ be a submanifold of M, then for any $p \in \Sigma$, we have the decomposition

$$T_pM = T_p\Sigma + N_p\Sigma,$$

where $N_p\Sigma$ is the orthogonal complement of $T_p\Sigma$. Then define the **normal bundle** of Σ to be

$$N\Sigma = \bigsqcup_{p \in \Sigma} N_p\Sigma.$$

It is similar to tangent bundle to give differentiable structure on $N\Sigma$. Let $\pi : N\Sigma \to \Sigma$ be the natural projection, we define

$$\Gamma(N\Sigma) = \{s : \Sigma \to N\Sigma |\ s \text{ smooth and } \pi \circ s = \mathrm{id}\}.$$

Thus we can define the second fundamental form of a submanifold.

Definition. Let Σ be a submanifold of M, then we define the **second fundamental form** of Σ to be

$$\begin{aligned} \mathrm{II} : \mathfrak{X}(\Sigma) \times \mathfrak{X}(\Sigma) &\to \Gamma(N\Sigma) \\ (X,Y) &\mapsto (\nabla_X Y)^{\perp}. \end{aligned}$$

Since $\mathrm{II}(X,Y) - \mathrm{II}(Y,X) = (\nabla_X Y)^{\perp} - (\nabla_Y X)^{\perp} = ([X,Y])^{\perp} = 0$, the second fundamental form is symmetric. Moreover, we can define the shape operator.

Definition. For a fixed $\xi \in \Gamma(N\Sigma)$, we define

$$\begin{aligned} S_\xi : \mathfrak{X}(\Sigma) &\to \mathfrak{X}(\Sigma) \\ X &\mapsto \nabla_X \xi - (\nabla_X \xi)^{\perp}. \end{aligned}$$

We have the following Weingarten formula.

Proposition 7.2. *For $\xi \in \Gamma(N\Sigma)$ and $X,Y \in \mathfrak{X}(\Sigma)$, we have*

$$\langle S_\xi(X), Y\rangle = \langle \mathrm{II}(X,Y), -\xi\rangle.$$

In particular, S_ξ is symmetric.

Proof. We have the calculation

$$\begin{aligned} \langle S_\xi(X), Y\rangle &= \langle \nabla_X \xi - (\nabla_X \xi)^{\perp}, Y\rangle \\ &= \langle \nabla_X \xi, Y\rangle - \langle (\nabla_X \xi)^{\perp}, Y\rangle \\ &= X\langle \xi, Y\rangle - \langle \xi, \nabla_X Y\rangle \\ &= \langle -\xi, \nabla_X Y\rangle = \langle -\xi, (\nabla_X Y)^{\perp}\rangle \\ &= \langle \mathrm{II}(X,Y), -\xi\rangle. \end{aligned}$$

□

We mention here the Gauss–Codazzi equations. The proof is pure calculation, so we omit the proof and without having effect on our course.

Theorem 7.3. *Let $\Sigma \subset M$ be Riemannian manifolds, then we have*
Gauss equation

$$R^{\Sigma}(X,Y,Z,W) = R^{M}(X,Y,Z,W) + \langle \mathrm{II}(X,Z), \mathrm{II}(Y,W)\rangle - \langle \mathrm{II}(X,W), \mathrm{II}(Y,Z)\rangle;$$

Codazzi equation

$$R^M(X,Y,Z,\xi) = (\nabla_Y \mathrm{II})(X,Z,\xi) - (\nabla_X \mathrm{II})(Y,Z,\xi),$$

where $\mathrm{II}(X,Y,\xi) = \langle \mathrm{II}(X,Y), \xi \rangle$.

Lastly, we define the mean curvature of a submanifold.

Definition. Let Σ be a submanifold of M, with second fundamental form II. Then the **mean curvature vector** of Σ is defined as

$$\vec{H}(p) = \mathrm{tr}\,\mathrm{II}|_p = \sum_{i=1}^{k} \mathrm{II}(e_i, e_i),$$

where $\{e_i\}$ is an orthonormal basis at $p \in \Sigma$.

In particular, if Σ is a hypersurface (i.e. of codimension 1) and two-sided (i.e. there is a normal unit vector field $N \in \Gamma(N\Sigma)$), then we have

$$\vec{H} = HN, \ \mathrm{II} = hN,$$

where H is a function on Σ, and h is a symmetric $(0,2)$-tensor on Σ.

Example 7.4. Consider $\mathbb{S}^{n-1} \subset \mathbb{R}^n$. Choose the inward unit normal vector field $N(x) = -x$, then we have

$$\nabla_X N = X(-x) = -X.$$

Hence we have $h_{ij} = \delta_{ij}$, and $H = n-1$.

Definition. Let $\Sigma \subset M$ be an two-sided hypersurface, N be a unit normal vector field of Σ. Then the eigenvalues of S_N are called **principal curvatures**.

7.2 The theorem

Before we state the theorem, we need some preparation.

Let $\Sigma^{n-1} \subset (M^n, g)$ be an embedded closed hypersurface, with $\mathrm{Ric}^M \geq (n-1)Kg$. Let Ω be a region enclosed by Σ, and N be an inward unit normal vector field on Σ. Consider for $p \in \Sigma$, let $\gamma_{N(p)}(t) = \exp_p(tN(p))$. Define

$$\tau(p) = \sup\{t \in (0,+\infty) |\ d(\gamma_{N(p)}, \Sigma) = t\},$$

then Kasue showed in 1980s that

$$\mathrm{Cut}(\Sigma) = \{q \in M |\ q = \exp_p(\tau(p)N(p))\}$$

is of zero-measure.

Then we state the theorem.

Theorem 7.5 (Heintze–Karcher)**.** *Let (M^n,g) be a Riemannian manifold with* $\mathrm{Ric}\geq (n-1)Kg$*. Let $\Sigma\subset M$ be an embedded closed hypersurface that encloses Ω. Then*

$$\mathrm{Vol}(\Omega)\leq \int_\Sigma\int_0^{\tau(p)}\left(\mathrm{sn}_K'(r)-\frac{H(p)}{n-1}\mathrm{sn}_K(r)\right)^{n-1}\mathrm{d}r\,\mathrm{d}\,\mathrm{Vol}_\Sigma(p),$$

where

$$\mathrm{sn}_K(r)=\begin{cases}\frac{1}{\sqrt{K}}\sin(\sqrt{K}r), & K>0,\\ r, & K=0,\\ \frac{1}{\sqrt{-K}}\sinh(\sqrt{-K}r), & K<0.\end{cases}$$

The equality holds if and only if $\partial\Omega$ is umbilical.

Proof. Under Fermi coordinate $(q,r)\mapsto \exp_p(rN(p))$, we have the metric $g=\mathrm{d}r^2+g_\Sigma$, and the volume form of M can be written

$$\mathrm{d}\,\mathrm{Vol}_g(p,r)=\mathscr{J}(p,r)\,\mathrm{d}r\,\mathrm{d}\,\mathrm{Vol}_\Sigma(p).$$

Let d_Σ be the distance function to Σ, then one can show similarly to Proposition 5.23 that

$$\Delta d_\Sigma(p,r)=\frac{\partial}{\partial r}\log\mathscr{J}(p,r).$$

Denote $\Phi=\mathscr{J}^{\frac{1}{n-1}}$, similarly there holds $\frac{\partial^2}{\partial r^2}\Phi(p,r)+K\Phi(p,r)\leq 0$ by $\mathrm{Ric}\geq(n-1)Kg$ and Bochner formula. We need to calculate $\frac{\partial}{\partial r}\Phi(p,0)$. We know that $\Phi(p,0)=1$, then we have (we omit p)

$$\begin{aligned}\Phi'(0)&=\frac{1}{n-1}\mathscr{J}^{\frac{2-n}{n-1}}(0)\mathscr{J}'(0)\\&=\frac{1}{n-1}\mathscr{J}'(0)\\&=\frac{1}{n-1}\frac{\partial}{\partial r}\bigg|_{r=0}\sqrt{\det((g_\Sigma)_{ij})}\\&=\frac{1}{n-1}\left(\frac{1}{2}\cdot\frac{1}{\sqrt{\det((g_r)_{ij})}}\det((g_r)_{ij})g^{kl}(\langle\nabla_{\partial_r}\partial_l,\partial_k\rangle+\langle\partial_k,\nabla_{\partial_r}\partial_l\rangle)\right)\\&=\frac{1}{n-1}\left(\mathscr{J}(0)g^{kl}\langle\nabla_{\partial_r}\partial_l,\partial_k\rangle\right)\\&=\frac{1}{n-1}\left(g^{kl}\langle\nabla_{\partial_l}\partial_r,\partial_k\rangle\right)\\&=\frac{1}{n-1}\left(g^{kl}\langle S_{\partial_r}(\partial l),\partial_k\rangle\right)\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{n-1}\left(g^{kl}(-h_{lk})\right) \\
&= -\frac{1}{n-1}\operatorname{tr} h = -\frac{H(p)}{n-1}.
\end{aligned}
$$

Then let Φ_K satisfy $\frac{\partial^2}{\partial r^2}\Phi_K(p,r)+K\Phi_K(p,r)=0$, $\Phi_K(p,0)=0$, $\Phi_K'(p,0)=-\frac{1}{n-1}H(p)$. Then we have

$$
\Phi_K(p,r)=\mathrm{sn}_K'(r)-\frac{1}{n-1}H(p)\,\mathrm{sn}_K(r),
$$

and the theorem follows from Sturm–Liouville comparison theorem. If the equality holds, we trace back where $\ddot{\Phi}+K\Phi=0$ holds. This is in the argument at Bochner formula part which asserts that $\nabla^2 d_\Sigma(p,r)$ has equal eigenvalues. However, since musical isomorphism commutes with covariant derivative, we just need to check bilinear form

$$
\langle \nabla_X(\nabla d_\Sigma),Y\rangle=\langle \nabla_X\,\partial_r,Y\rangle=\langle S_{\partial_r}(X),Y\rangle=-h(X,Y)
$$

for $X,Y\in\mathfrak{X}(\Sigma)$. That is, the eigenvalues of second fundamental form of $\partial\Omega$ is equal. Hence $\partial\Omega$ is umbilical. $\square$

Remark 7.6. (1) More precise estimation can show that

$$
\mathrm{Vol}(\Omega)\le\int_\Sigma\int_0^{\tau(p)}\prod_{i=1}^{n-1}(\mathrm{sn}_K'(r)-\kappa_i(p)\,\mathrm{sn}_K(r))\,\mathrm{d}r\,\mathrm{d}\,\mathrm{Vol}_\Sigma(p),
$$

where κ_i are the principal curvatures. This implies Heintze–Karcher's estimate by AM-GM inequality.

(2) $\Phi_K^{n-1}(p,r)\,\mathrm{d}r\,\mathrm{d}\,\mathrm{Vol}_\Sigma$ cannot be regarded as volume form of M_K.

7.3 Alexandorv's Theorem

The first application of Heintze–Karcher volume estimate is Alexandorv's theorem on CMC hypersurface.

Before we start to discuss the theorem, we first need to discuss the first variation formula of area functional (we use area to denote the volume of a submanifold), in order to introduce the notion of minimal submanifolds.

Proposition 7.7 (First variation formula for area)**.** [1] *Let $\varphi:\Sigma^k\hookrightarrow(M^n,g)$ is an isometric immersion. Let area functional be*

$$
A(\Sigma^k)=\int_{\Sigma^k}\mathrm{d}A_{\varphi^*g}.
$$

[1]This version is different from which on class. We introduce the notion of proper variation, instead of $\operatorname{supp}X$ being compact.

Let $f : \Sigma \times (-\varepsilon, \varepsilon) \to M$ be a family of immersions, denote $f(\Sigma, t) = \Sigma_t$, and suppose $f(p, 0) = p$ for all $p \in M$. Let $X := \frac{\partial}{\partial t}\Big|_{t=0} f \in TM$ be the variation vector field. Then we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} A(\Sigma_t) = -\int_\Sigma \langle \vec{H}, X \rangle \, \mathrm{d}A_\Sigma.$$

Proof. Let $f_t^* g = g_t$, and

$$\mathscr{J}(p,t) = \frac{\sqrt{\det g_t}}{\sqrt{\det g_0}}.$$

Then we have

$$A_t(p) = \mathscr{J}(p,t) A_0(x),$$

so we just need to calculate $\partial_t \mathscr{J}(p,t)$ at $t = 0$. We have (omit p)

$$\begin{aligned}
\frac{\partial}{\partial t}\Big|_{t=0} \mathscr{J}(t) &= \frac{1}{\sqrt{\det g_0}} \frac{\partial}{\partial t}\Big|_{t=0} \sqrt{\det g_t} \\
&= \frac{1}{\det g_0} \cdot \frac{1}{2} \cdot \sqrt{\det g_0} \cdot g_0^{ij} \frac{\partial}{\partial t}\Big|_{t=0} g_{ij} \\
&= \frac{1}{2} g_0^{ij} \left(\left\langle \nabla_{\partial_t} \frac{\partial f}{\partial x_i}, \frac{\partial f}{\partial x_j} \right\rangle + \left\langle \frac{\partial f}{\partial x_i}, \nabla_{\partial_t} \frac{\partial f}{\partial x_j} \right\rangle \right) \\
&= g_0^{ij} \left\langle \nabla_{\partial_i} \frac{\partial f}{\partial t}, \frac{\partial f}{\partial x^j} \right\rangle \\
&= g_0^{ij} \left\langle S_X \left(\frac{\partial f}{\partial x^i} \right), \frac{\partial f}{\partial x^j} \right\rangle \\
&= -\left\langle g_0^{ij} \,\mathrm{II}\left(\frac{\partial f}{\partial x^i}, \frac{\partial f}{\partial x^j} \right), X \right\rangle \\
&= -\langle \vec{H}, X \rangle.
\end{aligned}$$

Take integral we obtain the required formula. $\square$

Corollary 7.8. *If $\Sigma \subset M$ minimizes area in any small variation, then Σ has mean curvature* 0.

Thus we can have the notion of minimal submanifold.

Definition. Let Σ be a submanifold of M, if Σ has constant mean curvature 0, then Σ is called a **minimal submanifold**.

Remark 7.9. We do not demand a minimal submanifold actually minimizes area.

Now we state Alexandorv's theorem.

Theorem 7.10 (Alexandorv)**.** *Any embedded closed constant mean curvature (CMC) hypersurface in $\mathbb{R}^n$ is a round sphere.*

Remark 7.11. The original proof of Alexandorv in 1958 used the celebrated *moving plane method* in theory of PDE. However, we will use a geometric approach to prove the theorem, following the work of Montiel and Ros in 1991.

Before we prove Alexandorv's theorem, we need Ros' theorem and Minkowski's formula as preparation.

Theorem 7.12 (Ros)**.** *Let (M,g) be a compact Riemannian manifold with* $\mathrm{Ric} \geq 0$*. Let $\Sigma = \partial M$, and assume Σ is a CMC submanifold with $H_\Sigma > 0$. Then*

$$\mathrm{Vol}(M) \leq \frac{n}{n+1} \int_\Sigma \frac{1}{H}\,\mathrm{d}A.$$

The equality holds if and only if Σ is umbilical.

Proof. In the proof of Heintze–Karcher comparison theorem, we can compare M to Euclidean space and obtain

$$0 \leq \Phi(r) \leq 1 - \frac{H(p)}{n-1} r,$$

where r equals to the distance function d_p. This implies

$$r \leq \frac{n-1}{H(p)}.$$

Hence by Heintze–Karcher comparison theorem again, we have

$$\begin{aligned}
\mathrm{Vol}(M) &\leq \int_\Sigma \int_0^{\tau(p)} \left(1 - \frac{H(p)}{n-1} r\right)^{n-1} \mathrm{d}r\,\mathrm{d}\,\mathrm{Vol}_\Sigma(p) \\
&\leq \int_\Sigma \int_0^{(n-1)/H(p)} \left(1 - \frac{H(p)}{n-1} r\right)^{n-1} \mathrm{d}r\,\mathrm{d}\,\mathrm{Vol}_\Sigma(p) \\
&= \int_\Sigma \frac{n}{n+1} \cdot \frac{1}{H(p)}\,\mathrm{d}\,\mathrm{Vol}_\Sigma(p).
\end{aligned}$$

The equality holds if and only if the equality in Heintze–Karcher theorem holds, that is, Σ is umbilical. $\square$

Theorem 7.13 (Minkowski's formula)**.** *Let $\Sigma^{n-1} \hookrightarrow \mathbb{R}^n$ be a closed hypersurface. Then*

$$\int_\Sigma (H\langle x, N\rangle + n - 1)\,\mathrm{d}A = 0,$$

where x is the position function, and N is the inward unit normal vector field.

Proof. First, we have

$$\operatorname{div}_\Sigma x = \sum_{i=1}^{n-1} \langle (\nabla_{e_i} x)^\top, e_i \rangle = n-1,$$

where $\{e_i\}$ is an orthonormal basis of $T_p\Sigma$, and equality holds since $\nabla_Y x = Y$ for any $Y \in T_p\mathbb{R}^n$. Moreover, we have

$$\begin{aligned}
\operatorname{div}_\Sigma(x^\top) &= \operatorname{div}_\Sigma(x) - \operatorname{div}_\Sigma(x^\perp) \\
&= n-1+\sum_{i=1}^{n-1} \langle \nabla_{e_i} x^\perp, e_i \rangle \\
&= n-1+\sum_{i=1}^{n-1} \langle S_{x^\perp}(e_i), e_i \rangle \\
&= n-1+\sum_{i=1}^{n-1} \langle \mathrm{II}(e_i,e_i), -x^\perp \rangle \\
&= n-1+H\langle x,N\rangle.
\end{aligned}$$

Then by Stokes formula, we have

$$0 = \int_\Sigma \operatorname{div}_\Sigma(x^\top) = \int_\Sigma (n-1+H\langle x,N\rangle)\,\mathrm{d}A. \qquad \square$$

Proof of Alexandorv's theorem. Let Σ be a closed CMC hypersurface in $\mathbb{R}^n$, and the region Ω satisfy $\partial\Omega = \Sigma$. We first show if $H(p)$ is constant, then $H(p) > 0$. Since Σ is closed, it is enclosed by a sufficiently large sphere. Shrink the sphere until it is tangent to Σ at a point. Then at the point, every principal eigenvalue of Σ is greater than the sphere, hence is positive. Therefore $H(p)$ is positive on Σ.

Now we complete the proof. We have

$$\begin{aligned}
nH\operatorname{Vol}(\Omega) &= H\int_\Omega n \\
&= H\int_\Omega \operatorname{div}_{\mathbb{R}^n} x \\
&= -H\int_\Sigma \langle x,N\rangle && \text{(Divergence theorem)} \\
&= (n-1)|\Sigma|. && \text{(Minkowski's formula)}
\end{aligned}$$

Then $\operatorname{Vol}(\Omega) = \frac{n-1}{n}\int_\Sigma \frac{1}{H}$. This means the equality in Ros' theorem holds, then Σ is umbilical in $\mathbb{R}^n$. Hence Σ must be a round sphere. $\square$

Remark 7.14. If $\Sigma \hookrightarrow \mathbb{R}^3$ is an immersion, then there exists closed CMC immersion which is not a round shpere. A first counterexample was given by Wente in 1986, called Wente's torus. Kapouleas constructed CMC immersions for any topology in 1990.

7.4 Levy–Gromov Isoperimetric Inequality

We introduce another application of Heintze–Karcher volume estimate, which is the isoperimetric inequality of Levy and Gromov.

We first recall isoperimetric problem. In $\mathbb{R}^n$, among C^1 bounded domain with fixed volume, round ball minimizes the boundary area. Schmidt proved in 1930s, that in $\mathbb{S}^n$ the same result holds.

First we need a conception.

Definition. Let (M^n, g) be a Riemannian manifold, $v \in (0, \operatorname{Vol}(M))$. Then we define the **isoperimetric profile** to be

$$I_M(v) = \inf\{A(\partial\Omega) \mid \Omega \subset M^n,\ \operatorname{Vol}(\Omega) = v\}.$$

Example 7.15. The isoperimetric profile of $\mathbb{R}^n$ is easy, one can calculate

$$I_{\mathbb{R}^n}(v) = n|B_1|^{1/n} v^{(n-1)/n}.$$

By abuse of notation, we have the following geometric quantity: Take $\beta \in (0,1]$, consider $B(\beta)$ be a geodesic ball on $\mathbb{S}^n$ such that $\operatorname{Vol}(B(\beta)) = \beta \operatorname{Vol}(\mathbb{S}^n)$. Then we denote

$$I(\beta) = \frac{A(\partial B(\beta))}{\operatorname{Vol}(\mathbb{S}^n)}.$$

Now we can state Levy and Gromov's isometric inequality.

Theorem 7.16 (Levy–Gromov)**.** *Let (M^n, g) be a Riemannian manifold with* $\operatorname{Ric} \geq (n-1)Kg > 0$*. Let $\Omega \subset M$ be a bounded region such that* $\operatorname{Vol}(\Omega) = \beta \operatorname{Vol}(M)$*. Then*

$$\frac{A(\partial\Omega)}{\operatorname{Vol}(M)} \geq I(\beta) = \frac{A(\partial B(\beta))}{\operatorname{Vol}(\mathbb{S}^n)}.$$

That is, $I_M(\beta) \geq I(\beta)$. The equality holds if and only if M is isometric to $\mathbb{S}^n(1/\sqrt{K})$.

Remark 7.17. Befoe proving the theorem, we need some remark. Let Ω be an isoperimetric region, that is, Ω minimizes $A(\partial\Omega)$. We want to ask its existence and regularity problems. Firstly, since M is positively curved, Bonnet–Myers theorem implies M is compact, and calculus of variation shows isoperimetric region exists. Secondly, geometric measure theory shows $H_{\partial\Omega}$ is constant for regular points. These discussions are far beyond our scope, so we just mention them here.

Proof of Theorem 7.16. As usual, we scale the metric to let $K = 1$. Compare M to M_1, we have

$$0 \leq \Phi(r) \leq \cos r - \frac{H_0}{n-1} \sin r.$$

Hence the distance function must satisfy

$$\tau(p) \leq \cot^{-1}\left(\frac{H_0}{n-1}\right) =: r_0.$$

Then by Heintze–Karcher volume estimate, we have

$$\mathrm{Vol}(\Omega) \leq \int_{\partial\Omega}\int_0^{r_0}\left(\cos r - \frac{H_0}{n-1}\sin r\right)^{n-1} \mathrm{d}r\,\mathrm{d}\,\mathrm{Vol}_{\partial\Omega}.$$

The integrand of outer integral is a constant by Remark 7.17, hence we have

$$\begin{aligned}\mathrm{Vol}(\Omega) &\leq A(\partial\Omega)\int_0^{r_0}\left(\cos r - \frac{H_0}{n-1}\sin r\right)^{n-1} \mathrm{d}r \\ &= A(\partial\Omega)\frac{\mathrm{Vol}(B_{r_0})}{A(\partial B_{r_0})},\end{aligned}$$

where B_{r_0} is the geodesic ball of radius r_0 in $\mathbb{S}^n$. Set $a(r_0) = \frac{A(\partial B_{r_0})}{\mathrm{Vol}(B_{r_0})}$, the above inequality can be written

$$\frac{A(\partial\Omega)}{\mathrm{Vol}(M)} \geq \beta a(r_0).$$

Same work on Ω^c shows

$$\frac{A(\partial\Omega)}{\mathrm{Vol}(M)} \geq (1-\beta)a(\pi - r_0).$$

Hence

$$\frac{A(\partial\Omega)}{\mathrm{Vol}(M)} \geq \inf\{\beta a(r_0), (1-\beta)a(\pi - r_0)\},$$

and monotonicity shows the infimum is attained when $\beta a(t_0) = (1-\beta)a(\pi - t_0)$ for some t_0, and in this situation we can calculate that $\beta a(r_0) = I(\beta)$. Hence we have the inequality

$$\frac{A(\partial\Omega)}{\mathrm{Vol}(M)} \geq I(\beta).$$

If the equality holds, then the equality holds in Heintze–Karcher volume estimate, this shows M is isometric to $\mathbb{S}^n$. □

Remark 7.18. Bayle gave an alternative proof in his doctoral thesis in 2004. He uses the function

$$f(v) = I_M^{\frac{n}{n-1}}(v)$$

and uses second variation formula for area functional to obtain

$$f''(v) \leq nf(v)^{-\frac{n-2}{n}}.$$

Moreover, if $M = \mathbb{S}^n$, then the equality holds. Thus one can use comparison theorem in ODE theory to obtain the result.

Klatag (2017) applies this method to metric measure space $\mathrm{RCD}(K,n)$ using optimal transport.

As a finale, we quote some remarkable works of Brendle.

Theorem 7.19 (Brendle)**.** *If M is a noncompact, complete Riemannian manifold with* $\mathrm{Ric} \geq 0$*. Assume M has Euclidean volume growth. Let $\Omega \subset M$ bounded, then*

$$A(\partial\Omega)^{\frac{n}{n-1}} \geq n|B_1|^{1/n}\,\mathrm{Vol}(\Omega)^{\frac{n-1}{n}}\,\theta^{1/n},$$

where

$$\theta = \lim_{r\to\infty} \frac{\mathrm{Vol}(B_r)}{|B_1|r^n}$$

is the asymptotic volume ratio of M. The equality holds if and only if $\Omega \subset \mathbb{R}^n$.

Cabre used ABP (Alexandorv–Bakelman–Pucci) method to prove isoperimetric inequality in $\mathbb{R}^n$, Brendle generalized Cabre's proof to the following theorem.

Theorem 7.20 (Brendle)**.** *Let $\Sigma^n \subset \mathbb{R}^{n+k}$ be a minimal submanifold. Then*

$$|\,\partial\Sigma| \geq n|B_1|^{1/n}\,\mathrm{Vol}(\Sigma)^{\frac{n-1}{n}}$$

when $k \leq 2$. The equality holds if and only if $\Sigma \cong \mathbb{B}^n \subset \mathbb{R}^n$.

Our last theorem is

Theorem 7.21 (Brendle)**.** *We have*

$$\int_\Sigma H + |\,\partial\Sigma| \geq n|B_1|^{1/n}|\Sigma|^{\frac{n-1}{n}}.$$

Please refer to *The Isoperimetric Inequality* (Brendle–Eichmair, 2024) for detailed discussion.

Chapter 8

A Little Thing on Bernstein Problem

In our last topic class we discuss the Bernstein problem for minimal surfaces. We first introduce some motivation for the problem.

Let $u : \Omega \subset \mathbb{R}^{n-1} \to \mathbb{R}$ be a function, then its graph $(x^1, \cdots, x^{n-1}, u(x^1, \cdots, x^{n-1}))$ is a minimal surface if and only if it satisfies the following **minimal surface equation** (MSE):

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0.$$

When $n = 3$, MSE is found by Lagrange in 1762.

If we do not restrict us on graph, then we have the following Plateau problem.

Problem (Plateau). Let $\Gamma \subset \mathbb{R}^3$ be a simple closed curve, find an area-minimizing surface $\Sigma \subset \mathbb{R}^3$ such that $\partial\Sigma = \Gamma$.

This was solved by Douglas–Rado in 1930s. Please refer to Struwe's *Calculus of Variation* for more discussion.

But for minimal graph (i.e. a minimal surface that is a graph), we have the following Bernstein's theorem.

Theorem 8.1 (Bernstein, 1916)**.** *Let $u : \mathbb{R}^2 \to \mathbb{R}$ be a function that satisfies MSE (called **entire graph**), then u is an affine function, that is*

$$u(x) = \vec{a} \cdot x + b.$$

Then we have the Bernstein problem.

Problem (Bernstein). Let $u : \mathbb{R}^{n-1} \to \mathbb{R}$ be an entire graph in $\mathbb{R}^n$. Is the graph of u an affine graph? In other words, is entire minimal graph in $\mathbb{R}^n$ a hyperplane?

Bernstein problem is also solved. The timeline is

- Bernstein solved $n = 3$ case in 1916.
- Fleming gave an important alternative method in 1960s.
- De Giorgi solved $n = 4$ case enlightened by Fleming.
- Almgren solved $n = 5$ case in 1966.
- J. Simons solved $n \leq 8$ cases in 1968.
- The problem is false when $n \geq 9$. Counterexample was given by Bombien–de Giorgi–Ginsti in 1969.

We have the following fact.

Proposition 8.2. *A minimal graph in $\mathbb{R}^n$ is area-minimizing.*

Proof. We need to show if Σ_u is a minimal graph, Σ is a surface such that $\partial\Sigma = \partial\Sigma_u$, then $|\Sigma_u| \leq |\Sigma|$. Define a vector field $X \in \mathfrak{X}(\mathbb{R}^n)$, such that

$$X(x^1, \cdots, x^n) = \frac{(-\nabla u, 1)}{\sqrt{1+|\nabla u|^2}}.$$

Since X has nothing to do with x^n, we have

$$\operatorname{div} X = \operatorname{div}_{\mathbb{R}^{n-1}} \left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \right) = 0.$$

Let Ω be the domain enclosed by Σ_u and Σ, ν_Σ and ν_{Σ_u} be the outward unit normal vector field of Σ and Σ_u. Then by divergence theorem, we have

$$\begin{aligned} 0 &= \int_\Omega \operatorname{div}_{\mathbb{R}^n} X \\ &= \int_\Sigma \langle X, \nu_\Sigma \rangle + \int_{\Sigma_u} \langle X, \nu_{\Sigma_u} \rangle. \end{aligned}$$

Since $\nu_{\Sigma_u} = \frac{(\nabla u, -1)}{\sqrt{1+|\nabla u|^2}}$, we have $\langle X, \nu_{\Sigma_u} \rangle = -1$. Thus by Cauchy–Schwarz inequality, we have

$$|\Sigma_u| \leq \int_\Sigma \langle X, \nu_\Sigma \rangle \leq \int_\Sigma |X||\nu_\Sigma| = |\Sigma|.$$ □

Remark 8.3. This fact is closely related to calibrated geometry. We refer to Harvey and Lawson's *Calibrated Geometries* (1982).

Corollary 8.4. *Let Σ_u be an entire minimal graph, and $B_R \subset \mathbb{R}^n$. Then*

$$A(\Sigma_n \cap B_r) \leq \frac{1}{2}\omega_{n-1}R^{n-1},$$

that is, Σ_u has Euclidean area growth.

Proof. $\Sigma \cap B_R$ separates B_R into upper part B_R^1 and lower part B_R^2. Then we have

$$|\Sigma_u| \leq \min\{|\partial B_R^1|, |\partial B_R^2|\} \leq \frac{1}{2}\omega_{n-1}R^{n-1}. \qquad \square$$

We now discuss the conception of stability. This is used to solve the problem that when minimal surfaces actually minimizes area.

Definition (Stability)**.** We call the surface Σ is **stable** if for any proper variation $f : \Sigma \times (-\varepsilon, \varepsilon) \to M$ the second variation is nonnegative, that is,

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\bigg|_{t=0} |\Sigma_t| = \int_\Sigma (|\nabla^\perp X|^2 - |\langle \mathrm{II}, X\rangle|^2 - \mathrm{tr}_\Sigma \langle R^M(\bullet, X)\bullet, X\rangle) \geq 0,$$

where X is the variation vector field. When Σ has codimension 1, let $X = \varphi N$ with N be a unit normal vector field, $\varphi \in C_c^\infty(\Sigma)$. Then the condition is equivalent to

$$\int_\Sigma (|\nabla\varphi|^2 - |h|^2\varphi^2 - \mathrm{Ric}(N,N)\varphi^2) \geq 0$$

or

$$\int_\Sigma -\varphi(\Delta + |h|^2 + \mathrm{Ric}(N,N))\varphi \geq 0,$$

where $(\Delta + |h|^2 + \mathrm{Ric}(N,N))$ is the **Jacobi operator**.

About stability, we have the following theorem.

Theorem 8.5 (Fischer–Colbrie–Schoen)**.** *A surface Σ is stable if and only if there is a $u \in C^\infty(\Sigma)$ with $u > 0$, such that*

$$(\Delta + |h|^2 + \mathrm{Ric}(N,N))u = 0.$$

We now give a proof of 2-dimensional Bernstein problem. This is a corollary of following theorem.

Theorem 8.6. *Let $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ be a minimal graph, $B_R^2 \subset \Omega$. Then*

$$\int_{B^3_{\sqrt{R}} \cap \Sigma_u} |h|^2 \leq \frac{C}{\log R}.$$

Proof. Since minimal graph minimizes area, the second variation of area must be positive, and thus minimal graph is stable. Therefore for any $\varphi \in C_c^\infty(\Sigma)$

$$\int_\Sigma |h|^2\varphi^2 \le \int_\Sigma |\nabla_\Sigma \varphi|^2.$$

Choose

$$\varphi(x) = \begin{cases} 1, & |x| \le \sqrt{R}, \\ 2\left(1 - \frac{\log|x|}{\log R}\right), & \sqrt{R} \le |X| \le R, \\ 0, & |x| \ge R, \end{cases}$$

then we have an estimate

$$|\nabla_\Sigma \varphi| \le |\nabla_{\mathbb{R}^3}\varphi| \le \frac{1}{\log R} \cdot \frac{1}{|x|}.$$

Therefore we have

$$\int_{\Sigma \cap B_R^3} |\nabla_\Sigma \varphi|^2 \le \sum_{k=\log R/2}^{\log R} \int_{\Sigma \cap (B^3_{e^k} \setminus B^3_{e^{k-1}})} \frac{1}{(\log R)^2} \cdot \frac{1}{|x|^2}.$$

(We are mainly estimating the order of h, so we can adjust R to make $\log R$ an integer.) But we have

$$|\Sigma \cap B^3_{e^k}| \le ce^{2k}, \ \frac{1}{(\log R)^2} \cdot \frac{1}{|x|^2} \le e^{-2(k-1)},$$

hence

$$\begin{aligned} \int_{\Sigma \cap B_R^3} |\nabla_\Sigma \varphi|^2 &\le ce^2 \log R \cdot \frac{1}{(\log R)^2} \\ &= \frac{C}{\log R}. \end{aligned}$$

□

Remark 8.7. This argument is called logarithm cutoff argument.

Before we procced our discuss, we need the following monotonicity formula.

Theorem 8.8 (Monotonicity formula)**.** *Let $\Sigma^k \subset \mathbb{R}^n$ be a minimal submanifold, $x_0 \in \mathbb{R}^n$. Then for $r > s$ we have*

$$\frac{|\Sigma^k \cap B_r^n(x_0)|}{r^k} - \frac{|\Sigma^k \cap B_s^n(x_0)|}{r^k} = \int_{(B_r^n(x_0) \setminus B_s^n(x_0)) \cap \Sigma} \frac{|(x-x_0)^\perp|^2}{|x-x_0|^{k+2}} \, \mathrm{d}A.$$

Proof. Without loss of generality we assume $x_0 = 0$. We have

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{|\Sigma^k \cap B_r|}{r^k}\right) = r^{-k} \int_{\Sigma \cap \partial B_r} \frac{1}{|\nabla_\Sigma |x||} - kr^{-k-1}|\Sigma \cap B_r|,$$

here we use the coarea formula

$$|\Sigma^k \cap B_r| = |\{x \in \Sigma |\ |x| \le r\} = \int_0^r \int_{\Sigma \cap \partial B_r} \frac{1}{|\nabla_\Sigma |x||},$$

a proof can be found on Mei Jiaqiang's *Mathematical Analysis*. Since Σ^k is minimal, we have $\mathrm{div}_{\Sigma \cap B_r}(x^\perp) = 0$ as in Minkowski's formula. Then

$$\begin{aligned}
k|\Sigma \cap B_r| &= \int_{\Sigma \cap B_r} \mathrm{div}_\Sigma(x) \\
&= \int_{\Sigma \cap B_r} \mathrm{div}_\Sigma(x^\top) \\
&= \int_{\Sigma \cap \partial B_r} \langle x^\top, \mu \rangle \quad (\mu \text{ is the outward unit normal vector field}) \\
&= \int_{\Sigma \cap \partial B_r} |x^\top| \quad (\mu = x^\top / |x^\top|) \\
&= r \int_{\Sigma \cap \partial B_r} \frac{|\nabla_\Sigma |x||^2}{|\nabla_\Sigma |x||} \quad (|x^\top| = |\nabla_\Sigma |x|| \cdot |x|) \\
&= \frac{1}{r} \int_{\Sigma \cap \partial B_r} \frac{|x|^2 - |x^\perp|^2}{|\nabla_\Sigma |x||} \\
&= r \int_{\Sigma \cap \partial B_r} \frac{1}{|\nabla_\Sigma |x||} - \frac{1}{r} \int_{\Sigma \cap \partial B_r} \frac{|x^\perp|^2}{|\nabla_\Sigma |x||}.
\end{aligned}$$

Hence

$$\frac{\mathrm{d}}{\mathrm{d}r} \left(\frac{|\Sigma \cap B_r|}{r^k} \right) = r^{-k-2} \int_{\Sigma \cap \partial B_r} \frac{|x^\perp|^2}{|\nabla_\Sigma |x||}. \qquad \square$$

Monotonicity formula is useful with the notion of cone. We now introduce the conception of cone.

Definition. Let $\Sigma^{k-1} \subset \mathbb{S}^{n-1}$ be a $(k-1)$-dimensional submanifold in $\mathbb{S}^{n-1}$. Then the **cone** over Σ is defined as

$$\mathscr{C}_\Sigma := \left\{ x \in \mathbb{R}^n \,\middle|\, \frac{x}{|x|} \in \Sigma \right\}.$$

There is a fact that

Proposition 8.9. $\Sigma \subset \mathbb{S}^{n-1}$ *is minimal if and only if* $\mathscr{C}_\Sigma$ *is a minimal cone.*

One can observe that for a cone $\mathscr{C}$, $\frac{|\mathscr{C} \cap B_r|}{r^k}$ is constant. Conversely, by monotonicity formula, for a minimal submanifold $\Sigma \subset \mathbb{R}^n$ the following are equivalent:

(1) $\frac{|\Sigma \cap B_r|}{r^k}$ is constant;

(2) $x^\perp \equiv 0$;

(3) Σ is a cone.

Now we can outline the proof of Bernstein problem.

1. Fleming: If every area-minimizing cone is flat, then the area-minimizing hypersurface in $\mathbb{R}^n$ is flat. This is proved by blow-down argument. Choose $p \in \Sigma$, let $\Sigma_r := \frac{1}{r}(\Sigma \backslash \{p\})$. Let $r \to \infty$, Σ_r converges to $\mathscr{C}$. By geometric measure theory, $\mathscr{C}$ is a set of finite perimeter. We need to show $\mathscr{C}$ is area-minimizing. We have

$$\frac{|\mathscr{C} \cap B_r|}{R^{n-1}} = \lim_{r\to\infty} \frac{|\Sigma_r \cap B_R|}{R^{n-1}} = \lim_{r\to\infty} \frac{|\Sigma \cap B_{Rr}|}{(Rr)^{n-1}} = \theta_R = \text{const}.$$

Thus $\mathscr{C}$ is a cone. By assumption, $\mathscr{C}$ is flat, then $\frac{|\mathscr{C}\cap B_R|}{R^{n-1}} = \omega_{n-1}$. Therefore

$$\omega_{n-1} \le \frac{|\Sigma \cap B_r|}{r^{n-1}} \le \lim_{r\to\infty} \frac{\Sigma \cap B_r}{r^{n-1}} = \frac{|\mathscr{C} \cap B_r|}{r^{n-1}} = \omega_{n-1},$$

hence Σ is a cone, and therefore is flat.

2. Simons 1968: Area-minimizing cone with possible singularity at 0 in $\mathbb{R}^n$ ($n \le 7$) is flat. Roughly speaking, the proof is analysing stability and the following Simons' inequality[1]

$$\Delta_\Sigma |h|^2 \ge -2|h|^4 + 2\left(1 + \frac{2}{n-1}\right) |\nabla_\Sigma |h||^2.$$

3. De Giorgi then solved $n = 8$ case.[2]

4. In $\mathbb{R}^8$ we have the Simons cone $\{(x,y) \in \mathbb{R}^4 \times \mathbb{R}^4 | \, |x| = |y|\}$ being area-minimizing but not flat.

Finally we discuss the stable Bernstein problem.

Problem (Stable Bernstein). Let $\Sigma^{n-1} \subset \mathbb{R}^n$ ($n \le 7$) be a stable complete two-sided minimal hypersurface. Is Σ flat?

We have the following progress.

- $n = 3$ case is solved by Fischer–Colbrie–Schoen–do Carmo–Peng in 1980.
- $n = 4$ case is solved by Chodosh–Li in 2021. Catino–Mastrolia–Roncoroni gave an alternative approach in 2023.
- $n = 5$ case is solved by Chodosh–Li–Stryker in 2024/01.

[1] We think Prof. Xia mistakenly wrote "identity".

[2] Prof. Xia provided a vague argument, so we omit it here.

- $n = 6$ case is solved by Mazet in 2024/05.
- $n = 7$ case is still open.

On the other hand, we have the following results.

- Schoen–Simon–Yau 1976: Assume $|\Sigma \cap B_R| \leq CR^{n-1}|$, then stable Bernstein problem is true for $n \leq 6$.
- Bellettini 2023: the $n = 7$ case of extensions of above theorem is true.

These two results uses de Giorgi–Mozer iteration on Simons inequality.

At last we give some reference books.

- Colding–Minicozi: *A Course in Minimal Surface*.
- L. Simon: *Geometric Measure Theory*.
- F. Maggi: *Geometric Measure Theory*.
- Y. Xin: *Geometry of Submanifold*.
- Chodosh: Lecture notes.

Appendix A

Some Background on Tensors

We now supply some background materials on tensors.

A.1 Basic Notions

First we introduce the notion of tensors and differential forms. In this section, we will not give proof of any proposition, please refer to any book on differentiable manifolds for proofs.

Definition. Let M be a differentiable manifold. An (r,s)**-tensor** T is a $C^\infty(M)$-multilinear map

$$T : \underbrace{\mathfrak{X}(M)^* \times \cdots \times \mathfrak{X}(M)^*}_{r} \times \underbrace{\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M)}_{s} \to C^\infty(M),$$

where $\mathfrak{X}(M)^*$ denotes the dual module of $\mathfrak{X}(M)$ over $C^\infty(M)$. $(0,s)$-tensors are called **covariant tensors**, and $(r,0)$-tensors are called **contravariant tensors**.

Remark A.1. We often define $(0,0)$-tensors to be smooth functions, and identify $(1,0)$-tensors with vector fields.

Definition. Let $\omega, \eta \in \mathfrak{X}(M)^*$. The **tensor product** of ω, η, denoted by $\omega \otimes \eta$, is defined as

$$\begin{aligned} \omega \otimes \eta : \mathfrak{X}(M) \times \mathfrak{X}(M) &\to C^\infty(M) \\ (X,Y) &\mapsto \omega(X) \cdot \eta(Y). \end{aligned}$$

Clearly tensor product is associative, and distributive to addition. We can also define tensor product for contravariant tensors.

Proposition A.2. *Let $\mathfrak{T}^{r,s}(M)$ be the $C^\infty(M^n)$ module of (r,s)-tensors. Then on a local chart (U,φ), $\mathfrak{T}^{r,s}|_U$ is free with basis*

$$\left\{\frac{\partial}{\partial x^{i_1}}\otimes\cdots\otimes\frac{\partial}{\partial x^{i_r}}\otimes \mathrm{d}x^{j_1}\otimes\cdots\otimes\mathrm{d}x^{j_s}\right\}$$

for all indices $1\le i_1,\cdots,i_r\le n$, $1\le j_1,\cdots,j_s\le n$, where

$$\mathrm{d}x^i\left(\frac{\partial}{\partial x^j}\right)=\delta_{ij}.$$

Definition. A **differential form** on M of order r is a skew-symmetric $(0,r)$-tensor. The space of differential form of order r is denoted by $\bigwedge^r M$.

Proposition A.3. *There is a map $\pi:\mathfrak{T}^r(M)\to\bigwedge^r M$ as a quotient map of $C^\infty(M)$ modules given by*

$$\omega(X_1,\cdots,X_r)\mapsto\frac{1}{r!}\sum_{\sigma\in S_r}\operatorname{sgn}(\sigma)\omega\left(X_{\sigma(1)},\cdots,X_{\sigma(r)}\right)$$

for any $(0,r)$-tensor ω and $X_i\in\mathfrak{X}(M)$, $i=1,\cdots,r$.

Definition. We have a bilinear map $\wedge:\bigwedge^r(M)\times\bigwedge^s(M)\to\bigwedge^{r+s}(M)$ defined by the commutative diagram

$$\begin{tikzcd}
\mathfrak{T}^r(M)\times\mathfrak{T}^s(M) \arrow[r, "\otimes"] \arrow[d, "\pi\times\pi"] & \mathfrak{T}^{r+s}(M) \arrow[d, "\pi"] \\
\bigwedge^r(M)\times\bigwedge^s(M) \arrow[r, "\wedge"] & \bigwedge^{r+s}(M)
\end{tikzcd}$$

Proposition A.4. *Under a local chart (U,φ), $\bigwedge^r(M)|_U$ has a basis*

$$\{\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_r}\}$$

for indices $1\le i_1<\cdots<i_r\le n$.

Definition. We define the **exterior differential** of a form ω by writing

$$\omega=\omega_{i_1\cdots i_r}\,\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_r}$$

on a local chart and define

$$\begin{aligned}
\mathrm{d}\omega &:= \mathrm{d}\omega_{i_1\cdots i_r}\wedge\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_r}\\
&:= \frac{\partial\omega_{i_1\cdots i_r}}{\partial x^i}\,\mathrm{d}x^i\wedge\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_r}.
\end{aligned}$$

One can check this definition does not depend on the choice of local chart.

A.2 Musical Isomorphisms

Let (M^n,g) be a Riemannian manifold. For simplicity, we only discuss the musical isomorphisms of $(1,0)$- and $(0,1)$-tensors.

Definition. Let X be a vector field (i.e. $(1,0)$-tensor), and on a local chart we have $X = X^i\partial_i$. Then we define a $(0,1)$-tensor $X^\flat$ by

$$X^\flat_j = g_{ij}X^i.$$

One can check this definition does not depend on the choice of local chart. Then $X \mapsto X^\flat$ gives an isomorphism of $\mathfrak{X}(M) \to \mathfrak{X}(M)^*$.

Definition. Let ω be a 1-form, we define $\omega^\sharp$ to be the vector field such that for any 1-form η, we have

$$\eta(\omega^\sharp) = \omega(\eta^\flat).$$

Then $\omega \mapsto \omega^\sharp$ gives an isomorphism of $\mathfrak{X}(M)^* \to \mathfrak{X}(M)$.

Locally, if $\omega = \omega_i\,\mathrm{d}x^i$, we have $(\omega^\sharp)^j = g^{ij}\omega_i$.

Notice that the indices of $X^\flat$ are lowered and of $\omega^\sharp$ are raised, this explains why we use musical notation to define these isomorphisms.

Using raising index, we can define the gradient of a function.

Definition. Let $f \in C^\infty(M)$. The **gradient** of f is defined as $\nabla f := (\mathrm{d}f)^\sharp$.

By definition, ∇f has local expression

$$\nabla f = g^{ij}\frac{\partial f}{\partial x^i}\frac{\partial}{\partial x^j}.$$

A.3 Contraction and Trace

Definition. Let T be a (r,s)-tensor, by **contract** i,j **indices of** T $(1 \le i \le r,\ 1 \le j \le s)$ we mean a tensor $\mathrm{tr}_{ij}T$ defined by

$$\begin{aligned}&(\mathrm{tr}_{ij}T)_p(\omega_1,\cdots,\widehat{\omega}_i,\cdots,\omega_r,X_1,\cdots,\widehat{X}_j,\cdots,X_s)\\ =&\sum_{k=1}^n T_p(\omega_1,\cdots,\omega_{i-1},\eta_k,\omega_{i+1},\cdots,\omega_r,X_1,\cdots,X_{j-1},e_k,X_{j+1},\cdots,X_s)\end{aligned}$$

with $\{e_k\}_{k=1}^n$ be an orthonormal basis of T_pM and $\{\eta_k\}_{k=1}^n$ be the dual basis.

Respective to a local chart, if the tensor has local expression

$$T = T^{\nu_1\cdots\nu_r}_{\mu_1\cdots\mu_s}\frac{\partial}{\partial x^{\nu_1}}\otimes\cdots\otimes\frac{\partial}{\partial x^{\nu_r}}\otimes \mathrm{d}x^{\mu_1}\otimes\cdots\otimes \mathrm{d}x^{\mu_s},$$

then $\mathrm{tr}_{ij}\,T$ has local expression

$$(\mathrm{tr}_{ij}\,T)^{\nu_1\cdots\widehat{\nu_i}\cdots\nu_r}_{\mu_1\cdots\widehat{\mu_j}\cdots\mu_s} = T^{\nu_1\cdots\nu_{i-1}m\nu_{i+1}\cdots\nu_r}_{\mu_1\cdots\mu_{j-1}m\mu_{j+1}\cdots\mu_s}$$

Definition. If L is a $(1,1)$-tensor, we define its **trace** by $\mathrm{tr}\,L = \mathrm{tr}_{11}\,L$.

We know that a symmetric $(0,2)$-tensor S (i.e. a bilinear form) is one-to-one corresponding to a $(1,1)$-tensor L (i.e. a linear transformation), the correspondence is given by

$$S(x,x) = g(x,L(x))$$

and on a local chart

$$L^i_j = g^{ik}S_{jk}.$$

Then we have the definition

Definition. The trace of a symmetric $(0,2)$-tensor is defined as the trace of its corresponding $(1,1)$-tensor.

We look at the trace of symmetric $(0,2)$-tensor locally. We have

$$\mathrm{tr}\,S = L^i_i = g^{ij}S_{ij}.$$

We use trace to define the norm of a symmetric $(0,2)$-tensor.

Definition. Let S be a symmetric $(0,2)$-tensor, with corresponding linear transformation L. Then we define its norm by $|S|^2 := \mathrm{tr}(L\circ L)$.

Locally, we have

$$(L\circ L)^i_j = L^i_k L^k_j$$

then contract the indices we have

$$\mathrm{tr}(L\circ L) = L^i_j L^j_i = (g^{ik}S_{kj})(g^{jl}S_{li}) = g^{ik}g^{jl}S_{li}S_{kj}.$$

A.4 Covariant Derivative

Definition. Let T be a $(0,r)$-tensor ($r\geq 0$), then we define the **covariant derivative** of T as

$$\nabla_X T(Y_1,\cdots,Y_r) = XT(Y_1,\cdots,Y_r) - \sum_{i=1}^{r} T(Y_1,\cdots,\nabla_X Y_i,\cdots,Y_r)$$

for any vector fields $Y_1, \cdots, Y_r$. Notice that if T is a function (i.e., $(0,0)$-tensor), the minus summation term does not exist. Moreover, we define the **covariant differential** of T as

$$\nabla T(Y_1, \cdots, Y_r, X) = \nabla_X T(Y_1, \cdots, Y_r).$$

One can check ∇T is indeed tensorial, and satisfies Leibniz rule, that is

$$\nabla(T \otimes S) = (\nabla T) \otimes S + T \otimes (\nabla S).$$

We look at covariant derivative locally. Let

$$T = T_{i_1 \cdots i_r} \, \mathrm{d}x^{i_1} \otimes \cdots \otimes \mathrm{d}x^{i_r},$$

and write

$$\nabla_{\partial_k} T = T_{i_1 \cdots i_r ; k} \, \mathrm{d}x^{i_1} \otimes \cdots \otimes \mathrm{d}x^{i_r}.$$

Let's compute the coefficient. First we compute

$$\nabla_{\partial_i} \mathrm{d}x^j = a_k \, \mathrm{d}x^k,$$

then

$$\begin{aligned} a_k &= \nabla_{\partial_i} \mathrm{d}x^j (\partial_k) \\ &= -\mathrm{d}x^j (\nabla_{\partial_i} \partial_k) \\ &= -\mathrm{d}x^j (\Gamma_{ik}^l \partial_l) \\ &= -\Gamma_{ik}^j, \end{aligned}$$

that is,

$$\nabla_{\partial_i} dx^j = -\Gamma_{ik}^j dx^k.$$

Using this, we can compute

$$\begin{aligned} &\nabla_{\partial_k} (T_{i_1 \cdots i_r} \, \mathrm{d}x^{i_1} \otimes \cdots \otimes \mathrm{d}x^{i_r}) \\ =& \partial_k T_{i_1 \cdots i_r} \, \mathrm{d}x^{i_1} \otimes \cdots \otimes \mathrm{d}x^{i_r} + T_{i_1 \cdots i_r} \sum_{l=1}^{r} \mathrm{d}x^{i_1} \otimes \cdots \otimes (\nabla_{\partial_k} \mathrm{d}x^{i_j}) \otimes \cdots \otimes \mathrm{d}x^{i_r} \\ =& \partial_k T_{i_1 \cdots i_r} \, \mathrm{d}x^{i_1} \otimes \cdots \otimes \mathrm{d}x^{i_r} + T_{i_1 \cdots i_r} \sum_{l=1}^{r} \mathrm{d}x^{i_1} \otimes \cdots \otimes (-\Gamma_{kp}^{i_j} \, \mathrm{d}x^p) \otimes \cdots \otimes \mathrm{d}x^{i_r} \\ =& \partial_k T_{i_1 \cdots i_r} \, \mathrm{d}x^{i_1} \otimes \cdots \otimes \mathrm{d}x^{i_r} - \sum_{l=1}^{r} T_{i_1 \cdots i_{l-1} p i_{l+1} \cdots i_r} \Gamma_{k i_l}^p \, \mathrm{d}x^{i_1} \otimes \cdots \otimes \mathrm{d}x^{i_r}, \end{aligned}$$

hence we have

$$T_{i_1 \cdots i_r ; k} = \partial_k T_{i_1 \cdots i_r} - \sum_{l=1}^{r} T_{i_1 \cdots i_{l-1} p i_{l+1} \cdots i_r} \Gamma_{k i_l}^p.$$

Moreover, we can consider second covariant derivative.

Notation A.5. We use the symbol $\nabla^2_{X,Y}T$ to denote the tensor

$$\nabla^2_{X,Y}T(Y_1,\cdots,Y_r)=\nabla(\nabla T)(Y_1,\cdots,Y_r,Y,X).$$

Proposition A.6. *We have*

$$\nabla^2_{X,Y}T=\nabla_X(\nabla_Y T)-\nabla_{\nabla_X Y}T. \tag{A.1}$$

Proof. We have

$$\begin{aligned}
&\nabla^2_{X,Y}T(Y_1,\cdots,Y_r)\\
=&\nabla(\nabla T)(Y_1,\cdots,Y_r,Y,X)\\
=&\nabla_X(\nabla T)(Y_1,\cdots,Y_r,Y)\\
=&X(\nabla T)(Y_1,\cdots,Y_r,Y)-\sum_{i=1}^r(\nabla T)(Y_1,\cdots,\nabla_X Y_i,\cdots,Y_r,Y)\\
&-(\nabla T)(Y_1,\cdots,Y_r,\nabla_X Y)\\
=&X(\nabla_Y T)(Y_1,\cdots,Y_r)-\sum_{i=1}^r(\nabla_Y T)(Y_1,\cdots,\nabla_X Y_i,\cdots,Y_r)\\
&-(\nabla_{\nabla_X Y}T)(Y_1,\cdots,Y_r)\\
=&\nabla_X(\nabla_Y T)(Y_1,\cdots,Y_r)-(\nabla_{\nabla_X Y}T)(Y_1,\cdots,Y_r).
\end{aligned}$$

□

Next we discuss curvature of tensors.

Definition (Curvature operator)**.** Let X,Y be vector fields, define an endomorphism $R(X,Y)$ of $(0,r)$-tensors as

$$R(X,Y)T=\nabla_Y\nabla_X T-\nabla_X\nabla_Y T+\nabla_{[X,Y]}T.$$

Proposition A.7 (Ricci identity)**.** *For a $(0,r)$-tensor T and vector fields X,Y, we have*

$$(\nabla^2_{Y,X}-\nabla^2_{X,Y})T=R(X,Y)T=-\sum_{i=1}^n T(Y_1,\cdots,R(X,Y)Y_i,\cdots,Y_r)$$

Proof. Using equation A.1, we have

$$\begin{aligned}
(\nabla^2_{Y,X}-\nabla^2_{X,Y})T&=\nabla_Y\nabla_X T-\nabla_{\nabla_Y X}T-\nabla_X\nabla_Y T+\nabla_{\nabla_X Y}T\\
&=\nabla_Y\nabla_X T-\nabla_X\nabla_Y T+\nabla_{(\nabla_X Y-\nabla_Y X)}T\\
&=\nabla_Y\nabla_X T-\nabla_X\nabla_Y T+\nabla_{[X,Y]}T \quad \text{(torsion-freeness)}\\
&=R(X,Y)T.
\end{aligned}$$

The latter equality holds since $R(X,Y)$ satisfies Leibniz law, then we have

$$\begin{aligned}(R(X,Y)T)(Y_1,\cdots,Y_r) &= R(X,Y)T(Y_1,\cdots,Y_r) - \sum_{i=1}^{n} T(Y_1,\cdots,R(X,Y)Y_i,\cdots,Y_r)\\ &= -\sum_{i=1}^{n} T(Y_1,\cdots,R(X,Y)Y_i,\cdots,Y_r),\end{aligned}$$

the second equality holds since $T(Y_1,\cdots,Y_r)$ is a function. □

Now we introduce some differential operators.

Definition. Let (M,g) be a Riemannian manifold, $f \in C^\infty(M)$. Define the **Hessian** of f to be a $(0,2)$-tensor $\nabla^2 f$. Ricci identity shows $\nabla^2 f$ is symmetric, hence we define its trace to be the **Laplacian** of f, denoted by Δf.

Proposition A.8. *Let (M,g) be a Riemannian manifold, $f \in C^\infty(M)$. Then we have*

$$\nabla^2 f(X,Y) = Y(Xf) - (\nabla_Y X)(f).$$

Proof. This is just the equation A.1. □

Definition. Let M be a differentiable manifold, $X \in \mathfrak{X}(M)$. Then we define the **divergence** of X to be $\mathrm{tr}(Y \mapsto \nabla_Y X)$.

Theorem A.9 (Divergence theorem)**.** *Let Ω be a region in $\mathbb{R}^n$ with $\Sigma := \partial\Omega$ smooth. Let $X \in \mathfrak{X}(\Omega)$, then we have*

$$\int_\Sigma \langle X,N\rangle\,\mathrm{d}A = \int_\Omega \mathrm{div}\,X\,\mathrm{d}\,\mathrm{Vol},$$

where N is the outward unit normal vector field of Σ.

Please refer to any book on calculus for the proof.

Index