

2024 BICMR Summer School on Differential Geometry

Riemannian Geometry

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Contents

1	Differentiable Manifolds	1
1.1	Differentiable Manifolds and Maps	1
1.2	Tangent Spaces and Tangent Maps	3
1.3	Tangent Bundles and Vector Fields	5
2	Metric and Connection	7
2.1	Riemannian Metric	7
2.2	Metric Structure	9
2.3	Levi–Civita Connection	10
2.4	Covariant Derivative	11
3	Geodesics and Curvature	13
3.1	Geodesics and Exponential Maps	13
3.2	Curvature	15
3.3	Jacobi Fields	17
3.4	Some Local Charts	19
3.5	Conjugate Points	21
4	Variation Formula and Index Form	23
4.1	First Variation Formula	24
4.2	Second Variation Formula	25
4.3	Index Form	26
5	Comparison Theorems	31
5.1	Rauch Comparison Theorem	31

Lecture 1

Differentiable Manifolds

In this lecture, we review some basic notions of differentiable manifolds.

1.1 Differentiable Manifolds and Maps

Definition. Let M^n be a Hausdorff space with countable topological basis. If there exists an open cover $\{U_\alpha\}$ of M , and homeomorphisms $\varphi_\alpha : U_\alpha \rightarrow \varphi_\alpha(U_\alpha)$ onto its image $\varphi_\alpha(U_\alpha) \subset \mathbb{R}^n$ open, such that

- (1) $M = \bigcup_\alpha U_\alpha$,
- (2) if $U_\alpha \cap U_\beta \neq \emptyset$, then $\varphi_\alpha^{-1} \circ \varphi_\beta : \varphi_\beta(U_\alpha \cap U_\beta) \rightarrow \varphi_\alpha(U_\alpha \cap U_\beta)$ is differentiable (we mean C^∞ here),

then M is called an **n -dimensional differentiable manifold**.

Moreover, we call $(U_\alpha, \varphi_\alpha)$ a **local chart**, $\{(U_\alpha, \varphi_\alpha)\}$ an **atlas**, and we say the atlas induces a **differentiable structure** on M .

Remark 1.1. We often assume the atlas is *maximal*, that is, there is no more local chart being compatible with the atlas.

Example 1.2. We illustrate some examples of differentiable manifolds.

- (1) \mathbb{R}^n itself is a differentiable manifolds, with single local chart $(\mathbb{R}^n, \text{id})$.
- (2) $\mathbb{S}^n := \{x \in \mathbb{R}^{n+1} \mid (x^1)^2 + \dots + (x^{n+1})^2 = 1\}$. We use stereographic projection as local chart. Define the stereographic projection from north pole

$$\varphi_N : \mathbb{S}^n \setminus \{(0, \dots, 0, 1)\} \rightarrow \mathbb{R}^n$$

$$x \mapsto \left(\frac{x^1}{1 + x^{n+1}}, \dots, \frac{x^n}{1 + x^{n+1}} \right)$$

Similarly define φ_S to be stereographic projection from the south pole. Then we have

$$\begin{aligned}\varphi_S \circ \varphi_N^{-1} : \mathbb{R} \setminus \{0\} &\rightarrow \mathbb{R} \setminus \{0\} \\ (y^1, \dots, y^n) &\mapsto \left(\frac{y^1}{\sum_i (y^i)^2}, \dots, \frac{y^n}{\sum_i (y^i)^2} \right)\end{aligned}$$

is clearly differentiable.

- (3) Let $M_1^{n_1}, M_2^{n_2}$ be differentiable manifolds, then $M_1 \times M_2$ has the **product manifold** structure. To be precise, let M_1, M_2 have atlas $\{U_\alpha, \varphi_\alpha\}, \{V_\beta, \psi_\beta\}$, then $M_1 \times M_2$ has atlas $(U_\alpha \times V_\beta, \varphi_\alpha \times \psi_\beta)$. In particular, we have

- (flat) n -torus $\mathbb{T}^n = \mathbb{S}^1 \times \dots \times \mathbb{S}^1$ (n times);
- cylinder $\mathbb{S}^1 \times \mathbb{R}$ (or generally $\mathbb{S}^k \times \mathbb{R}^{n-k}$).

- (4) Real projective space \mathbb{RP}^n . Let equivalence relation \sim on $\mathbb{R}^{n+1} \setminus \{0\}$ be $x \sim y \iff x = \lambda y, \lambda \neq 0$. Then define $\mathbb{RP}^n = (\mathbb{R}^{n+1} \setminus \{0\}) / \sim$. We now define the differentiable structure on \mathbb{RP}^n . Let $U_i = \{x \in \mathbb{RP}^n : x = [x^1, \dots, x^{n+1}], x^i \neq 0\}$, and

$$\begin{aligned}\varphi_i : U_i &\rightarrow \mathbb{R}^n \\ x &\mapsto \left(\frac{x^1}{x^i}, \dots, \frac{x^{i-1}}{x^i}, \frac{x^{i+1}}{x^i}, \dots, \frac{x^{n+1}}{x^i} \right)\end{aligned}$$

We check $\varphi_j \circ \varphi_i^{-1}$ on $U_i \cap U_j$. We may assume $i < j$, then

$$\begin{aligned}\varphi_j \circ \varphi_i^{-1}(y^1, \dots, y^n) &= \varphi_j([y^1, \dots, y^{i-1}, 1, y^{i+1}, \dots, y^n]) \\ &= \left(\frac{y^1}{y^j}, \dots, \frac{y^{i-1}}{y^j}, \frac{1}{y^j}, \frac{y^{i+1}}{y^j}, \dots, \frac{y^n}{y^j} \right)\end{aligned}$$

is differentiable.

We now give the definition of differentiable maps.

Definition. A map $f : M \rightarrow N$ is **differentiable** at $p \in M$ if there exists local chart (U, φ) of p and (V, ψ) of $f(p)$, such that $\psi \circ f \circ \varphi^{-1}$ is differentiable at $\varphi(p)$.

Remark 1.3. (1) If $\tilde{\varphi}, \tilde{\psi}$ are another chart at p and $f(p)$, then we have

$$\tilde{\psi} \circ f \circ \tilde{\varphi}^{-1} = (\tilde{\psi} \circ \psi^{-1}) \circ (\psi \circ f \circ \varphi^{-1}) \circ (\varphi \circ \tilde{\varphi})$$

is still differentiable at p by the compatibility of charts, so differentiable maps are well-defined.

(2) When $N = \mathbb{R}$, f is also called a **differentiable function**.

Notation 1.4. We use $C^\infty(M, N)$ to denote the \mathbb{R} -vector space of differentiable maps between M and N , $C^\infty(M)$ to denote the \mathbb{R} -algebra of differentiable functions on M . We use $C_p^\infty(M)$ to denote the \mathbb{R} -algebra of germs of differentiable functions at p . We often use $\gamma: I \subset \mathbb{R} \rightarrow M$ to denote a **differentiable curve** on M .

1.2 Tangent Spaces and Tangent Maps

Definition. Let $\gamma: I \rightarrow M$ be a curve, $\gamma(0) = p$. We define the **tangent vector along γ at p** as a mapping $\dot{\gamma}(0): C_p^\infty(M) \rightarrow \mathbb{R}$, $\dot{\gamma}(0)f = \left. \frac{d}{dt} \right|_{t=0} (f \circ \gamma)(t)$. Then we define the **tangent space at p**

$$T_p M := \{ \dot{\gamma}(0) \mid \gamma: I \rightarrow M \text{ differentiable, } \gamma(0) = p \}.$$

Proposition 1.5. We have the **Leibniz rule** $\dot{\gamma}(0)(fg) = (\dot{\gamma}(0)g)f(p) + (\dot{\gamma}(0)f)g(p)$. So a tangent vector is a derivative on $C_p^\infty(M)$.

We now calculate the local representation of a tangent vector. Fix a chart $\varphi = (x^1, \dots, x^n)$, we have

$$\begin{aligned} \dot{\gamma}(0)f &= \left. \frac{d}{dt} \right|_{t=0} (f \circ \gamma)(t) \\ &= \left. \frac{d}{dt} \right|_{t=0} (f \circ \varphi^{-1}) \circ (\varphi \circ \gamma)(t) \\ &= \sum_{i=1}^n \left. \frac{\partial}{\partial x^i} \right|_{\varphi(p)} (f \circ \varphi^{-1}) \left. \frac{d}{dt} \right|_{t=0} x^i(\gamma(t)) \quad (\text{Chain rule}) \end{aligned} \tag{1.1}$$

Using equation (1.1), we can describe $T_p M$ as a vector space.

Proposition 1.6. $T_p M$ is a real vector space of dimension n . Moreover, given a local chart $\varphi = (x^1, \dots, x^n)$, we have

$$T_p M = \text{Span} \left\{ \left. \frac{\partial}{\partial x^i} \right|_p \right\}$$

where $\left. \partial / \partial x^i \right|_p$ is the tangent vector of $\sigma_i(t) = \varphi^{-1}(\varphi(p) + te_i)$, $e_i = (0, \dots, 1, \dots, 0)$ with only i -th component being 1. Thus we have

$$\left. \frac{\partial}{\partial x^i} \right|_p f = \left. \frac{\partial}{\partial x^i} (f \circ \varphi^{-1}) \right|_{\varphi(p)}$$

Proof. Clearly $T_p M$ has natural vector space structure. Thus by the definition of $\left. \frac{\partial}{\partial x^i} \right|_p$'s, $\text{Span} \left\{ \left. \frac{\partial}{\partial x^i} \right|_p \right\} \subset T_p M$. For the converse inclusion, let $v \in T_p M$, then there is a curve $\gamma : I \rightarrow M$ with $\dot{\gamma}(0) = v$. Then by (1.1), $\dot{\gamma}(0)$ is a linear combination of $\left. \frac{\partial}{\partial x^i} \right|_p$'s, hence $T_p M \subset \text{Span} \left\{ \left. \frac{\partial}{\partial x^i} \right|_p \right\}$. \square

Definition (Tangent maps). Let $f : M \rightarrow N$ be a differentiable map, we define $f_{*p} : T_p M \rightarrow T_{f(p)} N$ as

$$f_{*p}(v)(g) = v(g \circ f)$$

for any $g \in C_{f(p)}^\infty N$. In particular, if $N = \mathbb{R}$, given $v \in T_p M$, let $\dot{\gamma}(0) = v$, then $f_{*p}(v) = \left. \frac{d}{dt} \right|_{t=0} (f \circ \gamma)(t)$.

Again we can look at the local representation of f_{*p} . Let $\phi = (x^1, \dots, x^n)$, $\psi = (y^1, \dots, y^m)$ be local charts containing p and $f(p)$. Let $v = \sum_{i=1}^n v^i \left. \frac{\partial}{\partial x^i} \right|_p = \dot{\sigma}(0)$, then $\left. \frac{d}{dt} \right|_{t=0} (\phi \circ \sigma)(t) = (v^1, \dots, v^n)$. Thus we have

$$\begin{aligned} f_{*p}(v)(g) &= \left. \frac{d}{dt} \right|_{t=0} (g \circ f \circ \sigma)(t) \\ &= \sum_{i,j} \left. \frac{\partial}{\partial y^j} (g \circ \psi^{-1}) \right|_{\psi \circ f(p)} \left. \frac{\partial}{\partial x^i} (\psi \circ f \circ \phi^{-1})^j \right|_{\phi(p)} \left. \frac{d}{dt} \right|_{t=0} (\phi \circ \sigma)^i(t) \\ &= \sum_{i,j} v^i \left. \frac{\partial}{\partial x^i} (\psi \circ f \circ \phi^{-1})^j \right|_{\phi(p)} \left. \frac{\partial}{\partial y^j} \right|_{f(p)} g \end{aligned}$$

In particular, we have

$$f_{*p} \left(\left. \frac{\partial}{\partial x^i} \right|_p \right) = \sum_j \left. \frac{\partial}{\partial x^i} (\psi \circ f \circ \phi^{-1})^j \right|_{\phi(p)} \left. \frac{\partial}{\partial y^j} \right|_{f(p)}$$

We can easily verify the following chain rule:

Proposition 1.7. Let $f : M \rightarrow N$, $g : N \rightarrow P$ be differentiable maps, then we have $(g \circ f)_{*p} = g_{*f(p)} \circ f_{*p}$.

Definition (Diffeomorphism). A map $f : M \rightarrow N$ is called a **diffeomorphism** if f is bijective, and f, f^{-1} are both differentiable.

Proposition 1.8. If $f : M \rightarrow N$ is a diffeomorphism, then $f_{*p} : T_p M \rightarrow T_{f(p)} N$ is an isomorphism.

This proposition can be easily proved by chain rule.

Remark 1.9. (1) The Proposition 1.8 shows that dimension of a manifold is well-defined in the category (differentiable manifolds, differentiable maps).

(2) Since we can do calculus locally on manifolds, the *Inverse function theorem* is valid on differentiable manifolds. That is, if $f_{*p} : T_p M \rightarrow T_{f(p)} N$ is an isomorphism, then $f : M \rightarrow N$ is a local diffeomorphism at p .

1.3 Tangent Bundles and Vector Fields

Definition (Tangent bundle). Assume differentiable manifold M^n has atlas $\{U_\alpha, \varphi_\alpha\}$, define

$$TM := \bigsqcup_{p \in M} T_p M$$

$$\pi : TM \rightarrow M, (p, v) \mapsto p$$

We give an atlas of TM to make it into a $2n$ -dimensional differentiable manifold. Let

$$\Phi_\alpha : \bigsqcup_{p \in U_\alpha} T_p M \rightarrow \mathbb{R}^{2n}$$

$$(p, v) \mapsto (\varphi_\alpha(p), (v^1, \dots, v^n))$$

where $v = \sum_{i=1}^n v^i \frac{\partial}{\partial x^i} \Big|_p$. Let's check

$$\Phi_\beta \circ \Phi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \times \mathbb{R}^n \rightarrow \varphi_\beta(U_\alpha \cap U_\beta) \times \mathbb{R}^n$$

$$(x^1, \dots, x^n, v^1, \dots, v^n) \mapsto \left(\varphi_\beta \circ \varphi_\alpha^{-1}(x), \sum_{i=1}^n \frac{\partial(\varphi_\beta \circ \varphi_\alpha^{-1})^1}{\partial x^i} v^i, \dots, \sum_{i=1}^n \frac{\partial(\varphi_\beta \circ \varphi_\alpha^{-1})^n}{\partial x^i} v^i \right)$$

Clearly it is differentiable, then $\{(\pi^{-1}(U_\alpha), \Phi_\alpha)\}$ induces a differentiable structure on TM .

We call $T_p M$ a **fiber** over p , and $\pi : TM \rightarrow M$ the projection.

Definition (Vector field). A **vector field** is a differentiable map $X : M \rightarrow TM$ such that $X(p) \in T_p M$.

Notation 1.10. We use $\mathfrak{X}(M)$ to denote the collection of vector fields on M .

Proposition 1.11. $X \in \mathfrak{X}(M^n)$ if and only if in any local chart (U, φ) , we have $X(p) = \sum_{i=1}^n X^i(p) \frac{\partial}{\partial x^i} \Big|_p$ for $X^i \in C^\infty(U)$, $i = 1, 2, \dots, n$.

This proposition is equivalent to X can be a mapping $C^\infty(M) \rightarrow C^\infty(M)$ defined by $Xf(p) = X(p)f$.

Definition (Lie bracket). For $X, Y \in \mathfrak{X}(M)$, define $[X, Y] = XY - YX$, then $[X, Y] \in \mathfrak{X}(M)$.

Remark 1.12. We explain the definition more explicitly. If we act two vector fields on the product of two functions, we have

$$\begin{aligned} (XY)_p(fg) &= X_p(Y(fg)) = X_p(gYf + fYg) \\ &= \boxed{X_p g \cdot Y_p f + X_p f \cdot Y_p g} + g(p)X_p Yf + f(p)X_p Yg \end{aligned}$$

The boxed thing is bad, it spoils Leibniz rule. But if we subtract $YX_p(fg)$, the boxed thing is cancelled. So $XY - YX \in \mathfrak{X}(M)$.

Proposition 1.13. *On some local chart, we have $\left[\frac{\partial}{\partial x^i} \Big|_p, \frac{\partial}{\partial x^j} \Big|_p \right] = 0$.*

Proof. This is equivalent to mixed partial derivative is commutative for smooth functions in \mathbb{R}^n . □

Lecture 2

Metric and Connection

In this lecture we introduce the Riemannian metric on a differentiable manifold, and the connection compatible with metric, i.e., Levi-Civita connection. Moreover, we introduce the covariant derivative of a vector field along a curve, and parallel transport of vectors along a curve.

Notation 2.1. From now on we adopt *Einstein summation convention*: any index appear twice as both upper index and lower index means taking summation respective to the index. For example, a vector field on a local chart can be expressed as

$$X = X^i \frac{\partial}{\partial x^i} = \sum_{i=1}^n X^i \frac{\partial}{\partial x^i}$$

2.1 Riemannian Metric

Definition. Let M^n be a differentiable manifold. A **Riemannian metric** on M is a smooth assignment on each $T_p M$, $\forall p \in M$, a symmetric positive definite bilinear form g_p , that is for $X, Y \in T_p M$ we have

1. $g_p(X, Y) = g_p(Y, X)$;
2. $g_p(X, X) \geq 0$, $g_p(X, X) = 0 \iff X = 0$.

“Smooth” means in any local chart (U, φ) we have

$$g_{ij}(p) = g \left(\left. \frac{\partial}{\partial x^i} \right|_p, \left. \frac{\partial}{\partial x^j} \right|_p \right)$$

is smooth about p for any indices i, j . Then g is a symmetric positive definite $(0, 2)$ -tensor

$$g = g_{ij} dx^i \otimes dx^j$$

Proposition 2.2. Any differentiable manifold M admits a Riemannian metric.

Proof. We use partition of unity. Let $\{U_\alpha, x_\alpha^i\}$ be a locally finite atlas of M , $\{\phi_\alpha\}$ be a partition of unity subordinate to $\{U_\alpha\}$, i.e., $\text{supp } \phi_\alpha \subset \subset U_\alpha$ and $\sum_\alpha \phi_\alpha = 1$. On the local chart (U_α, x_α^i) , let $g_\alpha = \sum_{i=1}^n dx_\alpha^i \otimes dx_\alpha^i$. Set

$$g = \sum_\alpha \phi_\alpha g_\alpha,$$

we can check g is indeed a Riemannian metric on M . □

Remark 2.3. For an n -form $\omega \in \wedge^n M$ with $\text{supp } \omega$ compact, we can define its integral as

$$\int_M \omega = \sum_\alpha \int_M \phi_\alpha \omega$$

One can check the definition is independent from the choice of partition of unity.

- Example 2.4.** (1) \mathbb{R}^n has the Euclidean metric $g = \sum_{i=1}^n dx^i \otimes dx^i$, $g_{ij} = \delta_{ij}$.
- (2) Let $f : M \rightarrow (N, h)$ be an immersion, then we define $f^*h(X, Y)|_p = h(f_*pX, f_*pY)$, f^*h is the induced metric from h by immersion.
- (3) Let $f : (M, g) \rightarrow (N, h)$ be an immersion, if $g = f^*h$, then f is called a **local isometry**. If $f : M \rightarrow N$ is a diffeomorphism, then f is called an **isometry**.
- (4) The standard metric on \mathbb{S}^n is the induced metric by the embedding $i : \mathbb{S}^n \hookrightarrow (\mathbb{R}^{n+1}, \delta_{ij})$.
- (5) If $(M_1, g_1), (M_2, g_2)$ are Riemannian manifolds, then their product manifold has product metric $g_1 \times g_2$. In particular, equip \mathbb{S}^1 with standard metric g , $(\mathbb{T}^n, g^n) = (\mathbb{S}^1 \times \cdots \times \mathbb{S}^1, g \times \cdots \times g)$ is the flat torus (we will explain the word “flat” later).
- (6) Let $f : M \rightarrow (N, g)$ be a covering map, then f^*g is a Riemannian metric on M , called Riemannian covering map. In particular, let $\text{Isom}(M) := \{f : M \rightarrow M \mid f \text{ is isometry}\}$ denote the isometry group of M , $\Gamma \subset \text{Isom}(M)$ be a subgroup, then M/Γ is a manifold, and $f : M \rightarrow M/\Gamma$ is a Riemannian covering map. Examples of this manner are

- $\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$;
- $\mathbb{RP}^n = \mathbb{S}^n / \{\text{id}, A\}$, where A is the antipodal map.

2.2 Metric Structure

Definition. Let $\gamma: [0, 1] \rightarrow M$ be a curve, define its **length** to be

$$\begin{aligned} L(\gamma) &:= \int_0^1 |\dot{\gamma}(t)| \, dt \\ &= \int_0^1 \sqrt{g(\gamma(t))(\dot{\gamma}(t), \dot{\gamma}(t))} \, dt \end{aligned}$$

Let $p, q \in M$, define their **distance** to be

$$d(p, q) = \inf_{\gamma \in C_{p,q}} L(\gamma)$$

where $C_{p,q}$ denotes the collection of all smooth curve joining p and q .

Proposition 2.5. *The distance function $d: M \times M \rightarrow \mathbb{R}$ has the following properties:*

- (1) $d(p, q) \geq 0$, and $d(p, q) = 0 \iff p = q$;
- (2) $d(p, q) = d(q, p)$;
- (3) $d(p, r) \leq d(r, q) + d(p, q)$.

Thus the distance function makes M into a metric space.

Proof. Only need to show $d(p, q) = 0 \iff p = q$, all else are trivial. We assume $p \neq q$, need to show $d(p, q) > 0$. Let $\gamma: [0, 1] \rightarrow M$ be any curve joining p and q . Choose a local chart (U, φ) such that $\varphi(U) = B_r(0)$, $q \notin U$. By Jordan–Brouwer Separation Theorem, γ must intersect ∂U at $s := \gamma(c)$. Then we have

$$L(\gamma) \geq L(\gamma|_{[0,c]}) = \int_0^c \sqrt{g_{ij} \dot{x}^i(\gamma(t)) \dot{x}^j(\gamma(t))} \, dt$$

Regarding $g: \bar{U} \times \mathbb{S}^{n-1} \rightarrow \mathbb{R}$, g is a continuous function on a compact set, thus it attains its minimum $g(x)(v, v) \geq m$, and $m > 0$ since $v \in \mathbb{S}^{n-1} \neq 0$. Thus we have

$$L(\gamma|_{[0,c]}) \geq m \int_0^c |\dot{x}(\gamma(t))| \, dt \geq mr > 0$$

mr does not depend on γ , hence $d(p, q) \geq mr > 0$. □

Proposition 2.6. *(M, d) with metric topology coincides with its original topology.*

For a proof, we refer to John Lee's *Introduction to Smooth Manifolds*, 2nd ed., Theorem 13.29.

2.3 Levi–Civita Connection

Definition. (Affine connection) Let M be a differentiable manifold, if the operator $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)$, denoting $\nabla_X Y$, satisfies

- (1) $\nabla_{(fX+gY)}Z = f\nabla_X Z + g\nabla_Y Z$ for $f, g \in C^\infty(M)$,
- (2) $\nabla_X(Y+Z) = \nabla_X Y + \nabla_X Z$,
- (3) $\nabla_X(fY) = (Xf)Y + f\nabla_X Y$,

then ∇ is called an **affine connection** on M .

Definition. An affine connection ∇ on Riemannian manifold (M, g) is called **Levi–Civita connection** if it satisfies

$$(LC1) \quad Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z),$$

$$(LC2) \quad \nabla_X Y - \nabla_Y X - [X, Y] = 0.$$

Proposition 2.7. *On a Riemannian manifold (M, g) there exists a unique Levi–Civita connection.*

Proof. We have the Koszul formula:

$$\begin{aligned} 2g(\nabla_X Y, Z) = & Xg(Y, Z) + Yg(X, Z) - Zg(X, Y) \\ & + g([X, Y], Z) - g([X, Z], Y) - g([Y, Z], X) \end{aligned}$$

The formula shows Levi–Civita connection is unique, and can be used as the definition of Levi–Civita connection. \square

We check Levi–Civita connection locally. First we introduce the Christoffel symbols:

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma_{ij}^k \frac{\partial}{\partial x^k}$$

Then (LC1) is equivalent to

$$\frac{\partial g_{ij}}{\partial x^k} = \Gamma_{ki}^l g_{lj} + \Gamma_{kj}^l g_{li}$$

(LC2) is equivalent to

$$\Gamma_{ij}^k = \Gamma_{ji}^k$$

Using (LC1) and (LC2), we obtain

Proposition 2.8. *We have the expression of Γ_{ij}^k :*

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl} \left(\frac{\partial g_{il}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l} \right)$$

Let vector fields $X = X^i \frac{\partial}{\partial x^i}$, $Y = Y^j \frac{\partial}{\partial x^j}$, then we have

$$\nabla_X Y = X^i \left(\frac{\partial Y^k}{\partial x^i} + Y^j \Gamma_{ij}^k \right) \frac{\partial}{\partial x^k} \quad (2.1)$$

This shows that $\nabla_X Y(p)$ depends only on $X(p)$ and the value of Y along the curve $\gamma(t)$ with $\gamma(0) = p, \dot{\gamma}(0) = X(p)$. Using this, we can introduce the covariant derivative of vector fields along curves.

2.4 Covariant Derivative

Definition. Let $\gamma: [0, 1] \rightarrow M$ be a curve, Y is a vector field along γ . Then define

$$\frac{\nabla}{dt} Y := \nabla_{\dot{\gamma}(t)} Y$$

We look at covariant derivative locally. Choose a local chart (U, ϕ) , let $Y = Y^i \frac{\partial}{\partial x^i}$, $\dot{\gamma}(t) = \dot{\gamma}^j(t) \frac{\partial}{\partial x^j}$, then by equation (2.1), we have

$$\begin{aligned} \frac{\nabla}{dt} Y &= \dot{\gamma}^j(t) \left(\frac{\partial Y^k(t)}{\partial x^j} + Y^i(t) \Gamma_{ij}^k(\gamma(t)) \right) \frac{\partial}{\partial x^k} \\ &= \left(\dot{Y}^k(t) + Y^j(t) \dot{\gamma}^i(t) \Gamma_{ij}^k(\gamma(t)) \right) \frac{\partial}{\partial x^k} \quad (\text{chain rule}) \end{aligned} \quad (2.2)$$

Definition (Parallel transport). Let $\gamma: [0, 1] \rightarrow M$ be a curve, Y is a vector field along γ . If $\nabla Y/dt = 0$, then we call Y is **parallel** along γ .

Proposition 2.9. Given a curve $\gamma: [0, 1] \rightarrow M$ and an initial vector $Y_{\gamma(0)} \in T_{\gamma(0)}M$, then there exists a unique parallel vector field along γ with initial vector $Y_{\gamma(0)}$.

Proof. Parallel transport satisfies (2.2), and it is a second order ordinary differential equation. By the unique existence theorem of solution of ODEs, the proposition is proved. \square

Definition. Let $\gamma: [0, 1] \rightarrow M$ be a curve, $\gamma(0) = p, \gamma(1) = q$, we define a mapping $P_\gamma: T_p M \rightarrow T_q M$ as follows: Let $Y_0 \in T_p M$, there exists a unique parallel vector field Y along γ with $Y(0) = Y_0$, then we define $P_\gamma(Y_0) = Y(1)$. Clearly P_γ is linear.

Proposition 2.10. P_γ is an isometry, hence an isomorphism.

Proof. Using notation above, let $X_0, Y_0 \in T_p M$, X, Y are parallel vector field along γ with $X(0) = X_0, Y(0) = Y_0$. Then we have

$$\frac{d}{dt} g(X, Y) = g(\nabla_{\dot{\gamma}(t)} X, Y) + g(X, \nabla_{\dot{\gamma}(t)} Y) = 0,$$

since X, Y are parallel. Thus $g(X, Y)$ is constant, we have $g(X_0, Y_0) = g(P_\gamma(X_0), P_\gamma(Y_0))$. \square

The last proposition reveals the meaning of the word “connection”, it means ∇ “connects” different tangent spaces.

Proposition 2.11. *Let $\gamma: [0, 1] \rightarrow M$ be a curve with $\gamma(0) = p$, X, Y be vector fields with $X(p) = \dot{\gamma}(0)$. Then $\nabla_X Y(p) = \left. \frac{d}{dt} \right|_{t=0} P_\gamma^{-1}(Y(\gamma(t)))$.*

Proof. Let $\{e_1(t), \dots, e_n(t)\}$ be a parallel frame along γ , then $\nabla_{\dot{\gamma}(t)} e_i(t) = 0$, in particular, $\nabla_{\dot{\gamma}(0)} e_i(0) = 0$. Let $Y(\gamma(t)) = Y^i(\gamma(t))e_i(t)$, thus

$$\begin{aligned} \nabla_X Y(p) &= \nabla_{\dot{\gamma}(0)} Y^i(\gamma(0))e_i(0) \\ &= \dot{\gamma}(0)Y^i(\gamma(0))e_i(0) + Y^i(0)\nabla_{\dot{\gamma}(0)} e_i(0) \\ &= \left(\left. \frac{d}{dt} \right|_{t=0} Y^i(\gamma(t)) \right) e_i \end{aligned}$$

On the other hand, parallel transport gives

$$P_\gamma^{-1}(Y(\gamma(t))) = Y^i(\gamma(t))e_i(0),$$

hence

$$\begin{aligned} \left. \frac{d}{dt} \right|_{t=0} P_\gamma^{-1}(Y(\gamma(t))) &= \left(\left. \frac{d}{dt} \right|_{t=0} Y^i(\gamma(t)) \right) e_i \\ &= \nabla_X Y(p) \end{aligned} \quad \square$$

Remark 2.12. The symbol $\frac{\nabla}{dt}$ is rarely used in literature, so we will simply use $\nabla_{\dot{\gamma}(t)} X$ to denote covariant derivative.

Lecture 3

Geodesics and Curvature

In this lecture, we first introduce the concepts of geodesics and exponential maps. By differentiating exponential map, we can introduce the concept of curvature and Jacobi fields. Using exponential map, we can also introduce geodesic normal coordinate and geodesic polar coordinate. As an application, we will use geodesic polar coordinate to show geodesics are locally length-minimizing. Finally, we will introduce the notion of conjugate points.

3.1 Geodesics and Exponential Maps

Definition. A curve $\gamma: [0, 1] \rightarrow M$ is called a **geodesic** if $\nabla_{\dot{\gamma}(t)} \dot{\gamma}(t) = 0$.

Remark 3.1. Geodesics are constant speed, i.e., $|\dot{\gamma}(t)| \equiv \text{const}$. This can be shown by $\frac{d}{dt} \langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle = 2 \langle \nabla_{\dot{\gamma}(t)} \dot{\gamma}(t), \dot{\gamma}(t) \rangle = 0$.

In a local chart (U, φ) , let $\varphi \circ \gamma(t) = (x^1(t), \dots, x^n(t))$, then $\dot{\gamma}(t) = \dot{x}^i(t) \frac{\partial}{\partial x^i} \Big|_{\gamma(t)}$. Thus the geodesic equation is

$$\nabla_{\dot{\gamma}(t)} \dot{\gamma}(t) = 0 \iff \ddot{x}^k(t) + \Gamma_{ij}^k(\gamma(t)) \dot{x}^i(t) \dot{x}^j(t) = 0, \quad k = 1, \dots, n,$$

with $\gamma(0) = p, \dot{\gamma}(0) = v \in T_p M$.

Since the solution of an ODE relies continuously on initial value, we have the following proposition.

Proposition 3.2. *For any $p \in M$, there exists a neighborhood V of p , such that there exists $\delta > 0, \varepsilon > 0$ and a differentiable map $\gamma: (-\delta, \delta) \times \mathcal{U} \rightarrow M$, where $\mathcal{U} = \{(q, v) \in TV \mid q \in V, v \in T_p M, |v| < \varepsilon\}$, such that $\gamma(t; q, v)$ is a geodesic with $\gamma(0) = q, \dot{\gamma}(0) = v$.*

The idea of the proof is the ODE theorem we mentioned above, but the proof is not simply using only ODE theory. A proof of an equivalent proposition can be found in Wu Hung–Hsi, et. al.'s *Introduction to Riemannian Geometry* Chapter 3, Lemma 1.

Observe that $\gamma(\lambda t; p, v) = \gamma(t; p, \lambda v)$. Denote $\gamma(t; p, v) = \gamma_v(t)$, then above observation can be written as $\gamma_{\lambda v}(t) = \gamma_v(\lambda t)$. Therefore, we can shorten the initial vector to lengthen the domain of geodesic.

Definition (Exponential map). Let $U \subset T_p M$ be a neighborhood of origin, such that for any $v \in U$, $\gamma_v(1)$ is defined (such neighborhood exists by Proposition 3.2). We define the **exponential map** at p to be

$$\begin{aligned} \exp_p : U &\rightarrow M \\ v &\mapsto \gamma_v(1) \end{aligned}$$

Remark 3.3. We scale the initial vector and can obtain

$$\exp_p(v) = \gamma_v(1) = \gamma_{v/|v|}(|v|)$$

This means the exponential map act on v is moving forward distance $|v|$ along the geodesic with initial direction $v/|v|$.

Proposition 3.4. $\exp_{p*}|_0 : T_0(T_p M) \rightarrow T_p M$ is identity (we identify $T_0(T_p M)$ with $T_p M$).

Proof. We have

$$\exp_{p*}|_0(v) = \left. \frac{d}{dt} \right|_{t=0} \exp_p(tv) = v. \quad \square$$

Corollary 3.5. There is a ball $B_\epsilon(0) \subset T_p M$ such that $\exp_p : B_\epsilon(0) \rightarrow M$ is a diffeomorphism onto its image.

Proof. Since $\exp_{p*}|_0$ is identity, it is nondegenerate, the corollary follows by Inverse Function Theorem. \square

Example 3.6. (1) We know that the geodesics on \mathbb{S}^n are great circles, hence \exp_p is defined on the whole $T_p M$. But \exp_p is not injective, since $\exp_p(0) = \exp_p(2\pi v) = p$ for unit vector v in $T_p M$.

(2) Let $M = \mathbb{S}^1 \times \mathbb{R}$ be the cylinder. We know from elementary differential geometry that the geodesics on cylinder are directrix circles, helices and generatrix lines. Then in local chart $(e^{2\pi i t}, s) \mapsto (t, s)$, we know the \exp_p is not injective in the direction $(1, 0)$, and injective in other directions.

Definition. If \exp_p can be defined on whole $T_p M$ for any $p \in M$, we say M is **geodesically complete**.

We have the following important theorem.

Theorem 3.7 (Hopf–Rinow). *Let M be a Riemannian manifold, the following are equivalent:*

- (1) M is geodesically complete.
- (2) $\exp_p : T_p M \rightarrow M$ is well-defined for some $p \in M$.
- (3) The Heine–Borel property holds, that is, any closed bounded set is compact on M .
- (4) M is complete as a metric space, that is, any Cauchy sequence converges.

We will not prove Hopf–Rinow theorem here. For a proof, one can refer to Peter Petersen's *Riemannian Geometry*, 3rd ed., Theorem 5.7.1.

3.2 Curvature

We know $\exp_{p*}|_0$ is identity, and we want to ask:

Question. What is $\exp_{p*}|_v : T_v(T_p M) \rightarrow T_{\exp_p(v)} M$?

To calculate $\exp_{p*}|_v(\xi)$, we choose a line $v + s\xi$, and then

$$\exp_{p*}|_v(\xi) = \left. \frac{d}{ds} \right|_{s=0} \exp_p(v + s\xi)$$

Now we can introduce a family of geodesics $\gamma(t, s) = \gamma_s(t) = \exp_p(t(v + s\xi))$, and denote $\gamma(t) = \gamma(t, 0)$. Let $J_s(t) = \frac{\partial}{\partial s} \gamma(t, s)$, then $J_s(t) = \nabla_{\dot{\gamma}_s(t)} \frac{\partial \gamma}{\partial s}$. Since $\nabla_{\dot{\gamma}_s(t)} \frac{\partial \gamma}{\partial t} = 0$, we have

$$\begin{aligned} \dot{J}_s(t) &= \nabla_{\dot{\gamma}_s(t)} \nabla_{\dot{\gamma}_s(t)} \frac{\partial \gamma}{\partial s} \\ &= \nabla_{\dot{\gamma}_s(t)} \nabla_{\frac{\partial \gamma}{\partial s}} \frac{\partial \gamma}{\partial t} \quad (\text{torsion-freeness}) \\ &= \nabla_{\frac{\partial \gamma}{\partial t}} \nabla_{\frac{\partial \gamma}{\partial s}} \frac{\partial \gamma}{\partial t} - \nabla_{\frac{\partial \gamma}{\partial s}} \nabla_{\frac{\partial \gamma}{\partial t}} \frac{\partial \gamma}{\partial t}. \end{aligned}$$

Denote

$$R\left(\frac{\partial \gamma}{\partial t}, \frac{\partial \gamma}{\partial s}\right) = \nabla_{\frac{\partial \gamma}{\partial s}} \nabla_{\frac{\partial \gamma}{\partial t}} \frac{\partial \gamma}{\partial t} - \nabla_{\frac{\partial \gamma}{\partial t}} \nabla_{\frac{\partial \gamma}{\partial s}} \frac{\partial \gamma}{\partial t} + \nabla\left[\frac{\partial \gamma}{\partial t}, \frac{\partial \gamma}{\partial s}\right],$$

then we have

$$\frac{\partial^2}{\partial t^2} J_s(t) + R\left(\frac{\partial \gamma}{\partial t}, \frac{\partial \gamma}{\partial s}\right) \frac{\partial \gamma}{\partial t} = 0.$$

Let $s = 0$, we have

$$\ddot{J}(t) + R(\dot{\gamma}(t), J(t))\dot{\gamma}(t) = 0.$$

We make it into a definition.

Definition (Riemann curvature tensor). Let $R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)$ defined by

$$R(X, Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X, Y]} Z.$$

We also look at Riemann curvature tensor locally. Tedious calculation will show Riemann curvature tensor is truly tensorial. Using a more simple notation $\partial_i = \partial/\partial x^i$, we have

$$\begin{aligned} R(\partial_i, \partial_j) \partial_k &= \nabla_{\partial_j} \nabla_{\partial_i} \partial_k - \nabla_{\partial_i} \nabla_{\partial_j} \partial_k \\ &= \nabla_{\partial_j} (\Gamma_{ik}^l \partial_l) - \nabla_{\partial_i} (\Gamma_{jk}^l \partial_l), \end{aligned}$$

and

$$\begin{aligned} R_{ijk}^l &= (\partial_j \Gamma_{ik}^l - \partial_i \Gamma_{jk}^l + \Gamma_{ik}^m \Gamma_{jm}^l - \Gamma_{jk}^m \Gamma_{im}^l) \partial_l \\ &= \partial^2 g + \partial g * \partial g \end{aligned}$$

We also define $R_{ijkl} = R_{ijk}^m g_{ml}$, or $R(X, Y, Z, W) = g(R(X, Y)Z, W)$.

Example 3.8. (\mathbb{R}, δ) has $R \equiv 0$. Any metric admits zero curvature is call **flat**.

Proposition 3.9. *Riemann curvature tensor has following symmetric properties: For $X, Y, Z, W \in \mathfrak{X}(M)$, we have*

- (1) $R(X, Y, Z, W) = -R(Y, X, Z, W) = -R(X, Y, W, Z) = R(Z, W, X, Y)$;
- (2) $R(X, Y, Z, W) + R(Y, Z, X, W) + R(Z, X, Y, W) = 0$ (First Bianchi Identity).

Proof. Tedious calculation. □

Definition (Sectional curvature). Let $p \in M$, $\pi \subset T_p M$ be a 2-plane, $\pi = \text{Span}\{X, Y\}$. Then define the sectional curvature at p of π as

$$K_p(\pi) := \frac{R(X, Y, X, Y)}{|X|^2 |Y|^2 - \langle X, Y \rangle^2}.$$

Remark 3.10. One can show sectional curvature does not depend on the choice of basis. A proof can be found in Manfredo do Carmo's *Riemannian Geometry*, Proposition 3.1.

Proposition 3.11. *Let M be a Riemannian manifold. $R(X, Y, Z, W)$ as a $(0, 4)$ -tensor is determined by all $K_p(\pi)$.*

For a proof, see Wu Hung-Hsi et. al., *Introduction to Riemannian Geometry*, Chapter 2 Lemma 2.

We mention one little observation. If all sectional curvature at p is constant K_p , then

$$R_p(X, Y, Z, W) = K_p(\langle X, Z \rangle \langle Y, W \rangle - \langle X, W \rangle \langle Y, Z \rangle)$$

For a theorem about constant sectional curvature, we mention here Schur's Theorem.

Theorem 3.12 (Schur). *Let (M^n, g) be a Riemannian manifold, $n \geq 3$. If $K_p(\pi)$ is independent of $\pi \subset T_p M$ for any $p \in M$, then M has constant sectional curvature.*

We define another two important curvature.

Definition (Ricci curvature). Let (M^n, g) be a Riemannian manifold. Define **Ricci curvature tensor** $\text{Ric} : \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)$ as

$$\text{Ric}_p(X) = \sum_{i=1}^n R_p(e_i, X)e_i,$$

where $\{e_i\}$ is an orthonormal frame around p . It's easy to check the definition is independent from the choice of orthonormal frame, and Ric is self-adjoint.

Definition (Scalar curvature). Let (M^n, g) be a Riemannian manifold. We define **Scalar curvature** $\text{Scal} \in C^\infty(M)$ as

$$\text{Scal}(p) = \sum_{i=1}^n \langle \text{Ric}_p(e_i), e_i(p) \rangle$$

for an orthonormal frame $\{e_i\}$ around p .

Definition. Let (M, g) be a Riemannian manifold, if $\text{Ric} = \lambda(p)g$ for a $\lambda \in C^\infty(M)$, we call M an **Einstein manifold**.

We also mention here

Theorem 3.13 (Schur). *Let M^n be an Einstein manifold with $n \geq 3$, then M has constant scalar curvature.*

3.3 Jacobi Fields

Definition. Let γ be a geodesic, a vector field J along γ is called a **Jacobi field** if $\ddot{J} + R(\dot{\gamma}, J)\dot{\gamma} = 0$.

Let $\{e_i(t)\}$ be a parallel orthonormal frame along γ , $J(t) = \sum_i f_i(t)e_i(t)$. Define $a_{ij}(t) = \langle R(\dot{\gamma}(t), e_i(t))\dot{\gamma}(t), e_j(t) \rangle$, then the equation for Jacobi field is equivalent to

$$\ddot{f}_i(t) + \sum_j a_{ij}(t)f_j(t) = 0, \quad i = 1, 2, \dots, n.$$

By ODE theory, given $f_i(0), \dot{f}_i(0)$, $i = 1, 2, \dots, n$, the $f_i(t)$'s are uniquely determined. Translating into language of Jacobi field, we have a Jacobi field is uniquely determined by $J(0)$ and $\dot{J}(0)$.

Notation 3.14. Let γ be a geodesic on a Riemannian manifold M , the vector space of Jacobi fields along γ is denoted by $\mathcal{J}(\gamma)$.

Above discussion can be summarized as the following proposition.

Proposition 3.15. *Let M^n be a Riemannian manifold, γ be a geodesic, then $\dim \mathcal{J}(\gamma) = 2n$.*

The next proposition shows only the Jacobi field at tangential direction are interesting.

Proposition 3.16. *Let J be a Jacobi field along γ , we have the decomposition $J(t) = J^\perp(t) + (at + b)\dot{\gamma}(t)$, where $J^\perp(t) \perp \dot{\gamma}(t)$.*

Proof. We have

$$\begin{aligned} \frac{d^2}{dt^2} \langle J(t), \dot{\gamma}(t) \rangle &= \langle \ddot{J}(t), \dot{\gamma}(t) \rangle \\ &= -\langle R(\dot{\gamma}, J)\dot{\gamma}, \dot{\gamma} \rangle \\ &= 0. \end{aligned} \quad \square$$

We have the following Gauss Lemma.

Proposition 3.17 (Gauss Lemma). $\langle \exp_{p*}|_v(\xi), \dot{\gamma}_v(1) \rangle = \langle \gamma, v \rangle$.

Proof. Let one-parameter geodesic family $\gamma(t, s) = \exp_p(t(v + s\xi))$, then using the calculation in the beginning of section 3.2, we have

$$\begin{aligned} J(t) &= \left. \frac{\partial}{\partial s} \right|_{s=0} \gamma(t, s) = \exp_{p*}|_{tv}(t\xi) \\ &= t \exp_{p*}|_{tv}(\xi). \end{aligned}$$

Moreover, we have $\langle J(t), \dot{\gamma}_v(t) \rangle = at + b$ (Proposition 3.16),

$$\begin{aligned} a &= \left. \frac{d}{dt} \right|_{t=0} \langle J(t), \dot{\gamma}_v(t) \rangle = \langle J(0), \dot{\gamma}_v(0) \rangle + \langle J(0), \nabla_v \dot{\gamma}_v(t) \rangle \\ &= \langle J(0), v \rangle = \left\langle \left(\exp_{p*}|_{tv}(\xi) + t \frac{d}{dt} \exp_{p*}|_{tv}(\xi) \right) \right|_{t=0}, v \rangle \\ &= \langle \exp_{p*}|_0(\xi), v \rangle = \langle \xi, v \rangle, \end{aligned}$$

and $b = \langle J(0), v \rangle = 0$. But we have

$$t \langle \exp_{p*}|_{tv}(\xi), \dot{\gamma}_v(t) \rangle = \langle J(t), \dot{\gamma}_v(t) \rangle = \langle \xi, v \rangle t.$$

Let $t = 1$ we obtain the conclusion. \square

Example 3.18. We calculate the Jacobi field on Riemannian manifolds admit a constant sectional curvature. By scaling the metric, we can assume $K = 0, 1, -1$. The corresponding simply connected complete Riemannian manifolds are called **space forms**, which are $\mathbb{R}^n, \mathbb{S}^n, \mathbb{H}^n$. The Jacobi field equation is

$$\begin{cases} \ddot{J}(t) + KJ(t) = 0, \\ J(0) = 0, \dot{J}(0) = \xi. \end{cases}$$

Let $\xi(t)$ be the parallel transport along a geodesic with $\xi(0) = \xi$, then solving the equation by eigenvalue method, we obtain

$$J(t) = \begin{cases} t\xi(t), & K = 0, \\ \sin(t)\xi(t), & K = 1, \\ \sinh(t)\xi(t), & K = -1. \end{cases}$$

3.4 Some Local Charts

In this section, we adopt the traditional terminology “coordinate” to mean chart.

First we introduce the geodesic normal coordinate. Given a Riemannian manifold (M, g) and $p \in M$, there exists an $\varepsilon > 0$ such that $\exp_p : B_\varepsilon(0) \rightarrow \exp_p(B_\varepsilon(0)) =: B_\varepsilon(p)$ is an diffeomorphism. Let $\{e_i\}$ be an orthonormal basis of Euclidean space $(T_p M, g_p)$, $\{\alpha^i\}$ be the dual basis of $\{e_i\}$, then we construct the **geodesic normal coordinate**

$$q \in B_\varepsilon(p) \mapsto (\alpha^1(\exp_p^{-1}(q)), \dots, \alpha^n(\exp_p^{-1}(q))).$$

Proposition 3.19. *Under geodesic normal coordinate, we have*

$$g_{ij}(p) = \delta_{ij}, \Gamma_{ij}^k(p) = 0$$

Proof. Since \exp_p is a diffeomorphism, we have $\frac{\partial}{\partial x^i} \Big|_p = \exp_{p*}|_0(e_i) = e_i$, hence $g_{ij} = g_p(e_i, e_j) = \delta_{ij}$. Moreover, let $x(t) = ty$ for $y \in T_p M \setminus \{0\}$, then $x(t)$ is the coordinate of some geodesic in $B_\varepsilon(p)$, thus it satisfies the equation

$$\ddot{x}^k(t) + \Gamma_{ij}^k(x(t))\dot{x}^i(t)\dot{x}^j(t) = 0.$$

Since $\ddot{x}^k(t) = 0$, $\dot{x}^i(t) = y^i \neq 0$, we must have $\Gamma_{ij}^k(ty) = 0$. Let $y \rightarrow 0$ and we obtain the conclusion. \square

Next we introduce the geodesic polar coordinate. Let $(r, \theta^1, \dots, \theta^{n-1})$ be the polar coordinate on Euclidean space $(T_p M, g_p)$, and we define the **geodesic polar coordinate** by

$$q \in B_\varepsilon(p) \setminus \{p\} \mapsto (r(\exp_p^{-1}(q)), \theta^1(\exp_p^{-1}(q)), \dots, \theta^{n-1}(\exp_p^{-1}(q))).$$

Proposition 3.20. *Under geodesic polar coordinate, we have*

$$g\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) = 1, \quad g\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta^i}\right) = 0$$

Proof. To make things clear, we write the inverse of geodesic polar coordinate as

$$F : (r, \omega) \mapsto \exp_p(r\omega)$$

for $r \in (0, +\infty)$, $\omega \in \mathbb{S}^{n-1}$. Then we use $\partial_r, \partial_{\theta^1}, \dots, \partial_{\theta^{n-1}}$ to denote the tangent vectors in $(0, +\infty) \times \mathbb{S}^{n-1}$, we have

$$\begin{aligned} \frac{\partial}{\partial r} &= F_*(\partial_r) \\ \frac{\partial}{\partial \theta^i} &= F_*(\partial_{\theta^i}), \quad i = 1, \dots, n. \end{aligned}$$

First we know ∂_r is the tangent vector of direction $r\omega$, hence $\partial/\partial r$ is the tangent vector of a unit-speed radial geodesic, that is

$$g\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) = 1.$$

Moreover, we have

$$\begin{aligned} \frac{\partial}{\partial r} g\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta^i}\right) &= g\left(\nabla_{\frac{\partial}{\partial r}} \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta^i}\right) + g\left(\frac{\partial}{\partial r}, \nabla_{\frac{\partial}{\partial r}} \frac{\partial}{\partial \theta^i}\right) \\ &= g\left(\frac{\partial}{\partial r}, \nabla_{\frac{\partial}{\partial r}} \frac{\partial}{\partial \theta^i}\right) \\ &= g\left(\frac{\partial}{\partial r}, \nabla_{\frac{\partial}{\partial \theta^i}} \frac{\partial}{\partial r}\right) \quad (\text{torsion-freeness}) \\ &= \frac{1}{2} \frac{\partial}{\partial \theta^i} g\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) \\ &= 0, \end{aligned}$$

hence $g\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta^i}\right)$ is constant. But if we let $r \rightarrow 0$, we have $\partial/\partial \theta^i \rightarrow 0$, therefore

$$g\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta^i}\right) = 0$$

□

Corollary 3.21. *Under geodesic polar coordinate, the metric tensor has local expression*

$$g = dr^2 + g_{ij}(r, \theta) d\theta^i \otimes d\theta^j$$

As an application, we prove geodesics are locally length-minimizing.

Proposition 3.22. *Let $\gamma: [0, 1] \rightarrow M$ be a geodesic in $B_\varepsilon(p)$, $\tilde{\gamma}: [0, 1] \rightarrow M$ be any curve in $B_\varepsilon(p)$ with $\gamma(0) = \tilde{\gamma}(0) = p$, $\gamma(1) = \tilde{\gamma}(1) = q$. Then $L(\gamma) \leq L(\tilde{\gamma})$.*

Proof. Let $q = \exp_p(v)$, φ be the geodesic polar coordinate, then we have

$$\gamma(t) = (tr_0, \omega_0), \quad \tilde{\gamma}(t) = (r(t), \omega(t))$$

such that $\omega_0, \omega(t) \in \mathbb{S}^{n-1}$, $r(1) = r_0$. Therefore

$$\begin{aligned} L(\gamma) &= \int_0^1 g_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t)) dt \\ &= \int_0^1 |v| dt = r_0 \\ L(\tilde{\gamma}) &= \int_0^1 (|\dot{r}(t)|^2 + g_{ij} \dot{\theta}^i(t) \dot{\theta}^j(t))^{1/2} dt \\ &\geq \int_0^1 |\dot{r}(t)| dt \\ &\geq \int_0^1 \dot{r}(t) dt = r_0 \end{aligned} \quad \square$$

3.5 Conjugate Points

Definition (Conjugate points). If $\exp_{p*}|_{\dot{\gamma}_v(t_0)}$ is degenerate at $\dot{\gamma}_v(t_0)$, then we call $\gamma_v(t_0)$ a **conjugate point** of p along γ .

Proposition 3.23. *We have \exp_p is degenerate at $\gamma(t_0)$ if and only if there is a Jacobi field J not identically equal to 0 such that $J(0) = J(t_0) = 0$.*

Proof. $\exp_{p*}|_{\dot{\gamma}_v(t_0)}(\xi) = 0$ if and only if $J(t) = \frac{\partial}{\partial s} \Big|_{s=0} \exp_p(t(v + s\xi))$ satisfies $J(0) = J(t) = 0$. \square

Remark 3.24. This proposition shows conjugate is symmetric.

Lecture 4

Variation Formula and Index Form

In this lecture we introduce the variation of energy. The variation formulae are closely related to minimizing property of geodesics. As an application of second variation formula, we introduce the Bonnet–Myers Theorem. Regarding the second variation formula as a quadric form, we have the notion of index form. We will explain the relation between index form and conjugate points. Finally, we will briefly mention the Morse Index Theorem.

Let $\gamma: [0, a] \rightarrow M$ be a curve, we define two functional

$$L(\gamma) = \int_0^a |\dot{\gamma}(t)| dt$$
$$E(\gamma) = \int_0^a \frac{1}{2} |\dot{\gamma}(t)|^2 dt.$$

Then by Cauchy–Schwarz inequality, we have

$$L(\gamma)^2 \leq 2aE(\gamma),$$

with equality holds if and only if $|\dot{\gamma}(t)| = \text{const.}$

Proposition 4.1. *If γ is a length-minimizing geodesic, then γ is energy-minimizing.*

Proof. Let $\tilde{\gamma}$ be another curve, then

$$2aE(\gamma) = L^2(\gamma) \leq L^2(\tilde{\gamma}) \leq 2aE(\tilde{\gamma}) \quad \square$$

Our aim is to prove the converse.

Proposition 4.2. *If γ is energy-minimizing, then γ is a length-minimizing geodesic.*

4.1 First Variation Formula

Definition (Variation). Let $\gamma_0 : [0, a] \rightarrow M$ be a curve, a **variation** of γ_0 is a differentiable map $\gamma : [0, a] \times (-\varepsilon, \varepsilon) \rightarrow M$ such that $\gamma(t, 0) = \gamma_0(t)$. If $\gamma(0, s) = \gamma_0(0)$, $\gamma(a, s) = \gamma_0(a)$ for any $s \in (-\varepsilon, \varepsilon)$, then we call it a **proper variation**. We call $\frac{\partial}{\partial s} \Big|_{s=0} \gamma(t, s) =: V(t)$ the **variation vector field**.

Proposition 4.3 (First variation formula). *Let $\gamma(t, s)$ be a variation, its energy $E(s) = \int_0^a \frac{1}{2} \left| \frac{\partial}{\partial t} \gamma(t, s) \right|^2 dt$, then we have*

$$E'(0) = \langle V, \dot{\gamma} \rangle_0^a - \int_0^a \langle V(t), \nabla_{\dot{\gamma}_0(t)} \dot{\gamma}_0(t) \rangle dt.$$

Proof. We calculate

$$\begin{aligned} \frac{d}{ds} E(s) &= \int_0^a \left\langle \frac{\partial \gamma}{\partial t}, \nabla_{\frac{\partial}{\partial s}} \frac{\partial \gamma}{\partial t} \right\rangle dt \\ &= \int_0^a \left\langle \frac{\partial \gamma}{\partial t}, \nabla_{\frac{\partial}{\partial t}} \frac{\partial \gamma}{\partial s} \right\rangle dt \end{aligned} \quad (\text{LC2})$$

$$= \int_0^a \left(\frac{\partial}{\partial t} \left\langle \frac{\partial \gamma}{\partial t}, \frac{\partial \gamma}{\partial s} \right\rangle - \left\langle \frac{\partial \gamma}{\partial s}, \nabla_{\frac{\partial}{\partial t}} \frac{\partial \gamma}{\partial t} \right\rangle \right) dt. \quad (\text{LC1})$$

Take $s = 0$, we obtain

$$\begin{aligned} E'(0) &= \int_0^a \left(\frac{\partial}{\partial t} \langle V(t), \dot{\gamma}_0(t) \rangle - \langle V(t), \nabla_{\dot{\gamma}_0(t)} \dot{\gamma}_0(t) \rangle \right) dt \\ &= \langle V, \dot{\gamma} \rangle_0^a - \int_0^a \langle V(t), \nabla_{\dot{\gamma}_0(t)} \dot{\gamma}_0(t) \rangle dt. \end{aligned} \quad \square$$

Corollary 4.4. $E'(0) = 0$ for all proper variation if and only if $\nabla_{\dot{\gamma}_0(t)} \dot{\gamma}_0(t) = 0$, that is, γ_0 is a geodesic.

Now we can give a proof of energy-minimizing curves are length-minimizing geodesics.

Proof of Proposition 4.2. Let $\gamma : [0, a] \rightarrow M$ be a curve such that for any $\tilde{\gamma} : [0, 1] \rightarrow M$ with $\gamma(0) = \tilde{\gamma}(0)$, $\gamma(1) = \tilde{\gamma}(1)$, the inequality $E(\gamma) \leq E(\tilde{\gamma})$ holds, we show that $L(\gamma) \leq L(\tilde{\gamma})$. Let $\gamma(t, s)$ be any variation with $\gamma(t, 0) = \gamma$, then γ is a critical point of $E(s)$. Hence by Corollary 4.4, γ is a geodesic. Then we can reparameterize $\tilde{\gamma}$ into arc-length, obtaining $\hat{\tilde{\gamma}}$. Therefore

$$L^2(\gamma) = 2aE(\gamma) \leq 2aE(\hat{\tilde{\gamma}}) = L^2(\hat{\tilde{\gamma}}) = L^2(\tilde{\gamma}) \quad \square$$

4.2 Second Variation Formula

Since we concentrate on critical points of variation of energy, we define second variation formula only for geodesics.

Proposition 4.5 (Second variation formula). *Let $\gamma_0 : [0, a] \rightarrow M$ be a geodesic, then*

$$E''(0) = \int_0^a (|\dot{V}(t)|^2 - \langle R(\dot{\gamma}_0(t), V(t))\dot{\gamma}_0(t), V(t) \rangle) dt + \boxed{\langle \nabla_{V(t)} V(t), \dot{\gamma}_0(t) \rangle|_0^a}.$$

The boxed term is called **boundary term**, and it vanishes when the variation is proper.

Proof. We take the expression of $E'(s)$ from the proof of first variation formula and differentiate

$$\begin{aligned} E''(s) &= \int_0^a \frac{\partial}{\partial s} \left\langle \frac{\partial \gamma}{\partial t}, \nabla_{\frac{\partial \gamma}{\partial s}} \frac{\partial \gamma}{\partial t} \right\rangle dt \\ &= \int_0^a \left(\left\langle \nabla_{\frac{\partial \gamma}{\partial s}} \frac{\partial \gamma}{\partial t}, \nabla_{\frac{\partial \gamma}{\partial s}} \frac{\partial \gamma}{\partial t} \right\rangle + \left\langle \frac{\partial \gamma}{\partial t}, \nabla_{\frac{\partial \gamma}{\partial s}} \nabla_{\frac{\partial \gamma}{\partial s}} \frac{\partial \gamma}{\partial t} \right\rangle \right) dt, \end{aligned}$$

where

$$\left\langle \nabla_{\frac{\partial \gamma}{\partial s}} \frac{\partial \gamma}{\partial t}, \nabla_{\frac{\partial \gamma}{\partial s}} \frac{\partial \gamma}{\partial t} \right\rangle = \left\langle \nabla_{\frac{\partial \gamma}{\partial t}} \frac{\partial \gamma}{\partial s}, \nabla_{\frac{\partial \gamma}{\partial t}} \frac{\partial \gamma}{\partial s} \right\rangle = |\dot{V}(t)|^2$$

and

$$\nabla_{\frac{\partial \gamma}{\partial s}} \nabla_{\frac{\partial \gamma}{\partial s}} \frac{\partial \gamma}{\partial t} = \nabla_{\frac{\partial \gamma}{\partial s}} \nabla_{\frac{\partial \gamma}{\partial t}} \frac{\partial \gamma}{\partial s} = \nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial \gamma}{\partial s}} \frac{\partial \gamma}{\partial s} - R\left(\frac{\partial \gamma}{\partial s}, \frac{\partial \gamma}{\partial t}\right) \frac{\partial \gamma}{\partial s}$$

Thus

$$E''(s) = \int_0^a \left(|\dot{V}(t)|^2 - \langle R(\dot{\gamma}, V)\dot{\gamma}, V \rangle + \frac{\partial}{\partial t} \left\langle \frac{\partial \gamma}{\partial t}, \nabla_{\frac{\partial \gamma}{\partial s}} \frac{\partial \gamma}{\partial s} \right\rangle - \left\langle \nabla_{\frac{\partial \gamma}{\partial t}} \frac{\partial \gamma}{\partial t}, \nabla_{\frac{\partial \gamma}{\partial s}} \frac{\partial \gamma}{\partial s} \right\rangle \right) dt,$$

the last term is 0, hence second variation formula holds by taking $s = 0$. \square

We now use second variation formula to prove the famous Bonnet–Myers Theorem.

Theorem 4.6 (Bonnet–Myers). *Let (M^n, g) be a complete Riemannian manifold with $\text{Ric} \geq (n-1)Kg > 0$, then $\text{diam}(M, g) \leq \pi/\sqrt{K}$. In particular, M is compact.*

Proof. We can scale the metric and assume $K = 1$. We prove the theorem by contradiction.

Assume there exists $p, q \in M$ joined by length-minimizing geodesic $\gamma : [0, 1] \rightarrow M$, with $d(p, q) = L(\gamma) > \pi$. Since γ is length-minimizing, $E''(0)(V, V) \geq 0$ for any proper variation vector field V . Let $\{e_1(t), \dots, e_{n-1}(t), \dot{\gamma}(t)/|\dot{\gamma}(t)|\}$ be a parallel orthonormal

frame along γ , and set $V_i(t) = \sin(\pi t)e_i(t)$, $i = 1, \dots, n-1$. Then $V_i(0) = V_i(1) = 0$ for each $i = 1, \dots, n-1$, V_i 's are proper variation vector field. Thus we have

$$\begin{aligned} E''(0)(V_i, V_i) &= \int_0^1 (|\dot{V}_i(t)|^2 - \langle R(\dot{\gamma}(t), V(t))\dot{\gamma}(t), V(t) \rangle) dt \\ &= \int_0^1 (-\langle V_i, \ddot{V}_i \rangle - \langle R(\dot{\gamma}(t), V(t))\dot{\gamma}(t), V(t) \rangle) dt \quad (\text{integration by parts}) \\ &= \int_0^1 (\pi^2 \sin^2(\pi t) - \sin^2(\pi t)L^2(\gamma)K(e_n, e_i)) dt \end{aligned}$$

Take summation we have

$$\begin{aligned} \sum_{i=1}^{n-1} E''(0)(V_i, V_i) &= \int_0^1 \sin^2(\pi t)((n-1)\pi^2 - L^2(\gamma)\langle \text{Ric}(e_n), e_n \rangle) dt \\ &\leq \int_0^1 (n-1) \sin^2(\pi t)(\pi^2 - L^2(\gamma)) dt \\ &< 0 \end{aligned}$$

Then there must exist a V_i such that $E''(0)(V_i, V_i) < 0$, contradiction! Hence we proved $\text{diam}(M) \leq \pi$. Moreover, by Hopf–Rinow theorem, M is bounded implies M is compact (M is automatically closed as a topological space). \square

Involving some theory of covering spaces, we have the following corollary.

Corollary 4.7. *The universal covering $\tilde{M} \rightarrow M$ is compact. Moreover, $\pi_1(M)$ is finite.*

Proof. We can lift g to \tilde{M} to make $\pi : \tilde{M} \rightarrow M$ a Riemannian covering, then π^*g also admits a Ricci curvature bounded below. By Bonnet–Myers theorem, \tilde{M} is compact. For the next claim, let $p \in M$, then $\pi^{-1}(p)$ is a discrete closed set in \tilde{M} , hence must be finite. Then the covering map is of finite sheet, $\pi_1(\tilde{M})$ has finite index in $\pi_1(M)$. But $\pi_1(\tilde{M})$ is trivial, $\pi_1(M)$ must be finite. \square

Remark 4.8. (1) We cannot weaken the condition to $K = 0$, in fact, even $\text{Sect} > 0$ is not enough. The surface $z = x^2 + y^2$ in \mathbb{R}^3 is a counterexample.

(2) If (M^n, g) satisfies $\text{Ric} \geq (n-1)g$ and $\text{diam}(M, g) = \pi$, then M must be isometric to \mathbb{S}^n . This is Cheng's Maximal Diameter Theorem.

4.3 Index Form

Definition (Index form). Let $\gamma : [0, a] \rightarrow M$ be a geodesic. The **index form** of γ is a bilinear form on $\mathcal{V} := \{\text{vector fields along } \gamma\}$ defined by

$$I(X, Y) = \int_0^a (\langle \ddot{X}, \dot{Y} \rangle - \langle R(\dot{\gamma}, X)\dot{\gamma}, Y \rangle) dt$$

Lemma 4.9. *Let $U \in \mathcal{V}_0 := \{Y \in \mathcal{V} \mid Y(0) = Y(a) = 0\}$, then U is a Jacobi field if and only if $I(U, Y) = 0$ for any $Y \in \mathcal{V}_0$.*

Proof. First we observe

$$I(U, Y) = - \int_0^a \langle \ddot{V} + R(\dot{\gamma}, V)\dot{\gamma}, Y \rangle dt$$

If V is a Jacobi field, then $\ddot{V} + R(\dot{\gamma}, V)\dot{\gamma} = 0$, which implies $I(U, Y) = 0$. Conversely, if $I(U, Y) = 0$ for any $Y \in \mathcal{V}_0$, we choose $Y = \ddot{V} + R(\dot{\gamma}, V)\dot{\gamma}$, then

$$0 = - \int_0^a |\ddot{V} + R(\dot{\gamma}, V)\dot{\gamma}|^2 dt,$$

this implies $\ddot{V} + R(\dot{\gamma}, V)\dot{\gamma} = 0$, that is, V is a Jacobi field. \square

We observe if a Jacobi field J is not in \mathcal{V}_0 , the above integration by parts is changed into

$$\begin{aligned} I(Y, J) &= \langle Y, J \rangle|_0^a - \int_0^a \langle \ddot{V} + R(\dot{\gamma}, V)\dot{\gamma}, Y \rangle \\ &= \langle Y, J \rangle|_0^a \end{aligned} \quad (4.1)$$

This will be useful in some calculation.

Next theorem shows the positive definiteness of I is related to conjugate points.

Theorem 4.10. *Let $\gamma[0, a] \rightarrow M$ be a geodesic, I be its index form.*

- (1) *If γ has no conjugate points of $\gamma(0)$, then I is positive definite on \mathcal{V}_0 .*
- (2) *If $\gamma(a)$ is the only conjugate point of $\gamma(0)$, then I is positive semidefinite but not positive definite.*
- (3) *If there is a $t_0 < a$ such that $\gamma(t_0)$ is conjugate to $\gamma(0)$, then there is a $U \in \mathcal{V}$ such that $I(U, U) < 0$.*

Proof of Theorem 4.10 (1). ¹ Let $\gamma(0) = p$, $\tilde{\gamma}$ is the radial line in $T_p M$ defined by $\tilde{\gamma}(t) = \dot{\gamma}(0)t$. Since γ has no conjugate points of $\gamma(0)$, the exponential map \exp_p is not degenerate on the whole $\tilde{\gamma}$. Hence there is a neighborhood U of $\tilde{\gamma}([0, a])$ such that $\exp_p : U \rightarrow M$ is an immersion. Now by carefully modifying the proof of Proposition 3.22, we have γ is the length-minimizing curve in $\exp_p(U)$. Hence by Proposition 4.1, γ also minimizes energy. Let $\gamma(t, s) : [0, a] \times (-\varepsilon, \varepsilon) \rightarrow M$ be any proper variation of γ , by taking ε small enough we can assume every γ_s is in $\exp_p(U)$. Then we have

$$E''(0) = \lim_{s \rightarrow 0} \frac{E(-s) + E(s) - 2E(0)}{s^2} \geq 0.$$

¹This proof is not the proof provided on class.

Since $E''(0)(V, V) = I(V, V)$ for any variation vector field V , we have $I(V, V) \geq 0$ for all $V \in \mathcal{V}_0$. Now we must show that $I(V, V) = 0$ implies $V = 0$. Let $I(V, V) = 0$, $X \in \mathcal{V}_0$ and $\delta > 0$, we have

$$0 \leq I(V + \delta X, V + \delta X) = I(V, V) + 2\delta I(V, X) + \delta^2 I(X, X),$$

this implies

$$2I(V, X) + \delta I(X, X) \geq 0,$$

let $\delta \rightarrow 0$, we obtain $I(V, X) \geq 0$ for all $X \in \mathcal{V}_0$. Similarly, consider $I(V - \delta X, V - \delta X)$, we obtain $I(V, X) \leq 0$ for all $X \in \mathcal{V}_0$. This means $I(V, X) = 0$ for all $X \in \mathcal{V}_0$. By Proposition 4.9, V is a Jacobi field. But γ has no conjugate points of $\gamma(0)$, V must identically equal to 0. This proves I being positive definite. \square

Before proving the rest of Theorem 4.10, we need the *Index Lemma*.

Proposition 4.11 (Index Lemma). *Assume γ is a geodesic without conjugate points. Let $U \in \mathcal{V}$ with $U(0) = 0$, J be a Jacobi field such that $J(0) = 0$, $J(a) = U(a)$, then $I(J, J) \leq I(U, U)$. The equality holds if and only if $U = J$.*

Proof. Since $U - J \in \mathcal{V}_0$, by Theorem 4.10 (1), we have $I(U - J, U - J) \geq 0$ with equality holds if and only if $U = J$. Then

$$I(U - J, U - J) = I(U, U) - 2I(U, J) + I(J, J)$$

But $I(U, J) = \langle U, \dot{J} \rangle|_0^a = \langle J, \dot{J} \rangle|_0^a = I(J, J)$, thus

$$I(U, U) \geq I(J, J) \quad \square$$

Proof of the rest of Theorem 4.10. (2) For any $0 < t_0 < a$, let $I_{[0, t_0]}$ denote

$$I_{[0, t_0]}(X, Y) = \int_0^{t_0} (\langle \ddot{X}, \dot{Y} \rangle + \langle R(\dot{Y}, X) \dot{Y}, Y \rangle) dt.$$

Then for any $X \in \mathcal{V}_0|_{[0, t_0]}$, by (1) we have $I_{[0, t_0]}(X, X) \geq 0$. We now construct $\tau_{t_0}(U)$ for any $U \in \mathcal{V}_0$. Let $\{e_i(t)\}$ be a parallel frame, $U(t) = \sum_{i=1}^n f_i(t) e_i(t)$. Then we define

$$\tau_{t_0}(U)(t) = \sum_{i=1}^n f_i\left(\frac{a}{t_0}t\right) e_i\left(\frac{a}{t_0}t\right) \in \mathcal{V}_0|_{[0, t_0]}.$$

Thus we have

$$I_{[0, t_0]}(\tau_{t_0}(U), \tau_{t_0}(U)) \geq 0,$$

let $t_0 \rightarrow a$ then we obtain the conclusion. Moreover, since $\gamma(a)$ is conjugate to $\gamma(0)$, there is a Jacobi field J with $J(0) = J(a) = 0$, $J \not\equiv 0$, and $I(J, J) = \langle J, \dot{J} \rangle|_0^a = 0$. This shows I is positive semidefinite but not positive definite.

(3) Since $\gamma(t_0)$ is conjugate to $\gamma(0)$, there exists a Jacobi field J_1 along $\gamma|_{[0,t_0]}$ such that $J_1(0) = J_1(t_0) = 0$. Let

$$V(t) = \begin{cases} J_1(t), & t \in [0, t_0], \\ 0, & t \in [t_0, a]. \end{cases}$$

Let $\delta > 0$ so small that $\gamma|_{[t_0-\delta, t_0+\delta]}$ is contained in a normal neighborhood of $\gamma(t_0)$, then there exists a Jacobi field J_2 along $\gamma|_{[t_0-\delta, t_0+\delta]}$ with $J_2(t_0 - \delta) = J_1(t_0 - \delta)$, $J_2(t_0 + \delta) = 0$. Define

$$U(t) = \begin{cases} J_1(t), & t \in [0, t_0 - \delta], \\ J_2(t), & t \in [t_0 - \delta, t_0 + \delta], \\ 0, & t \in [t_0 + \delta, a]. \end{cases}$$

Then we have $I(V, V) = I_{[0, t_0]}(J_1, J_1) = 0$, and²

$$\begin{aligned} I(U, U) &= I_{[0, t_0 - \delta]}(J_1, J_1) + I_{[t_0 - \delta, t_0 + \delta]}(J_2, J_2) \\ &< I_{[0, t_0 - \delta]}(V, V) + I_{[t_0 - \delta, t_0 + \delta]}(V, V) \quad (\text{Index Lemma}) \\ &= I(V, V) \\ &= 0 \end{aligned} \quad \square$$

Translating Theorem 4.10 into language of geometry, we have the following Jacobi Theorem.

Theorem 4.12 (Jacobi). *Let $\gamma: [0, a] \rightarrow M$ be a geodesic, then*

- (1) *If γ has no conjugate points of $\gamma(0)$, then γ is length-minimizing under any small proper variation.*
- (2) *If there is a $t_0 \in (0, a)$ such that $\gamma(t_0)$ is conjugate to $\gamma(0)$, then there is a proper variation $\gamma_s(t)$ such that $L(\gamma_s) < L(\gamma)$ for all s .*

Finally we mention something about *Morse index*. Define Morse index

$$\text{ind}(\gamma) = \max\{\dim S \mid S \leq \mathcal{V}_0, I \text{ is negative definite on } S\}$$

The famous *Morse Index Theorem* claims

$$\text{ind}(\gamma) = \#\{\text{conjugate points (counting multiplicity)}\} < +\infty$$

For more information, one can refer to John Milnor's *Morse Theory*.

²Actually we allow piecewise smooth curve in variation and index form.

Lecture 5

Comparison Theorems

5.1 Rauch Comparison Theorem

We first state and prove the Rauch comparison theorem.

Theorem 5.1 (Rauch comparison). *Let M^n, \tilde{M}^n be Riemannian manifolds, $\gamma: [0, a] \rightarrow M$, $\tilde{\gamma}: [0, a] \rightarrow \tilde{M}$ be unit speed geodesics, and J, \tilde{J} be Jacobi fields along $\gamma, \tilde{\gamma}$ respectively, such that $J(0) = \tilde{J}(0) = 0$, $\langle J(0), \dot{\gamma}(0) \rangle = \langle \tilde{J}(0), \dot{\tilde{\gamma}}(0) \rangle$, $|J(0)| = |\tilde{J}(0)|$. Assume that*

- (1) γ has no conjugate points of $\gamma(0)$ along γ ;
- (2) $K_\gamma(\dot{\gamma}, v) \geq K_{\tilde{\gamma}}(\dot{\tilde{\gamma}}, \tilde{v})$ with $|v| = |\tilde{v}| = 1$.

Then $|J(t)| \leq |\tilde{J}(t)|$ for all $t \in [0, a]$.

Remark 5.2. Before giving the proof of the theorem, we notice that when $\dim M = \dim \tilde{M} = 2$, the theorem reduces to the Liouville–Sturm comparison theorem in ODE theory.

Proof of Theorem 5.1. Decompose J, \tilde{J} into $J^\perp(t) + (at + b)\dot{\gamma}(t)$, $\tilde{J}^\perp(t) + (\tilde{a}t + \tilde{b})\dot{\tilde{\gamma}}(t)$ respectively. Since $J(0) = \tilde{J}(0) = 0$, we have $b = \tilde{b} = 0$; Moreover, since $\langle J(0), \dot{\gamma}(0) \rangle = \langle \tilde{J}(0), \dot{\tilde{\gamma}}(0) \rangle$, we have $a = \tilde{a}$. Since $|J^\perp|^2 = |J|^2 - (at)^2$, $|\tilde{J}^\perp|^2 = |\tilde{J}|^2 - (\tilde{a}t)^2$, we can just compare J^\perp and \tilde{J}^\perp . So we assume without loss of generality that J, \tilde{J} are normal Jacobi fields, that is, $J(0) \perp \dot{\gamma}(0)$, $\tilde{J}(0) \perp \dot{\tilde{\gamma}}(0)$.

Let $f(t) = |J(t)|^2$, $\tilde{f}(t) = |\tilde{J}(t)|^2$. By l'Hospital's rule, we have

$$\lim_{\varepsilon \rightarrow 0} \frac{\tilde{f}(\varepsilon)}{f(\varepsilon)} = 1,$$

hence only need to show \tilde{f}/f is monotonically increasing, which is equivalent to

$$\frac{\dot{\tilde{f}}}{\tilde{f}} \geq \frac{\dot{f}}{f} \iff \frac{\langle \dot{\tilde{J}}, \tilde{J} \rangle}{|\tilde{J}|^2} \geq \frac{\langle \dot{J}, J \rangle}{|J|^2}.$$

We can check the last inequality pointwisely, so we can scale the Jacobi fields to make $|\tilde{J}| = |J|$. Then we write the inequality into a lemma. \square

Lemma 5.3. *If J, \tilde{J} are Jacobi fields along $\gamma, \tilde{\gamma}$ respectively, with $J(0) = \tilde{J}(0) = 0$, $\dot{J}(0) \perp \dot{\gamma}(0)$, $\dot{\tilde{J}}(0) \perp \dot{\tilde{\gamma}}(0)$, and $|J(c)| = |\tilde{J}(c)|$. Moreover, the conditions (1) and (2) in Theorem 5.1 hold, then $\langle \dot{\tilde{J}}(c), \tilde{J}(c) \rangle \geq \langle \dot{J}(c), J(c) \rangle$.*

Proof. First we notice that

$$\langle \dot{J}(c), J(c) \rangle = I_{[0,c]}(J, J), \quad \langle \dot{\tilde{J}}(c), \tilde{J}(c) \rangle = I_{[0,c]}(\tilde{J}, \tilde{J}).$$

From now on we will simply use I as $I_{[0,c]}$. (Here we abuse the notation, using same I to denote index form on different curves. We will write the curve in subscript in case of ambiguous if necessary.) Let $\{e_1(t), \dots, e_{n-1}(t), \dot{\gamma}(t)\}$, $\{\tilde{e}_1(t), \dots, \tilde{e}_{n-1}(t), \dot{\tilde{\gamma}}(t)\}$ be parallel orthonormal frames along $\gamma, \tilde{\gamma}$ respectively, such that $J(c) = \alpha e_1(c)$, $\tilde{J}(c) = \alpha \tilde{e}_1(c)$. Let

$$J(t) = \sum_{i=1}^{n-1} h_i(t) e_i(t), \quad \tilde{J}(t) = \sum_{i=1}^{n-1} \tilde{h}_i(t) \tilde{e}_i(t),$$

then $h_i(c) = \tilde{h}_i(c) = 0$, $i = 2, \dots, n-1$. Define $U(t) = \sum_{i=1}^{n-1} \tilde{h}_i(t) e_i(t)$ along γ , then $U(c) = J(c)$, $U(0) = J(0) = 0$, $|U(t)| = |\tilde{J}(t)|$. Thus by Index Lemma, $I_\gamma(J, J) \leq I_\gamma(U, U)$. However, we have

$$\begin{aligned} I_\gamma(U, U) &= \int_0^c (|\dot{U}|^2 - \langle R(\dot{\gamma}, U) \dot{\gamma}, U \rangle) dt \\ &= \int_0^c \left(|\dot{\tilde{J}}|^2 - |U|^2 K(\dot{\gamma}, U/|U|) \right) dt \\ &\leq \int_0^c \left(|\dot{\tilde{J}}|^2 - |\tilde{J}| K(\dot{\tilde{\gamma}}, \tilde{J}/|\tilde{J}|) \right) dt \\ &= I_{\tilde{\gamma}}(\tilde{J}, \tilde{J}) \end{aligned}$$

Thus we have $\langle \dot{\tilde{J}}(c), \tilde{J}(c) \rangle \geq \langle \dot{J}(c), J(c) \rangle$. \square

Corollary 5.4. *If Riemannian manifold (M^n, g) satisfies $\text{Sect} \leq 0$, then $|J(t)| \geq t$, that is, $|\exp_{p*}|_v(w)| \geq |w|$.*

Proof. Compare (M^n, g) with Euclidean space (\mathbb{R}^n, δ) . □

The corollary implies the following important theorem.

Theorem 5.5. *Let M be a complete Riemannian manifold with $\text{Sect} \leq 0$, $p \in M$ be any point. Then M has no conjugate points of p .*

Theorem 5.5 is known as the first part of the celebrated Cartan–Hadamard Theorem. We state the rest part of the theorem as follows.

Theorem 5.6 (Cartan–Hadamard). *Let M^n be a complete Riemannian manifold with $\text{Sect} \leq 0$, then the universal cover of M is diffeomorphic to \mathbb{R}^n .*

Part of the Proof. Without loss of generality, we assume M is simply connected. Then we only need to show M is diffeomorphic to \mathbb{R}^n . Since M is complete, $\exp_p : T_p M \rightarrow M$ is surjective, and by Theorem 5.5, the \exp_p is nondegenerate. Thus \exp_p is a local isometry between $(T_p M, \exp_p^* g) \rightarrow (M, g)$. We show that $\exp_p^* g$ is complete. Let $\gamma_v(t) : [0, +\infty) \rightarrow T_p M, t \mapsto vt$ for any unit vector $v \in T_p M$, then $\exp_p(\gamma)$ is a geodesic in (M, g) . By the definition of $\exp_p^* g$, $\gamma_v(t)$ is a geodesic in $(T_p M, \exp_p^* g)$, hence the exponential map $\exp_0 : T_0(T_p M) \rightarrow T_p M$ is well-defined at $0 \in T_p M$. Hence by Hopf–Rinow Theorem, $\exp_p^* g$ is complete. Now the proof reduces to the following proposition. □

Proposition 5.7. *Let $f : M \rightarrow N$ be local isometry between Riemannian manifolds, with M complete. Then N is complete and f is a covering map.*

We stop here, and refer to Peter Peterson’s *Riemannian Geometry*, 3rd ed., Lemma 5.6.4.

We mention something more. Cartan–Hadamard Theorem can be used to prove the uniqueness of space forms, that is, simply connected complete constant sectional curvature Riemannian manifolds are unique up to an isometry. This can be found in Wu Hung–Hsi et. al.’s *Introduction to Riemannian Geometry*, Chapter 5.