2024 BICMR Summer School on Differential Geometry

Riemannian Geometry

Prof. Xia Chao
Notes taken by Zeng Mengchen

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Contents

Lecture 1
Differentiable Manifolds

In this lecture, we review some basic notions of differentiable manifolds.

1.1 Differentiable Manifolds and Maps

Definition. Let M^n be a Hausdorff space with countable topological basis. If there exists an open cover $\{U_\alpha\}$ of M, and homeomorphisms $\varphi_\alpha : U_\alpha \to \varphi_\alpha(U_\alpha)$ onto its image $\varphi_\alpha(U_\alpha) \subset \mathbb{R}^n$ open, such that

(1) $M = \bigcup_\alpha U_\alpha$,

(2) if $U_\alpha \cap U_\beta \neq \varnothing$, then $\varphi_\alpha^{-1} \circ \varphi_\beta : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$ is differentiable (we mean C^∞ here),

then M is called an n**-dimensional differentiable manifold**.

Moreover, we call $(U_\alpha, \varphi_\alpha)$ a **local chart**, $\{(U_\alpha, \varphi_\alpha)\}$ an **atlas**, and we say the atlas induces a **differentiable structure** on M.

Remark 1.1. We often assume the atlas is *maximal*, that is, there is no more local chart being compatible with the atlas.

Example 1.2. We illustrate some examples of differentiable manifolds.

(1) $\mathbb{R}^n$ itself is a differentiable manifolds, with single local chart $(\mathbb{R}^n, \mathrm{id})$.

(2) $\mathbb{S}^n := \{x \in \mathbb{R}^{n+1} | \ (x^1)^2 + \cdots + (x^{n+1})^2 = 1\}$. We use stereographic projection as local chart. Define the stereographic projection from north pole

$$\begin{aligned} \varphi_N : & \mathbb{S}^n \backslash \{(0, \cdots, 0, 1)\} \to \mathbb{R}^n \\ & x \mapsto \left(\frac{x^1}{1 + x^{n+1}}, \cdots, \frac{x^n}{1 + x^{n+1}} \right) \end{aligned}$$

Similarly define φ_S to be stereographic projection from the south pole. Then we have

$$\begin{aligned}\varphi_S \circ \varphi_N^{-1} : \mathbb{R}\backslash\{0\} &\to \mathbb{R}\backslash\{0\} \\ (y^1, \cdots, y^n) &\mapsto \left(\frac{y^1}{\sum_i (y^i)^2}, \cdots, \frac{y^n}{\sum_i (y^i)^2} \right)\end{aligned}$$

is clearly differentiable.

(3) Let $M_1^{n_1}, M_2^{n_2}$ be differentiable manifolds, then $M_1 \times M_2$ has the **product manifold** structure. To be precise, let M_1, M_2 have atlas $\{U_\alpha, \varphi_\alpha\}, \{(V_\beta, \psi_\beta)\}$, then $M_1 \times M_2$ has atlas $(U_\alpha \times V_\beta, \varphi_\alpha \times \psi_\beta)$. In particular, we have

- (flat) n-torus $\mathbb{T}^n = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$ (n times);
- cylinder $\mathbb{S}^1 \times \mathbb{R}$ (or generally $\mathbb{S}^k \times \mathbb{R}^{n-k}$).

(4) Real projective space $\mathbb{RP}^n$. Let equivalence relation $\sim$ on $\mathbb{R}^{n+1}\backslash\{0\}$ be $x \sim y \iff x = \lambda y,\ \lambda \neq 0$. Then define $\mathbb{RP}^n = (\mathbb{R}^{n+1}\backslash\{0\})/\sim$. We now define the differentiable structure on $\mathbb{RP}^n$. Let $U_i = \{x \in \mathbb{RP}^n :\ x = [x^1, \cdots, x^{n+1}],\ x^i \neq 0\}$, and

$$\begin{aligned}\varphi_i : U_i &\to \mathbb{R}^n \\ x &\mapsto \left(\frac{x^1}{x^i}, \cdots, \frac{x^{i-1}}{x^i}, \frac{x^{i+1}}{x^i}, \cdots, \frac{x^{n+1}}{x^i} \right)\end{aligned}$$

We check $\varphi_j \circ \varphi_i^{-1}$ on $U_i \cap U_j$. We may assume $i < j$, then

$$\begin{aligned}\varphi_j \circ \varphi_i^{-1}(y^1, \cdots, y^n) &= \varphi_j \left([y^1, \cdots, y^{i-1}, 1, y^{i+1}, \cdots, y^n] \right) \\ &= \left(\frac{y^1}{y^j}, \cdots, \frac{y^{i-1}}{y^j}, \frac{1}{y^j}, \frac{y^{i+1}}{y^j}, \cdots, \frac{y^n}{y^j} \right)\end{aligned}$$

is differentiable.

We now give the definition of differentiable maps.

Definition. A map $f : M \to N$ is **differentiable** at $p \in M$ if there exists local chart (U, φ) of p and (V, ψ) of $f(p)$, such that $\psi \circ f \circ \varphi^{-1}$ is differentiable at $\varphi(p)$.

Remark 1.3. (1) If $\tilde{\varphi}, \tilde{\psi}$ are another chart at p and $f(p)$, then we have

$$\tilde{\psi} \circ f \circ \tilde{\varphi}^{-1} = (\tilde{\psi} \circ \psi^{-1}) \circ (\psi \circ f \circ \varphi^{-1}) \circ (\varphi \circ \tilde{\varphi})$$

is still differentiable at p by the compatiblility of charts, so differentiable maps are well-defined.

(2) When $N = \mathbb{R}$, f is also called a **differentiable function**.

Notation 1.4. We use $C^\infty(M,N)$ to denote the $\mathbb{R}$-vector space of differentiable maps between M and N, $C^\infty(M)$ to denote the $\mathbb{R}$-algebra of differentiable functions on M. We use $C_p^\infty(M)$ to denote the $\mathbb{R}$-algebra of germs of differentiable functions at p. We often use $\gamma : I \subset \mathbb{R} \to M$ to denote a **differentiable curve** on M.

1.2 Tangent Spaces and Tangent Maps

Definition. Let $\gamma : I \to M$ be a curve, $\gamma(0) = p$. We define the **tangent vector along γ at** p as a mapping $\dot\gamma(0) : C_p^\infty(M) \to \mathbb{R}$, $\dot\gamma(0)f = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}(f\circ\gamma)(t)$. Then we define the **tangent space at** p

$$T_pM := \{\dot\gamma(0)|\ \gamma : I \to M \text{ differentiable},\ \gamma(0) = o = p\}.$$

Proposition 1.5. *We have the **Leibniz rule** $\dot\gamma(0)(fg) = (\dot\gamma(0)g)f(p) + (\dot\gamma(0)f)g(p)$. So a tangent vector is a* derivative *on $C_p^\infty(M)$.*

We now calculate the local representation of a tangent vector. Fix a chart $\varphi = (x^1, \cdots, x^n)$, we have

$$\begin{aligned}
\dot\gamma(0)f &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}(f\circ\gamma)(t) \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}(f\circ\varphi^{-1})\circ(\varphi\circ\gamma)(t) \qquad (1.1)\\
&= \sum_{i=1}^n \left.\frac{\partial}{\partial x^i}\right|_{\varphi(p)}(f\circ\varphi^{-1}) \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} x^i(\gamma(t)) \quad \text{(Chain rule)}
\end{aligned}$$

Using equation (1.1), we can describe T_pM as a vector space.

Proposition 1.6. *T_pM is a real vector space of dimension n. Moreover, given a local chart $\varphi = (x^1, \cdots, x^n)$, we have*

$$T_pM = \mathrm{Span}\left\{\left.\frac{\partial}{\partial x^i}\right|_p\right\}$$

where $\partial/\partial x^i|_p$ is the tangent vector of $\sigma_i(t) = \varphi^{-1}(\varphi(p) + te_i)$, $e_i = (0, \cdots, 1, \cdots, 0)$ with only i-th component being 1. *Thus we have*

$$\left.\frac{\partial}{\partial x^i}\right|_p f = \left.\frac{\partial}{\partial x^i}(f\circ\varphi^{-1})\right|_\varphi (p)$$

Proof. Clearly T_pM has natural vector space structure. Thus by the definition of $\frac{\partial}{\partial x^i}\Big|_p$'s, $\mathrm{Span}\left\{\frac{\partial}{\partial x^i}\Big|_p\right\} \subset T_pM$. For the converse inclusion, let $v \in T_pM$, then there is a curve $\gamma : I \to M$ with $\dot{\gamma}(0) = v$. Then by (1.1), $\dot{\gamma}(0)$ is a linear combination of $\frac{\partial}{\partial x^i}\Big|_p$'s, hence $T_pM \subset \mathrm{Span}\left\{\frac{\partial}{\partial x^i}\Big|_p\right\}$. □

Definition (Tangent maps)**.** Let $f : M \to N$ be a differentiable map, we define $f_{*p} : T_pM \to T_{f(p)}M$ as

$$f_{*p}(v)(g) = v(g \circ f)$$

for any $g \in C^\infty_{f(p)}N$. In particular, if $N = \mathbb{R}$, given $v \in T_pM$, let $\dot{\gamma}(0) = v$, then $f_{*p}(v) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}(f \circ \gamma)(t)$.

Again we can look at the local representation of f_{*p}. Let $\varphi = (x^1, \cdots, x^n)$, $\psi = (y^1, \cdots, y^m)$ be local charts containing p and $f(p)$. Let $v = \sum_{i=1}^n v^i \frac{\partial}{\partial x^i}\Big|_p = \dot{\sigma}(0)$, then $\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}(\varphi \circ \sigma)(t) = (v^1, \cdots, v^n)$. Thus we have

$$\begin{aligned} f_{*p}(v)(g) &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0}(g \circ f \circ \sigma)(t) \\ &= \sum_{i,j} \frac{\partial}{\partial y^j}(g \circ \psi^{-1})\bigg|_{\psi \circ f(p)} \frac{\partial}{\partial x^i}(\psi \circ f \circ \varphi^{-1})^j\bigg|_{\varphi(p)} \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0}(\varphi \circ \sigma)^i(t) \\ &= \sum_{i,j} v^i \frac{\partial}{\partial x^i}(\psi \circ f \circ \varphi^{-1})^j\bigg|_{\varphi(p)} \frac{\partial}{\partial y^j}\bigg|_{f(p)} g \end{aligned}$$

In particular, we have

$$f_{*p}\left(\frac{\partial}{\partial x^i}\bigg|_p\right) = \sum_j \frac{\partial}{\partial x^i}(\psi \circ f \circ \varphi^{-1})^j\bigg|_{\varphi(p)} \frac{\partial}{\partial y^j}\bigg|_{f(p)}$$

We can easy to verity the following chain rule:

Proposition 1.7. *Let $f : M \to N$, $g : N \to P$ be differentiable maps, then we have $(g \circ f)_{*p} = g_{*f(p)} \circ f_{*p}$.*

Definition (Diffeomorphism)**.** A map $f : M \to N$ is called a **diffeomorphism** if f is bijective, and f, f^{-1} are both differentiable.

Proposition 1.8. *If $f : M \to N$ is a diffeomorphism, then $f_{*p} : T_pM \to T_{f(p)}N$ is an isomorphism.*

This proposition can be easily proved by chain rule.

Remark 1.9. (1) The Proposition 1.8 shows that dimension of a manifold is well-defined in the category (differentiable manifolds, differentiable maps).

(2) Since we can do calculus locally on manifolds, the *Inverse function theorem* is valid on differentiable manifolds. That is, if $f_{*p}: T_pM \to T_{f(p)}N$ is an isomorphism, then $f: M \to N$ is a local diffeomorphism at p.

1.3 Tangent Bundles and Vector Fields

Definition (Tangent bundle)**.** Assume differentiable manifold M^n has atlas $\{U_\alpha, \varphi_\alpha\}$, define

$$TM := \sqcup_{p\in M} T_pM$$
$$\pi : TM \to M,\ (p,v) \mapsto p$$

We give an atlas of TM to make it into a $2n$-dimensional differentiable manifold. Let

$$\begin{aligned}\Phi_\alpha : \sqcup_{p\in U_\alpha} T_pM &\to \mathbb{R}^{2n}\\ (p,v) &\mapsto (\varphi_\alpha(p),(v^1,\cdots,v^n))\end{aligned}$$

where $v = \sum_{i=1}^n v^i \left.\frac{\partial}{\partial x^i}\right|_p$. Let's check

$$\Phi_\beta \circ \Phi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \times \mathbb{R}^n \to \varphi_\beta(U_\alpha \cap U_\beta) \times \mathbb{R}^n$$
$$(x^1,\cdots,x^n,v^1,\cdots,v^n) \mapsto \left(\varphi_\beta\circ\varphi_\alpha^{-1}(x), \sum_{i=1}^n \frac{\partial(\varphi_\beta\circ\varphi_\alpha^{-1})^1}{\partial x^i} v^i, \cdots, \sum_{i=1}^n \frac{\partial(\varphi_\beta\circ\varphi_\alpha^{-1})^n}{\partial x^i} v^i\right)$$

Clearly it is differentiable, then $\{(\pi^{-1}(U_\alpha),\Phi_\alpha)\}$ induces a differentiable structure on TM.

We call T_pM a **fiber** over p, and $\pi : TM \to M$ the projection.

Definition (Vector field)**.** A **vector field** is a differentiable map $X : M \to TM$ such that $X(p) \in T_pM$.

Notation 1.10. We use $\mathfrak{X}(M)$ to denote the collection of vector fields on M.

Proposition 1.11. *$X \in \mathfrak{X}(M^n)$ if and only if in any local chart (U,φ), we have $X(p) = \sum_{i=1}^n X^i(p) \left.\frac{\partial}{\partial x^i}\right|_p$ for $X^i \in C^\infty(U)$, $i = 1,2,\cdots,n$.*

This proposition is equivalent to X can be a mapping $C^\infty(M) \to C^\infty(M)$ defined by $Xf(p) = X(p)f$.

Definition (Lie bracket)**.** For $X,Y \in \mathfrak{X}(M)$, define $[X,Y] = XY - YX$, then $[X,Y] \in \mathfrak{X}(M)$.

Remark 1.12. We explain the definition more explicitly. If we act two vector fields on the product of two functions, we have

$$\begin{aligned}(XY)_p(fg) &= X_p(Y(fg)) = X_p(gYf + fYg)\\ &= \boxed{X_pg \cdot Y_pf + X_pf \cdot Y_pg} + g(p)X_pYf + f(p)X_pYg\end{aligned}$$

The boxed thing is bad, it spoils Leibniz rule. But if we substract $YX_p(fg)$, the boxed thing is cancelled. So $XY - YX \in \mathfrak{X}(M)$.

Proposition 1.13. *On some local chart, we have* $\left[\frac{\partial}{\partial x^i}\Big|_p, \frac{\partial}{\partial x^j}\Big|_p \right] = 0.$

Proof. This is equivalent to mixed partial derivative is commutative for smooth functions in $\mathbb{R}^n$. $\square$

Lecture 2

Metric and Connection

In this lecture we introduce the Riemannian metric on a differentiable manifold, and the connection compatible with metric, i.e., Levi–Civita connection. Moreover, we introduce the covariant derivative of a vector field along a curve, and parallel transport of vectors along a curve.

Notation 2.1. From now on we adopt *Einstein summation convention*: any index appear twice as both upper index and lower index means taking summation respective to the index. For example, a vector field on a local chart can be expressed as

$$X = X^i \frac{\partial}{\partial x^i} = \sum_{i=1}^{n} X^i \frac{\partial}{\partial x^i}$$

2.1 Riemannian Metric

Definition. Let M^n be a differentiable manifold. A **Riemannian metric** on M is a smooth assignment on each T_pM, $\forall p \in M$, a symmetric positive definite bilinear form g_p, that is for $X, Y \in T_pM$ we have

1. $g_p(X,Y) = g_p(Y,X)$;
2. $g_p(X,X) \geq 0$, $g_p(X,X) = 0 \iff X = 0$.

"Smooth" means in any local chart (U, φ) we have

$$g_{ij}(p) = g\left(\left.\frac{\partial}{\partial x^i}\right|_p, \left.\frac{\partial}{\partial x^j}\right|_p \right)$$

is smooth about p for any indices i, j. Then g is a symmetric positive definite $(0,2)$-tensor

$$g = g_{ij}\, \mathrm{d}x^i \otimes \mathrm{d}x^j$$

Proposition 2.2. *Any differentiable manifold M admits a Riemannian metric.*

Proof. We use partition of unity. Let $\{U_\alpha, x^i_\alpha\}$ be a locally finite atlas of M, $\{\phi_\alpha\}$ be a partition of unity subordinate to $\{U_\alpha\}$, i.e., $\operatorname{supp}\phi_\alpha \subset\subset U_\alpha$ and $\sum_\alpha \phi_\alpha = 1$. On the local chart (U_α, x^i_α), let $g_\alpha = \sum_{i=1}^n \mathrm{d}x^i_\alpha \otimes \mathrm{d}x^i_\alpha$. Set

$$g = \sum_\alpha \phi_\alpha g_\alpha,$$

we can check g is indeed a Riemannian metric on M. □

Remark 2.3. For an n-form $\omega \in \bigwedge^n M$ with $\operatorname{supp}\omega$ compact, we can define its integral as

$$\int_M \omega = \sum_\alpha \int_M \phi_\alpha \omega$$

One can check the definition is independent from the choice of partition of unity.

Example 2.4. (1) $\mathbb{R}^n$ has the Euclidean metric $g = \sum_{i=1}^n \mathrm{d}x^i \otimes \mathrm{d}x^i$, $g_{ij} = \delta_{ij}$.

(2) Let $f : M \to (N, h)$ be an immersion, then we define $f^*h(X,Y)|_p = h(f_{*p}X, f_{*p}Y)$, f^*h is the induced metric from h by immersion.

(3) Let $f : (M, g) \to (N, h)$ be an immersion, if $g = f^*h$, then f is called a **local isometry**. If $f : M \to N$ is a diffeomorphism, then f is called an **isometry**.

(4) The standard metric on $\mathbb{S}^n$ is the induced metric by the embedding $i : \mathbb{S}^n \hookrightarrow (\mathbb{R}^{n+1}, \delta_{ij})$.

(5) If $(M_1, g_1), (M_2, g_2)$ are Riemannian manifolds, then their product manifold has product metric $g_1 \times g_2$. In particular, equip $\mathbb{S}^1$ with standard metric g, $(\mathbb{T}^n, g^n) = (\mathbb{S}^1 \times \cdots \times \mathbb{S}^1, g \times \cdots \times g)$ is the flat torus (we will explain the word "flat" later).

(6) Let $f : M \to (N, g)$ be a covering map, then f^*g is a Riemannian metric on M, called Riemannian covering map. In particular, let $\operatorname{Isom}(M) := \{f : M \to M \mid f \text{ is isometry}\}$ denote the isometry group of M, $\Gamma \subset \operatorname{Isom}(M)$ be a subgroup, then M/Γ is a manifold, and $f : M \to M/\Gamma$ is a Riemannian covering map. Examples of this manner are

- $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$;
- $\mathbb{RP}^n = \mathbb{S}^n/\{\mathrm{id}, A\}$, where A is the antipodal map.

2.2 Metric Structure

Definition. Let $\gamma : [0,1] \to M$ be a curve, define its **length** to be

$$\begin{aligned} L(\gamma) &:= \int_0^1 |\dot{\gamma}(t)| \,\mathrm{d}t \\ &= \int_0^1 \sqrt{g(\gamma(t))(\dot{\gamma}(t), \dot{\gamma}(t))} \,\mathrm{d}t \end{aligned}$$

Let $p, q \in M$, define their **distance** to be

$$d(p,q) = \inf_{\gamma \in C_{p,q}} L(\gamma)$$

where $C_{p,q}$ denotes the collection of all smooth curve joining p and q.

Proposition 2.5. *The distance function $d : M \times M \to \mathbb{R}$ has the following properties:*

(1) $d(p,q) \geq 0$*, and* $d(p,q) = 0 \iff p = q$*;*

(2) $d(p,q) = d(q,p)$*;*

(3) $d(p,r) \leq d(r,q) + d(p,q)$*.*

Thus the distance function makes M into a metric space.

Proof. Only need to show $d(p,q) = 0 \iff p = q$, all else are trivial. We assume $p \neq q$, need to show $d(p,q) > 0$. Let $\gamma : [0,1] \to M$ be any curve joining p and q. Choose a local chart (U, φ) such that $\varphi(U) = B_r(0)$, $q \notin U$. By Jordan–Brouwer Separation Theorem, γ must intersect ∂U at $s := \gamma(c)$. Then we have

$$L(\gamma) \geq L(\gamma|_{[0,c]}) = \int_0^c \sqrt{g_{ij} \dot{x}^i(\gamma(t)) \dot{x}^j(\gamma(t))} \,\mathrm{d}t$$

Regarding $g : \overline{U} \times \mathbb{S}^{n-1} \to \mathbb{R}$, g is a continuous function on a compact set, thus it attains its minimum $g(x)(v,v) \geq m$, and $m > 0$ since $v \in \mathbb{S}^{n-1} \neq 0$. Thus we have

$$L(\gamma|_{[0,c]}) \geq m \int_0^c |\dot{x}(\gamma(t))| \,\mathrm{d}t \geq mr > 0$$

mr does not depend on γ, hence $d(p,q) \geq mr > 0$. $\square$

Proposition 2.6. (M, d) *with metric topology coincides with its original topology.*

For a proof, we refer to John Lee's *Introduction to Smooth Manifolds*, 2nd ed., Theorem 13.29.

2.3 Levi–Civita Connection

Definition. (Affine connection) Let M be a differentiable manifold, if the operator $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$, denoting $\nabla_X Y$, satisfies

(1) $\nabla_{(fX+gY)}Z = f\nabla_X Z + g\nabla_Y Z$ for $f, g \in C^\infty(M)$,

(2) $\nabla_X(Y+Z) = \nabla_X Y + \nabla_X Z$,

(3) $\nabla_X(fY) = (Xf)Y + f\nabla_X Y$,

then ∇ is called an **affine connection** on M.

Definition. An affine connection ∇ on Riemannian manifold (M, g) is called **Levi–Civita connection** if it satisfies

(LC1) $Xg(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$,

(LC2) $\nabla_X Y - \nabla_Y X - [X,Y] = 0$.

Proposition 2.7. *On a Riemannian manifold (M, g) there exists a unique Levi–Civita connection.*

Proof. We have the Koszul formula:

$$\begin{aligned} 2g(\nabla_X Y, Z) =& Xg(Y,Z) + Yg(X,Z) - Zg(X,Y) \\ &+ g([X,Y],Z) - g([X,Z],Y) - g([Y,Z],X) \end{aligned}$$

The formula shows Levi–Civita connection is unique, and can be used as the definition of Levi–Civita connection. $\square$

We check Levi–Civita connection locally. First we introduce the Christoffel symbols:

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma_{ij}^k \frac{\partial}{\partial x^k}$$

Then (LC1) is equivalent to

$$\frac{\partial g_{ij}}{\partial x^k} = \Gamma_{ki}^l g_{lj} + \Gamma_{kj}^l g_{li}$$

(LC2) is equivalent to

$$\Gamma_{ij}^k = \Gamma_{ji}^k$$

Using (LC1) and (LC2), we obtain

Proposition 2.8. *We have the expression of Γ_{ij}^k:*

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl} \left(\frac{\partial g_{il}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l} . \right)$$

Let vector fields $X = X^i \frac{\partial}{\partial x^i}, Y = Y^i \frac{\partial}{\partial x^i}$, then we have

$$\nabla_X Y = X^i \left(\frac{\partial Y^k}{\partial x^i} + Y^j \Gamma_{ij}^k \right) \frac{\partial}{\partial x^k} \tag{2.1}$$

This shows that $\nabla_X Y(p)$ depends only on $X(p)$ and the value of Y along the curve $\gamma(t)$ with $\gamma(0) = p, \dot{\gamma}(0) = X(p)$. Using this, we can introduce the covariant derivative of vector fields along curves.

2.4 Covariant Derivative

Definition. Let $\gamma : [0,1] \to M$ be a curve, Y is a vector field along γ. Then define

$$\frac{\nabla}{\mathrm{d}t} Y := \nabla_{\dot{\gamma}(t)} Y$$

We look at covariant derivative locally. Choose a local chart (U, φ), let $Y = Y^i \frac{\partial}{\partial x^i}$, $\dot{\gamma}(t) = \dot{\gamma}^i(t) \frac{\partial}{\partial x^i}$, then by equation (2.1), we have

$$\begin{aligned} \frac{\nabla}{\mathrm{d}t} Y &= \dot{\gamma}^i(t) \left(\frac{\partial Y^k(t)}{\partial x^i} + Y^j(t) \Gamma_{ij}^k(\gamma(t)) \right) \frac{\partial}{\partial x^k} \\ &= \left(\dot{Y}^k(t) + Y^j(t) \dot{\gamma}^i(t) \Gamma_{ij}^k(\gamma(t)) \right) \frac{\partial}{\partial x^k} \quad \text{(chain rule)} \end{aligned} \tag{2.2}$$

Definition (Parallel transport)**.** Let $\gamma : [0,1] \to M$ be a curve, Y is a vector field along γ. If $\nabla Y / \mathrm{d}t = 0$, then we call Y is **parallel** along γ.

Proposition 2.9. *Given a curve $\gamma : [0,1] \to M$ and an initial vector $Y_{\gamma(0)} \in T_{\gamma(0)} M$, then there exists a unique parallel vector field along γ with initial vector $Y_{\gamma(0)}$.*

Proof. Parallel transport satisfies (2.2), and it is a second order ordinary differential equation. By the unique existence theorem of solution of ODEs, the proposition is proved. □

Definition. Let $\gamma : [0,1] \to M$ be a curve, $\gamma(0) = p, \gamma(1) = q$, we define a mapping $P_\gamma : T_p M \to T_q M$ as follows: Let $Y_0 \in T_p M$, there exsits a unique parallel vector field Y along γ with $Y(0) = Y_0$, then we define $P_\gamma(Y_0) = Y(1)$. Clearly P_γ is linear.

Proposition 2.10. *P_γ is an isometry, hence an isomorphism.*

Proof. Using notation above, let $X_0, Y_0 \in T_p M$, X, Y are parallel vector field along γ with $X(0) = X_0, Y(0) = Y_0$. Then we have

$$\frac{\mathrm{d}}{\mathrm{d}t} g(X,Y) = g(\nabla_{\dot{\gamma}(t)} X, Y) + g(X, \nabla_{\dot{\gamma}(t)} Y) = 0,$$

since X, Y are parallel. Thus $g(X,Y)$ is constant, we have $g(X_0,Y_0) = g(P_\gamma(X_0), P_\gamma(Y_0))$. $\square$

The last proposition reveals the meaning of the word "connection", it means ∇ "connects" different tangent spaces.

Proposition 2.11. *Let $\gamma : [0,1] \to M$ be a curve with $\gamma(0) = p$, X, Y be vector fields with $X(p) = \dot\gamma(0)$. Then $\nabla_X Y(p) = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} P_\gamma^{-1}(Y(\gamma(t)))$.*

Proof. Let $\{e_1(t), \cdots, e_n(t)\}$ be a parallel frame along γ, then $\nabla_{\dot\gamma(t)} e_i(t) = 0$, in particular, $\nabla_{\dot\gamma(0)} e_i(0) = 0$. Let $Y(\gamma(t)) = Y^i(\gamma(t)) e_i(t)$, thus

$$\begin{aligned}
\nabla_X Y(p) &= \nabla_{\dot\gamma(0)} Y^i(\gamma(0)) e_i(0) \\
&= \dot\gamma(0) Y^i(\gamma(0)) e_i(0) + Y^i(0) \nabla_{\dot\gamma(0)} e_i(0) \\
&= \left(\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} Y^i(\gamma(t)) \right) e_i
\end{aligned}$$

On the other hand, parallel transport gives

$$P_\gamma^{-1}(Y(\gamma(t))) = Y^i(\gamma(t)) e_i(0),$$

hence

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} P_\gamma^{-1}(Y(\gamma(t))) &= \left(\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} Y^i(\gamma(t)) \right) e_i \\
&= \nabla_X Y(p)
\end{aligned}$$
$\square$

Remark 2.12. The symbol $\frac{\nabla}{\mathrm{d}t}$ is rarely used in literature, so we will simply use $\nabla_{\dot\gamma(t)} X$ to denote covariant derivative.

Lecture 3
Geodesics and Curvature

In this lecture, we first introduce the concepts of geodesics and exponential maps. By differentiating exponential map, we can introduce the concept of curvature and Jacobi fields. Using exponential map, we can also introduce geodesic normal coordinate and geodesic polar coordinate. As an application, we will use geodesic polar coordinate to show geodesics are locally length-minimizing. Finally, we will introduce the notion of conjugate points.

3.1 Geodesics and Exponential Maps

Definition. A curve $\gamma\colon[0,1]\to M$ is called a **geodesic** if $\nabla_{\dot\gamma(t)}\dot\gamma(t)=0$.

Remark 3.1. Geodesics are constant speed, i.e., $|\dot\gamma(t)|\equiv\text{const}$. This can be shown by $\frac{\mathrm{d}}{\mathrm{d}t}\langle\dot\gamma(t),\dot\gamma(t)\rangle=2\langle\nabla_{\dot\gamma(t)}\dot\gamma(t),\dot\gamma(t)\rangle=0$.

In a local chart (U,φ), let $\varphi\circ\gamma(t)=(x^1(t),\cdots,x^n(t))$, then $\dot\gamma(t)=\dot x^i(t)\left.\frac{\partial}{\partial x^i}\right|_{\gamma(t)}$. Thus the geodesic equation is

$$\nabla_{\dot\gamma(t)}\dot\gamma(t)=0\iff\ddot x^k(t)+\Gamma^k_{ij}(\gamma(t))\dot x^i(t)\dot x^j(t)=0,\ k=1,\cdots,n,$$

with $\gamma(0)=p,\dot\gamma(0)=v\in T_pM$.

Since the solution of an ODE relies continuously on initial value, we have the following proposition.

Proposition 3.2. *For any $p\in M$, there exists a neighborhood V of p, such that there exists $\delta>0,\varepsilon>0$ and a differentiable map $\gamma\colon(-\delta,\delta)\times\mathscr{U}\to M$, where $\mathscr{U}=\{(q,v)\in TV\,|\ q\in V,\ v\in T_pM,\ |v|<\varepsilon\}$, such that $\gamma(t;q,v)$ is a geodesic with $\gamma(0)=q,\dot\gamma(0)=v$.*

The idea of the proof is the ODE theorem we mentioned above, but the proof is not simply using only ODE theory. A proof of an equivalent proposition can be found in Wu Hung–Hsi, et. al.'s *Introduction to Riemannian Geometry* Chapter 3, Lemma 1.

Observe that $\gamma(\lambda t; p, v) = \gamma(t; p, \lambda v)$. Denote $\gamma(t; p, v) = \gamma_v(t)$, then above observation can be written as $\gamma_{\lambda v}(t) = \gamma_v(\lambda t)$. Therefore, we can shorten the initial vector to lenghten the domain of geodesic.

Definition (Exponential map)**.** Let $U \subset T_pM$ be a neighborhood of origin, such that for any $v \in U$, $\gamma_v(1)$ is defined (such neighborhood exists by Proposition 3.2). We define the **exponential map** at p to be

$$\begin{aligned} \exp_p : U &\to M \\ v &\mapsto \gamma_v(1) \end{aligned}$$

Remark 3.3. We rescale the initial vector and can obtain

$$\exp_p(v) = \gamma_v(1) = \gamma_{v/|v|}(|v|)$$

This means the exponential map act on v is moving forward distance $|v|$ along the geodesic with initial direction $v/|v|$.

Proposition 3.4. $\exp_{p*}|_0 : T_0(T_pM) \to T_pM$ *is identity (we identify* $T_0(T_pM)$ *with* T_pM*).*

Proof. We have

$$\exp_{p*}|_0(v) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \exp_p(tv) = v. \qquad \square$$

Corollary 3.5. *There is a ball* $B_\varepsilon(0) \subset T_pM$ *such that* $\exp_p : B_\varepsilon(0) \to M$ *is a diffeomorphism onto its image.*

Proof. Since $\exp_{p*}|_0$ is identity, it is nondegenerate, the corollary follows by Inverse Function Theorem. $\square$

Example 3.6.
(1) We know that the geodesics on $\mathbb{S}^n$ are great circles, hence $\exp_p$ is defined on the whole T_pM. But $\exp_p$ is not injective, since $\exp_p(0) = \exp_p(2\pi v) = p$ for unit vector v in T_pM.

(2) Let $M = \mathbb{S}^1 \times \mathbb{R}$ be the cylinder. We know from elementary differential geometry that the geodesics on cylinder are directrix circles, helices and generatrix lines. Then in local chart $(e^{2\pi it}, s) \mapsto (t, s)$, we know the $\exp_p$ is not injective in the direction $(1, 0)$, and injective in other directions.

Definition. If $\exp_p$ can be defined on whole T_pM for any $p \in M$, we say M is **geodesically complete**.

We have the following important theorem.

Theorem 3.7 (Hopf–Rinow). *Let M be a Riemannian manifold, the following are equivalent:*

(1) M is geodesically complete.

(2) $\exp_p : T_pM \to M$ is well-defined for some $p \in M$.

(3) The Heine–Borel property *holds, that is, any closed bounded set is compact on M.*

(4) M is complete as a metric space, that is, any Cauchy sequence converges.

We will not prove Hopf–Rinow theorem here. For a proof, one can refer to Peter Petersen's *Riemannian Geometry*, 3rd ed., Theorem 5.7.1.

3.2 Curvature

We know $\exp_{p*}|_0$ is identity, and we want to ask:

Question. What is $\exp_{p*}|_v : T_v(T_pM) \to T_{\exp_p(v)}M$?

To calculate $\exp_{p*}|_v(\xi)$, we choose a line $v + s\xi$, and then

$$\exp_{p*}|_v(\xi) = \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \exp_p(v + s\xi)$$

Now we can introduce a family of geodesics $\gamma(t,s) = \gamma_s(t) = \exp_p(t(v+s\xi))$, and denote $\gamma(t) = \gamma(t,0)$. Let $J_s(t) = \frac{\partial}{\partial s}\gamma(t,s)$, then $\dot{J}_s(t) = \nabla_{\dot{\gamma}_s(t)} \frac{\partial \gamma}{\partial s}$. Since $\nabla_{\dot{\gamma}_s(t)} \frac{\partial \gamma}{\partial t} = 0$, we have

$$\begin{aligned}
\ddot{J}_s(t) &= \nabla_{\dot{\gamma}_s(t)} \nabla_{\dot{\gamma}_s(t)} \frac{\partial \gamma}{\partial s} \\
&= \nabla_{\dot{\gamma}_s(t)} \nabla_{\frac{\partial \gamma}{\partial s}} \frac{\partial \gamma}{\partial t} \quad \text{(torsion-freeness)} \\
&= \nabla_{\frac{\partial \gamma}{\partial t}} \nabla_{\frac{\partial \gamma}{\partial s}} \frac{\partial \gamma}{\partial t} - \nabla_{\frac{\partial \gamma}{\partial s}} \nabla_{\frac{\partial \gamma}{\partial t}} \frac{\partial \gamma}{\partial t}.
\end{aligned}$$

Denote

$$R\left(\frac{\partial \gamma}{\partial t}, \frac{\partial \gamma}{\partial s}\right) = \nabla_{\frac{\partial \gamma}{\partial s}} \nabla_{\frac{\partial \gamma}{\partial t}} - \nabla_{\frac{\partial \gamma}{\partial t}} \nabla_{\frac{\partial \gamma}{\partial s}} + \nabla_{\left[\frac{\partial \gamma}{\partial t}, \frac{\partial \gamma}{\partial s}\right]},$$

then we have

$$\frac{\partial^2}{\partial t^2} J_s(t) + R\left(\frac{\partial \gamma}{\partial t}, \frac{\partial \gamma}{\partial s}\right) \frac{\partial \gamma}{\partial t} = 0.$$

Let $s = 0$, we have

$$\ddot{J}(t) + R(\dot{\gamma}(t), J(t))\dot{\gamma}(t) = 0.$$

We make it into a definition.

Definition (Riemann curvature tensor)**.** Let $R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ defined by

$$R(X,Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z.$$

We also look at Riemann curvature tensor locally. Tedious calculation will show Riemann curvature tensor is truly tensorial. Using a more simple notation $\partial_i = \partial/\partial x^i$, we have

$$\begin{aligned} R(\partial_i, \partial_j)\,\partial_k &= \nabla_{\partial_j} \nabla_{\partial_i} \partial_k - \nabla_{\partial_i} \nabla_{\partial_j} \partial_k \\ &= \nabla_{\partial_j} (\Gamma^l_{ik}\,\partial_l) - \nabla_{\partial_i} (\Gamma^l_{jk}\,\partial_l), \end{aligned}$$

and

$$\begin{aligned} R^l_{ijk} &= (\partial_j \Gamma^l_{ik} - \partial_i \Gamma^l_{jk} + \Gamma^m_{ik} \Gamma^l_{jm} - \Gamma^m_{jk} \Gamma^l_{im})\,\partial_l \\ &= \partial^2 g + \partial g * \partial g \end{aligned}$$

We also define $R_{ijkl} = R^m_{ijk} g_{ml}$, or $R(X,Y,Z,W) = g(R(X,Y)Z, W)$.

Example 3.8. $(\mathbb{R}, \delta)$ has $R \equiv 0$. Any metric admits zero curvature is call **flat**.

Proposition 3.9. *Riemann curvature tensor has following symmetric properties: For $X, Y, Z, W \in \mathfrak{X}(M)$, we have*

(1) $R(X,Y,Z,W) = -R(Y,X,Z,W) = -R(X,Y,W,Z) = R(Z,W,X,Y)$*;*

(2) $R(X,Y,Z,W) + R(Y,Z,X,W) + R(Z,X,Y,W) = 0$ *(First Bianchi Identity).*

Proof. Tedious calculation. □

Definition (Sectional curvature)**.** Let $p \in M$, $\pi \subset T_pM$ be a 2-plane, $\pi = \mathrm{Span}\{X, Y\}$. Then define the sectional curvature at p of π as

$$K_p(\pi) := \frac{R(X,Y,X,Y)}{|X|^2|Y|^2 - \langle X, Y\rangle^2}.$$

Remark 3.10. One can show sectional curvature does not depend on the choice of basis. A proof can be found in Manfredo do Carmo's *Riemannian Geometry*, Proposition 3.1.

Proposition 3.11. *Let M be a Riemannian manifold. $R(X,Y,Z,W)$ as a $(0,4)$-tensor is determined by all $K_p(\pi)$.*

For a proof, see Wu Hung–Hsi et. al., *Introduction to Riemannian Geometry*, Chapter 2 Lemma 2.

We mention one little observation. If all sectional curvature at p is constant K_p, then

$$R_p(X,Y,Z,W) = K_p(\langle X, Z\rangle \langle Y, W\rangle - \langle X, W\rangle \langle Y, Z\rangle)$$

For a theorem about constant sectoinal curvature, we mention here Schur's Theorem.

Theorem 3.12 (Schur)**.** *Let (M^n, g) be a Riemannian manifold, $n \geq 3$. If $K_p(\pi)$ is independent of $\pi \subset T_pM$ for any $p \in M$, then M has constant sectional curvature.*

We define another two important curvature.

Definition (Ricci curvature)**.** Let (M^n, g) be a Riemannian manifold. Define Ric : $\mathfrak{X}(M) \to \mathfrak{X}(M)$ as

$$\mathrm{Ric}_p(X) = \sum_{i=1}^{n} R_p(e_i, X)e_i,$$

where $\{e_i\}$ is an orthonormal frame around p. It's easy to check the definition is independent from the choice of orthonormal frame, and Ric is self-adjoint.

Definition (Scalar curvature)**.** Let (M^n, g) be a Riemannian manifold. We define Scal $\in C^\infty(M)$ as

$$\mathrm{Scal}(p) = \sum_{i=1}^{n} \langle \mathrm{Ric}_p(e_i), e_i(p) \rangle$$

for an orthonormal frame $\{e_i\}$ around p.

Definition. Let (M, g) be a Riemannian manifold, if Ric $= \lambda(p)g$ for a $\lambda \in C^\infty(M)$, we call M an **Einstein manifold**.

We also mention here

Theorem 3.13 (Schur)**.** *Let M^n be an Einstein manifold with $n \geq 3$, then M has constant scalar curvature.*

3.3 Jacobi Fields

Definition. Let γ be a geodesic, a vector field J along γ is called a **Jacobi field** if $\ddot{J} + R(\dot{\gamma}, J)\dot{\gamma} = 0$.

Let $\{e_i(t)\}$ be a parallel orthonormal frame along γ, $J(t) = \sum_i f_i(t)e_i(t)$. Define $a_{ij}(t) = \langle R(\dot{\gamma}(t), e_i(t))\dot{\gamma}(t), e_j(t) \rangle$, then the qeuation for Jacobi field is equivalent to

$$\ddot{f}_i(t) + \sum_j a_i j(t) f_j(t) = 0, \ i = 1, 2, \cdots, n.$$

By ODE theory, given $f_i(0), \dot{f}_i(0)$, $i = 1, 2, \cdots, n$, the $f_i(t)$'s are uniquely determined. Translating into language of Jacobi field, we have a Jacobi field is uniquely determined by $J(0)$ and $\dot{J}(0)$.

Notation 3.14. Let γ be a geodesic on a Riemannian manifold M, the vector space of Jacobi fields along γ is denoted by $\mathscr{J}(\gamma)$.

Above discussion can be summarized as the following proposition.

Proposition 3.15. *Let M^n be a Riemannian manifold, γ be a geodesic, then* $\dim \mathscr{J}(\gamma) = 2n$.

The next proposition shows only the Jacobi field at tangential direction are interesting.

Proposition 3.16. *Let J be a Jacobi field along γ, we have the decomposition $J(t) = J^{\perp}(t) + (at+b)\dot{\gamma}(t)$, where $J^{\perp}(t) \perp \dot{\gamma}(t)$.*

Proof. We have

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\langle J(t), \dot{\gamma}(t)\rangle &= \langle \ddot{J}(t), \dot{\gamma}(t)\rangle \\ &= -\langle R(\dot{\gamma}, J)\dot{\gamma}, \dot{\gamma}\rangle \\ &= 0.\end{aligned}$$

□

We have the following Gauss Lemma.

Proposition 3.17 (Gauss Lemma)**.** $\langle \exp_{p*}|_v(\xi), \dot{\gamma}_v(1)\rangle = \langle \gamma, v\rangle$.

Proof. Let one-parameter geodesic family $\gamma(t,s) = \exp_p(t(v+s\xi))$, then using the calculation in the beginning of section 3.2, we have

$$\begin{aligned}J(t) = \left.\frac{\partial}{\partial s}\right|_{s=0} \gamma(t,s) &= \exp_{p*}|_{tv}(t\xi) \\ &= t\exp_{p*}|_{tv}(\xi).\end{aligned}$$

Moreover, we have $\langle J(t), \dot{\gamma}_v(t)\rangle = at+b$ (Proposition 3.16),

$$\begin{aligned}a &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\langle J(t), \dot{\gamma}_v(t)\rangle = \langle \dot{J}(0), \dot{\gamma}_v(0)\rangle + \langle J(0), \nabla_v \dot{\gamma}_v(t)\rangle \\ &= \langle \dot{J}(0), v\rangle = \left\langle \left.\left(\exp_{p*}|_{tv}(\xi) + t\frac{\mathrm{d}}{\mathrm{d}t}\exp_{p*}|_{tv}(\xi)\right)\right|_{t=0}, v\right\rangle \\ &= \langle \exp_{p*}|_0(\xi), v\rangle = \langle \xi, v\rangle,\end{aligned}$$

and $b = \langle J(0), v\rangle = 0$. But we have

$$t\langle \exp_{p*}|_{tv}(\xi), \dot{\gamma}_v(t)\rangle = \langle J(t), \dot{\gamma}_v(t)\rangle = \langle \xi, v\rangle t.$$

Let $t = 1$ we obtain the conclusion. □

Example 3.18. We calculate the Jacobi field on Riemannian manifolds admit a constant sectional curvature. By scaling the metric, we can assume $K = 0, 1, -1$. The corresponding simply connected complete Riemannian manifolds are called **space forms**, which are $\mathbb{R}^n, \mathbb{S}^n, \mathbb{H}^n$. The Jacobi field equation is

$$\begin{cases} \ddot{J}(t) + KJ(t) = 0, \\ J(0) = 0, \dot{J}(0) = \xi. \end{cases}$$

Let $\xi(t)$ be the parallel transport along a geodesic with $\xi(0) = \xi$, then solving the equation by eigenvalue method, we obtain

$$J(t) = \begin{cases} t\xi(t), & K = 0, \\ \sin(t)\xi(t), & K = 1, \\ \sinh(t)\xi(t), & K = -1. \end{cases}$$

3.4 Some Local Charts

In this section, we adopt the traditional terminology "coordinate" to mean chart.

First we introduce the geodesic normal coordinate. Given a Riemannian manifold (M, g) and $p \in M$, there exists an $\varepsilon > 0$ such that $\exp_p : B_\varepsilon(0) \to \exp_p(B_\varepsilon(0)) =: B_\varepsilon(p)$ is an diffeomorphism. Let $\{e_i\}$ be an orthonormal basis of Euclidean space (T_pM, g_p), $\{\alpha^i\}$ be the dual basis of $\{e_i\}$, then we construct the **geodesic normal coordinate**

$$q \in B_\varepsilon(p) \mapsto (\alpha^1(\exp_p^{-1}(q)), \cdots, \alpha^n(\exp_p^{-1}(q))).$$

Proposition 3.19. *Under geodesic normal coordinate, we have*

$$g_{ij}(p) = \delta_{ij},\ \Gamma_{ij}^k(p) = 0$$

Proof. Since $\exp_p$ is a diffeomorphism, we have $\left.\frac{\partial}{\partial x^i}\right|_p = \exp_{p*}|_0(e_i) = e_i$, hence $g_{ij} = g_p(e_i, e_j) = \delta_{ij}$. Moreover, let $x(t) = ty$ for $y \in T_pM \backslash \{0\}$, then $x(t)$ is the coordinate of some geodesic in $B_\varepsilon(p)$, thus it satisfies the equation

$$\ddot{x}^k(t) + \Gamma_{ij}^k(x(t))\dot{x}^i(t)\dot{x}^j(t) = 0.$$

Since $\ddot{x}^k(t) = 0$, $\dot{x}^i(t) = y^i \neq 0$, we must have $\Gamma_{ij}^k(ty) = 0$. Let $y \to 0$ and we obtain the conclusion. □

Next we introduce the geodesic polar coordinate. Let $(r, \theta^1, \cdots, \theta^{n-1})$ be the polar coordinate on Euclidean space (T_pM, g_p), and we define the **geodesic polar coordinate** by

$$q \in B_\varepsilon(p) \backslash \{p\} \mapsto (r(\exp_p^{-1}(q)), \theta^1(\exp_p^{-1}(q)), \cdots, \theta^{n-1}(\exp_p^{-1}(q))).$$

Proposition 3.20. *Under geodesic polar coordinate, we have*

$$g\left(\frac{\partial}{\partial r},\frac{\partial}{\partial r}\right)=1,\ g\left(\frac{\partial}{\partial r},\frac{\partial}{\partial \theta^i}\right)=0$$

Proof. To make things clear, we write the inverse of geodesic polar coordinate as

$$F:(r,\omega)\mapsto \exp_p(r\omega)$$

for $r\in(0,+\infty),\omega\in\mathbb{S}^{n-1}$. Then we use $\partial_r,\partial_{\theta^1},\cdots,\partial_{\theta^{n-1}}$ to denote the tangent vectors in $(0,+\infty)\times\mathbb{S}^{n-1}$, we have

$$\begin{aligned}\frac{\partial}{\partial r}&=F_*(\partial_r)\\ \frac{\partial}{\partial\theta^i}&=F_*(\partial_\theta^i),\ i=1,\cdots,n.\end{aligned}$$

First we know ∂_r is the tangent vector of direction $r\omega$, hence $\partial/\partial r$ is the tangent vector of a unit-speed radial geodesic, that is

$$g\left(\frac{\partial}{\partial r},\frac{\partial}{\partial r}\right)=1.$$

Moreover, we have

$$\begin{aligned}\frac{\partial}{\partial r}g\left(\frac{\partial}{\partial r},\frac{\partial}{\partial\theta^i}\right)&=g\left(\nabla_{\frac{\partial}{\partial r}}\frac{\partial}{\partial r},\frac{\partial}{\partial\theta^i}\right)+g\left(\frac{\partial}{\partial r},\nabla_{\frac{\partial}{\partial r}}\frac{\partial}{\partial\theta^i}\right)\\ &=g\left(\frac{\partial}{\partial r},\nabla_{\frac{\partial}{\partial r}}\frac{\partial}{\partial\theta^i}\right)\\ &=g\left(\frac{\partial}{\partial r},\nabla_{\frac{\partial}{\partial\theta^i}}\frac{\partial}{\partial r}\right)\quad\text{(torsion-freeness)}\\ &=\frac{1}{2}\frac{\partial}{\partial\theta^i}g\left(\frac{\partial}{\partial r},\frac{\partial}{\partial r}\right)\\ &=0,\end{aligned}$$

hence $g\left(\frac{\partial}{\partial r},\frac{\partial}{\partial\theta^i}\right)$ is constant. But if we let $r\to 0$, we have $\partial/\partial\theta^i\to 0$, therefore

$$g\left(\frac{\partial}{\partial r},\frac{\partial}{\partial\theta^i}\right)=0$$

□

Corollary 3.21. *Under geodesic polar coordinate, the metric tensor has local expression*

$$g=\mathrm{d}r^2+g_{ij}(r,\theta)\,\mathrm{d}\theta^i\otimes\mathrm{d}\theta^j$$

As an application, we prove geodesics are locally length-minimizing.

Proposition 3.22. *Let $\gamma : [0,1] \to M$ be a geodesic in $B_\varepsilon(p)$, $\tilde{\gamma} : [0,1] \to M$ be any curve in $B_\varepsilon(p)$ with $\gamma(0) = \tilde{\gamma}(0) = p$, $\gamma(1) = \tilde{\gamma}(1) = q$. Then $L(\gamma) \le L(\tilde{\gamma})$.*

Proof. Let $q = \exp_p(v)$, φ be the geodesic polar coordinate, then we have

$$\gamma(t) = (tr_0, \omega_0),\ \tilde{\gamma}(t) = (r(t), \omega(t))$$

such that $\omega_0, \omega(t) \in \mathbb{S}^{n-1}$, $r(1) = r_0$. Therefore

$$\begin{aligned}
L(\gamma) &= \int_0^1 g_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))\,\mathrm{d}t \\
&= \int_0^1 |v|\,\mathrm{d}t = r_0 \\
L(\tilde{\gamma}) &= \int_0^1 (|\dot{r}(t)|^2 + g_{ij}\dot{\theta}^i(t)\dot{\theta}^j(t))^{1/2}\,\mathrm{d}t \\
&\ge \int_0^1 |\dot{r}(t)|\,\mathrm{d}t \\
&\ge \int_0^1 \dot{r}(t) = r_0
\end{aligned}$$

□

3.5 Conjugate Points

Definition (Conjugate points)**.** If $\exp_{p*}|_{\dot{\gamma}_v(t_0)}$ is degenerate at $\dot{\gamma}_v(t_0)$, then we call $\gamma_v(t_0)$ a **conjugate point** of p along γ.

Proposition 3.23. *We have* $\exp_p$ *is degenerate at* $\gamma(t_0)$ *if and only if there is a Jacobi field J not identically equal to* 0 *such that* $J(0) = J(t_0) = 0$.

Proof. $\exp_{p*}|_{\dot{\gamma}_v(t_0)}(\xi) = 0$ if and only if $J(t) = \left.\frac{\partial}{\partial s}\right|_{s=0} \exp_p(t(v + s\xi))$ satisfies $J(0) = J(t) = 0$. □

Remark 3.24. This proposition shows conjugate is symmetric.