2024 BICMR Summer School on Differential Geometry

Riemannian Geometry

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Lecture 1

Differentiable Manifolds

In this lecture, we review some basic notions of differentiable manifolds.

1.1 Differentiable Manifolds and Maps

Definition. Let M^n be a Hausdorff space with countable topological basis. If there exists an open cover $\{U_\alpha\}$ of M, and homeomorphisms $\varphi_\alpha : U_\alpha \to \varphi_\alpha(U_\alpha)$ onto its image $\varphi_\alpha(U_\alpha) \subset \mathbb{R}^n$ open, such that

(1) $M = \bigcup_\alpha U_\alpha$,

(2) if $U_\alpha \cap U_\beta \neq \varnothing$, then $\varphi_\alpha^{-1} \circ \varphi_\beta : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$ is differentiable (we mean C^∞ here),

then M is called an ***n*-dimensional differentiable manifold**.

Moreover, we call $(U_\alpha, \varphi_\alpha)$ a **local chart**, $\{(U_\alpha, \varphi_\alpha)\}$ an **atlas**, and we say the atlas induces a **differentiable structure** on M.

Remark 1.1. We often assume the atlas is *maximal*, that is, there is no more local chart being compatible with the atlas.

Example 1.2. We illustrate some examples of differentiable manifolds.

(1) $\mathbb{R}^n$ itself is a differentiable manifolds, with single local chart $(\mathbb{R}^n, \mathrm{id})$.

(2) $\mathbb{S}^n := \{x \in \mathbb{R}^{n+1} | \, (x^1)^2 + \cdots + (x^{n+1})^2 = 1\}$. We use stereographic projection as

local chart. Define the stereographic projection from north pole

$$\begin{aligned}\varphi_N :&\mathbb{S}^n\backslash\{(0,\cdots,0,1)\} \to \mathbb{R}^n \\ x &\mapsto \left(\frac{x^1}{1+x^{n+1}},\cdots,\frac{x^n}{1+x^{n+1}}\right)\end{aligned}$$

Similarly define φ_S to be stereographic projection from the south pole. Then we have

$$\begin{aligned}\varphi_S\circ\varphi_N^{-1} &: \mathbb{R}\backslash\{0\} \to \mathbb{R}\backslash\{0\} \\ (y^1,\cdots,y^n) &\mapsto \left(\frac{y^1}{\sum_i (y^i)^2},\cdots,\frac{y^n}{\sum_i (y^i)^2}\right)\end{aligned}$$

is clearly differentiable.

(3) Let $M_1^{n_1}, M_2^{n_2}$ be differentiable manifolds, then $M_1\times M_2$ has the **product manifold** structure. To be precise, let M_1, M_2 have atlas $\{U_\alpha,\varphi_\alpha\},\{(V_\beta,\psi_\beta)\}$, then $M_1\times M_2$ has atlas $(U_\alpha\times V_\beta,\varphi_\alpha\times\psi_\beta)$. In particular, we have

- (flat) n-torus $\mathbb{T}^n = \mathbb{S}^1\times\cdots\times\mathbb{S}^1$ (n times);
- cylinder $\mathbb{S}^1\times\mathbb{R}$ (or generally $\mathbb{S}^k\times\mathbb{R}^{n-k}$).

(4) Real projective space $\mathbb{RP}^n$. Let equivalence relation $\sim$ on $\mathbb{R}^{n+1}\backslash\{0\}$ be $x\sim y \iff x=\lambda y,\ \lambda\neq 0$. Then define $\mathbb{RP}^n = (\mathbb{R}^{n+1}\backslash\{0\})/\sim$. We now define the differentiable structure on $\mathbb{RP}^n$. Let $U_i = \{x\in\mathbb{RP}^n:\ x=[x^1,\cdots,x^{n+1}],\ x^i\neq 0\}$, and

$$\begin{aligned}\varphi_i &: U_i \to \mathbb{R}^n \\ x &\mapsto \left(\frac{x^1}{x^i},\cdots,\frac{x^{i-1}}{x^i},\frac{x^{i+1}}{x^i},\cdots,\frac{x^{n+1}}{x^i}\right)\end{aligned}$$

We check $\varphi_j\circ\varphi_i^{-1}$ on $U_i\cap U_j$. We may assume $i<j$, then

$$\begin{aligned}\varphi_j\circ\varphi_i^{-1}(y^1,\cdots,y^n) &= \varphi_j\left([y^1,\cdots,y^{i-1},1,y^{i+1},\cdots,y^n]\right) \\ &= \left(\frac{y^1}{y^j},\cdots,\frac{y^{i-1}}{y^j},\frac{1}{y^j},\frac{y^{i+1}}{y^j},\cdots,\frac{y^n}{y^j}\right)\end{aligned}$$

is differentiable.

We now give the definition of differentiable maps.

Definition. A map $f:M\to N$ is **differentiable** at $p\in M$ if there exists local chart (U,φ) of p and (V,ψ) of $f(p)$, such that $\psi\circ f\circ\varphi^{-1}$ is differentiable at $\varphi(p)$.

Remark 1.3. (1) If $\tilde{\varphi}, \tilde{\psi}$ are another chart at p and $f(p)$, then we have

$$\tilde{\psi} \circ f \circ \tilde{\varphi}^{-1} = (\tilde{\psi} \circ \psi^{-1}) \circ (\psi \circ f \circ \varphi^{-1}) \circ (\varphi \circ \tilde{\varphi})$$

is still differentiable at p by the compatiblility of charts, so differentiable maps are well-defined.

(2) When $N = \mathbb{R}$, f is also called a **differentiable function**.

Notation 1.4. We use $C^\infty(M,N)$ to denote the $\mathbb{R}$-vector space of differentiable maps between M and N, $C^\infty(M)$ to denote the $\mathbb{R}$-algebra of differentiable functions on M. We use $C_p^\infty(M)$ to denote the $\mathbb{R}$-algebra of germs of differentiable functions at p. We often use $\gamma : I \subset \mathbb{R} \to M$ to denote a **differentiable curve** on M.

1.2 Tangent Spaces and Tangent Maps

Definition. Let $\gamma : I \to M$ be a curve, $\gamma(0) = p$. We define the **tangent vector along γ at** p as a mapping $\dot{\gamma}(0) : C_p^\infty(M) \to \mathbb{R}$, $\dot{\gamma}(0)f = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}(f \circ \gamma)(t)$. Then we define the **tangent space at** p

$$T_pM := \{\dot{\gamma}(0) |\ \gamma : I \to M \text{ differentiable},\ \gamma(0) = o = p\}.$$

Proposition 1.5. *We have the* **Leibniz rule** $\dot{\gamma}(0)(fg) = (\dot{\gamma}(0)g)f(p) + (\dot{\gamma}(0)f)g(p)$. *So a tangent vector is a* derivative *on* $C_p^\infty(M)$.

We now calculate the local representation of a tangent vector. Fix a chart $\varphi = (x^1, \cdots, x^n)$, we have

$$\begin{aligned}
\dot{\gamma}(0)f &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} (f \circ \gamma)(t) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} (f \circ \varphi^{-1}) \circ (\varphi \circ \gamma)(t) \\
&= \sum_{i=1}^{n} \frac{\partial}{\partial x^i}\bigg|_{\varphi(p)} (f \circ \varphi^{-1})\ \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} x^i(\gamma(t)) \quad \text{(Chain rule)}
\end{aligned} \tag{1.1}$$

Using equation (1.1), we can describe T_pM as a vector space.

Proposition 1.6. *T_pM is a real vector space of dimension n. Moreover, given a local chart $\varphi = (x^1, \cdots, x^n)$, we have*

$$T_pM = \operatorname{Span}\left\{ \frac{\partial}{\partial x^i}\bigg|_p \right\}$$

where $\partial/\partial x^i|_p$ is the tangent vector of $\sigma_i(t) = \varphi^{-1}(\varphi(p) + te_i)$, $e_i = (0, \cdots, 1, \cdots, 0)$ with only i-th component being 1*. Thus we have*

$$\left.\frac{\partial}{\partial x^i}\right|_p f = \left.\frac{\partial}{\partial x^i}(f \circ \varphi^{-1})\right|_\varphi (p)$$

Proof. Clearly T_pM has natural vector space structure. Thus by the definition of $\left.\frac{\partial}{\partial x^i}\right|_p$'s, $\operatorname{Span}\left\{\left.\frac{\partial}{\partial x^i}\right|_p\right\} \subset T_pM$. For the converse inclusion, let $v \in T_pM$, then there is a curve $\gamma : I \to M$ with $\dot{\gamma}(0) = v$. Then by (1.1), $\dot{\gamma}(0)$ is a linear combination of $\left.\frac{\partial}{\partial x^i}\right|_p$'s, hence $T_pM \subset \operatorname{Span}\left\{\left.\frac{\partial}{\partial x^i}\right|_p\right\}$. □

Definition (Tangent maps)**.** Let $f : M \to N$ be a differentiable map, we define $f_{*p} : T_pM \to T_{f(p)}M$ as

$$f_{*p}(v)(g) = v(g \circ f)$$

for any $g \in C^\infty_{f(p)}N$. In particular, if $N = \mathbb{R}$, given $v \in T_pM$, let $\dot{\gamma}(0) = v$, then $f_{*p}(v) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}(f \circ \gamma)(t)$.

Again we can look at the local representation of f_{*p}. Let $\varphi = (x^1, \cdots, x^n)$, $\psi = (y^1, \cdots, y^m)$ be local charts containing p and $f(p)$. Let $v = \sum_{i=1}^n v^i \left.\frac{\partial}{\partial x^i}\right|_p = \dot{\sigma}(0)$, then $\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}(\varphi \circ \sigma)(t) = (v^1, \cdots, v^n)$. Thus we have

$$\begin{aligned}
f_{*p}(v)(g) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}(g \circ f \circ \sigma)(t) \\
&= \sum_{i,j} \left.\frac{\partial}{\partial y^j}(g \circ \psi^{-1})\right|_{\psi \circ f(p)} \left.\frac{\partial}{\partial x^i}(\psi \circ f \circ \varphi^{-1})^j\right|_{\varphi(p)} \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}(\varphi \circ \sigma)^i(t) \\
&= \sum_{i,j} v^i \left.\frac{\partial}{\partial x^i}(\psi \circ f \circ \varphi^{-1})^j\right|_{\varphi(p)} \left.\frac{\partial}{\partial y^j}\right|_{f(p)} g
\end{aligned}$$

In particular, we have

$$f_{*p}\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) = \sum_j \left.\frac{\partial}{\partial x^i}(\psi \circ f \circ \varphi^{-1})^j\right|_{\varphi(p)} \left.\frac{\partial}{\partial y^j}\right|_{f(p)}$$

We can easy to verity the following chain rule:

Proposition 1.7. *Let $f : M \to N$, $g : N \to P$ be differentiable maps, then we have $(g \circ f)_{*p} = g_{*f(p)} \circ f_{*p}$.*

Definition (Diffeomorphism)**.** A map $f: M \to N$ is called a **diffeomorphism** if f is bijective, and f, f^{-1} are both differentiable.

Proposition 1.8. *If $f: M \to N$ is a diffeomorphism, then $f_{*p}: T_pM \to T_{f(p)}N$ is an isomorphism.*

This proposition can be easily proved by chain rule.

Remark 1.9. (1) The Proposition 1.8 shows that dimension of a manifold is well-defined in the category (differentiable manifolds, differentiable maps).

(2) Since we can do calculus locally on manifolds, the *Inverse function theorem* is valid on differentiable manifolds. That is, if $f_{*p}: T_pM \to T_{f(p)}N$ is an isomorphism, then $f: M \to N$ is a local diffeomorphism at p.

1.3 Tangent Bundles and Vector Fields

Definition (Tangent bundle)**.** Assume differentiable manifold M^n has atlas $\{U_\alpha, \varphi_\alpha\}$, define

$$TM := \sqcup_{p\in M} T_pM$$
$$\pi: TM \to M,\ (p,v) \mapsto p$$

We give an atlas of TM to make it into a $2n$-dimensional differentiable manifold. Let

$$\begin{aligned}\Phi_\alpha : \sqcup_{p\in U_\alpha} T_pM &\to \mathbb{R}^{2n} \\ (p,v) &\mapsto (\varphi_\alpha(p), (v^1, \cdots, v^n))\end{aligned}$$

where $v = \sum_{i=1}^n v^i \left.\frac{\partial}{\partial x^i}\right|_p$. Let's check

$$\Phi_\beta \circ \Phi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \times \mathbb{R}^n \to \varphi_\beta(U_\alpha \cap U_\beta) \times \mathbb{R}^n$$
$$(x^1, \cdots, x^n, v^1, \cdots, v^n) \mapsto \left(\varphi_\beta \circ \varphi_\alpha^{-1}(x), \sum_{i=1}^n \frac{\partial(\varphi_\beta \circ \varphi_\alpha^{-1})^1}{\partial x^i} v^i, \cdots, \sum_{i=1}^n \frac{\partial(\varphi_\beta \circ \varphi_\alpha^{-1})^n}{\partial x^i} v^i\right)$$

Clearly it is differentiable, then $\{(\pi^{-1}(U_\alpha), \Phi_\alpha)\}$ induces a differentiable structure on TM.

We call T_pM a **fiber** over p, and $\pi: TM \to M$ the projection.

Definition (Vector field)**.** A **vector field** is a differentiable map $X: M \to TM$ such that $X(p) \in T_pM$.

Notation 1.10. We use $\mathfrak{X}(M)$ to denote the collection of vector fields on M.

Proposition 1.11. *$X \in \mathfrak{X}(M^n)$ if and only if in any local chart (U, φ), we have $X(p) = \sum_{i=1}^n X^i(p) \left.\frac{\partial}{\partial x^i}\right|_p$ for $X^i \in C^\infty(U)$, $i = 1, 2, \cdots, n$.*

This proposition is equivalent to X can be a mapping $C^\infty(M) \to C^\infty(M)$ defined by $Xf(p) = X(p)f$.

Definition (Lie bracket)**.** For $X, Y \in \mathfrak{X}(M)$, define $[X, Y] = XY - YX$, then $[X, Y] \in \mathfrak{X}(M)$.

Remark 1.12. We explain the definition more explicitly. If we act two vector fields on the product of two functions, we have

$$\begin{aligned}(XY)_p(fg) &= X_p(Y(fg)) = X_p(gYf + fYg) \\ &= \boxed{X_p g \cdot Y_p f + X_p f \cdot Y_p g} + g(p)X_p Y f + f(p)X_p Y g\end{aligned}$$

The boxed thing is bad, it spoils Leibniz rule. But if we substract $YX_p(fg)$, the boxed thing is cancelled. So $XY - YX \in \mathfrak{X}(M)$.

Proposition 1.13. *On some local chart, we have* $\left[\left.\frac{\partial}{\partial x^i}\right|_p, \left.\frac{\partial}{\partial x^j}\right|_p\right] = 0.$

Proof. This is equivalent to mixed partial derivative is commutative for smooth functions in $\mathbb{R}^n$. □