2024 BICMR Summer School on Differential Geometry

Riemannian Geometry

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Lecture 1

Differentiable Manifolds

In this lecture, we review some basic notions of differentiable manifolds.

1.1 Differentiable Manifolds and Maps

Definition. Let M^n be a Hausdorff space with countable topological basis. If there exists an open cover $\{U_\alpha\}$ of M, and homeomorphisms $\varphi_\alpha : U_\alpha \to \varphi_\alpha(U_\alpha)$ onto its image $\varphi_\alpha(U_\alpha) \subset \mathbb{R}^n$ open, such that

(1) $M = \bigcup_\alpha U_\alpha$,

(2) if $U_\alpha \cap U_\beta \neq \varnothing$, then $\varphi_\alpha^{-1} \circ \varphi_\beta : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$ is differentiable (we mean C^∞ here),

then M is called an n**-dimensional differentiable manifold**.

Moreover, we call $(U_\alpha, \varphi_\alpha)$ a **local chart**, $\{(U_\alpha, \varphi_\alpha)\}$ an **atlas**, and we say the atlas induces a **differentiable structure** on M.

Remark 1.1. We often assume the atlas is *maximal*, that is, there is no more local chart being compatible with the atlas.

Example 1.2. We illustrate some examples of differentiable manifolds.

(1) $\mathbb{R}^n$ itself is a differentiable manifolds, with single local chart $(\mathbb{R}^n, \mathrm{id})$.

(2) $\mathbb{S}^n := \{x \in \mathbb{R}^{n+1} | \ (x^1)^2 + \cdots + (x^{n+1})^2 = 1\}$. We use stereographic projection as local chart. Define the stereographic projection from north pole

$$\begin{aligned} \varphi_N : & \mathbb{S}^n \backslash \{(0, \cdots, 0, 1)\} \to \mathbb{R}^n \\ & x \mapsto \left(\frac{x^1}{1+x^{n+1}}, \cdots, \frac{x^n}{1+x^{n+1}} \right) \end{aligned}$$

Similarly define φ_S to be stereographic projection from the south pole. Then we have

$$\begin{aligned}\varphi_S \circ \varphi_N^{-1} : \mathbb{R}\backslash\{0\} &\to \mathbb{R}\backslash\{0\} \\ (y^1, \cdots, y^n) &\mapsto \left(\frac{y^1}{\sum_i (y^i)^2}, \cdots, \frac{y^n}{\sum_i (y^i)^2}\right)\end{aligned}$$

is clearly differentiable.

(3) Let $M_1^{n_1}, M_2^{n_2}$ be differentiable manifolds, then $M_1 \times M_2$ has the **product manifold** structure. To be precise, let M_1, M_2 have atlas $\{U_\alpha, \varphi_\alpha\}, \{(V_\beta, \psi_\beta)\}$, then $M_1 \times M_2$ has atlas $(U_\alpha \times V_\beta, \varphi_\alpha \times \psi_\beta)$. In particular, we have

- (flat) n-torus $\mathbb{T}^n = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$ (n times);
- cylinder $\mathbb{S}^1 \times \mathbb{R}$ (or generally $\mathbb{S}^k \times \mathbb{R}^{n-k}$).

(4) Real projective space $\mathbb{RP}^n$. Let equivalence relation $\sim$ on $\mathbb{R}^{n+1}\backslash\{0\}$ be $x \sim y \iff x = \lambda y,\ \lambda \neq 0$. Then define $\mathbb{RP}^n = (\mathbb{R}^{n+1}\backslash\{0\})/\sim$. We now define the differentiable structure on $\mathbb{RP}^n$. Let $U_i = \{x \in \mathbb{RP}^n :\ x = [x^1, \cdots, x^{n+1}],\ x^i \neq 0\}$, and

$$\begin{aligned}\varphi_i : U_i &\to \mathbb{R}^n \\ x &\mapsto \left(\frac{x^1}{x^i}, \cdots, \frac{x^{i-1}}{x^i}, \frac{x^{i+1}}{x^i}, \cdots, \frac{x^{n+1}}{x^i}\right)\end{aligned}$$

We check $\varphi_j \circ \varphi_i^{-1}$ on $U_i \cap U_j$. We may assume $i < j$, then

$$\begin{aligned}\varphi_j \circ \varphi_i^{-1}(y^1, \cdots, y^n) &= \varphi_j\left([y^1, \cdots, y^{i-1}, 1, y^{i+1}, \cdots, y^n]\right) \\ &= \left(\frac{y^1}{y^j}, \cdots, \frac{y^{i-1}}{y^j}, \frac{1}{y^j}, \frac{y^{i+1}}{y^j}, \cdots, \frac{y^n}{y^j}\right)\end{aligned}$$

is differentiable.

We now give the definition of differentiable maps.

Definition. A map $f : M \to N$ is **differentiable** at $p \in M$ if there exists local chart (U, φ) of p and (V, ψ) of $f(p)$, such that $\psi \circ f \circ \varphi^{-1}$ is differentiable at $\varphi(p)$.

Remark 1.3. (1) If $\tilde{\varphi}, \tilde{\psi}$ are another chart at p and $f(p)$, then we have

$$\tilde{\psi} \circ f \circ \tilde{\varphi}^{-1} = (\tilde{\psi} \circ \psi^{-1}) \circ (\psi \circ f \circ \varphi^{-1}) \circ (\varphi \circ \tilde{\varphi})$$

is still differentiable at p by the compatiblility of charts, so differentiable maps are well-defined.

(2) When $N=\mathbb{R}$, f is also called a **differentiable function**.

Notation 1.4. We use $C^\infty(M,N)$ to denote the $\mathbb{R}$-vector space of differentiable maps between M and N, $C^\infty(M)$ to denote the $\mathbb{R}$-algebra of differentiable functions on M. We use $C_p^\infty(M)$ to denote the $\mathbb{R}$-algebra of germs of differentiable functions at p. We often use $\gamma: I\subset\mathbb{R}\to M$ to denote a **differentiable curve** on M.

1.2 Tangent Spaces and Tangent Maps

Definition. Let $\gamma: I\to M$ be a curve, $\gamma(0)=p$. We define the **tangent vector along γ at** p as a mapping $\dot\gamma(0): C_p^\infty(M)\to\mathbb{R}$, $\dot\gamma(0)f=\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}(f\circ\gamma)(t)$. Then we define the **tangent space at** p

$$T_pM := \{\dot\gamma(0)|\ \gamma: I\to M \text{ differentiable},\ \gamma(0)=o=p\}.$$

Proposition 1.5. *We have the* ***Leibniz rule*** $\dot\gamma(0)(fg)=(\dot\gamma(0)g)f(p)+(\dot\gamma(0)f)g(p)$. *So a tangent vector is a* derivative *on* $C_p^\infty(M)$.

We now calculate the local representation of a tangent vector. Fix a chart $\varphi=(x^1,\cdots,x^n)$, we have

$$\begin{aligned}\dot\gamma(0)f &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0}(f\circ\gamma)(t)\\ &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0}(f\circ\varphi^{-1})\circ(\varphi\circ\gamma)(t)\\ &= \sum_{i=1}^n \frac{\partial}{\partial x^i}\bigg|_{\varphi(p)}(f\circ\varphi^{-1})\ \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0}x^i(\gamma(t))\quad\text{(Chain rule)}\end{aligned}\tag{1.1}$$

Using equation (1.1), we can describe T_pM as a vector space.

Proposition 1.6. *T_pM is a real vector space of dimension n. Moreover, given a local chart $\varphi=(x^1,\cdots,x^n)$, we have*

$$T_pM=\mathrm{Span}\left\{\frac{\partial}{\partial x^i}\bigg|_p\right\}$$

where $\partial/\partial x^i|_p$ is the tangent vector of $\sigma_i(t)=\varphi^{-1}(\varphi(p)+te_i)$, $e_i=(0,\cdots,1,\cdots,0)$ with only i-th component being 1. *Thus we have*

$$\frac{\partial}{\partial x^i}\bigg|_p f=\frac{\partial}{\partial x^i}(f\circ\varphi^{-1})\bigg|_\varphi(p)$$

Proof. Clearly T_pM has natural vector space structure. Thus by the definition of $\frac{\partial}{\partial x^i}\Big|_p$'s, $\mathrm{Span}\left\{ \frac{\partial}{\partial x^i}\Big|_p \right\} \subset T_pM$. For the converse inclusion, let $v \in T_pM$, then there is a curve $\gamma : I \to M$ with $\dot\gamma(0) = v$. Then by (1.1), $\dot\gamma(0)$ is a linear combination of $\frac{\partial}{\partial x^i}\Big|_p$'s, hence $T_pM \subset \mathrm{Span}\left\{ \frac{\partial}{\partial x^i}\Big|_p \right\}$. □

Definition (Tangent maps)**.** Let $f : M \to N$ be a differentiable map, we define $f_{*p} : T_pM \to T_{f(p)}M$ as

$$f_{*p}(v)(g) = v(g \circ f)$$

for any $g \in C^\infty_{f(p)}N$. In particular, if $N = \mathbb{R}$, given $v \in T_pM$, let $\dot\gamma(0) = v$, then $f_{*p}(v) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}(f \circ \gamma)(t)$.

Again we can look at the local representation of f_{*p}. Let $\varphi = (x^1, \cdots, x^n)$, $\psi = (y^1, \cdots, y^m)$ be local charts containing p and $f(p)$. Let $v = \sum_{i=1}^n v^i \frac{\partial}{\partial x^i}\Big|_p = \dot\sigma(0)$, then $\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}(\varphi \circ \sigma)(t) = (v^1, \cdots, v^n)$. Thus we have

$$\begin{aligned} f_{*p}(v)(g) &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0}(g \circ f \circ \sigma)(t) \\ &= \sum_{i,j} \frac{\partial}{\partial y^j}(g \circ \psi^{-1})\bigg|_{\psi\circ f(p)} \frac{\partial}{\partial x^i}(\psi \circ f \circ \varphi^{-1})^j\bigg|_{\varphi(p)} \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0}(\varphi \circ \sigma)^i(t) \\ &= \sum_{i,j} v^i \frac{\partial}{\partial x^i}(\psi \circ f \circ \varphi^{-1})^j\bigg|_{\varphi(p)} \frac{\partial}{\partial y^j}\bigg|_{f(p)} g \end{aligned}$$

In particular, we have

$$f_{*p}\left(\frac{\partial}{\partial x^i}\bigg|_p \right) = \sum_j \frac{\partial}{\partial x^i}(\psi \circ f \circ \varphi^{-1})^j\bigg|_{\varphi(p)} \frac{\partial}{\partial y^j}\bigg|_{f(p)}$$

We can easy to verity the following chain rule:

Proposition 1.7. *Let $f : M \to N$, $g : N \to P$ be differentiable maps, then we have $(g \circ f)_{*p} = g_{*f(p)} \circ f_{*p}$.*

Definition (Diffeomorphism)**.** A map $f : M \to N$ is called a **diffeomorphism** if f is bijective, and f, f^{-1} are both differentiable.

Proposition 1.8. *If $f : M \to N$ is a diffeomorphism, then $f_{*p} : T_pM \to T_{f(p)}N$ is an isomorphism.*

This proposition can be easily proved by chain rule.

Remark 1.9. (1) The Proposition 1.8 shows that dimension of a manifold is well-defined in the category (differentiable manifolds, differentiable maps).

(2) Since we can do calculus locally on manifolds, the *Inverse function theorem* is valid on differentiable manifolds. That is, if $f_{*p} : T_pM \to T_{f(p)}N$ is an isomorphism, then $f : M \to N$ is a local diffeomorphism at p.

1.3 Tangent Bundles and Vector Fields

Definition (Tangent bundle)**.** Assume differentiable manifold M^n has atlas $\{U_\alpha, \varphi_\alpha\}$, define

$$TM := \sqcup_{p\in M} T_pM$$
$$\pi : TM \to M,\ (p,v) \mapsto p$$

We give an atlas of TM to make it into a $2n$-dimensional differentiable manifold. Let

$$\begin{aligned}\Phi_\alpha : \sqcup_{p\in U_\alpha} T_pM &\to \mathbb{R}^{2n}\\ (p,v) &\mapsto (\varphi_\alpha(p), (v^1, \cdots, v^n))\end{aligned}$$

where $v = \sum_{i=1}^n v^i \left.\frac{\partial}{\partial x^i}\right|_p$. Let's check

$$\Phi_\beta \circ \Phi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \times \mathbb{R}^n \to \varphi_\beta(U_\alpha \cap U_\beta) \times \mathbb{R}^n$$
$$(x^1, \cdots, x^n, v^1, \cdots, v^n) \mapsto \left(\varphi_\beta \circ \varphi_\alpha^{-1}(x), \sum_{i=1}^n \frac{\partial(\varphi_\beta \circ \varphi_\alpha^{-1})^1}{\partial x^i} v^i, \cdots, \sum_{i=1}^n \frac{\partial(\varphi_\beta \circ \varphi_\alpha^{-1})^n}{\partial x^i} v^i \right)$$

Clearly it is differentiable, then $\{(\pi^{-1}(U_\alpha), \Phi_\alpha)\}$ induces a differentiable structure on TM.

We call T_pM a **fiber** over p, and $\pi : TM \to M$ the projection.

Definition (Vector field)**.** A **vector field** is a differentiable map $X : M \to TM$ such that $X(p) \in T_pM$.

Notation 1.10. We use $\mathfrak{X}(M)$ to denote the collection of vector fields on M.

Proposition 1.11. *$X \in \mathfrak{X}(M^n)$ if and only if in any local chart (U, φ), we have $X(p) = \sum_{i=1}^n X^i(p) \left.\frac{\partial}{\partial x^i}\right|_p$ for $X^i \in C^\infty(U)$, $i = 1, 2, \cdots, n$.*

This proposition is equivalent to X can be a mapping $C^\infty(M) \to C^\infty(M)$ defined by $Xf(p) = X(p)f$.

Definition (Lie bracket)**.** For $X, Y \in \mathfrak{X}(M)$, define $[X,Y] = XY - YX$, then $[X,Y] \in \mathfrak{X}(M)$.

Remark 1.12. We explain the definition more explicitly. If we act two vector fields on the product of two functions, we have

$$\begin{aligned}(XY)_p(fg) &= X_p(Y(fg)) = X_p(gYf + fYg)\\ &= \boxed{X_pg \cdot Y_pf + X_pf \cdot Y_pg} + g(p)X_pYf + f(p)X_pYg\end{aligned}$$

The boxed thing is bad, it spoils Leibniz rule. But if we substract $YX_p(fg)$, the boxed thing is cancelled. So $XY - YX \in \mathfrak{X}(M)$.

Proposition 1.13. *On some local chart, we have* $\left[\left.\frac{\partial}{\partial x^i}\right|_p, \left.\frac{\partial}{\partial x^j}\right|_p \right] = 0.$

Proof. This is equivalent to mixed partial derivative is commutative for smooth functions in $\mathbb{R}^n$. $\square$

Lecture 2

Metric and Connection

In this lecture we introduce the Riemannian metric on a differentiable manifold, and the connection compatible with metric, i.e., Levi–Civita connection. Moreover, we introduce the covariant derivative of a vector field along a curve, and parallel transport of vectors along a curve.

Notation 2.1. From now on we adopt *Einstein summation convention*: any index appear twice as both upper index and lower index means taking summation respective to the index. For example, a vector field on a local chart can be expressed as

$$X = X^i \frac{\partial}{\partial x^i} = \sum_{i=1}^{n} X^i \frac{\partial}{\partial x^i}$$

2.1 Riemannian Metric

Definition. Let M^n be a differentiable manifold. A **Riemannian metric** on M is a smooth assignment on each T_pM, $\forall p \in M$, a symmetric positive definite bilinear form g_p, that is for $X, Y \in T_pM$ we have

1. $g_p(X,Y) = g_p(Y,X)$;
2. $g_p(X,X) \geq 0$, $g_p(X,X) = 0 \iff X = 0$.

"Smooth" means in any local chart (U, φ) we have

$$g_{ij}(p) = g\left(\left.\frac{\partial}{\partial x^i}\right|_p, \left.\frac{\partial}{\partial x^j}\right|_p \right)$$

is smooth about p for any indices i, j. Then g is a symmetric positive definite $(0,2)$-tensor

$$g = g_{ij}\,\mathrm{d}x^i \otimes \mathrm{d}x^j$$

Proposition 2.2. *Any differentiable manifold M admits a Riemannian metric.*

Proof. We use partition of unity. Let $\{U_\alpha, x^i_\alpha\}$ be a locally finite atlas of M, $\{\phi_\alpha\}$ be a partition of unity subordinate to $\{U_\alpha\}$, i.e., $\operatorname{supp}\phi_\alpha \subset\subset U_\alpha$ and $\sum_\alpha \phi_\alpha = 1$. On the local chart (U_α, x^i_α), let $g_\alpha = \sum_{i=1}^n \mathrm{d}x^i_\alpha \otimes \mathrm{d}x^i_\alpha$. Set

$$g = \sum_\alpha \phi_\alpha g_\alpha,$$

we can check g is indeed a Riemannian metric on M. □

Remark 2.3. For an n-form $\omega \in \bigwedge^n M$ with $\operatorname{supp}\omega$ compact, we can define its integral as

$$\int_M \omega = \sum_\alpha \int_M \phi_\alpha \omega$$

One can check the definition is independent from the choice of partition of unity.

Example 2.4. (1) $\mathbb{R}^n$ has the Euclidean metric $g = \sum_{i=1}^n \mathrm{d}x^i \otimes \mathrm{d}x^i$, $g_{ij} = \delta_{ij}$.

(2) Let $f : M \to (N,h)$ be an immersion, then we define $f^*h(X,Y)|_p = h(f_{*p}X, f_{*p}Y)$, f^*h is the induced metric from h by immersion.

(3) Let $f : (M,g) \to (N,h)$ be an immersion, if $g = f^*h$, then f is called a **local isometry**. If $f : M \to N$ is a diffeomorphism, then f is called an **isometry**.

(4) The standard metric on $\mathbb{S}^n$ is the induced metric by the embedding $i : \mathbb{S}^n \hookrightarrow (\mathbb{R}^{n+1}, \delta_{ij})$.

(5) If $(M_1, g_1), (M_2, g_2)$ are Riemannian manifolds, then their product manifold has product metric $g_1 \times g_2$. In particular, equip S^1 with standard metric g, $(\mathbb{T}^n, g^n) = (\mathbb{S}^1 \times \cdots \times \mathbb{S}^1, g \times \cdots \times g)$ is the flat torus (we will explain the word “flat” later).

(6) Let $f : M \to (N,g)$ be a covering map, then f^*g is a Riemannian metric on M, called Riemannian covering map. In particular, let $\mathrm{Isom}(M) := \{f : M \to M | \ f \text{ is isometry}\}$ denote the isometry group of M, $\Gamma \subset \mathrm{Isom}(M)$ be a subgroup, then M/Γ is a manifold, and $f : M \to M/\Gamma$ is a Riemannian covering map. Examples of this manner are

- $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$;
- $\mathbb{RP}^n = \mathbb{S}^n/\{\mathrm{id}, A\}$, where A is the antipodal map.

2.2 Metric Structure

Definition. Let $\gamma : [0,1] \to M$ be a curve, define its **length** to be

$$\begin{aligned} L(\gamma) &:= \int_0^1 |\dot{\gamma}(t)| \, \mathrm{d}t \\ &= \int_0^1 \sqrt{g(\gamma(t))(\dot{\gamma}(t), \dot{\gamma}(t))} \, \mathrm{d}t \end{aligned}$$

Let $p, q \in M$, define their **distance** to be

$$d(p,q) = \inf_{\gamma \in C_{p,q}} L(\gamma)$$

where $C_{p,q}$ denotes the collection of all smooth curve joining p and q.

Proposition 2.5. *The distance function $d : M \times M \to \mathbb{R}$ has the following properties:*

(1) $d(p,q) \geq 0$*, and* $d(p,q) = 0 \iff p = q$*;*

(2) $d(p,q) = d(q,p)$*;*

(3) $d(p,r) \leq d(r,q) + d(p,q)$*.*

Thus the distance function makes M into a metric space.

Proof. Only need to show $d(p,q) = 0 \iff p = q$, all else are trivial. We assume $p \neq q$, need to show $d(p,q) > 0$. Let $\gamma : [0,1] \to M$ be any curve joining p and q. Choose a local chart (U, φ) such that $\varphi(U) = B_r(0)$, $q \notin U$. By Jordan–Brouwer Separation Theorem, γ must intersect ∂U at $s := \gamma(c)$. Then we have

$$L(\gamma) \geq L(\gamma|_{[0,c]}) = \int_0^c \sqrt{g_{ij} \dot{x}^i(\gamma(t)) \dot{x}^j(\gamma(t))} \, \mathrm{d}t$$

Regarding $g : \overline{U} \times \mathbb{S}^{n-1} \to \mathbb{R}$, g is a continuous function on a compact set, thus it attains its minimum $g(x)(v,v) \geq m$, and $m > 0$ since $v \in \mathbb{S}^{n-1} \neq 0$. Thus we have

$$L(\gamma|_{[0,c]}) \geq m \int_0^c |\dot{x}(\gamma(t))| \, \mathrm{d}t \geq mr > 0$$

mr does not depend on γ, hence $d(p,q) \geq mr > 0$. □

Proposition 2.6. (M, d) *with metric topology coincides with its original topology.*

For a proof, we refer to John Lee's *Introduction to Smooth Manifolds*, 2nd ed., Theorem 13.29.

2.3 Levi–Civita Connection

Definition. (Affine connection) Let M be a differentiable manifold, if the operator ∇ : $\mathfrak{X}(M)\times\mathfrak{X}(M)\to\mathfrak{X}(M)$, denoting $\nabla_X Y$, satisfies

(1) $\nabla_{(fX+gY)}Z = f\nabla_X Z + g\nabla_Y Z$ for $f,g\in C^\infty(M)$,

(2) $\nabla_X(Y+Z) = \nabla_X Y + \nabla_X Z$,

(3) $\nabla_X(fY) = (Xf)Y + f\nabla_X Y$,

then ∇ is called an **affine connection** on M.

Definition. An affine connection ∇ on Riemannian manifold (M,g) is called **Levi–Civita connection** if it satisfies

(LC1) $Xg(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$,

(LC2) $\nabla_X Y - \nabla_Y X - [X,Y] = 0$.

Proposition 2.7. *On a Riemannian manifold (M,g) there exists a unique Levi–Civita connection.*

Proof. We have the Koszul formula:

$$\begin{aligned}2g(\nabla_X Y, Z) =& Xg(Y,Z) + Yg(X,Z) - Zg(X,Y)\\ &+ g([X,Y],Z) - g([X,Z],Y) - g([Y,Z],X)\end{aligned}$$

The formula shows Levi–Civita connection is unique, and can be used as the definition of Levi–Civita connection. □

We check Levi–Civita connection locally. First we introduce the Christoffel symbols:

$$\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j} = \Gamma_{ij}^k\frac{\partial}{\partial x^k}$$

Then (LC1) is equivalent to

$$\frac{\partial g_{ij}}{\partial x^k} = \Gamma_{ki}^l g_{lj} + \Gamma_{kj}^l g_{li}$$

(LC2) is equivalent to

$$\Gamma_{ij}^k = \Gamma_{ji}^k$$

Let vector fields $X = X^i\frac{\partial}{\partial x^i}, Y = Y^i\frac{\partial}{\partial x^i}$, then we have

$$\nabla_X Y = X^i\left(\frac{\partial Y^k}{\partial x^i} + Y^j\Gamma_{ij}^k\right)\frac{\partial}{\partial x^k} \tag{2.1}$$

This shows that $\nabla_X Y(p)$ depends only on $X(p)$ and the value of Y along the curve $\gamma(t)$ with $\gamma(0) = p, \dot\gamma(0) = X(p)$. Using this, we can introduce the covariant derivative of vector fields along curves.

2.4 Covariant Derivative

Definition. Let $\gamma : [0,1] \to M$ be a curve, Y is a vector field along γ. Then define

$$\frac{\nabla}{\mathrm{d}t} Y := \nabla_{\dot{\gamma}(t)} Y$$

We look at covariant derivative locally. Choose a local chart (U, φ), let $Y = Y^i \frac{\partial}{\partial x^i}$, $\dot{\gamma}(t) = \dot{\gamma}^i(t) \frac{\partial}{\partial x^i}$, then by equation (2.1), we have

$$\begin{aligned} \frac{\nabla}{\mathrm{d}t} Y &= \dot{\gamma}^i(t) \left(\frac{\partial Y^k(t)}{\partial x^i} + Y^j(t) \Gamma_{ij}^k(\gamma(t)) \right) \frac{\partial}{\partial x^k} \\ &= \left(\dot{Y}^k(t) + Y^j(t) \dot{\gamma}^i(t) \Gamma_{ij}^k(\gamma(t)) \right) \frac{\partial}{\partial x^k} \quad \text{(chain rule)} \end{aligned} \tag{2.2}$$

Definition (Parallel transport)**.** Let $\gamma : [0,1] \to M$ be a curve, Y is a vector field along γ. If $\nabla Y / \mathrm{d}t = 0$, then we call Y is **parallel** along γ.

Proposition 2.8. *Given a curve $\gamma : [0,1] \to M$ and an initial vector $Y_{\gamma(0)} \in T_{\gamma(0)}M$, then there exists a unique parallel vector field along γ with initial vector $Y_{\gamma(0)}$.*

Proof. Parallel transport satisfies (2.2), and it is a second order ordinary differential equation. By the unique existence theorem of solution of ODEs, the proposition is proved. □

Definition. Let $\gamma : [0,1] \to M$ be a curve, $\gamma(0) = p, \gamma(1) = q$, we define a mapping $P_\gamma : T_pM \to T_qM$ as follows: Let $Y_0 \in T_pM$, there exsits a unique parallel vector field Y along γ with $Y(0) = Y_0$, then we define $P_\gamma(Y_0) = Y(1)$. Clearly P_γ is linear.

Proposition 2.9. *P_γ is an isometry, hence an isomorphism.*

Proposition 2.10. *Using notation above, let $X_0, Y_0 \in T_pM$, X, Y are parallel vector field along γ with $X(0) = X_0, Y(0) = Y_0$. Then we have*

$$\frac{\mathrm{d}}{\mathrm{d}t} g(X,Y) = g(\nabla_{\dot{\gamma}(t)} X, Y) + g(X, \nabla_{\dot{\gamma}(t)} Y) = 0,$$

since X, Y are parallel. Thus $g(X,Y)$ is constant, we have $g(X_0, Y_0) = g(P_\gamma(X_0), P_\gamma(Y_0))$.

The last proposition reveals the meaning of the word "connection", it means ∇ "connects" different tangent spaces.

Proposition 2.11. *Let $\gamma : [0,1] \to M$ be a curve with $\gamma(0) = p$, X, Y be vector fields with $X(p) = \dot{\gamma}(0)$. Then $\nabla_X Y(p) = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} P_\gamma^{-1}(Y(\gamma(t)))$.*

Proof. Let $\{e_1(t), \cdots, e_n(t)\}$ be a parallel basis along γ, then $\nabla_{\dot\gamma(t)} e_i(t) = 0$, in particular, $\nabla_{\dot\gamma(0)} e_i(0) = 0$. Let $Y(\gamma(t)) = Y^i(\gamma(t)) e_i(t)$, thus

$$\begin{aligned}
\nabla_X Y(p) &= \nabla_{\dot\gamma(0)} Y^i(\gamma(0)) e_i(0) \\
&= \dot\gamma(0) Y^i(\gamma(0)) e_i(0) + Y^i(0) \nabla_{\dot\gamma(0)} e_i(0) \\
&= \left(\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} Y^i(\gamma(t)) \right) e_i
\end{aligned}$$

On the other hand, parallel transport gives

$$P_\gamma^{-1}(Y(\gamma(t))) = Y^i(\gamma(t)) e_i(0),$$

hence

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} P_\gamma^{-1}(Y(\gamma(t))) &= \left(\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} Y^i(\gamma(t)) \right) e_i \\
&= \nabla_X Y(p)
\end{aligned}$$

□

Remark 2.12. The symbol $\frac{\nabla}{\mathrm{d}t}$ is rarely used in literature, so we will simply use $\nabla_{\dot\gamma(t)} X$ to denote covariant derivative.