A Note on Complex Manifolds

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Last compile: August 29, 2023

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Preface

This is a lecture note of a seminar on complex manifolds, by OM Society of School of Mathematical Scieces, Beijing Normal University. We mainly follow KODAIRA and MORROW's classic [MK06]. We shall cover the part of complex manifolds, sheaf cohomology and geometry of complex manifolds. Deformation theory will be skipped. Numbering of sections will not follow the textbook, but for some important theorems we shall give the name or original numbering on the textbook.

This note is unfinished and will update continuously, it will be post on GitHub. The repository name is matthewzenm/complex-manifolds-seminar. Many typos and grammar mistakes will occur in this note, since I'm writing on my local machine and without spell checker. If you find some typos or grammar mistakes, please contact me so that I can fix them.

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Aug 24, 2023

Chapter 1

Complex Manifolds

In this chapter we introduce the elements of several complex variables and the notion of complex manifolds. We also provide some examples of complex manifolds.

1.1 Holomorphic maps

Definition 1.1.1. A complex valued function $f(z)$ on a connected open subset $W \subset \mathbb{C}^n$ is called *holomorphic*, if for each $a = (a_1, \cdots, a_n) \in W$, $f(z)$ can be expanded as a convergent power series

$$f(z) = \sum_{k_1 \geq 0, \cdots, k_n \geq 0} c_{k_1 \cdots k_n} (z_1 - a_1)^{k_1} \cdots (z_n - a_n)^{k_n}$$

in some neighborhood of a.

From now on we shall use *domain* to denote a connected open set.

Proposition 1.1.2. *If $p(z) = \sum c_{k_1 \cdots k_n} (z_1 - a_1)^{k_1} \cdots (z_n - a_n)^{k_n}$ converges at $z = w$, then $p(z)$ converges for every z with $|z_k - a_k| < |w_k - a_k|$, $k = 1, \cdots, n$.*

Proof. Trivial. □

Definition 1.1.3. The neighborhood above is called a *polydisc* or *polycylinder*, and denoted by $P(a, r) = \{z \in \mathbb{C}^n : \; |z - a| < r\}$.

A complex valued function of n complex variables can be seen as a function of $2n$ real variables, thus we have the following definition.

Definition 1.1.4. A complex valued function of n complex variables is *continuous* or *differentiable*, if it is continuous or differentiable as a function of $2n$ real variables.

We have

Theorem 1.1.5 (Osgood). *Let $f(z_1, \cdots, z_n)$ be a continuous function on the domain $W \subset \mathbb{C}^n$, if f is holomorphic with respected to each z_k and other z_i's fixed, then f is holomorphic on W.*

Proof. Let $a \in W$ lie in the polydisc $\overline{P(a,r)} \subset W$, we use Cauchy's integral formula iteratively:

$$f(z_1,z_2,\cdots,z_n) = \frac{1}{2\pi\sqrt{-1}}\int_{|z_1-a_1|=r_1}\frac{f(w_1,z_2,\cdots,z_n)}{w_1-z_1}\,\mathrm{d}w_1$$
$$f(w_1,z_2,\cdots,z_n) = \frac{1}{2\pi\sqrt{-1}}\int_{|z_2-a_2|=r_2}\frac{f(w_1,w_2,\cdots,z_n)}{w_2-z_2}\,\mathrm{d}w_2$$
$$\cdots$$

Substituting, we have

$$\left(\frac{1}{2\pi\sqrt{-1}}\right)^n\int\cdots\int_{\partial P(a,r)}\frac{f(w_1,\cdots,w_n)}{(w_1-z_1)\cdots(w_n-z_n)}\,\mathrm{d}w_1\cdots\mathrm{d}w_n$$

Since

$$\left|\frac{z_k-a_k}{w_k-a_k}\right|<1$$

The series

$$\frac{1}{w_k-z_k} = \frac{1}{(w_k-a_k)-(z_k-a_k)} = \frac{1}{w_k-a_k}\cdot\frac{1}{1-(z_k-a_k)/(w_k-a_k)}$$
$$= \frac{1}{w_k-a_k}\sum_{i=0}^{\infty}\left(\frac{z_k-a_k}{w_k-a_k}\right)^i$$

converges absolutely in $P(a,r)$, hence integrate term by term we have

$$f(z_1,\cdots,z_n) = \sum_{k_0\geq 0,\cdots,k_n\geq 0} c_{k_1\cdots k_n}(z_1-a_1)^{k_1}\cdots(z_n-a_n)^{k_n}$$

where

$$c_{k_1\cdots k_n} = \left(\frac{1}{2\pi\sqrt{-1}}\right)^{k_1+\cdots+k_n}\int\cdots\int_{\partial P(a,r)}\frac{f(w_1,\cdots,w_n)\,\mathrm{d}w_1\cdots\mathrm{d}w_n}{(w_1-a_1)^{k_1+1}\cdots(w_n-a_n)^{k_n+1}}$$

Let $|f|\leq M$ on $\overline{P(a,r)}$, then we have

$$|c_{k_0\cdots k_n}| \leq \frac{M}{r_1^{k_1}\cdots r_n^{k_n}}$$

and for $z\in P(a,r)$, we have $|(z_k-a_k)/r_k|<1$, then

$$\left|\sum c_{k_1\cdots k_n}(z_1-a_1)^{k_1}\cdots(z_n-a_n)^{k_n}\right| \leq M\left|\sum\left(\frac{z_1-a_1}{r_1}\right)^{k_1}\cdots\left(\frac{z_n-a_n}{r_n}\right)^{k_n}\right|$$
$$= M\prod_{k=1}^{n}\left|\frac{1}{1-(z_k-a_k)/r_k}\right|$$

This shows the expansion is convergent for $z\in P(a,r)$. Since a is arbitrary, f is holomorphic. $\square$

We now introduce the Cauchy–Riemann equations.

Notation 1.1.6. Let f be a differentiable function on a domain $W \subset \mathbb{C}^n$, denote

$$\frac{\partial}{\partial z_k} = \frac{1}{2}\left(\frac{\partial}{\partial x_k} - \sqrt{-1}\frac{\partial}{\partial y_k}\right) \tag{1.1}$$

$$\frac{\partial}{\partial \overline{z_k}} = \frac{1}{2}\left(\frac{\partial}{\partial x_k} + \sqrt{-1}\frac{\partial}{\partial y_k}\right) \tag{1.2}$$

for $z_k = x_k + \sqrt{-1}y_k$ and $1 \le k \le n$.

Theorem 1.1.7. *Let f be a (continuously) differentiable function on the domain $W \subset \mathbb{C}^n$, then f is holomorphic on W if and only if $\partial f/\partial \overline{z_k} = 0$ for $k = 1, \cdots, n$.*

Proof. This is a corollary of Theorem 1.1.5 and classical results in complex analysis in one variable. ☐

Proposition 1.1.8 (Chain rule)**.** *Suppose $f(w_1, \cdots, w_m)$ and $g_k(z),\ k = 1, \cdots, m$ are differentiable, and the domain of f contains the range of $g = (g_1, \cdots, g_m)$, then $f \circ g$ is differentiable, and if $w_m = g_m(z)$, then*

$$\frac{\partial f(g(z)))}{\partial z_k} = \sum_{i=1}^{m}\left(\frac{\partial f(w)}{\partial w_i}\cdot\frac{\partial w_i}{\partial z_k} + \frac{\partial f(w)}{\partial \overline{w_i}}\cdot\frac{\partial \overline{w_i}}{\partial z_k}\right)$$

$$\frac{\partial f(g(z)))}{\partial \overline{z_k}} = \sum_{i=1}^{m}\left(\frac{\partial f(w)}{\partial w_i}\cdot\frac{\partial w_i}{\partial \overline{z_k}} + \frac{\partial f(w)}{\partial \overline{w_i}}\cdot\frac{\partial \overline{w_i}}{\partial \overline{z_k}}\right)$$

Proof. Direct calculation verifies the proposition. ☐

Corollary 1.1.9. *If $f(w)$ is holomorphic in $w = (w_1, \cdots, w_m)$ and $g_k(z),\ k = 1, \cdots, m$ are holomorphic in z, then $f \circ g$ is holomorphic in z.*

Corollary 1.1.10. *The set $\Gamma(\Omega, \mathscr{O}_{\mathbb{C}^n})$ of holomorphic functions on open set Ω forms a ring. (We use sheaf notation before we introduce what is a sheaf.)*

Definition 1.1.11. A map $f(z) = (f_1(z), \cdots, f_m(z))$ from $\mathbb{C}^n$ to $\mathbb{C}^m$ is a *holomorphic map* if each $f_k(z)$ is holomorphic, $k = 1, \cdots, m$. The matrix

$$\begin{bmatrix} \partial f_1/\partial z_1 & \cdots & \partial f_m/\partial z_1 \\ \vdots & \ddots & \vdots \\ \partial f_1/\partial z_n & \cdots & \partial f_m/\partial z_n \end{bmatrix} := \left[\frac{\partial f_i}{\partial z_j}\right]_{1 \le i \le m, 1 \le j \le n}$$

is called *Jacobian matrix*, and if $m = n$, the determinant $\det[\partial f_i/\partial z_j]$ is called the *Jacobian*.

Writing out $f_i = u_i + \sqrt{-1}v_i$ and $z_j = x_j + \sqrt{-1}y_j$, we denote briefly

$$\frac{\partial(u,v)}{\partial(x,y)} := \det\left(\frac{\partial(u_1, v_1, \cdots, u_n, v_n)}{\partial(x_1, y_1, \cdots, x_n, y_n)}\right)$$

And we have

Lemma 1.1.12. *If f is holomorphic, then $\partial(u,v)/\partial(x,y) = |\det[\partial f_i/\partial z_j]|^2 \ge 0$.*

Proof. Let

$$\begin{bmatrix}\partial/\partial z_1 & \partial/\partial \overline{z_1} & \cdots & \partial/\partial z_n & \partial/\partial \overline{z_n}\end{bmatrix} = A$$
$$\begin{bmatrix}\partial/\partial x_1 & \partial/\partial y_1 & \cdots & \partial/\partial x_n & \partial/\partial y_n\end{bmatrix} = B$$

Then

$$A = B\begin{bmatrix} 1/2 & 1/2 & & & \\ -\sqrt{-1}/2 & \sqrt{-1}/2 & & & \\ & & 1/2 & 1/2 & \\ & & -\sqrt{-1}/2 & \sqrt{-1}/2 & \\ & & & & \ddots \end{bmatrix}$$

Hence

$$\begin{aligned}\frac{\partial(u,v)}{\partial(x,y)} &= \det\begin{bmatrix}u_1\\ v_1\\ \vdots\\ u_n\\ v_n\end{bmatrix}\begin{bmatrix}\partial/\partial x_1 & \partial/\partial y_1 & \cdots & \partial/\partial x_n & \partial/\partial y_n\end{bmatrix}\\
&= \det\left(\frac{\sqrt{-1}}{2}\right)^n \det\begin{bmatrix}u_1\\ v_1\\ \vdots\\ u_n\\ v_n\end{bmatrix}\begin{bmatrix}\partial/\partial z_1 & \partial/\partial \overline{z_1} & \cdots & \partial/\partial z_n & \partial/\partial \overline{z_n}\end{bmatrix}\\
&= \det\left(\frac{\sqrt{-1}}{2}\right)^n \begin{bmatrix}\partial u_1/\partial z_1 & \partial u_1/\partial \overline{z_1} & \cdots & \partial u_1/\partial z_n & \partial u_1/\partial \overline{z_n}\\ \partial v_1/\partial z_1 & \partial v_1/\partial \overline{z_1} & \cdots & \partial v_1/\partial z_n & \partial v_1/\partial \overline{z_n}\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ \partial u_n/\partial z_1 & \partial u_n/\partial \overline{z_1} & \cdots & \partial u_n/\partial z_n & \partial u_n/\partial \overline{z_n}\\ \partial v_n/\partial z_1 & \partial v_n/\partial \overline{z_1} & \cdots & \partial v_n/\partial z_n & \partial v_n/\partial \overline{z_n}\end{bmatrix}\end{aligned}$$

Multiply $\sqrt{-1}$ on even rows, $1/2$ on odd rows, and add $2k$th row to $(2k-1)$st row, subtract $(2k-1)$st row to $2k$th row, we obtain

$$\frac{\partial(u,v)}{\partial(x,y)} = \det\begin{bmatrix}\partial f_1/\partial z_1 & & \cdots & \partial f_1/\partial z_n & \\ & \overline{\partial f_1/\partial z_1} & \cdots & & \overline{\partial f_1/\partial z_n}\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ \partial f_n/\partial z_1 & & \cdots & \partial f_n/\partial z_n & \\ & \overline{\partial f_n/\partial z_1} & \cdots & & \overline{\partial f_n/\partial z_n}\end{bmatrix}$$

We expand the determinant, extract odd rows and even rows respectively, and we get

$$\begin{aligned}\frac{\partial(u,v)}{\partial(x,y)} &= \sum_{\sigma,\tau\in S_n} \operatorname{sgn}\sigma \operatorname{sgn}\tau\left(\frac{\partial f_1}{\partial z_{\sigma(1)}}\cdots\frac{\partial f_n}{\partial z_{\sigma(n)}}\right)\left(\overline{\frac{\partial f_1}{\partial z_{\tau(1)}}}\cdots\overline{\frac{\partial f_n}{\partial z_{\tau(n)}}}\right)\\
&= \det[\partial f_i/\partial z_j]\overline{\det[\partial f_i/\partial z_j}\\
&= |\det[\partial f_i/\partial z_j]|^2\end{aligned}$$

□

Theorem 1.1.13 (Inverse function theorem)**.** *Let $U \subset \mathbb{C}^n$ open, $f : U \to \mathbb{C}^n$ be a holomorphic map, $a \in U$. If $\det[\partial f_i/\partial z_j]_{z=a} \neq 0$, then for a sufficiently small neighborhood N of a, f is bijective on N, $f(N)$ is open and $f^{-1}|_{f(N)}$ is holomorphic on $f(N)$.*

Proof. By Lemma 1.1.12, $\partial(u,v)/\partial(x,y) \neq 0$ at a, then by the inverse function theorem of real functions, f is bijective and differentiable, and $f(N)$ is open. We check that $f^{-1}|_{f(N)}$ is holomorphic. Set $\varphi(w) = f^{-1}(w)$, then $z_i = \varphi_i(f(z))$. Differentiate the equality gives

$$\begin{aligned}
0 = \frac{\partial z_i}{\partial \overline{z_k}} &= \sum_{j=1}^{n} \left(\frac{\partial \varphi_i(w)}{\partial w_j} \cdot \frac{\partial f_j(z)}{\partial \overline{z_k}} + \frac{\partial \varphi_i(w)}{\partial \overline{w_j}} \cdot \frac{\partial \overline{f_j(z)}}{\partial \overline{z_k}} \right) \\
&= \sum_{j=1}^{n} \frac{\partial \varphi_i(w)}{\partial \overline{w_j}} \cdot \frac{\partial \overline{f_j(z)}}{\partial \overline{z_k}}
\end{aligned}$$

Since $\det[\partial \overline{f_j(z)}/\partial \overline{z_k}] = \overline{\det}[\partial f_j(z)/z_k] \neq 0$, by linear algebra we have $\partial \varphi_i(w)/\partial \overline{w_j} = 0$ for each i, j, that is, φ is holomorphic. □

1.2 Complex Manifolds

Definition 1.2.1. Let M be a topological manifold. A *coordinate chart* is an open set $U \subset M$ and a continuous map $\varphi : U \to \mathbb{C}^n$ that maps U homeomorphically onto an open set of $\mathbb{C}^n$. An *atlas* is a collection $\{(U_i, \varphi_i)\}_{i\in I}$ of coordinate charts that $M = \bigcup_{i\in I} U_i$ and for any $U_i \cap U_j \neq \varnothing$, $\varphi_i \circ \varphi_j^{-1}$ and $\varphi_j \circ \varphi_i^{-1}$ are both holomorphic. A *complex structure* is a maximal atlas.

A *complex manifold* is a topological manifolds endowed with a complex structure.

Lemma 1.2.2. *Every complex manifold is paracompact, i.e. every open cover has a locally finite open refinement.*

Proof. [Lee11, Theorem 4.77]. □

In the rest of the section, we provide some examples and constructions of complex manifolds.

Construction 1.2.3. The *complex projective space* $\mathbb{P}^n$ is defined as the set of all 1-dimensional subspaces of $\mathbb{C}^{n+1}$. It can be realized as the sphere $\{z \in \mathbb{C}^{n+1} : |z| = 1\}$ quotient out antipodal points, so it is a compact topological space. We denote the elements of $\mathbb{P}^n$ by *homogeneous coordinate* $(p_0, p_1, \cdots, p_n)$. The complex projective space is a complex manifolds in the following manner: We define $U_j = \{p \in \mathbb{P}^n : p_j \neq 0\}$, then $\{U_j\}$ is an open covering of $\mathbb{P}^n$. Define $z_j(p) = (z_j^0, \cdots, z_j^{j-1}, z_j^{j+1}, \cdots, z_j^n)$, where $z_j^i = p_i/p_j$, then z_j maps each U_j homeomorphically onto $\mathbb{C}^n$. Moreover, we have $f_{jk} = z_j \circ z_k^{-1}$ given by

$$(x_1, \cdots, x_k, \cdots, x_n) \mapsto \left(\frac{x_1}{x_j}, \cdots, \frac{1}{x_j}, \cdots, \frac{x_n}{x_j} \right)$$

is holomorphic, and so is its inverse. Therefore (U_j, z_j) gives an atlas of $\mathbb{P}^n$.

In projective space, we have the notion of algebraic objects.

Definition 1.2.4. Let $\mathbb{P}^n$ has homogeneous coordinate $\zeta = (\zeta_0, \cdots, \zeta_n)$. A *projective algebraic variety* M is the common zero locus of a family of homogeneous polynomials, i.e. for some homogeneous polynomials $f_1, \cdots, f_m$,

$$M := \{\zeta \in \mathbb{P}^n : f_i(\zeta) = 0,\ i = 1, \cdots, m\}$$

If the the rank of $[\partial f_i/\partial \zeta_j]$ is independent from ζ, then M becomes a manifold, called *algebraic manifold*. If f_d is a homogeneous polynomial of degree d, then its zero locus M_d is called a *hypersurface* in $\mathbb{P}^n$ of order d. If for each $\zeta \in M_d$, at least one of $\partial f_d/\partial \zeta_i \neq 0$, then M_d is nonsingular.

Example 1.2.5. We provide some examples.

1. $M_d \subset \mathbb{P}^2$ a nonsingular plane curve of order d is a Riemann surface of genus $g = \frac{1}{2}(d-1)(d-2)$. (cf. [GH94, pp. 219–221])

2. A nonsingular $M_d \subset \mathbb{P}^3$. M_d is simply connected and has Euler number $\chi(M_d) = d(d^2-4d+6)$. (For Euler number, cf. [Hir95, Section 10.2, Equation (5)], and for simply connectivity, cf. [Mil63, Theorem 7.4])

3. Let $M \subset \mathbb{P}^3$ be defined by
$$M = \{\zeta \in \mathbb{P}^3 : \ \zeta_1\zeta_2 - \zeta_0\zeta_3 = 0, \zeta_0\zeta_2 - \zeta_1^2 = 0, \zeta_2^2 - \zeta_1\zeta_3 = 0\}$$
We claim that M is complex analytically homeomorphic to $\mathbb{P}^1$. One can easily check the map
$$\mu : \mathbb{P}^1 \to \mathbb{P}^3, t \mapsto (t_0^3, t_0^2 t_1, t_0 t_1^2, t_1^3)$$
is biholomorphic. This is the simplest case of *Veronese embedding*.

Next we consider *quotient spaces*.

Definition 1.2.6. An *analytic automorphism* of M is a biholomorphic map of M onto M. The set of all analytic automorphisms of M forms a group under composition.

Let G be a subgroup of analytic automorphisms. G is called a properly *discontinuous group* of analytic automorphisms of M, if for any pair of compact subsets K_1, K_2, the set $\{g \in G : \ gK_1 \cap K_2 \neq \varnothing\}$ is finite.

G has no fixed points if for all $g \in G, g \neq 1$, g has no fixed points.

Theorem 1.2.7. *If M is a connected complex manifold, G is properly discontinuous and has no fixed point, then the quotient space M/G is a complex manifold.*

Proof. Denote $M/G = M^*$, and the orbit of $p \in M$ by p^*. We shall show that for all $q \in M$, we can choose a neighborhood $U \ni q$ such that for all $p_1 \neq p_2 \in U$ we have $p_1^* \neq p_2^*$. In fact, we can choose U such that $gU \cap U = \varnothing$ for $g \in G, g \neq 1$. M is locally compact, so let $U_1 \supset U_2 \supset \cdots$ be a base of relatively compact neighborhoods at q. Then $F_m = \{g \in G : \ gU_m \cap U_m \neq \varnothing\}$ is finite, and $F_1 \supset F_2 \supset \cdots$. If there is a $g \neq 1$ such that $g \in F_m$ for all $m \geq 1$, then $U_m \to \{q\}$ gives $g(q) = q$, contradicting the nonexistence of fixed points. Hence the requiring U exists. We cover M by such U's, and $U \to U^*$ is one-to-one. We give U^* the complex structure that U has, then this gives a complex structure on M^*. ☐

Example 1.2.8 (Complex tori). Let $M = \mathbb{C}^n$, $\omega_1, \cdots, \omega_{2n}$ be $2n$ $\mathbb{R}$-linear independent vector, $\omega_k = (\omega_{k1}, \cdots, \omega_{kn}) \in \mathbb{C}^n$. Let $G = \mathbb{Z}\omega_1 + \cdots + \mathbb{Z}\omega_{2n}$ act on M naturally, then G is properly discontinuous and has no fixed points. The quotient space is called *compex torus* of dimension n, denoted by $\mathbb{T}^n$.

Let $n = 1$, we have the exponential map $\exp 2\pi\sqrt{-1} : \mathbb{C} \to \mathbb{C}^*$. Consider $G = \mathbb{Z} + \mathbb{Z}\omega$, then $\mathbb{T} = \mathbb{C}/G$. But let $\alpha = e^{2\pi\sqrt{-1}\omega}$, for $g = m_1 + m_2\omega$, we have the following commutative diagram

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\exp 2\pi\sqrt{-1}} & \mathbb{C}^* \\ \downarrow g & & \downarrow \alpha^{m_2} \\ \mathbb{C} & \xrightarrow{\exp 2\pi\sqrt{-1}} & \mathbb{C}^* \end{array}$$

Hence if we let $G^* = \mathbb{Z}\omega$ act on $\mathbb{C}^*$ by multiplication, then we have $\mathbb{T} = \mathbb{C}/G = \mathbb{C}^*/G^*$.

Example 1.2.9 (Hopf manifolds)**.** Let $W = \mathbb{C}^n \backslash \{0\}$ and

$$G = \{g^m : \ m \in \mathbb{Z}, g(w) = \lambda^n w, 0 < |\lambda| < 1\}$$

It is easy to see G is properly discontinuous and has no fixed points on W, so W/G is a complex manifold. Moreover, one can show that W/G is diffeomorphic to $\mathbb{S}^1 \times \mathbb{S}^{2n-1}$. In fact, using polar coordinate we have $W \cong \mathbb{R}_{>0} \times \mathbb{S}^{2n-1}$, scalar multiplication quotients $\mathbb{R}_{>0}$ gets $\mathbb{S}^1$ and preserves $\mathbb{S}^{2n-1}$, hence $W/G \cong \mathbb{S}^1 \times \mathbb{S}^{2n-1}$

Example 1.2.10. Let M be a algebraic surface defined by $M = \{\zeta \in \mathbb{P}^3 : \ \zeta_0^5 + \zeta_1^5 + \zeta_2^5 + \zeta_3^5 = 0\}$. Let

$$G = \{g^m : \ g(\zeta_0, \cdots, \zeta_3) = (\rho\zeta_0, \cdots, \rho^4\zeta_3), \rho = e^{2\pi\sqrt{-1}/5}\}$$

where $m = 0, 1, 2, 3, 4$. Then g is a biholomorphic map $\mathbb{P}^3 \to \mathbb{P}^3$ and the restriction of g^i on M is an analytic automorphism. Consider the fixed points of g^m on $\mathbb{P}^3$, they satisfy $(\rho^{m(i+1)} - 1)\zeta_i = 0$, $i = 0, 1, 2, 3$. So we have the fixed points of G are $(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)$. None of the fixed points lies on M, so G is properly discontinuous and has no fixed points on M, then M/G is a complex manifold.

Finally we discuss *surgeries*. Given a complex manifold M and a compact submanifold or subvariety $S \subset M$. Suppose we have neighborhood W of S and manifolds $S^* \subset W^*$, and we also have a biholomorphic map $f : W^* \backslash S^* \to W \backslash S$. Then we can replace W by W^* and get a new manifold $M^* = (M \backslash W) \cup W^*$. More precisely, $M^* = (M \backslash S) \cup W^*$, where each point $z^* \in W^* \backslash S^*$ is identified with $z = f(z^*)$.

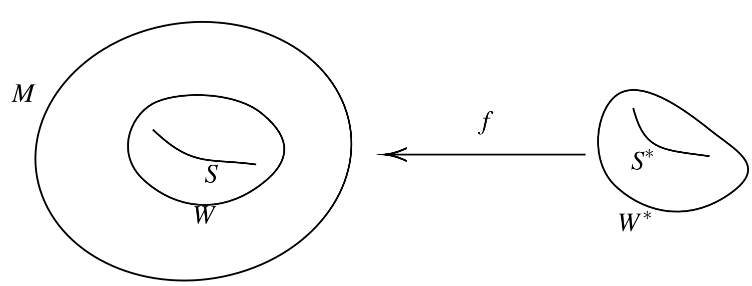

Figure 1.1: Surgery

Example 1.2.11 (Hirzebruch)**.** Let $M = \mathbb{P}^1 \times \mathbb{P}^1$. Since $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$, we can set $S = \{0\} \times \mathbb{P}^1$, $W = \{(z, \zeta) \in \mathbb{C} \times \mathbb{P}^1 : \ |z| < \varepsilon\}$ be a neighborhood of S in M. Let $W^* = \{(z, \zeta) \in \mathbb{C} \times (\mathbb{P}^1)^* : \ |z| < \varepsilon\}$ and $S = \{0\} \times (\mathbb{P}^1)^*$. Fix an integer $m > 0$ and define $f : W^* \backslash S^* \to W \backslash S$ by

$$f(z, \zeta^*) = (z, \zeta^*/z^m)$$

Then f is biholomorphic, let $M_m^* = (M\backslash S) \cup W^*$. Hirzebruch proved the following properties in [Hir51]:

1. M and M_m^* are topologically different if m is odd.
2. $M_m^* \ncong M_n^*$ as complex manifolds when $m \neq n$.
3. $M_{2m}^* = M$ topologically.
4. $M_{2m+1}^* = M_1^*$ topologically.

Example 1.2.12 (Blowing up)**.** First we discuss the case where M has complex dimension 2. Let p be any point on M, $S = \{p\}$ and $S^* = \mathbb{P}^1$. We define $M^* = (M\backslash S) \cup \mathbb{P}^1$ as follows: Choose a coordinate chart (W,z) such that $z(p) = 0$, $|z_1| < \varepsilon, |z_2| < \varepsilon$. We define a subvariety W^* of $W \times \mathbb{P}^1$ by

$$W^* := \{(z_1, z_2, (\zeta_1, \zeta_2)) \in W \times \mathbb{P}^1 : \ z_1\zeta_2 - z_2\zeta_1 = 0\}$$

Since $\partial f/\partial z_1 = \zeta_2, \partial f/\partial z_2 = -\zeta_1$ if $f = z_1\zeta_2 - z_2\zeta_1$, $(\partial f/\partial z_1, \partial f/\partial z_2) \neq 0$, hence W^* is a submanifold. Let $f^* : W^* \to W$ be the restriction of projection map $W \times \mathbb{P}^1 \to W$, then $W^* \supset 0 \times \mathbb{P}^1$, $f^* : S^* \to \{p\}$, and $f^* : W^*\backslash S^* \to W\backslash S$ is biholomorphic. That is because f^* has inverse $(z_1, z_2) \to (z_1, z_2, (z_1, z_2))$. By surgery we obtain $M^* = (M\backslash\{p\}) \cup \mathbb{P}^1$. We call M^* the *blowing up* of M at p, and denote $M^* = \mathrm{Bl}_p(M)$.

Blowing up can be complicated, a well-known fact in algebraic geometry is for six points $P_1, \cdots, P_6$ in "general position" (specified, no three points are colinear and no six points are on a conic), we have

$$\mathrm{Bl}_{P_6} \cdots \mathrm{Bl}_{P_1}(\mathbb{P}^2) \cong \{\zeta \in \mathbb{P}^3 : \ \zeta_0^3 + \zeta_1^3 + \zeta_2^3 + \zeta_3^3 = 0\} \subset \mathbb{P}^3$$

General case is similar, if $\dim_{\mathbb{C}} M = n$, let $p \in M$ and (W,z) be a coordinate chart as above. Define the submanifold $W^* := \{(z,\zeta) : \ z_i\zeta_j - z_j\zeta_i = 0, \ 1 \leq i < j \leq n\}$, and f^* be the restriction of projection map $W \times \mathbb{P}^1 \to W$. $f^* : (W^*\backslash\mathbb{P}^1) \to (W\backslash\{p\})$ is biholomorphic, so by surgery, we get $\mathrm{Bl}_p(M) = (M\backslash\{p\}) \cup \mathbb{P}^1$.

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