SEMINAR 1: FUNDAMENTAL GROUP

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1. Fundamental Group

1.1. **Definition.** We have learnt the notion of fundamental group in topology course, now we review it here.

A *homotopy* between maps $f, g : X \to Y$ is a map $h : I \times X \to Y$ such that $h(0, x) = f(x)$, $h(1, x) = g(x)$. Two paths $f, g : a \to b$ in X is defined to be equivalent if there exists a homotopy $h : I \times I \to X$ such that

$$h(t, 0) = a,\ h(t, 1) = b,\ h(0, s) = f(s),\ h(1, s) = g(s).$$

Denote the equivalence class of f by $[f]$. A *loop* is a path with $f(0) = f(1)$.

Define $\pi_1(X, x)$ to be the collection of all equivalent class of loops with starting point and end point x. We give a group structure on $\pi_1(X, x)$. Let $f : a \to b$, $g : b \to c$ be paths, define $g \cdot f$ to be the the path

$$g \cdot f(s) = \begin{cases} f(2s), & 0 \le s \le 1/2, \\ g(2s-1), & 1/2 \le s \le 1, \end{cases}$$

and define $[g][f] := [g \cdot f]$. We also define the inverse $g^{-1}(s) = g(1-s)$ and $[g]^{-1} = [g^{-1}]$. Moreover, the identity is defined to be the constant loop c_x. Clearly these definitions make $\pi_1(X, x)$ into a group.

1.2. **Choice of Base Point.** Let $a : x \to y$ be a path, we define a homomorphism from $\gamma[a] : \pi_1(X, x) \to \pi_1(X, y)$ to be $\gamma[a][f] = [a \cdot f \cdot a^{-1}]$. Clearly we have $\gamma[b \cdot a] = \gamma[b] \circ \gamma[a]$, hence $\gamma[a]$ is an isomorphism. Therefore, we have

Proposition 1.1. *For $x, y \in X$ in the same path connected component, $\pi_1(X, x)$ is (not necessarily canonical) isomorphic to $\pi_1(X, y)$.*

If $\pi_1(X, x)$ happens to be abelian, then let $a' : x \to y$ be another path, we have

$$\begin{aligned} \varphi[(a')^{-1}] \circ \varphi[a]([f]) &= [(a')^{-1} \cdot a][f][a^{-1} \cdot a'] \\ &= [f][(a')^{-1} \cdot a][a^{-1} \cdot a'] \\ &= [f], \end{aligned}$$

that is, the isomorphism between $\pi_1(X, x)$ and $\pi_1(X, y)$ does not depend on the choice of path, namely, canonical.

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1.3. **Homotopy Invariance.** Let $p : X \to Y$ be a map, define $p_* : \pi_1(X, x) \to \pi_1(Y, p(x))$ to be

$$p_*[f] = [p \circ f].$$

Clearly p_* is a homomorphism, and identity map $\mathrm{id} : X \to X$ induces the identity homomorphism. Moreover, if $p : X \to Y$, $q : Y \to Z$, then we have $q_* \circ p_* = (q \circ p)_*$.

Now suppose given two maps $p, q : X \to Y$ and a homotopy $h : p \simeq q$, a path $a(t) = h(x, t)$, we have

Proposition 1.2. *The following diagram is commutative:*

$$\begin{tikzcd} & \pi_1(X,x) \arrow[dl, "p_*"'] \arrow[dr, "q_*"] & \\ \pi_1(Y,p(x)) \arrow[rr, "\gamma[a]"] & & \pi_1(Y,q(x)). \end{tikzcd}$$

Proof. Let $f : I \to X$ be a loop, we must show $q \circ f$ is equivalent to $a \cdot (p \circ f) \cdot a^{-1}$. This is equivalent to show the constant loop $c_{p(x)}$ is equivalent to $a^{-1} \cdot (q \circ f)^{-1} \cdot a \cdot (p \circ f)$. Define $j : I \times I \to Y$ by $j(t, s) = h(t, f(s))$, then

$$j(0, s) = (p \circ f)(s),\ j(1, s) = (q \circ f)(s),\ j(t, 0) = a(t) = j(t, 1).$$

Thus we have a schematic diagram of j:

$$\begin{tikzcd} \cdot \arrow[r, "a"] & \cdot \\ \cdot \arrow[u, "p\circ f"] \arrow[r, "a"'] & \cdot \arrow[u, "q\circ f"'] \end{tikzcd}$$

Thus, going clockwise around the boundary starting at $(0, 0)$, we traverse $a^{-1} \cdot (q \circ f)^{-1} \cdot a \cdot (p \circ f)$. Since linear homotopy connects the boundary to origin, j induces a homotopy between the composite and $c_{p(x)}$. $\square$

1.4. **Calculations.**

Lemma 1.3. $\pi_1(\mathbb{R}, 0) = 0$.

Proof. We use linear homotopy $k : I \times \mathbb{R} \to \mathbb{R}$ by $k(t, s) = (1 - t)s$, then k connects identity and constant map at 0. Hence by homotopy invariance, we have $\pi_1(\mathbb{R}, 0) = 0$. $\square$

Proposition 1.4. $\pi_1(\mathbb{S}^1, 1) \cong \mathbb{Z}$.

Here we regard $\mathbb{S}^1 = \{z \in \mathbb{C} : \ |z| = 1\}$. We postpone the proof until we develop the theory of covering spaces.

2. Catagorical Language

2.1. **Categories.** A *category* consists of a collection of objects and a set $\hom(A, B)$ of morphisms for each pair of objects A and B. In each $\hom(A, B)$, there exists an identity morphism id and a composition law

$$\circ : \hom(B, C) \times \hom(A, B) \to \hom(A, C)$$

for each triple of objects A, B, C. Composition and identity morphisms must make the following diagrams commute:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "g\circ f"'] & B \arrow[d, "g"] \arrow[dr, "h\circ g"] & \\ & C \arrow[r, "h"] & D, \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "f"'] & B \arrow[d, "\mathrm{id}"] \\ & B, \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[d, "\mathrm{id}"'] \arrow[dr, "f"] & \\ A \arrow[r, "f"'] & B. \end{tikzcd}$$

A morphism $f \in \hom(A, B)$ is called an *isomorphism* if there exists a $g \in \hom(B, A)$ such that $g \circ f = \mathrm{id}_A$, $f \circ g = \mathrm{id}_B$.

We list some frequently used categories here.

- Set, category of sets, with morphisms to be mappings.
- Grpd, category of small groupoids. Small groupoids are defined to be small categories (i.e. whose collection of objects is a set) with all morphisms are isomorphisms, and morphisms between groupoids are functors (whose definition will be expalined later).
- Grp, category of groups. Groups are groupoids with single object.
- Ab, category of abelian groups.
- Top, category of topological spaces, with morphisms to be continuous mappings, i.e. maps.
- Top_*, category of pointed topological spaces, with morphisms to be maps that map the base point to base point.

We also define the *opposite category* of a category C (denoting C^{op}) consisting of same objects as C and $\hom_{\mathsf{C}^{\mathrm{op}}}(A, B) = \hom_{\mathsf{C}}(B, A)$.

2.2. **Functors and Natural Transformations.** We define a *covariant functor* F from category C to category D to be an assignment of each object A in C an object $F(A)$ in D, and of each morphism f in $\hom(A, B)$ a morphism $F(f) \in \hom(F(A), F(B))$. The functor $F : \mathsf{C} \to \mathsf{D}$ should be distributive to composition and preserve identity, that is, $F(g \circ f) = F(f) \circ F(g)$ for any $f \in \hom(A, B)$, $g \in \hom(B, C)$ and $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$. Also we define a *contravariant functor* from C to D to be a covariant functor from C^{op} to D.

A *natural transformation* from a functor $F : \mathsf{C} \to \mathsf{D}$ to a functor G is a family of morphisms $\varphi_A : F(A) \to G(A)$ making the following diagram commute for any object A, B and $f \in \hom(A, B)$

$$\begin{tikzcd} F(A) \arrow[r, "\varphi_A"] \arrow[d, "F(f)"] & G(A) \arrow[d, "G(f)"] \\ F(B) \arrow[r, "\varphi_B"] & G(B). \end{tikzcd}$$

Example 2.1. (1) The forgetful functor $\mathsf{Grp} \to \mathsf{Set}$, which maps a group to its underlying set.

(2) The free group functor $\mathsf{Set} \to \mathsf{Grp}$, which maps a set S to a free group $F(S)$ with basis S. A free group with basis S is characterized by the following universal property: any mapping from S to a group G factors through the natural embedding $\iota : S \to F(S)$, that is for $f : S \to G$, there exists a unique homomorphism $\tilde{f}$ making the following diagram

commute:

$$\begin{tikzcd} F(S) \arrow[r, dashed, "\tilde{f}"] & G \\ S. \arrow[u, "\iota"] \arrow[ur, "f"'] & \end{tikzcd}$$

If $g : S \to T$ is a mapping of sets, then $F(g)$ is induced by above universal property.

(3) The natural embedding of double dual in the category of vector spaces. Let Id be the identity functor, and denote the double dual of V by V^{**}. Then $v \mapsto (f \mapsto f(v))$ for any $v \in V$ and linear functional f defines an injection from V to V^{**}. One can check the family of injections consists a natural transformation from id to double dual. In particular, if V is finite dimensional, this shows V^{**} is canonically isomorphic to V.

2.3. **Colimits and Limits.** In category theory, an object A being initial means for any other objects B, $\hom(A, B)$ is a singleton. Being terminal reverses the morphisms, this means $\hom(B, A)$ is a singleton. A simple exercise is to show that initial objects and terminal objects are unique up to an isomorphism if they exist.

Let I be a small category. An I-shaped diagram in category C is a functor $F : \mathsf{I} \to \mathsf{C}$. A morphism between I-shaped diagrams is a natural transformation. Let A be an object in C, then there is a functor that maps all objects in I to A and all morphisms to id_A. Denote this I-diagram by $\underline{A}$.

Let F be an I-diagram, the *colimit* of F is the initial object among all morphisms having the form $\eta : F \to \underline{A}$, that is, η factors through $\iota : F \to \underline{\operatorname{colim} F}$. Conversely, the *limit* of F is the terminal object among all such morphisms, that is, $\varepsilon : \underline{A} \to F$ factors through $\pi : \underline{\lim F} \to F$. Expressing in diagrams, for each map $f : D \to D'$ in I, we have the commutative diagram for colimit:

$$\begin{tikzcd} F(D) \arrow[rr, "F(f)"] \arrow[dr, "\iota"] \arrow[ddr, bend right, "\eta"'] & & F(D') \arrow[dl, "\iota"'] \arrow[ddl, bend left, "\eta"] \\ & \operatorname{colim} F \arrow[d, dashed, "\tilde{\eta}"] & \\ & A, & \end{tikzcd}$$

and the commutative diagram for limit:

$$\begin{tikzcd} F(D) \arrow[rr, "F(f)"] & & F(D') \\ & \lim F \arrow[ul, "\pi"'] \arrow[ur, "\pi"] & \\ & A. \arrow[u, dashed, "\tilde{\varepsilon}"] \arrow[uul, bend left, "\varepsilon"] \arrow[uur, bend right, "\varepsilon"'] & \end{tikzcd}$$

Practically, if the diagram F consists of small category U, we usually write

$$\operatorname{colim} F = \operatorname*{colim}_{U \in \mathsf{U}} U,$$

and if I serves as an index set of $\{U_i\}_{i \in \mathsf{I}}$, we also write

$$\operatorname{colim} F = \operatorname*{colim}_{i \in \mathsf{I}} U_i.$$

For limits there are also similar notations.

Example 2.2. We list some notions here.

(1) Products and coproducts. If I is a discrete category, then the limits and colimits indexed on I are products and coproducts. The products in Set, Top, Grp are all Cartesian products with morphisms are projections. The coproducts in Set and Top are disjoint unions, and coproducts in Grp are free products (multiplications are adjunctions).
(2) Pushouts and pullbacks. This is the colimit of the diagram
$$e \longleftarrow d \longrightarrow f,$$
and the limit of the diagram
$$e \longrightarrow d \longleftarrow f,$$
respectively.
(3) Equalizers and coequalizers. This is the limit and colimit of the following diagram
$$d \rightrightarrows d'.$$

A category is said to be *complete* if it has all limits, and to be *cocomplete* if it has all colimits.

Proposition 2.3. *A category is cocomplete if and only if it has coproducts and coequalizers.*

Proof. One side is trivial, we prove that a category which has coproducts and coequalizers is cocomplete. We will abuse a lot of notation in the following proof.

Let $\{U_i\}_{i\in\mathsf{I}}$ be indexed by I. For any morphism $f : U_i \to U_j$, we compose $U_i \xrightarrow{f} U_j \xrightarrow{\iota} \bigsqcup_{j\in\mathsf{I}} U_j$, and obtain
$$\bigsqcup_{f:U_i\to U_j} U_i \xrightarrow{\varphi} \bigsqcup_{j\in\mathsf{I}} U_j.$$

Similarly, we obtain id between these two coproducts. Thus we have the coequalizer
$$\begin{tikzcd} \bigsqcup_{f:U_i\to U_j} U_i \arrow[rr, shift left, "\varphi"] \arrow[rr, shift right, "\mathrm{id}"'] \arrow[dr, "\eta"'] & & \bigsqcup_{j\in\mathsf{I}} U_j \arrow[dl, "\delta"] \\ & U. & \end{tikzcd}$$

We claim U is the colimit of $\{U_i\}_{i\in\mathsf{I}}$. We must verify the universal property. Suppose given a family of morphisms $f_i : U_i \to A$ compatible with morphisms in I, we check arbitrary commutative diagram
$$\begin{tikzcd} U_i \arrow[rr, "f"] \arrow[dr, "f_i"'] & & U_j \arrow[dl, "f_j"] \\ & A. & \end{tikzcd}$$

Let $\iota_i : U_i \to \bigsqcup_{f:U_i\to U_j} U_i$ embeds U_i to the one corresponds to f, $\iota_j : U_j \to \bigsqcup_{j\in\mathsf{I}} U_j$ be the natural embedding. Then f_j factors through ι_j, obtaining $f_j = \delta' \circ \iota_j$. For U_i corresponds to f, we have $f_j \circ f = f_i \circ \mathrm{id}$, by the universal property, this means $\delta' \circ \varphi = \delta' \circ \mathrm{id}$. Thus we can define

$\eta' = \delta' \circ \varphi$. Now we have a big commutative diagram:

$$\begin{tikzcd}
U_i \arrow[rr, "f"] \arrow[d, "\iota_i"] & & U_j \arrow[d, "\iota_j"] \\
\coprod_{f:U_i\to U_j} U_i \arrow[rr, shift left, "\varphi"] \arrow[rr, shift right, "\mathrm{id}"'] \arrow[dr, "\eta"] \arrow[ddr, "\eta'"'] & & \coprod_{j\in \mathsf{I}} U_j \arrow[dl, "\delta"'] \arrow[ddl, "\delta'"] \\
& U \arrow[d, dashed] & \\
& A, &
\end{tikzcd}$$

the last thing we need to do is to show $f_i = \gamma_i \circ \iota_i$, but this follows directly from the commutativity of whole diagram. □

Corollary 2.4. Grp *is cocomplete.*

Proof. We show that Grp verifies coequalizers. Let $\varphi : G \to H$ and $\psi : G \to H$ be two homomorphisms, set

$$K = H/\langle \varphi(g) \sim \psi(g) |\ g \in G\rangle.$$

It's clear that K satisfies the universal property. □

Corollary 2.5. *The pushout in* Grp $H \xleftarrow{\varphi} G \xrightarrow{\psi} K$ *is given explicitly by* $H * K$ *quotient out the normal subgroup generated by* $\varphi(g)\psi(g^{-1})$ *for all* $g \in G$.

Proof. By the construction of above proposition, the pushout is the coequalizer of

$$G * H * K * G * G \underset{\mathrm{id}}{\overset{f}{\rightrightarrows}} G * H * K.$$

Then by above corollary, the coequalizer must be

$$\begin{aligned}
G * H * K/\langle g_1 hk\varphi(g_2)\psi(g_3) \sim g_1 hk g_2 g_3\rangle &\cong G * H * K/\langle \varphi(g_2)\psi(g_3) \sim g_2 g_3\rangle \\
&\cong G * H * K/\langle \varphi(g) \sim \psi(g) \sim g\rangle \\
&\cong H * K/\langle \varphi(g) \sim \psi(g)\rangle.
\end{aligned}$$ □

By using opposite category formally, we can obtain

Proposition 2.6. *A category is complete if and only if it has products and equalizers.*

2.4. **Group Object.** Let C be a category which has products and terminal object $*$. A *group object* consists of an object G and a morphism $\mu : G \times G \to G$ called multiplication, a morphism $e : * \to G$ called unit element and a morphism $i : G \to G$ called inverse, satisfying

Associative law

$$\begin{tikzcd}
G \times G \times G \arrow[r, "\mu\times\mathrm{id}"] \arrow[d, "\mathrm{id}\times\mu"'] & G \times G \arrow[d, "\mu"] \\
G \times G \arrow[r, "\mu"] & G.
\end{tikzcd}$$

Identity law

$$\begin{tikzcd}
& G \arrow[dl, "\mathrm{id}\times e"'] \arrow[d, "\mathrm{id}"] \arrow[dr, "e\times\mathrm{id}"] & \\
G \times G \arrow[r, "\mu"] & G & G \times G. \arrow[l, "\mu"']
\end{tikzcd}$$

Inverse

$$\begin{tikzcd} & G \arrow[dl, "\mathrm{id}\times i"'] \arrow[d, "e"] \arrow[dr, "i\times \mathrm{id}"] & \\ G\times G \arrow[r, "\mu"] & G & G\times G. \arrow[l, "\mu"'] \end{tikzcd}$$

where e is the composition $G \to * \to G$.

Lemma 2.7. *If a functor preserves product, then it preserves group objects.*

Now we study topological groups. Topological groups with base point e are group objects in category Top_*, that is, G is a topological group if it is a topological space and multiplication and inverse are continuous. We want to show that

Proposition 2.8. *The fundamental group of a topological group (with base point e) is abelian.*

This proposition can be deduced by given direct homotopies, but we will adopt some "abstract nonsense" here.

Lemma 2.9. *Fundamental group functor $\pi_1 : \mathsf{Top}_* \to \mathsf{Grp}$ preserves product.*

Proof. We will check the universal properties elaborately. Let $\{X_\alpha\}_{\alpha\in A}$ be topological spaces, (X, p_α) be their product. Let G be a group, together with a family of homomorphisms $f_\alpha : G \to \pi_1(X_\alpha, x)$. Choose an element $g \in G$, suppose $f_\alpha(g) = [\gamma_\alpha] \in \pi_1(X_\alpha, x)$, $\alpha \in A$. Then $\gamma_\alpha : I \to X_\alpha$, by universal property, we have a $\gamma : I \to X$ such that $p_\alpha \circ \gamma = \gamma_\alpha$. Thus we define $f : G \to \pi_1(X, x)$ by $f(g) = [\gamma]$, then $p_{\alpha*}[\gamma] = [\gamma_\alpha]$. It's clear that such construction must be unique, hence $\pi_1(X, x)$ satisfies the universal property. Therefore

$$\pi_1\left(\prod_{\alpha\in A} X_\alpha, x\right) = \prod_{\alpha\in A} \pi_1(X_\alpha, x).$$ ☐

Lemma 2.10. *The group objects in Grp are abelian groups.*

Proof. The inverse is defined by $g \mapsto g^{-1}$, but it is a homomorphism, we have $(gh)^{-1} = g^{-1}h^{-1} = h^{-1}g^{-1}$, hence the group object is abelian. ☐

Now the proof can be reduced into one sentence.

Proof of Proposition 2.8. Since fundamental group functor respects products, it maps group objects to group objects, that is, topological groups to abelian groups. ☐

2.5. **Categorical Equivalence and Skeleton.** We introduce the notion of two categories being equivalent. Two categories C and D are *equivalent* if there are functors $F : \mathsf{C} \to \mathsf{D}$ and $G : \mathsf{D} \to \mathsf{C}$ and natural isomorphisms $G \circ F \to \mathrm{id}_{\mathsf{C}}$ and $F \circ G \to \mathrm{id}_{\mathsf{D}}$.

Also we introduce the notion of a *skeleton* $\operatorname{sk}\mathsf{C}$ of a category C. This is a "full" subcategory with one object from each isomorphism class of objects of C. "Full" means that the morphisms between two objects of $\operatorname{sk}\mathsf{C}$ are the morphisms between all the objects in two isomorphism classes.

$\operatorname{sk}\mathsf{C}$ is equivalent to C. The inclusion functor $J : \operatorname{sk}\mathsf{C} \to \mathsf{C}$ gives the equivalence. An inverse functor $F : \mathsf{C} \to \operatorname{sk}\mathsf{C}$ is obtained by letting $F(A)$ be the unique object in $\operatorname{sk}\mathsf{C}$ that is isomorphic to A, choosing an isomorphism $\alpha_A : A \to F(A)$ and defining $F(f) = \alpha_B \circ f \circ \alpha_A^{-1} : F(A) \to F(B)$ for a morphism $f \in \hom(A, B)$. We choose α_A to be identity if A is in $\operatorname{sk}\mathsf{C}$, and then $F \circ J = \mathrm{id}$, and by definition α_A's specify a natural isomorphism $\alpha : \mathrm{id} \to J \circ F$.

3. Fundamental Groupoids

Recall that a *groupoid* is a category whose morphisms are all isomorphisms. Let morphisms between groupoids be functors, we have the category Grpd of groupoids.

The *fundamental groupoid* is a functor $\Pi : \mathsf{Top} \to \mathsf{Grpd}$ that maps a space X to all the equivalence classes of paths in X.

We mention here the set of automorphisms of a point $x \in X$ is exactly the fundamental group $\pi_1(X, x)$. Therefore, we have

Proposition 3.1. *Let X be a path connected space. For each point $x \in X$, the inclusion $\pi_1(X, x) \to \Pi(X)$ is an equivalence of categories.*

Proof. We regard $\pi_1(X, x)$ as a category with a single object x, and it is a skeleton of $\Pi(X)$. To show this, we notice that any $y \in X$ can be connected with x by a path γ, and $[\gamma]$ is an isomorphism in $\Pi(X)$. □

4. The van Kampen Theorem

4.1. **Statement of the Theorem.** We state the van Kampen theorem as follows.

Theorem 4.1 (van Kampen)**.** *Let X be a path connected space, choose a base point $x \in X$. Let $\mathscr{O}$ be a cover of X by path connected open subsets such that the intersection of finite many subsets in $\mathscr{O}$ is still in $\mathscr{O}$, and x is in each $U \in \mathscr{O}$. Regard $\mathscr{O}$ as a category whose morphisms are inclusions of subsets, and observe that the functor $\pi_1(\cdot, x)$ restricted to the space and maps in $\mathscr{O}$, gives a diagram*

$$\pi_1|_{\mathscr{O}} : \mathscr{O} \to \mathsf{Grp}$$

of groups. Then we have

$$\pi_1(X, x) \cong \operatorname*{colim}_{U \in \mathscr{O}} \pi_1(U, x).$$

A useful special case of van Kampen theorem is that the space is covered by two open subsets.

Corollary 4.2. *Let $X = U \cup V$ with U, V open and path connected, $x \in U \cap V$. Then $\pi_1(X, x)$ is the pushout of the following diagram*

$$\begin{tikzcd} \pi_1(U \cap V, x) \arrow[r] \arrow[d] & \pi_1(U, x) \arrow[d, dashed] \\ \pi_1(V, x) \arrow[r, dashed] & \pi_1(X, x). \end{tikzcd}$$

4.2. **Examples of the van Kampen Theorem.** We give some examples of using van Kampen theorem to calculate fundamental group. We now drop the notation for the base point, writing $\pi_1(X)$ instead of $\pi_1(X, x)$.

Proposition 4.3. *Let $X = \bigvee_{i \in I} X_i$ with X_i's being path connected, and each of X_i contains a contractible neighborhood V_i for its base point. Then $\pi_1(X)$ is the coproduct (i.e. free product) of groups $\pi_1(X_i)$.*

Proof. Let U_i be the union of X_i and V_j for all $j \neq i$. Take $\mathscr{O}$ be the U_i and their finite intersections, and apply van Kampen theorem to $\mathscr{O}$. Since any intersection of two or more of the U_i is contractible, the intersections make no contribution to the colimit, and the colimit must be the coproduct. □

Corollary 4.4. *The fundamental group of a wedge sum of circles is a free group with one generator for each circle.*

We now give two examples of computing the fundamental group of compact surfaces. By the classification theorem of compact surfaces, any compact surface is homeomorphic to a sphere, or to a connected sum of tori $\mathbb{T}^2 := \mathbb{S}^1 \times \mathbb{S}^1$, or to a connect sum of projective planes $\mathbb{RP}^2 = \mathbb{S}^2/\mathbb{Z}_2$. We will see shortly that $\pi_1(\mathbb{RP}^2) = \mathbb{Z}_2$. Using this fact, we can compute the fundamental group of all compact surfaces by induction. We will illustrate by two examples.

Example 4.5. We compute the fundamental group of two-holed torus $\mathbb{T}^2\#\mathbb{T}^2$. We decompose the computation into several steps.

Step 1. $\pi_1(\mathbb{T}^2 - \{\mathrm{pt}\}) \cong \mathbb{Z} * \mathbb{Z}$. Linear homotopy shows $\mathbb{T}^2 - \{\mathrm{pt}\} \simeq \mathbb{S}^1 \vee \mathbb{S}^1$, the conclusion follows from Corollary 4.4.

Step 2. Let $A \subset \mathbb{T}^2 - \mathbb{D}^2$ be an annulus with inner boundary being $\partial\mathbb{D}^2$, A' also be an annulus on another punched torus chosen similarly, then $\mathbb{T}^2\#\mathbb{T}^2 \cong (\mathbb{T}^2 - \mathbb{D}^2) \sqcup_f (\mathbb{T}^2 - \mathbb{D}^2)$ with a suitable $f : A \to A'$. We compute $\pi_1(A) \hookrightarrow \pi_1(\mathbb{T}^2 - \mathbb{D}^2)$. First, we notice that $\mathbb{T}^2 - \mathbb{D}^2 \simeq \mathbb{T}^2 - \{\mathrm{pt}\}$. Moreover, since annulus is homeomorphic to $\mathbb{S}^1 \times I$, a loop in I does not contribute to $\pi_1(A)$, hence we check a loop in the outer boundary of A. Let a, b be the two generators of each circle in $\mathbb{S}^1 \vee \mathbb{S}^1$, then the linear homotopy sends a loop with n rounds to $(aba^{-1}b^{-1})^n$ (this can be shown by the polyhedral representation of $\mathbb{T}^2$). Thus we have

$$\begin{aligned} \pi_1(A) \cong \mathbb{Z} &\hookrightarrow \pi_1(\mathbb{T}^2 - \mathbb{D}^2) \cong \mathbb{Z} * \mathbb{Z} \\ n &\mapsto (aba^{-1}b^{-1})^n. \end{aligned}$$

Step 3. Apply van Kampen theorem to the cover $\{\mathbb{T}^2 - \mathbb{D}^2, \mathbb{T}^2 - \mathbb{D}^2, A\}$, we have $\pi_1(\mathbb{T}^2\#\mathbb{T}^2)$ is the pushout

$$\begin{tikzcd} \pi_1(A) \arrow[r] \arrow[d] & \pi_1(\mathbb{T}^2 - \mathbb{D}^2) \\ \pi_1(\mathbb{T}^2 - \mathbb{D}^2). & \end{tikzcd}$$

Let the fundamental group of two punched torus be $\langle a_1, b_1\rangle$ and $\langle a_2, b_2\rangle$ respectively, then we have

$$\pi_1(\mathbb{T}^2\#\mathbb{T}^2) \cong \langle a_1, b_1, a_2, b_2 |\ a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1}\rangle,$$

here the second inclusion reverses the orientation.

Example 4.6. The same process can be applied to Klein bottle $\mathbb{RP}^2\#\mathbb{RP}^2$. The key points are $\mathbb{RP}^2 - \{\mathrm{pt}\} \simeq \mathbb{S}^1$,

$$\begin{aligned} \pi_1(A) \cong \mathbb{Z} &\hookrightarrow \pi_1(\mathbb{RP}^2 - \mathbb{D}^2) \cong \mathbb{Z} \\ n &\mapsto a^{2n}, \end{aligned}$$

and hence $\pi_1(\mathbb{RP}^2\#\mathbb{RP}^2) \cong \langle a, b |\ a^2 = b^2\rangle$. Notice that this group is isomorphic to $\langle a, b |\ abab^{-1}\rangle$, which is computed by the polyhedral representation of Klein bottle.

5. Proof of van Kampen Theorem

We provide the proof of van Kampen theorem in this section. We first state the groupoid version of van Kampen theorem.

Theorem 5.1 (van Kampen)**.** *Let $\mathscr{O}$ be a cover of a space X by path connected open subsets such that the intersection of finite many subsets in $\mathscr{O}$ is again in $\mathscr{O}$. Regard $\mathscr{O}$ as a category whose morphisms are the inclusions of subsets and observe that the functor* Π*, restricted to the space and maps in $\mathscr{O}$, gives a diagram*

$$\Pi|_{\mathscr{O}} : \mathscr{O} \to \mathsf{Grpd}$$

of groupoids. Then we have

$$\Pi(X) \cong \operatorname*{colim}_{U \in \mathscr{U}} \Pi(U).$$

Proof. We must verify the universal property. Let C be a groupoid, and a morphism $\eta : \Pi|_{\mathscr{O}} \to \underline{\mathsf{C}}$ of $\mathscr{O}$-shaped diagrams of groupoids, we must construct a morphism, i.e. a functor $\tilde{\eta} : \Pi(X) \to \mathsf{C}$ of groupoids that restricts to η_U on $\Pi(U)$ for each $U \in \mathscr{O}$.

On objects, that is, the points of X, we must define $\tilde{\eta}(x) = \eta_U(x)$ for $x \in U$. This is independent of the choice of U since $\mathscr{O}$ is closed under finite intersections.

We now consider morphisms, that is, the homotopy classes of paths in X. If $f : x \to y$ lies entirely in a particular U, then we must define $\tilde{\eta}([f]) = \eta_U([f])$. Again, since $\mathscr{O}$ is closed under finite intersections, this specification is independent of the choice of U if f lies entirely in more than one U. By compactness, any path f is the composite of finite many paths f_i, each of which lies entirely in single U. Then we define $\tilde{\eta}([f])$ to be the composite of $\tilde{\eta}([f_i])$'s. We must show $\tilde{\eta}$ is well-defined. Suppose $h : f \simeq g$ is a homotopy, then the subdivision argument shows $\tilde{\eta}([f]) = \tilde{\eta}([g])$. Precisely, this means we subdivide $I \times I$ into sufficiently small subsquares, and by Lebesgue's lemma, each square falls into a single $U \in \mathscr{O}$. Then $[f] = [g]$ in $\Pi(X)$ is a consequence of a finite number of relations, each of which holds in one of the $\Pi(U)$. Therefore $\tilde{\eta}([f]) = \tilde{\eta}([g])$.

The above construction is clearly unique, hence this verifies the universal property and proves the theorem. □

The fundamental group version of van Kampen theorem (Theorem 4.1) is proved by categorical nonsense.

Lemma 5.2. *The van Kampen theorem (Theorem 4.1) holds when the cover $\mathscr{O}$ is finite.*

Proof. This is based on the categorical nonsense about skeleta of categories. Given a morphism between $\mathscr{O}$-shaped diagrams $\eta : \pi_1|_{\mathscr{O}} \to \underline{G}$, we must show that there is a unique homomorphism $\tilde{\eta} : \pi_1(X, x) \to G$ that restricts to η_U on $\pi_1(U, x)$.

Let $J_U : \pi_1(U, x) \to \Pi(U)$ be the inclusion functor, then J_U is a categorical equivalence. The inverse is given by for $y \in X$, the path c_x if $y = x$, so that $F_U \circ J_U = \mathrm{id}$; and a path lies entirely in U as long as $y \in U$, this is possible since $\mathscr{O}$ is finite, we can take U be the intersection of all open subsets in $\mathscr{O}$ that contains y. This ensures the chosen paths determine compatible inverse equivalence. Thus the functors $\pi(U) \xrightarrow{F_U} \pi_1(U, x) \xrightarrow{\eta_U} G$ specify an $\mathscr{O}$-shaped diagram of groupoids $\Pi|_{\mathscr{O}} \to \underline{G}$. By the fundamental groupoid version of van Kampen theorem, there is a unique map of groupoids $\xi : \Pi(X) \to G$ that restricts to $\eta_U \circ F_U$ on $\Pi(U)$ for each U. Then the composite $\pi_1(X, x) \xrightarrow{J} \Pi(X) \xrightarrow{\xi} G$ is the required homomorphism $\tilde{\eta}$. By the commutativity of the diagram

$$\begin{tikzcd}
& \pi_1(X,x) \arrow[dl, "\mathrm{id}"'] & \\
\pi_1(X,x) \arrow[r, "J"] & \Pi(X) \arrow[u, "F"'] \arrow[r, "\xi"] & G \\
\pi_1(U,x) \arrow[u, dashed] \arrow[r, "J_U"] \arrow[rr, bend right, "\mathrm{id}"'] & \Pi(U) \arrow[u] \arrow[r, "F_U"] & \pi_1(U,x), \arrow[u, "\eta_U"']
\end{tikzcd}$$

$\tilde{\eta}$ restricts to η_U on $\pi_1(U, x)$ for $U \in \mathscr{O}$. It is unique because ξ is unique. Hence we verified the universal property. □

Proof of van Kampen Theorem. Let $\mathscr{F}$ be the set of those finite subsets of the cover $\mathscr{O}$ that are closed under finite intersection. For $\mathscr{S} \in \mathscr{F}$, let $U_{\mathscr{S}}$ be the union of U's in $\mathscr{S}$. Then $\mathscr{S}$ is a cover of $U_{\mathscr{S}}$ to which the lemma applies. Thus

$$\operatorname*{colim}_{U \in \mathscr{S}} \pi_1(U, x) \cong \pi_1(U_{\mathscr{S}}, x).$$

Regard $\mathscr{F}$ as a category with a morphism $\mathscr{S} \to \mathscr{T}$ whenever $U_{\mathscr{S}} \subset U_{\mathscr{T}}$. We claim first that

$$\operatorname*{colim}_{\mathscr{S} \in \mathscr{F}} \pi_1(U_{\mathscr{S}}, x) \cong \pi_1(X, x).$$

In fact, by the subdivision argument, any loop $I \to X$ and any homotopy equivalence $h : I \times I \to X$ between loops has image in some $U_{\mathscr{S}}$. This implies that $\pi_1(X, x)$, together with the homomorphisms $\pi_1(U_{\mathscr{S}}, x) \to \pi_1(X, x)$, has the universal property that characterizes the claimed colimit. We claim next that

$$\begin{aligned} \operatorname*{colim}_{U \in \mathscr{O}} \pi_1(U, x) &\cong \operatorname*{colim}_{\mathscr{S} \in \mathscr{F}} \pi_1(U_{\mathscr{S}}, x) \\ &\cong \operatorname*{colim}_{\mathscr{S} \in \mathscr{F}} \operatorname*{colim}_{U \in \mathscr{S}} \pi_1(U, x), \end{aligned}$$

and this will complete the proof. By a comparison of universal properties, this iterated colimit is isomorphic to the single colimit

$$\operatorname*{colim}_{(U, \mathscr{S}) \in (\mathscr{O}, \mathscr{F})} \pi_1(U, x),$$

here the indexing category $(\mathscr{O}, \mathscr{F})$ has objects the pairs $(U, \mathscr{S})$ with $U \in \mathscr{S}$; a morphism $(U, \mathscr{S}) \to (V, \mathscr{T})$ whenever both $U \subset V$ and $U_{\mathscr{S}} \subset U_{\mathscr{T}}$. We show the isomorphism. We first give a general construction. For categories $\mathsf{C} \xrightarrow{F} \mathsf{D} \xrightarrow{G} \mathsf{E}$ with E cocomplete, we construct a natural morphism $F_* : \operatorname{colim}_{c \in \mathsf{C}} G \circ F(c) \to \operatorname{colim}_{d \in \mathsf{D}} G(d)$. Let $c \in \mathsf{C}$, then $F(c) \in \mathsf{D}$, we have a natural morphism $G \circ F(c) = G(F(c)) \to \operatorname{colim}_{d \in \mathsf{D}} G(d)$, and then by universal property, there is a unique $F_* : \operatorname{colim}_{c \in \mathsf{C}} G \circ F(c) \to \operatorname{colim}_{d \in \mathsf{D}} G(d)$. Now let functors $i : \mathscr{O} \to (\mathscr{O}, \mathscr{F})$, $U \mapsto (U, \{U\})$, and $p : (\mathscr{O}, \mathscr{F}) \to \mathscr{O}$, $(U, \mathscr{S}) \to U$. We omit the notation for base point, and the required isomorphism becomes

$$\operatorname*{colim}_{U \in \mathscr{O}} \pi_1(U) \cong \operatorname*{colim}_{(U, \mathscr{S}) \in (\mathscr{O}, \mathscr{F})} \pi_1 \circ p(U, \mathscr{S}).$$

Notice that $p \circ i = \mathrm{id}$, by the composite of functors

$$\mathscr{O} \xrightarrow{i} (\mathscr{O}, \mathscr{F}) \xrightarrow{p} \mathscr{O} \xrightarrow{\pi_1} \mathsf{Grp},$$

we have a commutative diagram

$$\begin{tikzcd} \pi_1 \circ p \circ i(U) \arrow[r, equal] \arrow[d] & \pi_1 \circ p(U, \{U\}) = \pi_1 \circ p(U, \mathscr{S}) \arrow[r, equal] \arrow[d] & \pi_1(U) \arrow[d] \\ \operatorname*{colim}_{U \in \mathscr{O}} \pi_1 \circ p \circ i(U) \arrow[r, "i_*"] \arrow[rr, dashed, bend right] & \operatorname*{colim}_{(U, \mathscr{S}) \in (\mathscr{O}, \mathscr{F})} \pi_1 \circ p(U, \mathscr{S}) \arrow[r, "p_*"] & \operatorname*{colim}_{U \in \mathscr{O}} \pi_1(U). \end{tikzcd}$$

The right arrows in second row are homomorphisms induced by functor i and p, hence the downward arrows are all natural homomorphisms. But since $p \circ i = \mathrm{id}$, $\operatorname{colim}_{U \in \mathscr{O}} \pi_1 \circ p \circ i(U) =$

$\operatorname{colim}_{U\in\mathscr{O}}\pi_1(U)$, the dashed arrow must be identity. Similarly, we have commutative diagram

$$\begin{tikzcd}
\pi_1\circ p\circ i\circ p(U,\mathscr{S}) \arrow[r, equal] \arrow[d] & \pi_1\circ p\circ i(U) \arrow[r, equal] \arrow[d] & \pi_1\circ p(U,\{U\})=\pi_1\circ p(U,\mathscr{S}) \arrow[d] \\
\operatorname*{colim}\limits_{(U,\mathscr{S})\in(\mathscr{O},\mathscr{F})}\pi_1\circ p\circ i\circ p(U,\mathscr{S}) \arrow[r, "p_*"] \arrow[rr, bend right, dashed] & \operatorname*{colim}\limits_{U\in\mathscr{O}}\pi_1\circ p\circ i(U) \arrow[r, "i_*"] & \operatorname*{colim}\limits_{(U,\mathscr{S})\in(\mathscr{O},\mathscr{F})}\pi_1\circ p(U,\mathscr{S}),
\end{tikzcd}$$

and the dashed homomorphism is identity. This shows $\operatorname{colim}_{U\in\mathscr{O}}\pi_1(U)\cong\operatorname{colim}_{(U,\mathscr{S})\in(\mathscr{O},\mathscr{F})}\pi_1\circ p(U,\mathscr{S})$. □