# SEMINAR 1: FUNDAMENTAL GROUP

ZENG MENGCHEN

## 1. Fundamental Group

1.1. **Definition.** We have learnt the notion of fundamental group in topology course, now we review it here.

A *homotopy* between maps $f, g : X \to Y$ is a map $h : I \times X \to Y$ such that $h(0, x) = f(x)$, $h(1, x) = g(x)$. Two paths $f, g : a \to b$ in $X$ is defined to be equivalent if there exists a homotopy $h : I \times I \to X$ such that

$$h(t, 0) = a,\ h(t, 1) = b,\ h(0, s) = f(s),\ h(1, s) = g(s).$$

Denote the equivalence class of $f$ by $[f]$. A *loop* is a path with $f(0) = f(1)$.

Define $\pi_1(X, x)$ to be the collection of all equivalent class of loops with starting point and end point $x$. We give a group structure on $\pi_1(X, x)$. Let $f : a \to b$, $g : b \to c$ be paths, define $g \cdot f$ to be the the path

$$g \cdot f(s) = \begin{cases} f(2s), & 0 \le s \le 1/2, \\ g(2s - 1), & 1/2 \le s \le 1, \end{cases}$$

and define $[g][f] := [g \cdot f]$. We also define the inverse $g^{-1}(s) = g(1 - s)$ and $[g]^{-1} = [g^{-1}]$. Moreover, the identity is defined to be the constant loop $c_x$. Clearly these definitions make $\pi_1(X, x)$ into a group.

1.2. **Choice of Base Point.** Let $a : x \to y$ be a path, we define a homomorphism from $\gamma[a] : \pi_1(X, x) \to \pi_1(X, y)$ to be $\gamma[a][f] = [a \cdot f \cdot a^{-1}]$. Clearly we have $\gamma[b \cdot a] = \gamma[b] \circ \gamma[a]$, hence $\gamma[a]$ is an isomorphism. Therefore, we have

**Proposition 1.1.** *For $x, y \in X$ in the same path connected component, $\pi_1(X, x)$ is (not necessarily canonical) isomorphic to $\pi_1(X, y)$.*

If $\pi_1(X, x)$ happens to be abelian, then let $a' : x \to y$ be another path, we have

$$\begin{aligned} \varphi[(a')^{-1}] \circ \varphi[a]([f]) &= [(a')^{-1} \cdot a][f][a^{-1} \cdot a'] \\ &= [f][(a')^{-1} \cdot a][a^{-1} \cdot a'] \\ &= [f], \end{aligned}$$

that is, the isomorphism between $\pi_1(X, x)$ and $\pi_1(X, y)$ does not depend on the choice of path, namely, canonical.

1.3. **Homotopy Invariance.** Let $p : X \to Y$ be a map, define $p_* : \pi_1(X, x) \to \pi_1(Y, p(x))$ to be

$$p_*[f] = [p \circ f].$$

Clearly $p_*$ is a homomorphism, and identity map $\mathrm{id} : X \to X$ induces the identity homomorphism. Moreover, if $p : X \to Y$, $q : Y \to Z$, then we have $q_* \circ p_* = (q \circ p)_*$.

Now suppose given two maps $p, q : X \to Y$ and a homotopy $h : p \simeq q$, a path $a(t) = h(x, t)$, we have

*Date*: September 6, 2024.

**Proposition 1.2.** *The following diagram is commutative:*

$$
\begin{tikzcd}
 & \pi_1(X,x) \arrow[dl, "p_*"'] \arrow[dr, "q_*"] & \\
\pi_1(Y,p(x)) \arrow[rr, "\gamma[a]"] & & \pi_1(Y,q(x)).
\end{tikzcd}
$$

*Proof.* Let $f : I \to X$ be a loop, we must show $q \circ f$ is equivalent to $a \cdot (p \circ f) \cdot a^{-1}$. This is equivalent to show the constant loop $c_{p(x)}$ is equivalent to $a^{-1} \cdot (q \circ f)^{-1} \cdot a \cdot (p \circ f)$. Define $j : I \times I \to Y$ by $j(t,s) = h(t, f(s))$, then

$$j(0,s) = (p \circ f)(s),\ j(1,s) = (q \circ f)(s),\ j(t,0) = a(t) = j(t,1).$$

Thus we have a schematic diagram of $j$:

$$
\begin{tikzcd}
\cdot \arrow[r, "a"] & \cdot \\
\cdot \arrow[u, "p\circ f"] \arrow[r, "a"'] & \cdot \arrow[u, "q\circ f"']
\end{tikzcd}
$$

Thus, going clockwise around the boundary starting at $(0,0)$, we traverse $a^{-1} \cdot (q \circ f)^{-1} \cdot a \cdot (p \circ f)$. Since linear homotopy connects the boundary to origin, $j$ induces a homotopy between the composite and $c_{p(x)}$. $\square$

1.4. **Calculations.**

**Lemma 1.3.** $\pi_1(\mathbb{R}, 0) = 0$.

*Proof.* We use linear homotopy $k : I \times \mathbb{R} \to \mathbb{R}$ by $k(t,s) = (1-t)s$, then $k$ connects identity and constant map at 0. Hence by homotopy invariance, we have $\pi_1(\mathbb{R}, 0) = 0$. $\square$

**Proposition 1.4.** $\pi_1(\mathbb{S}^1, 1) \cong \mathbb{Z}$.

Here we regard $\mathbb{S}^1 = \{z \in \mathbb{C} :\ |z| = 1\}$. We postpone the proof until we develop the theory of covering spaces.

## 2. Catagorical Language

2.1. **Categories.** A *category* consists of a collection of objects and a set $\hom(A,B)$ of morphisms for each pair of objects $A$ and $B$. In each $\hom(A,B)$, there exists an identity morphism id and a composition law

$$\circ : \hom(B,C) \times \hom(A,B) \to \hom(A,C)$$

for each triple of objects $A, B, C$. Composition and identity morphisms must make the following diagrams commute:

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, "g\circ f"'] & B \arrow[d, "g"] \arrow[dr, "h\circ g"] & \\
 & C \arrow[r, "h"] & D,
\end{tikzcd}
$$

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, "f"'] & B \arrow[d, "\mathrm{id}"] \\
 & B,
\end{tikzcd}
$$

$$\begin{tikzcd} A \arrow[d, "\mathrm{id}"'] \arrow[dr, "f"] & \\ A \arrow[r, "f"'] & B. \end{tikzcd}$$

A morphism $f \in \hom(A, B)$ is called an *isomorphism* if there exists a $g \in \hom(B, A)$ such that $g \circ f = \mathrm{id}_A$, $f \circ g = \mathrm{id}_B$.

We list some frequently used categories here.

- $\mathsf{Set}$, category of sets, with morphisms to be mappings.
- $\mathsf{Grpd}$, category of groupoids. Groupoids are defined to be small categories (i.e. whose collection of objects is a set) with all morphisms are isomorphisms, and morphisms between groupoids are functors (whose definition will be expalined later).
- $\mathsf{Grp}$, category of groups. Groups are groupoids with single object.
- $\mathsf{Ab}$, category of abelian groups.
- $\mathsf{Top}$, category of topological spaces, with morphisms to be continuous mappings, i.e. maps.
- $\mathsf{Top}^*$, category of pointed topological spaces, with morphisms to be maps that map the base point to base point.

We also define the *opposite category* of a category $\mathsf{C}$ (denoting $\mathsf{C}^{\mathrm{op}}$) consisting of same objects as $\mathsf{C}$ and $\hom_{\mathsf{C}^{\mathrm{op}}}(A, B) = \hom_{\mathsf{C}}(B, A)$.

2.2. **Functors and Natural Transformations.** We define a *covariant functor* $F$ from category $\mathsf{C}$ to category $\mathsf{D}$ to be an assignment of each object $A$ in $\mathsf{C}$ an object $F(A)$ in $\mathsf{D}$, and of each morphism $f$ in $\hom(A, B)$ a morphism $F(f) \in \hom(F(A), F(B))$. Also we define a *contravariant functor* from $\mathsf{C}$ to $\mathsf{D}$ to be a covariant functor from $\mathsf{C}^{\mathrm{op}}$ to $\mathsf{D}$.

A *natural transformation* from a functor $F : \mathsf{C} \to \mathsf{D}$ to a functor $G$ is a family of morphisms $\varphi_A : F(A) \to G(A)$ making the following diagram commute for any object $A, B$ and $f \in \hom(A, B)$

$$\begin{tikzcd} F(A) \arrow[r, "\varphi_A"] \arrow[d, "F(f)"] & G(A) \arrow[d, "G(f)"] \\ F(B) \arrow[r, "\varphi_B"] & G(B). \end{tikzcd}$$

**Example 2.1.** (1) The forgetful functor $\mathsf{Grp} \to \mathsf{Set}$, which maps a group to its underlying set.

(2) The free group functor $\mathsf{Set} \to \mathsf{Grp}$, which maps a set $S$ to a free group $F(S)$ with basis $S$. A free group with basis $S$ is characterized by the following universal property: any mapping from $S$ to a group $G$ factors through the natural embedding $\iota : S \to F(S)$, that is for $f : S \to G$, there exists a unique homomorphism $\tilde{f}$ making the following diagram commute:

$$\begin{tikzcd} F(S) \arrow[r, dashed, "\tilde{f}"] & G \\ S. \arrow[u, "\iota"] \arrow[ur, "f"] & \end{tikzcd}$$

If $g : S \to T$ is a mapping of sets, then $F(g)$ is induced by above universal property.

(3) The natural embedding of double dual in the category of vector spaces. Let Id be the identity functor, and denote the double dual of $V$ by $V^{**}$. Then $v \mapsto (f \mapsto f(v))$ for any $v \in V$ and linear functional $f$ defines an injection from $V$ to $V^{**}$. One can check the family of injections consists a

natural transformation from Id to double dual. In particular, if $V$ is finite dimensional, this shows $V^{**}$ is canonically isomorphic to $V$.

2.3. **Colimits and Limits.** In category theory, an object $A$ being initial means for any other objects $B$, $\hom(A, B)$ is a singleton. Being terminal reverses the morphisms, this means $\hom(B, A)$ is a singleton. A simple exercise is to show that initial objects and terminal objects are unique up to an isomorphism if they exist.

Let $\mathsf{I}$ be a small category. An $\mathsf{I}$-shaped diagram in category $\mathsf{C}$ is a functor $F : \mathsf{I} \to \mathsf{C}$. A morphism between $\mathsf{I}$-shaped diagrams is a natural transformation. Let $A$ be an object in $\mathsf{C}$, then there is a functor that maps all objects in $\mathsf{I}$ to $A$ and all morphisms to $\mathrm{id}_A$. Denote this $\mathsf{I}$-diagram by $\underline{A}$.

Let $F$ be an $\mathsf{I}$-diagram, the *colimit* of $F$ is the initial object among all morphisms having the form $\eta : F \to \underline{A}$, that is, $\eta$ factors through $\iota : F \to \underline{\operatorname{colim} F}$. Conversely, the *limit* of $F$ is the terminal object among all such morphisms, that is, $\varepsilon : F \to \underline{A}$ factors through $\pi : F \to \underline{\lim F}$. Expressing in diagrams, for each map $f : D \to D'$ in $\mathsf{I}$, we have the commutative diagram for colimit:

$$\begin{tikzcd}
F(D) \arrow[rr, "F(f)"] \arrow[dr, "\iota"] \arrow[ddr, "\eta"'] & & F(D') \arrow[dl, "\iota"'] \arrow[ddl, "\eta"] \\
& \operatorname{colim} F \arrow[d, dashed, "\tilde{\eta}"] & \\
& A, &
\end{tikzcd}$$

and the commutative diagram for limit:

$$\begin{tikzcd}
F(D) \arrow[rr, "F(f)"] & & F(D') \\
& \lim F \arrow[ul, "\pi"'] \arrow[ur, "\pi"] & \\
& A. \arrow[uul, "\varepsilon"] \arrow[uur, "\varepsilon"'] \arrow[u, dashed, "\tilde{\varepsilon}"] &
\end{tikzcd}$$

Practically, if the diagram $F$ consists of small category $\mathsf{U}$, we usually write

$$\operatorname{colim} F = \operatorname*{colim}_{U \in \mathsf{U}} U,$$

and if $\mathsf{I}$ serves as an index set of $\{U_i\}_{i \in \mathsf{I}}$, we also write

$$\operatorname{colim} F = \operatorname*{colim}_{i \in \mathsf{I}} U_i.$$

For limits there are also similar notations.

**Example 2.2.** We list some notions here.

(1) Products and coproducts. If $\mathsf{I}$ is a discrete category, then the limits and colimits indexed on $\mathsf{I}$ are products and coproducts. The products in $\mathsf{Set}$, $\mathsf{Top}$, $\mathsf{Grp}$ are all Cartesian products with morphisms are projections. The coproducts in $\mathsf{Set}$ and $\mathsf{Top}$ are disjoint unions, and coproducts in $\mathsf{Grp}$ are free products (multiplications are adjunctions).

(2) Pushouts and pullbacks. This is the colimit of the diagram

$$e \longleftarrow d \longrightarrow f,$$

and the limit of the diagram

$$e \longrightarrow d \longleftarrow f,$$

respectively.

(3) Equalizers and coequalizers. This is the limit and colimit of the following diagram

$$d \rightrightarrows d',$$

where the double right arrow indicates all nonidentity morphisms.

A category is said to be *complete* if it has all limits, and to be *cocomplete* if it has all colimits.

**Proposition 2.3.** *A category is cocomplete if and only if it has coproducts and coequalizers.*

*Proof.* One side is trivial, we prove that a category which has coproducts and coequalizers is cocomplete. We will abuse a lot of notation in the following proof.

Let $\{U_i\}_{i\in \mathsf{I}}$ be indexed by $\mathsf{I}$. For any morphism $f : U_i \to U_j$, we compose $U_i \xrightarrow{f} U_j \xrightarrow{\iota} \bigsqcup_{j\in\mathsf{I}} U_j$, and obtain

$$\bigsqcup_{f:U_i\to U_j} U_i \xrightarrow{\varphi} \bigsqcup_{j\in\mathsf{I}} U_j.$$

Similarly, we obtain id between these two coproducts. Thus we have the coequalizer

$$\begin{tikzcd}
\bigsqcup_{f:U_i\to U_j} U_i \arrow[rr, shift left, "\varphi"] \arrow[rr, shift right, "\mathrm{id}"'] \arrow[dr, "\eta"'] & & \bigsqcup_{j\in\mathsf{I}} U_j \arrow[dl, "\delta"] \\
& U. &
\end{tikzcd}$$

We claim $U$ is the colimit of $\{U_i\}_{i\in\mathsf{I}}$. We must verify the universal property. Suppose given a family of morphisms $f_i : U_i \to A$ compatible with morphisms in $\mathsf{I}$, we check arbitary commutative diagram

$$\begin{tikzcd}
U_i \arrow[rr, "f"] \arrow[dr, "f_i"'] & & U_j \arrow[dl, "f_j"] \\
& A. &
\end{tikzcd}$$

Let $\iota_i : U_i \to \bigsqcup_{f:U_i\to U_j} U_i$ embeds $U_i$ to the one corresponds to $f$, $\iota_j : U_j \to \bigsqcup_{j\in\mathsf{I}} U_j$ be the natural embedding. Then $f_j$ factors through $\iota_j$, obtaining $f_j = \delta' \circ \iota_j$. For $U_i$ corresponds to $f$, we have $f_j \circ f = f_i \circ \mathrm{id}$, by the universal property, this means $\delta' \circ \varphi = \delta' \circ \mathrm{id}$. Thus we can define $\eta' = \delta' \circ \varphi$. Now we have a big commutative diagram:

$$\begin{tikzcd}
U_i \arrow[rr, "f"] \arrow[d, "\iota_i"] & & U_j \arrow[d, "\iota_j"] \\
\bigsqcup_{f:U_i\to U_j} U_i \arrow[rr, shift left, "\varphi"] \arrow[rr, shift right, "\mathrm{id}"'] \arrow[dr, "\eta"] \arrow[ddr, "\eta'"'] & & \bigsqcup_{j\in\mathsf{I}} U_j \arrow[dl, "\delta"'] \arrow[ddl, "\delta'"] \\
& U \arrow[d, dashed] & \\
& A, &
\end{tikzcd}$$

the last thing we need to do is to show $f_i = \gamma_i \circ \iota_i$, but this follows directly from the commutativity of whole diagram. $\square$

By using opposite category formally, we can obtain

**Proposition 2.4.** *A category is complete if and only if it has products and equalizers.*

2.4. **Group Object.** Let $\mathsf{C}$ be a category which has products and initial object $*$. A *group object* consists of an object $G$ and a morphism $\mu : G \times G \to G$ called multiplication, a morphism $e : * \to G$ called unit element and a morphism $i : G \to G$ called inverse, satisfying

*Associative law*

$$\begin{tikzcd} G \times G \times G \arrow[r, "\mu \times \mathrm{id}"] \arrow[d, "\mathrm{id} \times \mu"'] & G \times G \arrow[d, "\mu"] \\ G \times G \arrow[r, "\mu"] & G. \end{tikzcd}$$

*Identity law*

$$\begin{tikzcd} & G \arrow[dl, "\mathrm{id} \times e"'] \arrow[d, "\mathrm{id}"] \arrow[dr, "e \times \mathrm{id}"] & \\ G \times G \arrow[r, "\mu"] & G & G \times G. \arrow[l, "\mu"'] \end{tikzcd}$$

*Inverse*

$$\begin{tikzcd} & G \arrow[dl, "\mathrm{id} \times i"'] \arrow[d, "e"] \arrow[dr, "i \times \mathrm{id}"] & \\ G \times G \arrow[r, "\mu"] & G & G \times G. \arrow[l, "\mu"'] \end{tikzcd}$$

where $e$ is the composition $G \to * \to G$.

**Lemma 2.5.** *If a functor preserves product, then it preserves group objects.*

Now we study topological groups. Topological groups are group objects in category $\mathsf{Top}$, that is, $G$ is a topological group if it is a topological space and multiplication and inverse are continuous. We want to show that

**Proposition 2.6.** *The fundamental group of a topological group is ableian.*

This proposition can be deduced by given direct homotopies, but we will adopt some "abstract nonsense" here.

**Lemma 2.7.** *Fundamental group functor* $\pi_1 : \mathsf{Top}^* \to \mathsf{Grp}$ *preserves product.*

*Proof.* We will check the universal properties elaborately. Let $\{X_\alpha\}_{\alpha \in A}$ be topological spaces, $(X, p_\alpha)$ be their product. Let $G$ be a group, together with a family of homomorphisms $f_\alpha : G \to \pi_1(X_\alpha, x)$. Choose an element $g \in G$, suppose $f_\alpha(g) = [\gamma_\alpha] \in \pi_1(X_\alpha, x)$, $\alpha \in A$. Then $\gamma_\alpha : I \to X_\alpha$, by universal property, we have a $\gamma : I \to X$ such that $p_\alpha \circ \gamma = \gamma_\alpha$. Thus we define $f : G \to \pi_1(X, x)$ by $f(g) = [\gamma]$, then $p_{\alpha *}[\gamma] = [\gamma_\alpha]$. It's clear that such construction must be unique, hence $\pi_1(X, x)$ satisfies the universal property. Therefore

$$\pi_1\left(\prod_{\alpha \in A} X_\alpha, x\right) = \prod_{\alpha \in A} \pi_1(X_\alpha, x). \qquad \square$$

**Lemma 2.8.** *The group objects in* $\mathsf{Grp}$ *are abelian groups.*

*Proof.* The inverse is defined by $g \mapsto g^{-1}$, but it is a homomorphism, we have $(gh)^{-1} = g^{-1}h^{-1} = h^{-1}g^{-1}$, hence the group object is abelian. $\square$

Now the proof can be reduced into one sentence.

*Proof of Proposition 2.6.* Since fundamental group functor respects products, it maps group objects to group objects, that is, topological groups to abelian groups. $\square$