SEMINAR 1: FUNDAMENTAL GROUP

ZENG MENGCHEN

1. Fundamental Group

1.1. **Definition.** We have learnt the notion of fundamental group in topology course, now we review it here.

A homotopy between maps $f,g:X\to Y$ is a map $h:I\times X\to Y$ such that $h(0,x)=f(x),\ h(1,x)=g(x).$ Two paths $f,g:a\to b$ in X is defined to be equivalent if there exists a homotopy $h:I\times I\to X$ such that

$$h(t,0) = a, h(t,1) = b, h(0,s) = f(s), h(1,s) = g(s).$$

Denote the equivalence class of f by [f]. A loop is a path with f(0) = f(1).

Define $\pi_1(X, x)$ to be the collection of all equivalent class of loops with starting point and end point x. We give a group structure on $\pi_1(X, x)$. Let $f: a \to b$, $g: b \to c$ be paths, define $g \cdot f$ to be the path

$$g \cdot f(s) = \begin{cases} f(2s), & 0 \le s \le 1/2, \\ g(2s-1), & 1/2 \le s \le 1, \end{cases}$$

and define $[g][f] := [g \cdot f]$. We also define the inverse $g^{-1}(s) = g(1-s)$ and $[g]^{-1} = [g^{-1}]$. Moreover, the identity is defined to be the constant loop c_x . Clearly these definitions make $\pi_1(X, x)$ into a group.

1.2. **Choice of Base Point.** Let $a: x \to y$ be a path, we define a homomorphism from $\gamma[a]: \pi_1(X,x) \to \pi_1(X,y)$ to be $\gamma[a][f] = [a \cdot f \cdot a^{-1}]$. Clearly we have $\gamma[b \cdot a] = \gamma[b] \circ \gamma[a]$, hence $\gamma[a]$ is an isomorphism. Therefore, we have

Proposition 1.1. For $x, y \in X$ in the same path connected component, $\pi_1(X, x)$ is (not necessarily canonical) isomorphic to $\pi_1(X, y)$.

If $\pi_1(X,x)$ happens to be abelian, then let $a':x\to y$ be another path, we have

$$\varphi[(a')^{-1}] \circ \varphi[a]([f]) = [(a')^{-1} \cdot a][f][a^{-1} \cdot a']$$
$$= [f][(a')^{-1} \cdot a][a^{-1} \cdot a']$$
$$= [f],$$

that is, the isomorphism between $\pi_1(X, x)$ and $\pi_1(X, y)$ does not depend on the choice of path, namely, canonical.

1.3. **Homotopy Invariance.** Let $p: X \to Y$ be a map, define $p_*: \pi_1(X, x) \to \pi_1(Y, p(x))$ to be

$$p_*[f] = [p \circ f].$$

Clearly p_* is a homomorphism, and identity map id: $X \to X$ induces the identity homomorphism. Moreover, if $p: X \to Y$, $q: Y \to Z$, then we have $q_* \circ p_* = (q \circ p)_*$.

Now suppose given two maps $p,q:X\to Y$ and a homotopy $h:p\simeq q$, a path a(t)=h(x,t), we have

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Proposition 1.2. The following diagram is commutative:

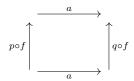
$$\pi_1(X,x)$$

$$\pi_1(Y,p(x)) \xrightarrow{\gamma[a]} \pi_1(Y,q(x)).$$

Proof. Let $f: I \to X$ be a loop, we must show $q \circ f$ is equivalent to $a \cdot (p \circ f) \cdot a^{-1}$. This is equivalent to show the constant loop $c_{p(x)}$ is equivalent to $a^{-1} \cdot (q \circ f)^{-1} \cdot a \cdot (p \circ f)$. Define $j: I \times I \to Y$ by j(t,s) = h(t,f(s)), then

$$j(0,s) = (p \circ f)(s), \ j(1,s) = (q \circ f)(s), \ j(t,0) = a(t) = j(t,1).$$

Thus we have a schematic diagram of j:



Thus, going clockwise around the boundary starting at (0,0), we traverse $a^{-1} \cdot (q \circ f)^{-1} \cdot a \cdot (p \circ f)$. Since linear homotopy connects the boundary to origin, j induces a homotopy between the composite and $c_{p(x)}$.

1.4. Calculations.

Lemma 1.3. $\pi_1(\mathbb{R},0) = 0$.

Proof. We use linear homotopy $k: I \times \mathbb{R} \to \mathbb{R}$ by k(t,s) = (1-t)s, then k connects identity and constant map at 0. Hence by homotopy invariance, we have $\pi_1(\mathbb{R},0) = 0$.

Proposition 1.4. $\pi_1(\mathbb{S}^1, 1) \cong \mathbb{Z}$.

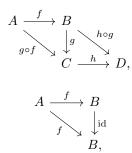
Here we regard $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$. We postpone the proof until we develop the theory of covering spaces.

2. Catagorical Language

2.1. Categories. A category consists of a collection of objects and a set hom(A, B) of morphisms for each pair of objects A and B. In each hom(A, B), there exists an identity morphism id and a composition law

$$\circ: \hom(B,C) \times \hom(A,B) \to \hom(A,C)$$

for each triple of objects A, B, C. Composition and identity morphisms must make the following diagrams commute:





A morphism $f \in \text{hom}(A, B)$ is called an *isomorphism* if there exists a $g \in \text{hom}(B, A)$ such that $g \circ f = \text{id}_A$, $f \circ g = \text{id}_B$.

We list some frequently used categories here.

- Set, category of sets, with morphisms to be mappings.
- Grpd, category of groupoids. Groupoids are defined to be small categories (i.e. whose collection of objects is a set) with all morphisms are isomorphisms, and morphisms between groupoids are functors (whose definition will be expalined later).
- Grp, category of groups. Groups are groupoids with single object.
- Ab, category of abelian groups.
- Top, category of topological spaces, with morphisms to be continuous mappings, i.e. maps.
- Top*, category of pointed topological spaces, with morphisms to be maps that map the base point to base point.

We also define the *opposite category* of a category C (denoting C^{op}) consisting of same objects as C and $hom_{C^{op}}(A, B) = hom_{C}(B, A)$.

2.2. Functors and Natural Transformations. We define a covariant functor F from category C to category D to be an assignment of each object A in C an object F(A) in D, and of each morphism f in hom(A,B) a morphism $F(f) \in hom(F(A),F(B))$. Also we define a contravariant functor from C to D to be a covariant functor from C^{op} to D.

A natural transformation from a functor $F: \mathsf{C} \to \mathsf{D}$ to a functor G is a family of morphisms $\varphi_A: F(A) \to G(A)$ making the following diagram commute for any object A, B and $f \in \mathrm{hom}(A, B)$

$$\begin{split} F(A) & \stackrel{\varphi_A}{\longrightarrow} G(A) \\ & \downarrow^{F(f)} & \downarrow^{G(f)} \\ F(B) & \stackrel{\varphi_B}{\longrightarrow} G(B). \end{split}$$

Example 2.1. (1) The forgetful functor $\mathsf{Grp} \to \mathsf{Set}$, which maps a group to its underlying set.

(2) The free group functor $\mathsf{Set} \to \mathsf{Grp}$, which maps a set S to a free group F(S) with basis S. A free group with basis S is characterized by the following universal property: any mapping from S to a group G factors through the natural embedding $\iota: S \to F(S)$, that is for $f: S \to G$, there exists a unique homomorphism \tilde{f} making the following diagram commute:

$$F(S) \xrightarrow{-\tilde{f}} G$$

$$\downarrow \uparrow \qquad f$$

$$S.$$

If $g: S \to T$ is a mapping of sets, then F(g) is induced by above universal property.

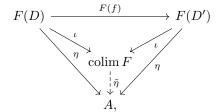
(3) The natural embedding of double dual in the category of vector spaces. Let Id be the identity functor, and denote the double dual of V by V^{**} . Then $v \mapsto (f \mapsto f(v))$ for any $v \in V$ and linear functional f defines an injection from V to V^{**} . One can check the family of injections consists a

natural transformation from Id to double dual. In particular, if V is finite dimensional, this shows V^{**} is canonically isomorphic to V.

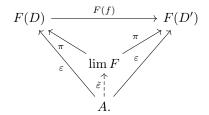
2.3. Colimits and Limits. In category theory, an object A being initial means for any other objects B, hom(A, B) is a singleton. Being terminal reverses the morphisms, this means hom(B, A) is a singleton. A simple exercise is to show that initial objects and terminal objects are unique up to an isomorphism if they exist.

Let I be a small category. An I-shaped diagram in category C is a functor $F: I \to C$. A morphism between I-shaped diagrams is a natural transformation. Let A be an object in C, then there is a functor that maps all objects in I to A and all morphisms to id_A. Denote this I-diagram by A.

Let F be an I-diagram, the *colimit* of F is the initial object among all morphisms having the form $\eta: F \to \underline{A}$, that is, η factors through $\iota: F \to \underline{\operatorname{colim}} F$. Conversely, the *limit* of F is the terminal object among all such morphisms, that is, $\varepsilon: F \to \underline{A}$ factors through $\pi: F \to \underline{\lim} F$. Expressing in diagrams, for each map $f: D \to D'$ in I, we have the commutative diagram for colimit:



and the commutative diagram for limit:



Practically, if the diagram F consists of small category U, we usually write

$$\operatorname{colim} F = \operatorname{colim}_{U \in \mathsf{U}} U,$$

and if I serves as an index set of $\{U_i\}_{i\in I}$, we also write

$$\operatorname{colim} F = \operatorname{colim}_{i \in \mathsf{I}} U_i.$$

For limits there are also similar notations.

Example 2.2. We list some notions here.

- (1) Products and coproducts. If I is a discrete category, then the limits and colimits indexed on I are products and coproducts. The products in Set, Top, Grp are all Cartesian products with morphisms are projections. The coproducts in Set and Top are disjoint unions, and coproducts in Grp are free products (multiplications are adjunctions).
- (2) Pushouts and pullbacks. This is the colimit of the diagram

$$e \longleftarrow d \longrightarrow f,$$

and the limit of the diagram

$$e \longrightarrow d \longleftarrow f$$

respectively.

(3) Equalizers and coequalizers. This is the limit and colimit of the following diagram

$$d \Longrightarrow d'$$

where the double right arrow indicates all nonidentity morphisms.

A category is said to be *complete* if it has all limits, and to be *cocomplete* if it has all colimits.

Proposition 2.3. A category is cocomplete if and only if it has coproducts and coequalizers.

Proof. One side is trivial, we prove that a category which has coproducts and coequalizers is cocomplete. We will abuse a lot of notation in the following proof.

Let $\{U_i\}_{i\in I}$ be indexed by I. For any morphism $f:U_i\to U_j$, we compose $U_i\stackrel{f}{\to} U_j\stackrel{\iota}{\to} \bigsqcup_{j\in I} U_j$, and obtain

$$\bigsqcup_{f:U_i\to U_j} U_i \xrightarrow{\varphi} \bigsqcup_{j\in \mathbf{I}} U_j.$$

Similarly, we obtain id between these two coproducts. Thus we have the coequalizer

$$\bigsqcup_{f:U_i \to U_j} U_i \xrightarrow{\varphi} \bigsqcup_{j \in I} U_j$$

$$U_i \xrightarrow{\operatorname{id}} \bigcup_{j \in I} U_j$$

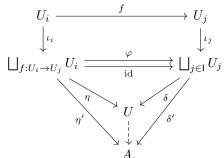
We claim U is the colimit of $\{U_i\}_{i\in I}$. We must verify the universal property. Suppose given a family of morphisms $f_i:U_i\to A$ compatible with morphisms in I, we check arbitary commutative diagram

$$U_i \xrightarrow{f} U_j$$

$$\downarrow f_i$$

$$A$$

Let $\iota_i: U_i \to \bigsqcup_{f:U_i \to U_j} U_i$ embeds U_i to the one corresponds to f, $\iota_j: U_j \to \bigsqcup_{j \in I} U_j$ be the natural embedding. Then f_j factors through ι_j , obtaining $f_j = \delta' \circ \iota_j$. For U_i corresponds to f, we have $f_j \circ f = f_i \circ \mathrm{id}$, by the universal property, this means $\delta' \circ \varphi = \delta' \circ \mathrm{id}$. Thus we can define $\eta' = \delta' \circ \varphi$. Now we have a big commutative diagram:



the last thing we need to do is to show $f_i = \gamma_i \circ \iota_i$, but this follows directly from the commutativity of whole diagram.

By using opposite category formally, we can obtain

Proposition 2.4. A category is complete if and only if it has products and equalizers.

2.4. **Group Object.** Let C be a category which has products and initial object *. A group object consists of an object G and a morphism $\mu: G \times G \to G$ called multiplication, a morphism $e: * \to G$ called unit element and a morphism $i: G \to G$ called inverse, satisfying

Associative law

$$\begin{array}{ccc} G \times G \times G & \xrightarrow{\mu \times \mathrm{id}} & G \times G \\ & & & \downarrow^{\mu} & & \downarrow^{\mu} \\ & & & & G \times G & \xrightarrow{\mu} & G. \end{array}$$

Identity law

$$G \times G \xrightarrow{\text{id} \times e} G \leftarrow G \times G$$

$$G \times G \xrightarrow{\mu} G \leftarrow G \times G$$

Inverse

$$G \times G \xrightarrow{\mu} G \xleftarrow{\mu} G \times G.$$

where e is the composition $G \to * \to G$.

Lemma 2.5. If a functor preserves product, then it preserves group objects.

Now we study topological groups. Topological groups are group objects in category Top , that is, G is a topological group if it is a topological space and multiplication and inverse are continuous. We want to show that

Proposition 2.6. The fundamental group of a topological group is ableian.

This proposition can be deduced by given direct homotopies, but we will adopt some "abstract nonsense" here.

Lemma 2.7. Fundamental group functor $\pi_1 : \mathsf{Top}^* \to \mathsf{Grp}$ preserves product.

Proof. We will check the universal properties elaborately. Let $\{X_{\alpha}\}_{\alpha\in A}$ be topological spaces, (X, p_{α}) be their product. Let G be a group, together with a family of homomorphisms $f_{\alpha}: G \to \pi_1(X_{\alpha}, x)$. Choose an element $g \in G$, suppose $f_{\alpha}(g) = [\gamma_{\alpha}] \in \pi_1(X_{\alpha}, x)$, $\alpha \in A$. Then $\gamma_{\alpha}: I \to X_{\alpha}$, by universal property, we have a $\gamma: I \to X$ such that $p_{\alpha} \circ \gamma = \gamma_{\alpha}$. Thus we define $f: G \to \pi_1(X, x)$ by $f(g) = [\gamma]$, then $p_{\alpha*}[\gamma] = [\gamma_{\alpha}]$. It's clear that such construction must be unique, hence $\pi_1(X, x)$ satisfies the universal property. Therefore

$$\pi_1\left(\prod_{\alpha\in A} X_{\alpha}, x\right) = \prod_{\alpha\in A} \pi_1(X_{\alpha}, x).$$

Lemma 2.8. The group objects in Grp are abelian groups.

Proof. The inverse is defined by $g \mapsto g^{-1}$, but it is a homomorphism, we have $(gh)^{-1} = g^{-1}h^{-1} = h^{-1}g^{-1}$, hence the group object is abelian.

Now the proof can be reduced into one sentence.

Proof of Proposition 2.6. Since fundamental group functor respects products, it maps group objects to group objects, that is, topological groups to abelian groups.