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Pricing Asian Options using Monte Carlo Methods

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Abstract

Asian options are of particular importance for commodity products which have low trading volumes (e.g. crude oil), since price manipulation is inhibited. Hence, the pricing of such options becomes one of the most interesting fields. Since there are no known closed form analytical solutions to arithmetic average Asian options, many numerical methods are applied.

This paper deals with pricing of arithmetic average Asian options with the help of Monte Carlo methods. We also investigate ways to improve the precision of the simulation estimates through the variation reduction techniques: the control variate and the antithetic variate methods. We then compare the results from these two methods.

1. Introduction

1.1 Financial Derivatives

Derivatives are financial contracts, or financial instruments, whose values are derived from the value of something else which is known as the underlying. The underlying value on which a derivative is based can be a traded asset, such as a stock; an index portfolio; a futures price; a commercial real estate; or some measurable state variable, such as the weather condition at some location. The payoff can involve various patterns of cash flows. Payments can be spread evenly through time, occur at specific dates, or a combination of the two. Derivatives are also referred to as contingent claims.
The main types of derivatives are forwards, futures, options, and swaps.

1.2 Options

An option is a contract between a buyer and a seller that gives the buyer the right—but not the obligation—to buy or to sell the underlying asset at an agreed price at a later date. There are two basic kinds of options: the call option and the put option. A call option gives the buyer the right to buy the underlying asset while a put gives the buyer to sell. The agreed price in the contract is known as the *strike price*; the date in the contract is known as the *expiry date*.
The vast majority of options are either European or American options. There are many other types of options such as barrier options; Bermudan options; Asian options; or look back options.

1.3 European and American Options

European options are the foundations of the options universe. Presenting itself as the most basic type of option contract, this type of option gives the holder or seller of the option the ability to exercise the option only at the expiry date. The pay-off is given by:

$\Phi(S) = \left(S(T) - K\right)^{+}$ for a European call option;

and by

$\Phi(S) = \left(K - S(T)\right)^{+}$ for a European put option,

where *S(T)* is the price of the underlying assets at the expiry date *T* and K is the strike price.
An American option, in contrast to the European option, may be exercised at any time prior to the expiry date. The pay-off is given by

$$\Phi(S) = \max_{t \le \tau \le T} \left(S(\tau) - K \right)^{+} \quad \text{for an American call option;}$$

and by

$$\Phi(S) = \max_{t \le \tau \le T} \left(K - S(\tau) \right)^{+} \quad \text{for an American put option,}$$

where τ is the exercise time.
It is obvious that the American option is much more complicated and interesting than the European option.

1.4 Why do people use options?

Options have two main uses: speculating and hedging.
Investing in options has a leverage compared to investing directly in the corresponding underlying assets for the speculators. For instance, if you believe that Ericsson shares are due to increase then you may speculate by becoming the holder of a suitable call option. Typically, you can make a greater profit relative to your original payout than you would do by simply purchasing the shares. That is why the options become more and more popular in the financial market.
On the other hand, it is a very useful tool for hedging. Especially in the view of the option writers, they may not be able to afford the huge potential risk just as the speculators could. They write options to gain some certain and less risky profits. The mechanism is like this: the writer can sell an option for more than it is worth and then hedge away all the risk she or he might be possible to take, and then make a locked gain. This idea is central to the theory and practice of option pricing.

1.5 Option Pricing

Although options have existed—at least in concept—since antiquity, it wasn't until publication of the Black-Scholes (1973) option pricing formula that a theoretically consistent framework for pricing options became available. Nowadays, option pricing plays a critical role in the research about the financial market. Due to the narrow range the Black-Scholes formula can apply to, some other option pricing methods are introduced and used to analyze the complicated options. There are three primary option pricing methods widely used: binomial methods, finite difference models and

Monte Carlo models.
Binomial methods involve the dynamics of the option's theoretical value for discrete time intervals over the option's duration. The model of this kind starts with a binomial tree of discrete future possible underlying stock prices. By constructing a riskless portfolio of an option and stock a simple formula can be used to find the option price at each node in the tree. Therefore they rely only indirectly on the Black-Scholes analysis through the assumption of risk neutrality. However, the binomial models are considered more accurate than Black-Scholes because they are more flexible, e.g. discrete future dividend payments can be modeled correctly at the proper forward time steps, and American options can be modeled as well as European ones.
The finite difference model can be derived, once the equations used to value options can be expressed in terms of partial differential equations. The idea underlying finite difference methods is to replace the partial derivatives occurring in partial differential equations by approximations based on Taylor series expansions of functions near the point or points of interest. One should note that binomial methods are particular cases of the explicit finite difference methods.
For many types of options, traditional valuation techniques are intractable due to the complexity of the instrument. In these cases, a Monte Carlo approach may often be useful. We will go into details in Section 4.

2. Financial Background

To begin with, we will enter the most resplendent star in our financial universe—the Black-Scholes world.
We usually use the Black and Scholes model to describe the price of an asset at time t. The Black-Scholes model consists of two assets with dynamics given by

$$dB(t) = rB(t)dt, \tag{2.1}$$

$$dS(t) = \mu S(t)dt + \sigma S(t)dW^P(t), \tag{2.2}$$

Where $B(t)$ and $S(t)$ are the prices of the risk-free asset and the risky asset respectively, $W(t)$ is a standard Brownian motion, r, μ and σ are deterministic constants.

The stochastic differential equation (2.2) is defined on a certain probability space (Ω, F, P). According to the Girsanov theorem, we know that there exists a probability measure Q such that under the Q measure

$$dS(t) = rS(t)dt + \sigma S(t)dW^Q(t), \tag{2.3}$$

where $W^Q(t) = W^P(t) + ((\mu - r)/\sigma)t$ which is a standard Brownian motion under Q.
The solution of (2.3) is

$$S(T) = S(t)\exp\left((r - \sigma^2/2)(T - t) + \sigma\left(W^Q(T) - W^Q(t)\right)\right). \tag{2.4}$$

Under Q we also have

$$\begin{aligned} d\left(e^{-rt}S(t)\right) &= e^{-rt}dS(t) + S(t)(de^{-rt}) \\ &= e^{-rt}\left(rS(t)dt + \sigma S(t)dW^Q(t)\right) + S(t)(-re^{-rt})dt \\ &= \sigma e^{-rt}S(t)dW^Q(t). \end{aligned} \tag{2.5}$$

It can be seen from (2.5) that $S(t)/B(t)$ is a martingale under the measure Q, so Q is called risk neutral measure.
These results can be found in any textbook on stochastic analysis such as Karatzas and Shreve (1988), Oksendal (1995)

Meanwhile, Black-Scholes equation gives us

$$F_t(t,S)+rSF_S(t,S)+\frac{1}{2}S^2\sigma^2F_{SS}(t,S)=rF(t,S), \tag{2.6}$$

$$F(T,S)=\Phi(S). \tag{2.7}$$

We also have

$$\begin{aligned} dF(t,S) &= F_t(t,S)dt+F_S(t,S)dS+\frac{1}{2}F_{SS}(t,S)(dS)^2 \\ &= \left(F_t(t,S)+rSF_S(t,S)+\frac{1}{2}S^2\sigma^2F_{SS}(t,S)\right)dt+\sigma SF_S(t,S)dW \\ &= rF(t,S)dt+\sigma SF_S(t,S)dW. \end{aligned} \tag{2.8}$$

Therefore, $F(t,S(t))/B(t)$ is a martingale under Q as well.

Hence,

$$\frac{F(t,S(t))}{B(t)}=E^Q_{t.S(t)}\left[\frac{F(T,S(T))}{B(T)}\right]=E^Q_{t.S(t)}\left[\frac{\Phi(S(T))}{B(T)}\right], \tag{2.9}$$

$$F(t,S(t))=\exp(-r(T-t))E^Q_{t.S(t)}\left(\Phi(S(T))\right). \tag{2.10}$$

As for the vanilla European call option, we get by (2.10)

$$C^K(t,S(t))=\exp(-r(T-t))E^Q_{t.S(t)}\left(S(T)-K\right)^+, \tag{2.11}$$

where $C^K(t,S(t))$ denotes the price of the vanilla European call option.

Then plug (2.4) into (2.11), we get

$$\begin{aligned} &C^K(t,S(t)) \\ &= \mathrm{e}^{-r(T-t)}E^Q_{t.S(t)}\left(S(t)e^{(r-\sigma^2/2)(T-t)+\sigma\left(W^Q(T)-W^Q(t)\right)}-K\right)^+ \end{aligned}$$

$$= \mathrm{e}^{-r(T-t)} \int_{\log\frac{K}{S(t)}}^{\infty} \left(S(t)e^{x} - K\right) \frac{1}{\sqrt{2\pi(T-t)}\sigma} \mathrm{e}^{-\frac{\left(x-\left(r-\frac{1}{2}\sigma^2\right)(T-t)\right)^2}{2\sigma^2(T-t)}} dx$$

$$= \mathrm{e}^{-r(T-t)} S(t) \int_{\log\frac{K}{S(t)}}^{\infty} \frac{1}{\sqrt{2\pi(T-t)}\sigma} \mathrm{e}^{-\frac{\left(x-\left(r-\frac{1}{2}\sigma^2\right)(T-t)\right)^2 - 2\sigma^2(T-t)x}{2\sigma^2(T-t)}} dx$$

$$- \mathrm{e}^{-r(T-t)} K \int_{\log\frac{K}{S(t)}}^{\infty} \frac{1}{\sqrt{2\pi(T-t)}\sigma} \mathrm{e}^{-\frac{\left(x-\left(r-\frac{1}{2}\sigma^2\right)(T-t)\right)^2}{2\sigma^2(T-t)}} dx$$

$$= \mathrm{e}^{-r(T-t)+r(T-t)} S(t) \int_{\log\frac{K}{S(t)}}^{\infty} \frac{1}{\sqrt{2\pi(T-t)}\sigma} \mathrm{e}^{-\frac{\left(x-\left(r+\frac{1}{2}\sigma^2\right)(T-t)\right)^2}{2\sigma^2(T-t)}} dx$$

$$- \mathrm{e}^{-r(T-t)} K \int_{\log\frac{K}{S(t)}}^{\infty} \frac{1}{\sqrt{2\pi(T-t)}\sigma} \mathrm{e}^{-\frac{\left(x-\left(r-\frac{1}{2}\sigma^2\right)(T-t)\right)^2}{2\sigma^2(T-t)}} dx$$

$$= S(t) \int_{\frac{\log\frac{K}{S(t)} - \left(r+\frac{1}{2}\sigma^2\right)(T-t)}{\sqrt{(T-t)}\sigma}}^{\infty} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{y^2}{2}} dy - \mathrm{e}^{-r(T-t)} K \int_{\frac{\log\frac{K}{S(t)} - \left(r-\frac{1}{2}\sigma^2\right)(T-t)}{\sqrt{(T-t)}\sigma}}^{\infty} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{y^2}{2}} dy$$

$$= S(t)\Phi(d_1) - \mathrm{e}^{-r(T-t)} K\Phi(d_2), \tag{2.12}$$

where $d_1 = \dfrac{\log\dfrac{S(t)}{K} + \left(r + \dfrac{1}{2}\sigma^2\right)(T-t)}{\sqrt{(T-t)}\sigma}, d_2 = \dfrac{\log\dfrac{S(t)}{K} + \left(r - \dfrac{1}{2}\sigma^2\right)(T-t)}{\sqrt{(T-t)}\sigma}$.

Note that

$$\left(S(T) - K\right)^{+} - \left(K - S(T)\right)^{+} = S(T) - K. \tag{2.13}$$

Take the expectation on both sides and discount back to time t, we get

$$C^K\left(t, S(t)\right) - P^K\left(t, S(t)\right)$$
$$= \exp\left(-r(T-t)\right)\left(E^Q_{t.S(t)}\left(S(T)-K\right)^{+} - E^Q_{t.S(t)}\left(K - S(T)\right)^{+}\right)$$
$$= \exp\left(-r(T-t)\right) E^Q_{t.S(t)}\left(S(T) - K\right)$$

$$= \exp\left(-r(T-t)\right) E^{Q}_{t,S(t)} S(T) - \exp\left(-r(T-t)\right) K$$

$$= S(t) - \exp\left(-r(T-t)\right) K \,, \tag{2.14}$$

where $P^K\ (t, S(t))$ denotes the price of the vanilla European put option.
This is the so-called *put-call parity* for the European options.
Hence, we can directly derive the price of the European put option from (2.14) together with (2.12):

$$P^K\left(t, S(t)\right)$$

$$= C^K\left(t, S(t)\right) - S(t) + \exp\left(-r(T-t)\right) K$$

$$= S(t)\left(\Phi(d_1) - 1\right) - \exp\left(-r(T-t)\right) K \left(\Phi(d_2) - 1\right)$$

$$= \exp\left(-r(T-t)\right) K \Phi(-d_2) - S(t)\Phi(-d_1) \tag{2.15}$$

3. Asian Options

3.1 What are Asian options?

Asian options are options in which the underlying variable is the average price over a period of time. Because of this fact, Asian options have a lower volatility and hence rendering them cheaper relative to their European counterparts (we will give the proof in 3.3). They are commonly traded on currencies and commodity products which have low trading volumes. They were originally used in 1987 when Banker's Trust Tokyo office used them for pricing average options on crude oil contracts; and hence the name "Asian" option.
There are some different types of Asian options:

Continuous arithmetic average Asian call or put with

$$\Phi(S)=\left(\frac{1}{T}\int_0^T S(t)dt-K\right)^+ \quad \text{or} \quad \Phi(S)=\left(K-\frac{1}{T}\int_0^T S(t)dt\right)^+ \quad .$$

Continuous geometric average Asian call or put with

$$\Phi(S)=\left(e^{\frac{1}{T}\int_0^T \log S(t)dt}-K\right)^+ \quad \text{or} \quad \Phi(S)=\left(K-e^{\frac{1}{T}\int_0^T \log S(t)dt}\right)^+ \quad .$$

Discrete arithmetic average Asian call or put with

$$\Phi(S)=\left(\frac{1}{m+1}\sum_{i=0}^{m} S(\frac{iT}{m})-K\right)^+ \quad \text{or} \quad \Phi(S)=\left(K-\frac{1}{m+1}\sum_{i=0}^{m} S(\frac{iT}{m})\right)^+ \quad .$$

Discrete geometric average Asian call or put with

$$\Phi(S)=\left(e^{\frac{1}{m+1}\sum_{i=0}^{m}\log S(\frac{iT}{m})}-K\right)^+ \quad \text{or} \quad \Phi(S)=\left(K-e^{\frac{1}{m+1}\sum_{i=0}^{m}\log S(\frac{iT}{m})}\right)^+ \quad .$$

Denote the price of the arithmetic average Asian call and put options at time 0 by $C^{K,a}\left(S_0,T\right)$ and $P^{K,a}\left(S_0,T\right)$, and denote the price of the geometric average Asian call and put options at time 0 by $C^{K,g}\left(S_0,T\right)$ and $P^{K,g}\left(S_0,T\right)$.

Proposition 3.1. The following inequalities hold in the discrete case:

$$C^{K,g}\left(S_0,T\right) \le C^{K,a}\left(S_0,T\right);$$

$$P^{K,g}\left(S_0,T\right) \ge P^{K,a}\left(S_0,T\right).$$

Proof: This directly follows from the inequality between the geometric and arithmetic mean

$$\left(\prod_{i=0}^{m} S(\frac{iT}{m})\right)^{1/(m+1)} \le \frac{1}{m+1}\sum_{i=0}^{m} S(\frac{iT}{m}),$$

together with the fact that $(X-K)^+$ is increasing in X while $(K-X)^+$ is decreasing in *X*.

3.2 Closed Form Solution for the Geometric Average Asian Options

We assume the price of the asset follows a log-normal distribution continuous in time, and the product of log-normal distributed random variables is also log-normal distributed, however the sum is not. Hence, we could expect that the pricing of geometric average Asian options should be easy to deal with, while for arithmetic average ones it may be relatively more complicated to handle. In fact, the pricing formula of geometric average Asian options can be derived in the Black-Scholes framework., see [6].
The payoff function for the discrete geometric average Asian call option is given by

$$\Phi(S) = \left(e^{\frac{1}{m+1}\sum_{i=0}^{m}\log S(\frac{iT}{m})} - K\right)^+$$

$$= \left(\left(\prod_{i=0}^{m} S(\frac{iT}{m})\right)^{1/(m+1)} - K\right)^+. \quad (3.1)$$

And we know

$$\prod_{i=0}^{m} S(t_i) = \frac{S(t_m)}{S(t_{m-1})}\left(\frac{S(t_{m-1})}{S(t_{m-2})}\right)^2\left(\frac{S(t_{m-2})}{S(t_{m-3})}\right)^3$$

$$\cdots\left(\frac{S(t_3)}{S(t_2)}\right)^{m-2}\left(\frac{S(t_2)}{S(t_1)}\right)^{m-1}\left(\frac{S(t_1)}{S_0}\right)^{m} S_0^{m+1} \qquad (3.2)$$

where $t_i = iT/m$ for $i = 1, 2, 3, \ldots, m$.

From (2.4) together with the property of the Brownian motion, it follows that

$$\frac{S(t_m)}{S(t_{m-1})} = \exp\left(\left((r-\sigma^2/2)T/m + \sigma\sqrt{T/m}X_1\right)\right),$$

$$\frac{S(t_{m-1})}{S(t_{m-2})} = \exp\left(\left((r-\sigma^2/2)T/m + \sigma\sqrt{T/m}X_2\right)\right),$$

$$\frac{S(t_{m-2})}{S(t_{m-3})} = \exp\left(\left((r-\sigma^2/2)T/m + \sigma\sqrt{T/m}X_3\right)\right),$$

$$\vdots$$

$$\frac{S(t_2)}{S(t_1)} = \exp\left(\left((r-\sigma^2/2)T/m + \sigma\sqrt{T/m}X_{m-1}\right)\right),$$

$$\frac{S(t_1)}{S_0} = \exp\left(\left((r-\sigma^2/2)T/m + \sigma\sqrt{T/m}X_m\right)\right).$$

where $\{X_i\}_{i=1}^{m}$ are independent, N(0,1)-distributed random variables.

Furthermore, we have

$$\log\left[\left(\prod_{i=0}^{m} S(t_i)\right)^{1/(m+1)} \Big/ S_0\right]$$

$$= \log\left[\left(\prod_{i=0}^{m} S(t_i) \Big/ S_0^{m+1}\right)^{1/(m+1)}\right]$$

$$= \frac{1}{m+1}\log\left(\prod_{i=0}^{m} S(t_i) \Big/ S_0^{m+1}\right)$$

$$
=\frac{1}{m+1}\log\left(\frac{S(t_m)}{S(t_{m-1})}\left(\frac{S(t_{m-1})}{S(t_{m-2})}\right)^2\cdots\left(\frac{S(t_2)}{S(t_1)}\right)^{m-1}\left(\frac{S(t_1)}{S_0}\right)^m\right)
$$

$$
=\frac{1}{m+1}\left(\log\left(\frac{S(t_m)}{S(t_{m-1})}\right)+2\log\left(\frac{S(t_{m-1})}{S(t_{m-2})}\right)+\right.
$$

$$
\left.\cdots+(m-1)\log\left(\frac{S(t_2)}{S(t_1)}\right)+m\log\left(\frac{S(t_1)}{S_0}\right)\right)
$$

$$
=\frac{1}{m+1}\Big(\Big((r-\sigma^2/2)T/m+\sigma\sqrt{T/m}X_1\Big)+
$$

$$
2\Big((r-\sigma^2/2)T/m+\sigma\sqrt{T/m}X_2\Big)+
$$

$$
\cdots+(m-1)\Big((r-\sigma^2/2)T/m+\sigma\sqrt{T/m}X_{m-1}\Big)+
$$

$$
m\Big((r-\sigma^2/2)T/m+\sigma\sqrt{T/m}X_m\Big)\Big)
$$

$$
=\frac{1}{m+1}\left((1+2+\cdots+m)(r-\sigma^2/2)T/m+\sigma\sqrt{T/m}\sum_{i=1}^{m}iX_i\right)
$$

$$
=\frac{(r-\sigma^2/2)T}{2}+\frac{\sigma\sqrt{T/m}\sum_{i=1}^{m}iX_i}{m+1}. \tag{3.3}
$$

By the ‘additive mean and variance’ property of independent normal random variables, we know that

$$
\frac{\sigma\sqrt{T/m}\sum_{i=1}^{m}iX_i}{m+1}\sim N\left(0,\frac{\sigma^2T(1^2+2^2+3^2+\cdots+m^2)}{m(m+1)^2}\right),
$$

i.e.,
$$
\frac{\sigma\sqrt{T/m}\sum_{i=1}^{m}iX_i}{m+1}\sim N\left(0,\frac{(2m+1)\sigma^2T}{6(m+1)}\right). \tag{3.4}
$$

Therefore, we have

$$
\frac{\sigma\sqrt{T/m}\sum_{i=1}^{m}iX_i}{m+1}=\sigma\sqrt{\frac{(2m+1)T}{6(m+1)}}Z \tag{3.5}
$$

where $Z \sim N(0,1)$.

Plugging (3.5) into (3.3), we obtain that

$$\log\left[\left(\prod_{i=0}^{m} S(t_i)\right)^{1/(m+1)} \Big/ S_0\right]$$

$$= (\rho - \sigma_Z^2/2)T + \sigma_Z\sqrt{T}Z \tag{3.6}$$

where

$$\sigma_Z = \sigma\sqrt{\frac{2m+1}{6(m+1)}}$$

and

$$\rho = \frac{(r - \sigma^2/2) + \sigma_Z^2}{2}.$$

Hence, we can obtain the price of the geometric average Asian call option by using the risk-neutral method:

$$C^{K,g}\left(S_0, T\right) = \exp(-rT)E\left(\left(\prod_{i=0}^{m} S(\frac{iT}{m})\right)^{1/(m+1)} - K\right)^+$$

$$= \exp\left((\rho - r)T\right)\exp(-\rho T)E\left(S_0\exp\left((\rho - \sigma_Z^2/2)T + \sigma_Z\sqrt{T}Z\right) - K\right)^+$$

$$= \exp\left((\rho - r)T\right)\tilde{C}^K(S_0, T) \tag{3.7}$$

where $\tilde{C}^K(S_0, T)$ denotes the price of a European call option with risk-free interest rate ρ and the volatility σ_Z. By the Black-Scholes formula, we get

$$C^{K,g}\left(S_0, T\right) = \exp\left((\rho - r)T\right)\tilde{C}^K(S_0, T)$$

$$= \exp\left((\rho - r)T\right)\left(S_0\Phi(\tilde{d}_1) - K\exp(-\rho T)\Phi(\tilde{d}_2)\right)$$

$$=\exp\left(-rT\right)\left(S_0\exp(\rho T)\Phi(\tilde{d}_1)-K\Phi(\tilde{d}_2)\right), \tag{3.8}$$

where $d_1=\dfrac{\log\dfrac{S_0}{K}+\left(\rho+\dfrac{1}{2}{\sigma_Z}^2\right)T}{\sqrt{T}\sigma_Z}$, $d_2=\dfrac{\log\dfrac{S_0}{K}+\left(\rho-\dfrac{1}{2}{\sigma_Z}^2\right)T}{\sqrt{T}\sigma_Z}$.

If $\tilde{P}^K(S_0,T)$ denotes the price of a European call option with the same parameters we also get:

$$\begin{aligned}P^{K,g}\left(S_0,T\right)&=\exp\left((\rho-r)T\right)\tilde{P}^K(S_0,T)\\&=\exp\left((\rho-r)T\right)\left(K\exp(-\rho T)\Phi(-\tilde{d}_2)-S_0\Phi(-\tilde{d}_1)\right)\\&=\exp\left(-rT\right)\left(K\Phi(-\tilde{d}_2)-S_0\exp(\rho T)\Phi(-\tilde{d}_1)\right)\end{aligned} \tag{3.9}$$

Since we know that the geometric average Asian options can be simply priced according to the Black–Scholes framework whereas the arithmetic ones cannot, we focus on the case of arithmetic average Asian options and on the pricing of this kind by using some other kind of pricing methods.

3.3 An Inequality between the European and Asian Option

Asian options are widely used in practice, and they are always the target which many scholars are focused on. There are mainly two reasons that could explain it: on one hand, the price of the underlying assets could be hard to manipulate near expiration date of the options due to their averaging effect; on the other, they are significantly cheaper than European options when used to hedge.

The first reason is much more obvious than the second. Asian options could offer the protection against the price manipulation. For example, if a standard European call option is based on a stock which remains low in price during a large part of the final time period and rises significantly at maturity, the option writer would have to face a massive loss; whereas the average option could avoid such manipulation.

Compared to European options, it costs less to hedge by picking Asian options. Let us take a look at the discrete arithmetic average Asian call option, please see [6].

Proposition 3.2. We have the following inequality:

$$C^{K,a}\left(t,S(t)\right)\le C^{K}\left(t,S(t)\right).$$

Why could this be true? Is it intuitively reasonable without the formal proof? It seems to be difficult to compare directly since there is no explicit formula of the price of an Asian option. However, we may look at the final value of the options. We know that

$$E\left(S(T)\right)=e^{r(T-t)}S(t)$$

$$\Rightarrow \quad E\left(S(\frac{iT}{m})\right)\le E\left(S(T)\right) \qquad (i=0,1,2,\dots,m)$$

$$\Rightarrow \quad E\left(\frac{1}{m+1}\sum_{i=0}^{m}S(\frac{iT}{m})\right)\le E\left(\frac{1}{m+1}\sum_{i=0}^{m}S(T)\right)=E\left(S(T)\right).$$

One can guess that maybe the following holds:

$$\begin{aligned}
&C^{K,a}\left(t,S(t)\right)\\
&=\exp\left(-r(T-t)\right)E^{Q}_{t.S(t)}\left(\frac{1}{m+1}\sum_{i=0}^{m}S(\frac{iT}{m})-K\right)^{+}\\
&\le \exp\left(-r(T-t)\right)E^{Q}_{t.S(t)}\left(S(T)-K\right)^{+}\\
&=C^{K}\left(t,S(t)\right).
\end{aligned}$$

Now we would like to give the formal proof the property of Asian options.

Proof. Let us define

$$\alpha_i=\frac{S(\frac{iT}{m})}{S(\frac{(i-1)T}{m})} \text{ and } \alpha_0=S(0),$$

combined with (2.4), it follows that

$$\alpha_i=\exp\left((r-\sigma^2/2)(\frac{T}{m})+\sigma\left(W^{Q}(\frac{iT}{m})-W^{Q}(\frac{(i-1)T}{m})\right)\right),$$

which is lognormal distributed with

$$E\alpha_i = \exp(\frac{rT}{m}) \geq 1.$$

Additionally, it is obvious that

$$S(\frac{iT}{m}) = \alpha_0 \cdot \alpha_1 \cdots \alpha_i,$$

$$S(T) = \alpha_0 \cdot \alpha_1 \cdots \alpha_m.$$

In order to prove

$$C^{K,a}\left(t, S(t)\right) \leq C^K\left(t, S(t)\right),$$

we only have to prove that

$$E^Q_{t.S(t)}\left(\frac{\alpha_0 + \alpha_0 \cdot \alpha_1 + \cdots + \alpha_0 \cdot \alpha_1 \cdots \alpha_m}{m+1} - K\right)^+$$

$$\leq E^Q_{t.S(t)}\left(\alpha_0 \cdot \alpha_1 \cdots \alpha_m - K\right)^+.$$

In fact, we have

$$\frac{E^Q_{t.S(t)}\left(\frac{\alpha_0 + \alpha_0 \cdot \alpha_1 + \cdots + \alpha_0 \cdot \alpha_1 \cdots \alpha_m}{m+1} - K\right)^+}{\alpha_0}$$

$$= E^Q_{t.S(t)}\left(\frac{1}{m+1} + \frac{\alpha_1 + \alpha_1\alpha_2 \cdots + \cdot \alpha_1 \cdots \alpha_m}{m+1} - \frac{K}{\alpha_0}\right)^+$$

$$= E^Q_{t.S(t)}\left(\frac{1}{m+1} + \frac{m}{m+1}Q - K'\right)^+,$$

where $Q = \frac{\alpha_1 + \alpha_1\alpha_2 \cdots + \cdot \alpha_1 \cdots \alpha_m}{m}$ with $EQ = E(\frac{\alpha_1 + \alpha_1\alpha_2 \cdots + \cdot \alpha_1 \cdots \alpha_m}{m}) \geq 1$ and

$$K' = \frac{K}{\alpha_0}$$

Note that $\frac{1}{m+1} + \frac{m}{m+1}Q - K' > 0$ if and only if $Q > K_0' = \frac{(m+1)K' - 1}{m}$

Now we have to distinguish two cases:

(a) $K_0^{'} \geq 1$, then we have

$$
\begin{aligned}
& E_{t.S(t)}^{Q}\left(\frac{1}{m+1}+\frac{m}{m+1}Q-K^{'}\right)^{+} \\
&= \int_{K_0^{'}}^{\infty}\left(\frac{1}{m+1}+\frac{m}{m+1}Q-K^{'}\right)p(Q)dQ \\
&\leq \int_{K_0^{'}}^{\infty}\left(Q-K^{'}\right)p(Q)dQ \\
&\leq \int_{K^{'}}^{\infty}\left(Q-K^{'}\right)p(Q)dQ \\
&= E_{t.S(t)}^{Q}\left(Q-K^{'}\right)^{+}.
\end{aligned}
$$

(b) For $K_0^{'} \leq 1$,

$$
\begin{aligned}
& E_{t.S(t)}^{Q}\left(\frac{1}{m+1}+\frac{m}{m+1}Q-K^{'}\right)^{+} \\
&= \int_{K_0^{'}}^{\infty}\left(\frac{1}{m+1}+\frac{m}{m+1}Q-K^{'}\right)p(Q)dQ \\
&= E_{t.S(t)}^{Q}\left(\frac{1}{m+1}+\frac{m}{m+1}Q-K^{'}\right)-\int_{0}^{K_0^{'}}\left(\frac{1}{m+1}+\frac{m}{m+1}Q-K^{'}\right)p(Q)dQ \\
&\leq E_{t.S(t)}^{Q}\left(\frac{1}{m+1}+\frac{m}{m+1}Q-K^{'}\right)-\int_{0}^{K_0^{'}}\left(Q-K^{'}\right)p(Q)dQ \\
&\leq E_{t.S(t)}^{Q}\left(Q-K^{'}\right)-\int_{0}^{K_0^{'}}\left(Q-K^{'}\right)p(Q)dQ \quad (EQ \geq 1) \\
&= \int_{K_0^{'}}^{\infty}\left(Q-K^{'}\right)p(Q)dQ \\
&\leq \int_{K^{'}}^{\infty}\left(Q-K^{'}\right)p(Q)dQ
\end{aligned}
$$

$$= E^{Q}_{t.S(t)}\left(Q - K^{'}\right)^{+}.$$

From above, we can conclude that in both cases

$$E^{Q}_{t.S(t)}\left(\frac{1}{m+1} + \frac{m}{m+1}Q - K^{'}\right)^{+} \leq E^{Q}_{t.S(t)}\left(Q - K^{'}\right)^{+}.$$

Besides, it follows that

$$E^{Q}_{t.S(t)}\left(Q - K^{'}\right)^{+} = E^{Q}_{t.S(t)}\left(\frac{\alpha_1 + \alpha_1\alpha_2 \cdots + \cdot\, \alpha_1 \cdots \alpha_m}{m} - K^{'}\right)^{+}$$

$$\leq E^{Q}_{t.S(t)}\left(\alpha_0 \cdot \alpha_1 \cdots \alpha_m - K\right)^{+},$$

where the inequality can be derived from the induction on the number of random variables which can be started at n=1. Therefore, we can easily get

$$C^{K,a}\left(t, S(t)\right) \leq C^{K}\left(t, S(t)\right),$$

which makes Asian options much more popular when people want to hedge away the risk.

4. Monte Carlo methods

4.1 The Monte Carlo Framework

Monte Carlo methods were initially applied to option pricing by Boyle in 1977. Nowadays, it has been more and more widely applied to price options with complicated features. In this section Monte Carlo framework will be described in a general setting.
Suppose we want to estimate some θ, and we have

$$\theta = E\left(g(X)\right).$$

where $g(X)$ is an arbitrary function such that $E\left(|g(X)|\right) < \infty$, then we could generate n independent random observations $X_1, X_2, \ldots, X_n$ from the probability function $f\left(X\right)$.

The estimator of θ is given by

$$\widehat{\theta} = \frac{1}{n}\sum_{i=1}^{n} g(X_i).$$

Since $E\left(|g(X)|\right) < \infty$, we can get by strong law of large number

$$\frac{1}{n}\sum_{i=1}^{n} g(X_i) \xrightarrow{a.s.} Eg\left(X\right) \quad \text{as} \quad n \to \infty$$

i.e. $$\widehat{\theta} \xrightarrow{a.s.} \theta \quad \text{as} \quad n \to \infty.$$

The Monte Carlo simulation is never exact, and one always has to take the sample variance into account. It can be expressed as

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(g(X_i) - \widehat{\theta}\right)^2.$$

Central limit theorem tells us that

$$\sqrt{n}\frac{\left(\widehat{\theta}-\theta\right)}{s}\xrightarrow{d}N(0,1) \quad \text{as} \quad n\to\infty .$$

We could also say that $\widehat{\theta}-\theta$ is approximately an standard normal variable scaled by $s/\sqrt{n}$, i.e. for large *n* we have

$$P\left(\widehat{\theta}-z_{\frac{\alpha}{2}}\frac{s}{\sqrt{n}}<\theta<\widehat{\theta}+z_{\frac{\alpha}{2}}\frac{s}{\sqrt{n}}\right)\approx 1-\alpha .$$

For example, an estimate of the 95%-confidence interval for θ is given by $\left(\widehat{\theta}-1.96\frac{s}{\sqrt{n}},\widehat{\theta}+1.96\frac{s}{\sqrt{n}}\right)$. Hence, we get a way to approximate θ and its variation.

We refer to the estimation of θ by $\widehat{\theta}$ as the crude Monte Carlo method. The disadvantage of the crude Monte Carlo method is its slow rate of convergence. Since

for large *n* we have $s\approx\sigma$, we have to enlarge our *n* by a factor of 100 to receive a reduction of the standard error $s/\sqrt{n}$ by factor 0.1. Thus more accuracy is endowed by much more calculation.

4.2 Variance Reduction Techniques

Instead of simply enlarging the number *n*, we can concentrate on reducing the size of *s* to narrow the confidence interval. Such techniques are known as variance reduction techniques. We would like to introduce some of such techniques which are frequently used, see [9].

The Control Variate Method
Suppose that we want to estimate θ: = E(Y) where Y = g(X). Suppose we can somehow find another random variable Z with known mean E(Z). Then we can construct many unbiased estimators of θ:

1. $\widehat{\theta}=Y$, which is our usual estimator;

2. $\widehat{\theta}_c=Y+c\left(Z-E(Z)\right)$

where c is some real number. And it is clear that

$$E(\widehat{\theta}_c) = E(Y) + c\left(E(Z) - E(Z)\right) = E(Y) = \theta \,,$$

hence we could apply Monte Carlo to the new estimator instead of the usual one. In this context, the random variable Z is called the *control variate*. Now the question is whether it has a lower variance or not. We have that

$$Var(\widehat{\theta}_c) = Var(Y) + c^2 Var(Z) + 2cCov(Y, Z)\,.$$

Since c can be any real number, we have to choose it to minimize the quadratic. Simple calculus implies that

$$c_{\min} = -\frac{Cov(Y, Z)}{Var(Z)}\,.$$

Plug this into the quadratic, then we get

$$\begin{aligned} Var(\widehat{\theta}_{c_{\min}}) &= Var(Y) + c_{\min}{}^2 Var(Z) + 2c_{\min} Cov(Y, Z) \\ &= Var(Y) - \frac{Cov(Y, Z)^2}{Var(Z)} \\ &= Var(\widehat{\theta}) - \frac{Cov(Y, Z)^2}{Var(Z)}\,. \end{aligned}$$

As long as $Cov(Y, Z) \neq 0$, we do achieve a variance reduction. Of course, on a general problem we typically do not know $Cov(Y, Z)$. However, it is possible to estimate $Cov(Y, Z)$*)* during a Monte Carlo simulation.

That is all the mechanism the control variate method involves.

The introduction of an appropriate control variate provides a very efficient variance reduction technique, however, in some problems it may be difficult to find a suitable control variate. The alternative method we now discuss, the antithetic variate method, is often easier to apply since it concentrates on the procedure used for generating the random deviates.

The Antithetic Variate Method

Suppose again that we would like to estimate θ = E(Y) = E[g(X)], and that we have generated two samples Y_1 and Y_2. Then an unbiased estimator of θ is given by

$$\widehat{\theta} = \frac{Y_1 + Y_2}{2}.$$

Hence, we have

$$Var(\widehat{\theta}) = \frac{Var(Y_1) + Var(Y_2) + 2Cov(Y_1, Y_2)}{4}.$$

It is obvious that we could get a variance reduction if we have the two samples negatively correlated. We now give the algorithm for the normal case.
Suppose that

$$\theta = E(Y) = E\left(g(X)\right), \qquad \text{where } X \sim N(0, 1),$$

the crude Monte Carlo estimate is

$$\widehat{\theta} = \frac{1}{n}\sum_{i=1}^{n} g(X_i), \qquad \text{with i.i.d. } X_i \sim N(0, 1),$$

and the estimate by the antithetic variate method is

$$\widehat{\theta}_a = \frac{1}{n}\sum_{i=1}^{n} \frac{g(X_i) + g(-X_i)}{2}, \qquad \text{with i.i.d. } X_i \sim N(0, 1),$$

where X_i and $-X_i$ are the so-called antithetic variates.

Clearly the two antithetic variates are negatively correlated. Thus, if the function g is monotonic, then we can achieve a variance reduction by using this method. For a discussion of this, please see [9].

4.3 Pricing a European call option

We are now in a position to use Monte Carlo for pricing the European option as an example. From (2.4), we know that

$$S(T) = S(t)\exp\left((r - \sigma^2/2)(T-t) + \sigma\sqrt{T-t}X\right), \text{ where } X \sim N(0,1).$$

For the sake of convenience, we simply let t = 0, then

$$S(T) = S_0 \exp\left((r - \sigma^2 / 2)T + \sigma\sqrt{T}X\right), \text{ where } \quad X \sim N(0,1).$$

Here we choose the European option, and then the relevant Monte Carlo algorithm is as following:

set sum = 0
for i = 1 to n
    generate $S(T)$
    set sum = sum + max(0, $S(T)$ - K)
end
set $\widehat{C^K} = \exp(-rT) sum / n$

To make the result much better, we would like to use the antithetic variate method to reduce the standard error.

Table 1: Crude Monte Carlo versus control variate method

for European call options with S_0 = 100, r = 0.05, σ = 0.2, n = 10000

K	T	Accurate option values	Option values by crude Monte Carlo	Standard error by crude Monte Carlo	Option values by antithetic variate	Standard error by antithetic variate
90	2	22.03338	21.97590	0.2523557	22.09341	0.1021036
	4	30.66385	30.56650	0.3688749	30.31988	0.1643223
	6	37.82558	38.28458	0.4829697	37.87642	0.2225738
	8	44.03010	43.83765	0.5645606	43.95125	0.2747861
100	2	16.12678	16.31694	0.2224956	16.04389	0.1105872
	4	25.21333	24.77212	0.3504406	25.22523	0.1791397
	6	32.77621	32.92166	0.4563211	33.11387	0.2346970
	8	39.35980	39.14650	0.5578259	39.52114	0.2929945
110	2	11.45546	11.62117	0.1990740	11.52594	0.1145844
	4	20.53958	20.62806	0.3256482	20.66553	0.1822201
	6	28.28893	29.41766	0.4466699	28.50783	0.2397748
	8	35.12053	35.81876	0.5560821	35.36624	0.3019171

Table 1 gives the results of the simulation method for the European call option with the given parameters. We can easily see that the option values we get by both the crude Monte Carlo method and the antithetic variate method are very close to the corresponding accurate option values by the Black-Scholes formula, which demonstrates that Monte Carlo methods can be used for option pricing. Besides, we get narrower confidence interval by the antithetic variate method.

5. Monte Carlo Methods for Arithmetic Average Asian Options

To get the price of an arithmetic average price option we have to use Monte Carlo techniques again. Here we choose arithmetic average price Asian call option as example. Firstly, we use the crude Monte Carlo method with 10000 simulated paths to price the Asian option. The algorithm is as following:

set *sum* = 0
for *i* = 1 to *n*
    generate $S(T/m)$, $S(2T/m)$,…, $S(T)$

    set *sum* = *sum* + max(0, $\dfrac{\sum_{i=0}^{m} S(iT/m)}{m+1}$ - K)

end

set $\widehat{C^{K,a}} = \exp(-rT) sum / n$

The results presented in Table 2, shows the limit of the crude Monte Carlo method. As K increases, although the standard errors decrease, the simulations become less efficient for the reason that the estimates go down much faster than the corresponding standard errors. On the other hand, when the time step number gets larger, the standard error does not become smaller at all. Since convergence of our schemes is where our interest lies, variance reduction technique is necessary.

Now we would like to use the control variate method in order to make the approximation more efficient. In fact, there are many possible control variate choices we could use. It is a nature guess to use geometric average Asian call option which is given by

$$V = e^{-rT}\left(e^{\frac{1}{m+1}\sum_{i=0}^{m} \log S(\frac{iT}{m})} - K \right)^{+}$$

as the control variate, and it might be the best control variate we can find. Besides, European call option given by

$$U = e^{-rT}\left(S(T) - K \right)^{+}$$

is another choice. To make things clear, we apply both to our scheme.

Table 2: Crude Monte Carlo method for arithmetic average

Asian call options with $S_0 = 100$, $r = 0.05$, $\sigma = 0.2$, $T = 1$, $n = 10000$

K	Accurate European call option values	m	Accurate geometric Asian call option values	Option values by crude Monte Carlo	Standard error by crude Monte Carlo
90	16.6994	10	12.2398	12.58826	0.101971
		20	12.2769	12.67178	0.103715
		50	12.3009	12.58716	0.105040
		100	12.3092	12.64952	0.105090
		200	12.3134	12.62127	0.104351
		500	12.3160	12.88644	0.105597
100	10.4506	10	5.4294	5.787441	0.079883
		20	5.4856	5.704657	0.078186
		50	5.5217	5.725524	0.078627
		100	5.5341	5.610714	0.078036
		200	5.5404	5.907945	0.081512
		500	5.5443	5.810678	0.079946
110	6.0401	10	1.7500	1.870005	0.045797
		20	1.7952	1.996965	0.048735
		50	1.8243	2.054536	0.050285
		100	1.8344	1.981087	0.048553
		200	1.8395	1.980732	0.048497
		500	1.8426	2.030355	0.049334

Tables 3 shows the figures we get by using the control variate methods. We can easily see that the option values by the two different control variates are approximately the same whereas the standard errors differ dramatically. The standard deviations of the estimates with the European option as the control variate have been reduced by approximately 50% - a very modest gain in efficiency. Observe that the introduction of the geometric average Asian option as the control variate greatly reduces the standard deviation of the estimates, which is in agreement with what we have guessed. For example the range of the 95 percent confidence limits in the case of the 20-timestep Asian option with a current stock price of 90 have been reduced from + 0.4066 to + 0.0104. To achieve the same reduction by increasing the number of trials would require about 15,306,000 trials instead of 10,000. The confidence limits by using the geometric average Asian option as the control variate would appear to be sufficiently accurate for most practical applications.

Table 3: different control variate methods for arithmetic average Asian call options with $S_0 = 100$, $r = 0.05$, $\sigma = 0.2$, $T = 1$, $n = 10000$

K	m	Option values with control variate U	Standard error with control variate U	Option values with control variate V	Standard error by control variate V
90	10	12.60476	0.049824	12.53939	0.002706
	20	12.57141	0.052173	12.56962	0.002651
	50	12.53681	0.053718	12.58879	0.002661
	100	12.68053	0.054246	12.58961	0.002584
	200	12.60594	0.054336	12.59134	0.002560
	500	12.70273	0.055229	12.59679	0.002560
100	10	5.685912	0.039633	5.667367	0.002301
	20	5.722458	0.041119	5.709748	0.002184
	50	5.697757	0.042278	5.742736	0.002152
	100	5.722463	0.042245	5.755471	0.002195
	200	5.802362	0.043767	5.760154	0.002226
	500	5.786874	0.043479	5.757984	0.002169
110	10	1.886487	0.026105	1.914058	0.001982
	20	1.972285	0.028739	1.948116	0.001991
	50	2.012795	0.029756	1.974184	0.001975
	100	1.998992	0.029442	1.977349	0.001951
	200	2.015098	0.029249	1.984055	0.001927
	500	1.989555	0.029637	1.989076	0.002056

It is of interest to examine one method of using antithetic variates in the present problem as well. The results of one such set of calculations are displayed in Table 4. Comparison with the appropriate figures in Table 2 together with Table 3 shows it is more efficient when using the antithetic variate method than using European option as the control variate; however it only achieves a relatively low gain compared to the other control variate method. This low gain in efficiency may be explained as follows. While X_i and $-X_i$ have perfect negative correlation, this does not hold for the corresponding functions of them. It is not enough to effect a significant reduction in the variance of the revised estimate by such a method.

Table 4: antitheticl variate method for arithmetic average Asian call options with $S_0 = 100$, $r = 0.05$, $\sigma = 0.2$, $T = 1$, $n = 10000$

K	m	Option values by antithetic variate	Standard error by antithetic variate
90	10	12.53248	0.024917
	20	12.57284	0.025130
	50	12.63573	0.026723
	100	12.59276	0.025457
	200	12.58078	0.025743
	500	12.63754	0.026040
100	10	5.735110	0.039218
	20	5.743603	0.038214
	50	5.678478	0.038413
	100	5.694155	0.038783
	200	5.799894	0.039700
	500	5.799978	0.039382
110	10	1.929926	0.030502
	20	1.986697	0.031415
	50	2.018194	0.032110
	100	1.986706	0.031493
	200	1.976423	0.031434
	500	2.029119	0.031606

Finally, simulation results in Table 2-4 together shows that no matter which method we choose among the crude Monte Carlo, the control variate as well as the antithetic variate method, the relationship

$$C^{K,g}(S_0,T) \le \widehat{C^{K,a}}(S_0,T) \le C^K(S_0,T)$$

always holds, which is in agreement with what we showed in Proposition 3.1 and Proposition 3.2.

6. Conclusion and Outlook

We investigated the problem of pricing arithmetic average Asian options using Monte Carlo simulation techniques. Before implementing such a method, we provided the background we may need. In terms of Monte Carlo simulation, the option values we get are just estimators, thus finding a judicious choice of control variates to enhance the pricing performance of simulation becomes the critical thing. Since it is quite easy to get the closed form solutions for the European option and the geometric average Asian option, we used these two kinds of options as the control variate. Our results suggest applying the geometric average Asian option as the control variate to the Monte Carlo approach, since it greatly improves the standard deviation result to provide a narrower confidence interval. We also checked the antithetic variate method, and it turned out that this variance reduction technique was less attractive than the geometric Asian option variate one in our case.

In this paper, we have included an argument showing that the arithmetic average Asian call option has a value less than or equal to the corresponding European call. Our numerical results show that this property holds in our simulations.

A big disadvantage of using Monte Carlo methods for path dependent options is the large number of calculations that are necessary to update the path dependent variables throughout the simulation. Even control variate methods come to the end of their capacity here. We should therefore also look for new innovations in the simulation theory, especially Quasi Monte Carlo methods as described in a financial setting by Joy, Boyle and Tan [5].

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Appendix

1. Matlab Code for 4.3

```
clear all

%Crude Monte Carlo for European call options

S0=100; K=90; T=8; r=0.05; sigma=0.2;
n=10000;
Z=sqrt(T)*randn(n,1);
ST=S0*exp((r-(sigma^2)/2)*T+sigma*Z);
Payoff=exp(-r*T)*max(0,ST-K);
C_mceu=mean(Payoff)
Stderr= std(Payoff)/sqrt(n)
CI_95=[C_mceu-1.96* Stderr,C_mceu+1.96* Stderr]


%Antithetic variate for European call options

ST_=S0*exp((r-(sigma^2)/2)*T+sigma*(-Z));
Payoff_=exp(-r*T)*max(0,ST_-K);
v=0.5*(Payoff+ Payoff_);
C_aveu=mean(v)
Stderrav= std(v)/sqrt(n)
CI_95av=[ C_aveu-1.96* Stderrav, C_aveu+1.96* Stderrav]


%Black-Scholes formula for European call options

d1_ = (log(S0/K)+(r+sigma^2/2)*T)/(sigma*sqrt(T));
d2_= (log(S0/K)+(r-sigma^2/2)*T)/(sigma*sqrt(T));
C_bseu= S0*normcdf(d1_)-K*exp(-r*T)*normcdf(d2_)
```

2. Code for 4.4

```
clear all
```

```
S0 = 100; K = 90; sigma = 0.2; r = 0.05; T = 1;
Dt = 0.1; m = T/Dt; n = 10000;

%Black-Scholes formula for European call options

d1_ = (log(S0/K)+(r+sigma^2/2)*T)/(sigma*sqrt(T));
d2_= (log(S0/K)+(r-sigma^2/2)*T)/(sigma*sqrt(T));
C_bseu= S0*normcdf(d1_)-K*exp(-r*T)*normcdf(d2_)

%Geometric average Asian call options valuation by Black-Schole formula

sigsqT= sigma^2*T*(2*m+1)/(6*m+6);
muT = 0.5*sigsqT + 0.5*(r - 0.5*sigma^2)*T;
d1=(log(S0/K) + (muT + 0.5*sigsqT))/(sqrt(sigsqT));
d2=d1 - sqrt(sigsqT);
geo=exp(-r*T)*( S0*exp(muT)*normcdf(d1)-K*normcdf(d2));

ranvec= randn(n,m);
Spath = S0*[ones(n,1),cumprod(exp((r-0.5*sigma^2)*Dt+sigma*sqrt(Dt)*
ranvec),2)];
ESpath= S0*[ones(n,1),cumprod(exp(r*Dt)*ones(n,m),2)];

% Crude Monte Carlo for arithmetic average Asian call options

arithave = mean(Spath,2);
Parith = exp(-r*T)*max(arithave-K,0);    % payoffs
Pmean = mean(Parith);
Pstderr = std(Parith)/sqrt(n)
confmc = [Pmean-1.96* Pstderr, Pmean+1.96* Pstderr]


% Control variate (European option) for arithmetic average Asian call options

europayoff = Spath(:,(m+1));
Peuro = exp(-r*T)*max(europayoff -K,0);          % European payoffs
cov_euro=cov(Parith, Peuro);
c_euro=-cov_euro (1,2)/ cov_euro (2,2);
E = Parith + c_euro*( Peuro - C_bseu);                        % control variate version
Emean = mean(E);
Estderr = std(E)/sqrt(n);
confcv = [Emean-1.96* Estderr, Emean+1.96* Estderr]

% Control variate (geometric Asian option) for arithmetic average Asian call options
```

```
geoave = exp((1/(m+1))*sum(log(Spath),2));
Pgeo = exp(-r*T)*max(geoave-K,0);            % geo payoffs
cov_geo=cov(Parith, Pgeo)
c_geo=-cov_geo(1,2)/ cov_geo(2,2)
Z = Parith +c_geo*( Pgeo-geo);                          % control variate version
Zmean = mean(Z);
Zstderr = std(Z)/sqrt(n);
confcv_ = [Zmean-1.96* Zstderr, Zmean+1.96* Zstderr]


%Antithetic variate for arithmetic average Asian call options

Spath_= S0*[ones(n,1),cumprod(exp((r-0.5*sigma^2)*Dt+sigma*sqrt(Dt)*
(-ranvec)),2)];
Uarithave = mean(Spath_,2);
Aarith = 0.5*exp(-r*T)*(max(arithave-K,0)+ max(Uarithave-K,0));    % payoffs
Amean = mean(Aarith);
Astderr = std(Aarith)/sqrt(n)
confav = [Amean-1.96* Astderr, Amean+1.96* Astderr]
```