VaR without Correlations for Portfolios of Derivative Securities

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We propose filtering historical simulation by GARCH processes to model the future distribution of assets and swap values. Options' price changes are computed by full reevaluation on the changing prices of underlying assets. Our methodology takes implicitly into account assets' correlations without restricting their values over time or computing them explicitly. VaR values for portfolios of derivative securities are obtained without linearising them. Historical simulation assigns equal probability to past returns, neglecting current market conditions. Our methodology is a refinement of historical simulation. © 1999 John Wiley & Sons, Inc. Jrl Fut Mark 19: 583–602, 1999

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INTRODUCTION

Current methods of evaluating the risk of portfolios of derivative securities are unsatisfactory. Delta-gamma hedging becomes unstable for large asset price changes or for options at the money with short maturities (Allen, 1997). Monte Carlo simulations assume a particular distributional form, imposing the structure of the risk that they were supposed to investigate. Moreover, they often use factorization techniques that are sensitive to the ordering of the data. Historical simulations usually sample from past data with equal probabilities. Therefore they are appropriate only if returns are i.i.d. (independently and identically distributed), an assumption violated by volatilities changing over time. This misspecification leads to inconsistent estimates of Value at Risk, as documented by Hendricks (1996) and McNeal and Frei (1998).

An overview of VaR (Value at Risk) estimation techniques is available in Davé and Stahl (1997). These authors show the effects of ignoring non-normality and volatility clustering in the computation of VaR. Even for the simple portfolios they consider, current VaR methodologies underestimate substantially the severity of losses. From their results, they infer that historical simulation modulated by a GARCH process is likely to be a better method. Such a technique is implemented with good results by Barone-Adesi, Bourgoin, and Giannopoulos (1998) for a portfolio replicating a stock market index.

We propose to extend the recent methodology of Barone-Adesi, Bourgoin, and Giannopoulos (1998) to portfolios with changing weights that may also include derivative securities. Following them, we model changes in asset prices to depend on current asset volatilities. Asset volatilities are simulated to depend on the most recently sampled portfolio returns. Our simulation is based on the combination of GARCH modeling (parametric) and historical portfolio returns (nonparametric). Historical residual returns are adapted to current market conditions by scaling them according to the ratio of current over past conditional volatility. By dividing historical residual returns by this volatility we standardize them for our simulation. These standardized residuals are then scaled by a volatility forecast that reflects current market conditions. Our simulated returns are based on these residuals.

The simulated returns are the basis for our study. To simulate a pathway of returns for each of a number of different assets over next 10 days, we select randomly 10 past sets or "strips" of returns, with each return in a strip corresponding to an asset's price change that occurred on a day in the past. Thus each strip of returns represents a sample of the comovements between asset prices. We compute residual returns from the returns. We then iteratively construct the daily volatilities for each asset that each of these strips of residuals implies according to the chosen GARCH model. We use the ratio of these volatilities over historical volatility to change the scale of each of our sampled residuals. The resulting simulated asset returns therefore reflect current market conditions rather than historical ones. Derivatives on the assets are simulated by full reevaluation at each point in time.

GARCH models are based on the assumption that residual asset returns follow a normal distribution. If residual returns are not normal, GARCH estimates may be consistent but inefficient. A better filter could then be selected. Following a large literature in financial econometrics, we will focus on GARCH.

In principle, any GARCH or other time series model is suitable for our methodology, provided it generates i.i.d. residuals from our return series. Therefore residual diagnostics as well as the R-square of the Pagan-Ullah regression are important criteria for our model selection. The high t-statistics of our model parameters suggest that our models are well specified. Missspecification would result in poor predictions of conditional variances, leading to poor backtesting results.

The core of our methodology is the historical returns of the data. The "raw" returns, however, are unsuitable for historical simulation because they do not fulfill the properties[1] necessary for reliable results.

Among others, Mandelbrot (1963) found that most financial series contain volatility clusters. In VaR analysis, volatility clusters imply that the probability of a specific incurred loss is not the same on each day. During days of higher volatility we will expect larger than usual losses.

SIMULATING A SINGLE PATHWAY

In our simulation we do not impose any theoretical distribution on the data. We use the empirical (historical) distribution of the return series. To render returns i.i.d. we need to remove any serial correlation and volatility clusters present in the data set. Serial correlations can be removed by adding an MA term in the conditional mean equation. To remove volatility clusters it is necessary to model the process that generates them. We propose to capture volatility clusters by modeling returns as GARCH processes (Bollerslev, 1986).[2] When appropriate we insert a moving av-

[1]For simulation, returns should be random numbers drawn from a stationary distribution—that is, they should be identically and independently distributed (i.i.d.).

[2]The particular form of GARCH process used for a series was determined by statistical testing. Although the GARCH(1,1) specification is suitable for most series, it may not be adequate for all the assets in the portfolio. Its failure may produce residuals that are not i.i.d. and do not satisfy the requirements of our historical simulation. We are currently investigating, in a different study, the relevance of GARCH mispecification on our VaR computations.

erage (MA) term in the conditional mean eq. (1) to remove any serial dependency. As an example an ARMA-GARCH(1,1) model can be written as:

$$r_t = \mu r_{t-1} + \theta\varepsilon_{t-1} + \varepsilon_t \qquad \varepsilon_t \sim \mathrm{N}(0, h_t \tag{1}$$

$$h_t = \omega + \alpha\,(\varepsilon_{t-1} + \gamma)^2 + \beta h_{t-1} \tag{2}$$

where μ is the AR(1) term, θ is the MA term, ω is a constant, and ε_t the random residual.

The GARCH(1,1) equation defines the volatility of ε_t as a function of the constant ω plus two terms reflecting the contributions of the most recent surprise ε_{t-1} and the last period's volatility h_{t-1}, respectively. The constants α and γ determine the influence of the last observation and its asymmetry.

To standardize residual returns we need to divide the estimated residual $\hat{\varepsilon}_t$ by the corresponding daily volatility estimate, $\sqrt{\hat{h}_t}$.[3] Thus the standardized residual return is given as:

$$e_t = \frac{\varepsilon_t}{\sqrt{\hat{h}_t}}$$

Under the GARCH hypothesis the set of standardized residuals are independently and identically distributed (i.i.d.) and therefore suitable for historical simulation. Empirical observations may depart from that to some degree.

As Barone-Adesi, Bourgoin, and Giannopoulos (1998) have shown, historical standardized innovations can be drawn randomly (with replacement) and, after being scaled with current volatility, may be used as innovations in the conditional mean (1) and variance (2) equations to generate pathways for future prices and variances respectively. Our methodology stands as follows:

- We draw standardized residual returns as a random vector ε_t of outcomes from a data set Θ:

$$e^* = \{e_1^*, e_2^*, \ldots, e_T^*\} \quad e_i^* \in \Theta \quad \text{where } i = 1, \ldots, 10 \text{ days.} \tag{3}$$

- To get the innovation forecast (simulated) value for period t + 1, z_{t+1}^*, we draw a random standardized residual return from the data set Θ and scale it with the volatility of period[4] $t + 1$:

[3]Henceforth, simply h and ε.

[4]The variance of period t+1 can be calculated at the end of period t as: $h_{t+1} = \hat{\omega} + \hat{\alpha}\,\varepsilon_t^2 + \hat{\beta} h_t$, in which ε_t is the latest estimated residual return in eq. (1).

$$z_{t+1}^* = e_1^* \cdot \sqrt{h_{t+1}} \tag{4}$$

- We begin simulation of the pathway of the asset's price from the currently known asset price, at period t. The simulated price p_{t+1}^* for $t + 1$ is given as

$$p_{t+1}^* = p_t + p_t(\hat{\mu} r_t + \hat{\theta} z_t^* + z_{t+1}^*) \tag{5}$$

where Z^* is estimated as in (4).

For $i = 2, 3 \ldots$ the volatility is unknown and must be simulated from the randomly selected rescaled residuals. In general $\sqrt{h_{t+i}^*}$, the (simulated) volatility estimate for period $t + i$, is obtained as:

$$\sqrt{h_{t+i}^*} = \sqrt{\hat{\omega} + \hat{\alpha}(z_{t+i+1}^*)^2 + \hat{\beta} h_{t+i+1}^*} \quad i \geq 2 \tag{6}$$

where z^* is estimated as in (4).

New elements ε_t^* are drawn from the data set Θ to form the simulated prices p_{t+i}^* as in (5).

The "empirical" distribution of simulated prices at the chosen time horizon (for example, $i = 10$) for a single asset is obtained by replicating the above procedure a large number of times—for example, 5000.

SIMULATING MULTIPLE PATHWAYS

To estimate risks for a portfolio of multiple assets, we need to preserve the multivariate properties of asset returns; however, methodologies that use the correlation matrix of asset returns encounter various problems with this. The use of conditional multivariate econometric models, which allow for correlations to change over time, is restricted to a few series at a time. The number of terms in a correlation matrix increases with the square of the number of assets in the portfolio. For large portfolios, the number of pairwise correlations becomes unmanageable.

When estimating time-varying correlation coefficients independently from each other, there is no guarantee that the resulting matrix satisfies the multivariate properties of the data. In practice, the resulting matrix may not be positive definite.

Additionally, the estimation of VaR from the correlation matrix requires knowledge of the probability distribution of each asset series. However, empirical distributions may not conform to any known distribution; often the empirical histograms are smoothed and forced to follow a known distribution convenient for the calculations. VaR measures that are based on arbitrary distributional assumptions may be unreliable; preliminary smoothing of data can cover up the non-normality of the data; VaR estimation, which is highly dependent on the good prediction of uncommon events, may be adversely affected from smoothing the data.

Finally, correlations measured from daily returns can be demonstrated to be unstable. Even their sign is ambiguous. Estimated correlation coefficients can be the subject of such great changes at any time, which even conditional models do not capture, that the successful forecast of portfolio losses may be seriously inhibited.

Our approach does not employ a correlation matrix. For a portfolio of multiple assets, we extend our simulation methodology[5] to simulate multiple pathways. We select a random date from the data set, which will have an associated set of residual returns. This "strip" of residual returns, derived at a common date in the past, is one sample from which we begin modeling the comovements between respective asset prices.

Thus for each asset for $i = 1 \ldots 10$ days we have the sampled residuals denoted by subscripts 1, 2, 3, . . . , for the different assets.

$$\text{Asset 1: } e_1^* = \{e_1, e_2, \ldots, e_{\mathrm{T}}\}_1 \tag{7}$$

$$\text{Asset 2: } e_2^* = \{e_1, e_2, \ldots, e_T\}_2 \tag{8}$$

$$\text{Asset 3: } e_3^* = \{e_1, e_2, \ldots, e_T\}_3 \tag{9}$$

with $e_i^* \in \Theta$ and so on for all the assets in the data set: $\Theta = \{\Theta_1, \ldots, \Theta_N\}$. From the data set Θ of historical standardized innovations, for $i = 1$, a date is randomly drawn and hence the associated residuals $e^*{}_1$, e_2^*, $e^*{}_3$ are selected. At $i = 2$ another date is drawn, with its corresponding residuals, and so on for $i = 3, 4 \ldots$ and so forth. Thus pathways for variances, h, and prices, p, are constructed for each asset that reflects the comovements between asset prices:

For $i = 1$ to 10:

$$\text{Asset 1: } h_{1,t+i}^* = \hat{\omega}_1 + \hat{a}_1(z_{1,t+i-1}^*)^2 + \hat{\beta}_1 h_{1,t+i-1}^* \tag{10}$$

$$p_{1,t+i}^* = p_{1,t+i-1}^* + p_{1,t+i-1}^*(\hat{\mu}_1 r_{1,t+i-1} + \hat{\theta}_1 z_{1,t+i-1}^* + z_{1,t+i}^* \tag{11}$$

$$\text{Asset 2: } h_{2,t+i}^* = \hat{\omega}_2 + \hat{a}_2(z_{2,t+i-1}^*)^2 + \hat{\beta}_2 h_{2,t+i-1}^* \tag{12}$$

$$p_{2,t+i}^* = p_{2,t+i-1}^* + p_{2,t+i-1}^*(\hat{\mu}_2 r_{2,t+i-1} + \hat{\theta}_2 z_{2,t+i-1}^* + z_{2,t+i}^* \tag{13}$$

$$\text{Asset 3: } h_{3,t+i}^* = \hat{\omega}_3 + \hat{a}_3(z_{3,t+i-1}^*)^3 + \hat{\beta}_3 h_{3,t+i-1}^* \tag{14}$$

[5]Additional reading about the this methodology can be found in Efron and Tibshirani (1993).

TABLE I

GARCH Estimates

Series	ω	α	β	γ	μ	θ	*R squared*	*ML*
A	0	0.07754	0.86421	−0.00292083	−0.431	0	0.381	−1383.99
Std Error		0.023	0.033	0.000767	0.044			
G	0	0.042527794	0.910057127	0.006027014		0	0.313	−1562.43
Std Error		0.01286	0.02324	0.00098939				
S	$1.797378*10^{-5}$	0.123744	0.791801			0	0.324	−2026.21
Std Error	$8.914*10^{-6}$	0.0298	0.065					

$$p^*_{3,t+i} = p^*_{3,t+i-1} + p^*_{3,t+i-1}(\hat{\mu}_3 r_{3,t+i-1} + \hat{\theta}_3 z^*_{3,t+i-1} + z^*_{3,t+i} \quad (14)$$

z^* where is estimated as in (4).

AN EMPIRICAL INVESTIGATION

We illustrate our methodology with a numerical example of a portfolio of three assets. Our hypothetical portfolio is invested across three LIFFE futures contracts and a call option on the Long Gilt future with net lots 2, −5, 10, and 7; lot conversion factors for the contracts are 2500, 500, 2500, and 500 respectively. Our historical data sets consists of two years of daily[6] prices, from 4 January 1994 until 27 December 1995, for three interest rate futures contracts, the 10-year German Government Bund (A), Long Gilt (G) and the three-month EuroSwiss Franc (S) contracts.[7]

Given the daily price, p_t, we obtain the daily returns r_t as

$$r_t = \ln\left(\frac{p_t}{p_{t-1}}\right) \quad (16)$$

and then we form continuous series of historical returns by rolling a few days before the expiration date to the next front month contract.

For each historical return series we fit the most suitable GARCH-ARMA specification, as in eqs. (1) and (2) to obtain i.i.d. residual returns. The parameter estimates together with standard errors and the likelihood value are shown in Table I.

The low standard errors as well as the residual statistics (not reported) support our parameterization choices. The equations are esti-

[6]All three contracts are traded on the London International Futures Exchange (LIFFE) at different delivery months.
[7]The price of the LIFFE EuroSwiss contract is derived by subtracting the appropriate forward-forward interest rate from 100. Hence pathway calculations are made using 100 minus the quoted price.

Table II

Close Prices and Conditional Volatility on 21 and 22 February

	Prices on Feb 21	*Return at Close of Business*	*Vol. p.a. (Feb 21)*	*Vol. p.a. (Feb 22)*
A	97.39	0.00446	0.10053	0.09347
G	107.219		0.10086	0.09623
S	97.48		0.37021	0.35436
Call option (G)	0.67169			

mated in four steps. First by OLS to get starting values, then by downhill simplex (because its robustness to bad starting values and discontinuities). The BHHH algorithm was then used to refine convergence and finally a quasi-Newton method, the BFGS, was used to get reliable standard errors.

As an example let the current close business be 21 February 1996; we want to estimate the portfolio VaR over the next two business days. The closing prices and annualised volatilities for the three futures on that date are reported in Table II.

The conditional volatility of the next date, that is, 22 February, is calculated by substituting the last trading date's residual error and variance into eq. (2). To simulate asset prices for 22 February we draw a random (with replacement) row[8] of historical (standardized) asset residual returns[9] and rescale them with the corresponding asset's volatility on 22 February to form a random surprise, ε_t, in eq. (1). In this way we generate parallel pathways for all linear assets in the portfolio without imposing the degree of cross correlation between the assets. By taking a row of random residuals we maintain the comovement between the assets when we generate the simulated forecasts.

Table III shows a sample of the standardized residuals for each asset used in our simulation.

- Let us assume that the random set of standardized residuals are: -1.15592, -1.13077, and 0.86704 for A, G, and S[10] contracts re-

[8]A row contains the—standardized—innovations that occur on a random date from the past across all contracts.

[9]Table I is an extract, for illustrative purposes, of standardized residual returns based on closing prices for three futures over a two-year period. We can have as many columns of residual returns as there are assets, or as in the case of swaps in a given currency, a set of columns of interest rate residual returns—for example, from 1 day to 10 years per currency, from which swap evaluations may be performed.

[10]This set corresponds to the 13.01.94.

TABLE III

Historical Standardized Residuals

Date	*A*	*G*	*S*
05/01/94	0.00000	0.00000	0.00000
06/01/94	−0.15123	0.08776	0.69159
07/01/94	0.85533	1.25962	0.00000
10/01/94	0.18241	−0.32852	0.96747
11/01/94	−0.24443	−0.94479	−0.58417
12/01/94	0.29110	0.27269	−0.41143
13/01/94	−1.15592	−1.13077	0.86704
14/01/94	−0.77676	−0.35823	0.65085
14/01/94	−0.38586	0.27006	−0.22329
18/01/94	0.32893	1.20579	−0.23623
13/11/95	0.93074	0.43796	−0.72107
21/02/96	0.40954	1.01243	0.085935

spectively.[11] At the first simulation run, the one date ahead rescaled residuals, z^*, for the three futures will be:

$$\text{A:} \qquad z^*_{1,t+1} = -1.15592^* \frac{9{,}98346}{\sqrt{252}} = -0.00680612$$

where $h_{1,t+1} = (0.09347/\sqrt{252})^2 = 0.00003467$

$$\text{G:} \qquad z^*_{2,t+1} = -1.13077^* \frac{0.09623}{\sqrt{252}} = -0.0068546$$

$$\text{S:} \qquad z^*_{3,t+1} = 0.86704^* \frac{0.35436}{\sqrt{252}} = 0.019354571$$

These are also the innovations for eq. (1). Recall from eq. (5) the i_{th} forecast for 22 February is given by:

$$p^*_{i,t+\tau} = p_{i,t} + p_{i,t}(\hat{\mu}_i r_t + \hat{\theta}^*_i z^*_{i,t} + z_{i,t+1})$$

where $(\hat{\mu}_i r_t + \hat{\theta}^*_i z^*_{i,t} + z^*_{i,t+1})$ is the simulated return. This gives us:

$$\text{A:} \quad p^*_{1,t+1} = 97.39 + 97.39(-0.43084^*0.00446 + -0.00680612)$$

$$= 97.39 + 97.39(-0.00872862) = 96.5399197$$

[11]As the random sampling is with replacement, we may draw the same date more than once during the simulation process.

G: $p^*_{2,t+1} = 107.219 + (107.219^* - 0.00685464) = 106.4840526$

S: $p^*_{3,t+1} = 100 - (2.52 + (2.52^*0.019354571) = 97.43122648$

$\rightarrow$ Working price $= 100 - 97.43122648 = 2.56877$

To produce the ith simulated volatility for the second date ahead we substitute ε_{t-1} with $z^*_{1,t+1}$, $z^*_{2,t+1}$, $z^*_{3,t+1}$, in (2). Hence the simulated variance for 23 February 1996 for contract A is:

$$h^*_{1,t+2} = \omega_1 + \alpha_1(z^*_{1,t+1} + \beta_1 h^*_{1,t+1} = 0 + 0.07754(-0.00680612 + -0.00292083)^2 + 0.86421^*0.00003467 = 0.0000373$$

Similarly, we calculate the i_{th} simulated variances for contracts G and S to be

$$h^*_{2,t+2} = \omega_2 + \alpha_2(z^*_{2,t+1} + \gamma_2)^2 + \beta_2 h^*_{2,t+1} = 0.0000405$$

$$h^*_{3,t+2} = \omega_3 + \alpha_3(z^*_{3,t+1} + \gamma_3)^2 + \beta_3 h^*_{3,t+1} = 0.000458881$$

We repeat the above calculations to get the N days ahead forecasts of the variances and prices for each of the three futures contracts. For example, to obtain the two-day ahead price forecasts, we sample randomly another row with historical standardized residuals, for each of the three contracts. Let us assume that this random set corresponds to 13 November 1995, and the values are 0.93074, 0.43796, −0.72107, for A, G, and S respectively. When these random historical standardized residuals are rescaled by the day 2 simulated volatilities, the following set of scaled residuals are produced:

A: $z^*_{1,t+2} = \sqrt{h^*_{1,t+2}}\ {}^*0.93074 = 0.006107194^*$

$0.93074 = 0.00568421$

G: $z^*_{2,t+2} = \sqrt{h^*_{2,t+2}}{}^*\ 0.4376 = 0.006363858^*$

$0.4376 = 0.002787115$

S: $z^*_{3,t+2} = \sqrt{h^*_{3,t+2}}\ {}^*-0.72107 = 0.021421503^*$

$-0.72107 = -0.015446403$

Hence $z^*_{1,t+2}$, $z^*_{2,t+2}$, $z^*_{3,t+2}$, are the simulated residuals for 23 February. Therefore the simulated set of prices for the same date will be:

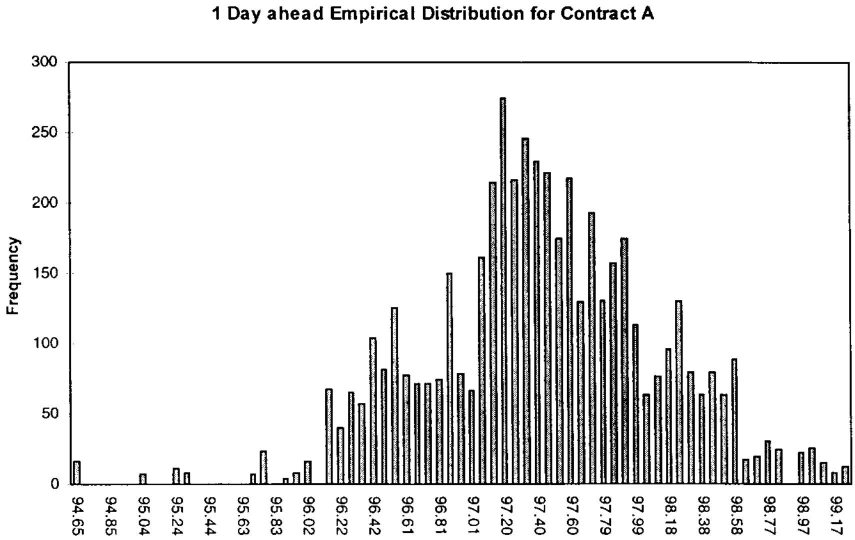

FIGURE 1
The 1-day ahead distribution of German Bund futures prices.

$$\text{A:}\quad p^*_{1,t+2} = 96.5399197 + 96.5399197^* (-0.43084^* - 0.00872862 - 0.00568421) = 97.45172459$$

$$\text{G:}\quad p^*_{2,t+2} = 106.4840526 + 106.4840526^* \; 0.002787115 = 106.780836$$

$$\text{S:}\quad p^*_{3,t+2} = 100 - (2.56877 + 2.56877^* - 0.015446403) = 97.47090479$$

Note μ_2 and $\mu_3 = 0$, so the AR term is absent in these equations.

The above steps can be repeated to produce the entire set of, let us say 5000, simulated values. Figure 1 illustrates examples of distributions of price pathways for 21.02.96, for the LIFFE German Bund financial futures contract.

Similarly, for longer VaR horizons, our steps can be repeated to obtain a simulated pathway for each date ahead. Figure 2 shows the distribution of the 5000 simulation runs for the 10th date ahead for the German Bund. The asymmetry of our simulated distribution is apparent.

Options

Options price paths are obtained from the corresponding asset price paths by using an options pricing model applied to each asset price in the path

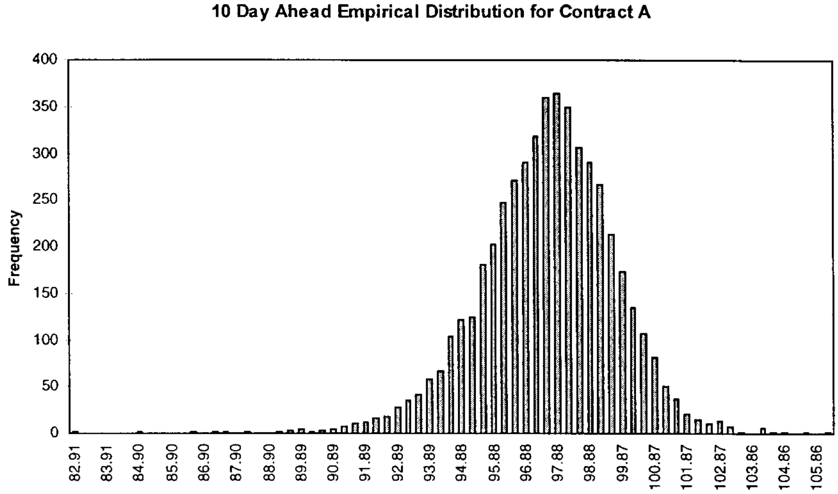

FIGURE 2
The 10-day ahead distribution of German Bund futures prices over 5000 runs.

and other relevant option pricing parameters—for example, implied volatility, σ, strike price, x, time to expiry, $T - t$, and interest rate, r. For the present we keep the values of these other parameters equal to their values at the start of simulation.

$$\text{Thus the call option price is denoted } c = f(p_i, X, \sigma, T - t, r) \quad (17)$$

where p_t is the underlying asset price at current time t. The price path for the call option on a given asset is:

$$\mathbf{c}_{t,t+1,t+i} = f(p_t, X, \sigma, T - t, r), f(p_{t+1}, X, \sigma, T - t + 1, r), \ldots =, f(p_{i+i}, X, \sigma, T - t + i, r) \quad (18)$$

Where $p_t, \ldots, p_{t+i}$ is the first vector (that is, for the first asset) from (15).

Additional option pathways use the asset prices from the corresponding asset price vectors in eq. (15). Figure 3 illustrates an example of the ten day ahead distribution of prices for an out-of-the money call option, for 5000 simulation runs on the LIFFE Long Gilt futures contract. The time to expiry was one and one-half months (expiry date 22/3/96), the strike price was 108 points, and the underlying futures price was 107.219. The option's market price was 0.670, and the ten-day median forecast price was 0.477. The minimum price was 0.00018 and the maximum 4.82152—thus illustrating the nonlinearity of option pricing.

Using the Black '76 model and the futures price path for contract G, the following price pathway was generated for the call option above.

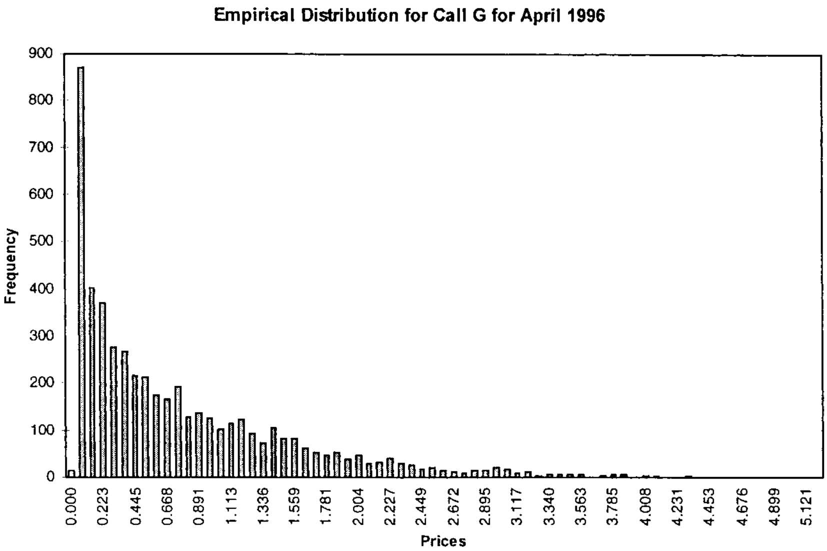

FIGURE 3
Empirical distribution for call G for April 1996.

TABLE IV

Option Pricing Model Input Values and Results

	Close of Business	*One Day Ahead*	*Two Days Ahead*
Futures Price Path	107.219	106.4841	106.7808
Strike Price	108.00	108.00	108.00
Implied Volatility	0.08	0.08	0.08
Time to Expiry	0.087302	0.083333	0.079365
Call Path (generated by Black '76 model)	0.67169	0.40956	0.47953

Aggregating Asset Pathways to Obtain Portfolio Pathways

For the first simulation we select the asset pathways that correspond to the contracts in the portfolio. These are the vectors

$$p_{1,(t+\tau)},\ p_{2,(t+\tau)},\ p_{3,(t+\tau)},\ \ldots,\ p_{n,(t+\tau)} \quad \tau = 0,\ 1\,..\,i \tag{19}$$

for n assets and a time horizon of i days. The position-weighted pathways in the portfolio are the vectors:

$$w_1 p_{1,(t+\tau}, \ w_2 p_{2,(t+\tau)},\ w_3 p_{3,(t+\tau)},\ \ldots,\ w_{\mathrm{n}} p_{n,(t+\tau)} \ \tau = 0,\ 1\,..\,i \tag{20}$$

where the scalars $w_1, w_2, w_3, \ldots, w_n$ are the weights of contracts in the portfolio.

The vectors of pathways are added to form the portfolio path $\pi_{t+\rho}$

$$\pi_{t+\tau} = w_1 p_{1,(t+\tau)} + w_2 p_{2,(t+\tau)} + w_3 p_{3,(t+\tau)} + , \ldots , + w_n p_{n,(t+\tau)} \; \tau = 0, 1 \,.\,.\, i \qquad (21)$$

The price pathways above are modified by weights derived by multiplying together the relevant number of lots, the lot conversion factor, and the currency rate (to Sterling). The exchange rate from DM is taken to be constant at 2.24, and the exchange rate from Swiss francs to Sterling taken constant at 1.82. The lot conversion factors are 2500 for the Euroswiss and Bund contracts and 500 for both Long Gilt contracts.

$$\text{A:} \quad w_1 p_{1\{t,,t+1,t+2\}} = 1/(2.24)^*2500^*2[97.3900, 96.5399, 97.4517]$$
$$= [£217388, £215491, £217526]$$

$$\text{G:} \quad w_2 p_{2,\{t,t+1,t+2\}} = 500^* - 5[107.2190, 106.4841, 106.7808]$$
$$= [-£268048, -£266210, -£266952]$$

$$\text{Call Option on G} = 500^*7[0.6717, 0.4096, 0.4759]$$
$$= [£2351, £1433, £1666]$$

$$\text{S:} \quad w_3 p_{3,\{t.t+1,1+2\}} = 1/(1.82)^*2500^*10[97.4800, 97.4312,97.4709]$$
$$= [£1292, £788, £915]$$

Thus the portfolio path based on prices is

$$\pi_{t,t+1,t+2} = w_1 p_{1\{t,,t+1,t+2\}} + w2 p_{2,\{t,t+1,t+2\}} + w_3 p_{3,\{t.t+1,1+2\}}$$
$$= [£217388, £215491, £217526] + [-£268048, -£266210, -£266952] + [£1292, £788, £915] + [£2351, £1433, £1666] = [-£47016, -£48498, -£46845]$$

The change in the portfolio's value after 2 days from its closing value is

$(-£46844.92496) - (-£47016.4787) = £171.5537$, which in this (first) simulation path is a gain in value.

By repeating the above procedure with different random values the em-

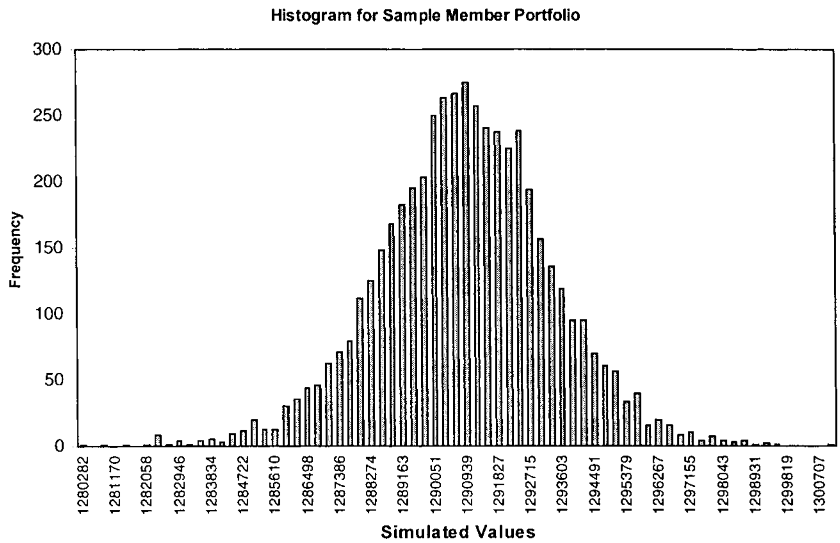

FIGURE 4
10-day ahead portfolio value distribution over 5000 simulations.

pirical distribution of portfolio values can be obtained. The representative "lowest value" of the portfolio, for example, for the 99th percentile, can be compared to the value of the portfolio at the start of simulation, to obtain the 99th percentile loss. A ten-day ahead multicontract portfolio example (a portfolio of futures and options in a variety of LIFFE contracts) is illustrated in Figure 4:

SWAPS

Our methodology can be applied to any type of asset. We may have a portfolio comprising exchange-traded futures and options, interest rate and currency swaps, and swaptions.

For example: A swap with three cash-flows remaining before it matures has its value denoted by an appropriate swap valuation function of zero coupon interest rates:

$$s = g(\iota_1, \iota_2, \iota_3, \phi) \tag{220}$$

where ϕ represents parameters defined in the swap contract necessary to value it (for example, coupon, floating and fixed interest rates, notional principal amount, payment dates of the cash-flows, maturity date, and so forth); ι_1, ι_2, and ι_3 are zero coupon interest rates (term structure) for dates corresponding to the future payment dates. The value of a swap at a given close of business will utilise the zero coupon rates (term structure) at this time.

We consider interest rate swaps to demonstrate how the methodology may be applied. A pathway of swap values is obtained by simulating zero coupon interest rates curves. For the first scenario, we simulate 10 zero coupon rates for each day of the holding period. This is replicated to obtain 5000 such simulations. To simulate a zero coupon rate curve, we need to define how we create it from the source interest rates—for example, money market rates, interest rate futures, and quoted swap rates for various maturities, for instance, to 10 years. These source rates, which could be depicted as a curve, allow a zero coupon rate curve to be created[12] from them; the zero coupon rate curve is defined by points of constant maturity that correspond to the maturities of the source rates.

We treat each of the source rates as an asset, and simulate a single pathway for each source rate, as described in the foregoing sections for futures pathways—that is, starting from logarithmic returns from historical time series of (constant maturity) source interest rates. We obtain a pathway for each source interest rate at the current close of business—that is, we simulate the source interest rate curve for each day of the holding period ($i = 10$). For each of these we apply the methodology, described by Hull (1997), to convert them to zero coupon interest rate curves. Replication of the process obtains 5000 zero coupon rate curves defined by small number (ten) constant maturity points.

Interest rate swaps are evaluated from each of the simulated yield curves. This necessitates interpolation between the constant maturity points. During the simulation process, we use linear interpolation because we believe this to be sufficiently accurate for simulation processes and much faster to compute than other methods (for example, cubic splines), given the number of simulations we require.

In this way we create pathways of swaps prices that correspond in order (a holding period of 10 days over 5000 scenarios) to the futures and options pathways. The 5000 simulated portfolio values for exchange traded instruments and interest rate derivatives together can therefore be estimated, regardless of type or currency of instrument.

Figure 5 is an example of the term structure of interest rates out to 10 years for Sterling prior to simulation, produced by linear interpolation.

For simplicity, if we consider that the three asset (interest rate) pathways from eqs. (11), (13), and (15) correspond to the cash-flow dates for

[12]The methodology for the creation of zero coupon rate curves is described in *Options, Futures and Other Derivatives* by John C. Hull (Prentice Hall, 1997).

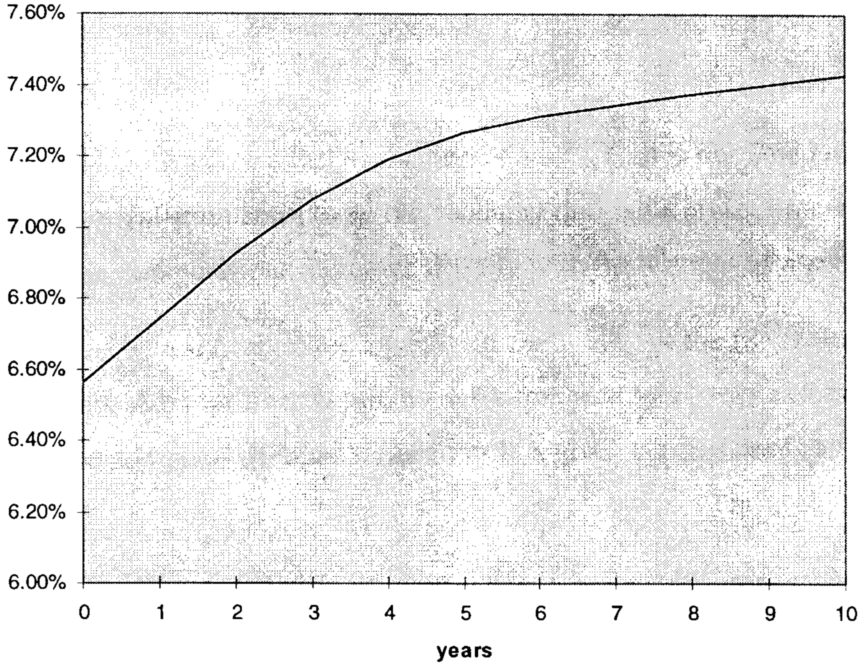

FIGURE 5
GBP term structure.

our swap (no interpolation of rates required), then writing ι^* for p^*, we depict the 10×3 matrix:

$$\mathbf{R} = \begin{bmatrix} \iota^*_{1,\iota+1} & \iota^*_{2,\iota+1} & \iota^*_{3,\iota+1} \\ \iota^*_{1,\iota+2} & \iota^*_{2,\iota+2} & \iota^*_{3,\iota+2} \\ \iota^*_{1,\iota+3} & \iota^*_{2,\iota+3} & \iota^*_{3,\iota+3} \\ \vdots & \vdots & \vdots \\ \iota^*_{1,\iota+i} & \iota^*_{2,\iota+i} & \iota^*_{3,\iota+i} \end{bmatrix} \tag{23}$$

where $i = 1$ to 10 days.

Each column of the matrix represents eqs. (11), (13), and (15) respectively—that is, they are the asset pathways to 10 days. To obtain a swap value pathway we require a row from the matrix for each day in the swap value path:

$$\mathbf{s}^*_{\iota+i} = g(\iota^*_{1,\iota+1}, \iota^*_{2,\iota+1}, \iota^*_{3,\iota+1}, \ldots, \iota^*_{1,\iota+i}, \iota^*_{2,\iota+i}, \iota^*_{3,\iota+i}, \phi) \tag{24}$$

For swap portfolios, the swap value pathways are aggregated as described generally for any set of assets, in eqs. (19) to (21); the net positions w_n for swaps can be represented as $+1$ or -1 for each swap, to describe the payment or receipt of fixed interest cash-flows respectively. Furthermore,

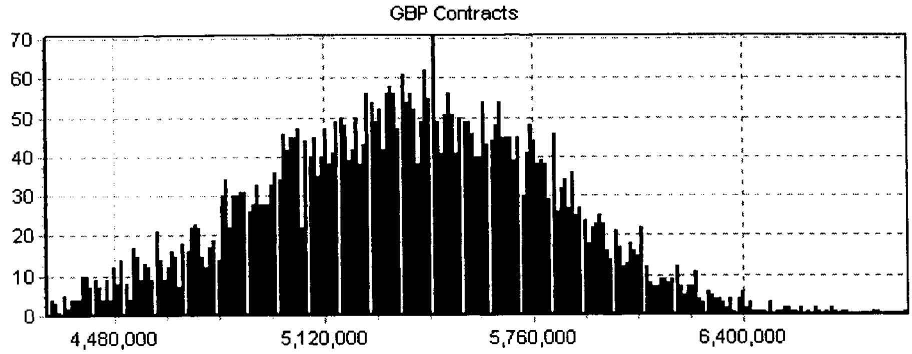

FIGURE 6
Random swaps portfolio values over 10 day holding period for 5000 simulation runs (y-axis is the count of values in each bar). Portfolio values in Sterling.

aggregated values for portfolios of swaps and futures and options contracts may be obtained with no fundamental change to our methodology. Five thousand simulation runs may be performed for portfolios of swaps, futures, and options, from which worst case losses can be obtained.[13]

In Figure 6, we simulate 5000 values of a random portfolio of "plain vanilla" interest rate swaps in Sterling, over a 10-day holding period. The 5000 portfolio values are obtained from 5000 simulated interest rate term structures.

The distribution of portfolio values is shown in the histogram; the lowest value, represented by 99th percentile, is compared to the median portfolio value. This is the "worst" loss for the portfolio, equal to £1,087,421, and is the difference between the least value at the 99th percentile of £4,280,410 and median value of £5,367,831.[14]

In Figure 7 we show the simulated linearly interpolated term structure from which the 99th percentile, 10-day holding period portfolio value is calculated. This simulated term structure is compared to the actual observed term structure 10 days on from the date at which simulation was started.

CONCLUSION

Our methodology simulates the returns of portfolios of derivative securities, taking into account information available on current market con-

[13]Appropriate currency exchange rates for the given close of business are currently used in the simulations where contracts are denominated in different currencies, to convert all values to a common currency.

[14]Alternatively the loss may be computed from the initial portfolio value as shown in the previous example, rather than the median. The two losses are the same in RiskMetrics because the median is assumed to be equal to the initial value in that methodology.

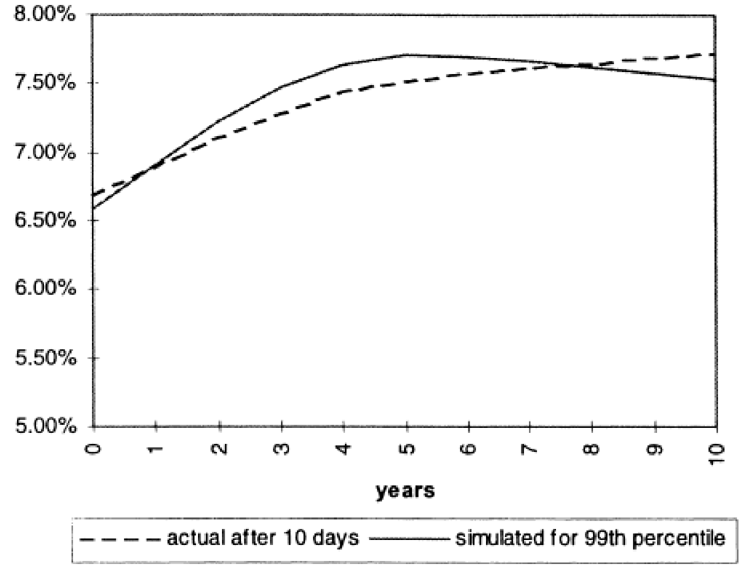

FIGURE 7
Sterling interest rate term structures: actual and after simulation, at 99th percentile and 10-day holding period.

ditions. We preserve the information on historical non-normalities of security returns and their comovements, without introducing the complexities and the noise associated with the computation of large covariance matrices.

Our methodology leads to a fast evaluation of VaR. That is possible because it requires a simple historical simulation to be run each day through a preset time-series filter. The number of our computations increases linearly with the number of assets.

The reliability of our evaluation depends on the quality of the filters used in our time series analysis. A better filter would by definition lead to a better assessment of risk. Therefore the adequacy of a particular filter in a given context needs to be verified through backtesting. In any event, the necessity of meeting the requirements of historical simulation must be recognized.

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