Fin404 Derivatives

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The VIX and related derivatives

The goal of this project is to study the construction of the volatility index (VIX) and to analyze different approaches to the modelling of volatility.

Part 1. Documentation

Provide a basic presentation (maximum 3 pages) of the market for derivatives related to the variance (or volatility) of the S&P500 index taking as reference the contracts traded on the Chicago Board Options Exchanges (CBOE). This part of your report is meant as an introductory briefing. As such it should not be technical, but it can include basic equations, numbers, and graphics as necessary. You are free to structure and organize your presentation as you see fit. However, your report should at the minimum address the following points:

- Short history of the market for variance derivatives
- Why do variance derivatives exist? What do they allow investors to achieve?
- What is the VIX volatility index?
- What are the main variance derivatives? What are possible use cases for these different contracts?
- What distinguishes the VIX from a variance swap?
- What distinguishes a variance swap from a variance futures?

You may assume throughout your presentation that the interest rate and the dividend yield on the index are constant.

Part 1. The Carr-Madan formula

Let $H : \mathbb{R}_+ \to \mathbb{R}$ be a function that is *piecewise* twice continuously differentiable function on the positive real line.

1. Combine the fundamental theorem of calculus and the integration by parts formula to show that

$$H(x) = H(x_0) + (x - x_0)H'(x_0) + \int_{x_0}^{x} H''(k)\,(x - k)\,dk$$

for all $(x, x_0) \in \mathbb{R}_+^2$ such that H' is continuous at x_0

2. Use the identity

$$(x - y) = (x - y)^+ - (y - x)^+$$

to show that

$$\begin{aligned} H(x) = {} & H(x_0) + H'(x_0)\,(x - x_0) \\ & + \int_0^{x_0} H''(k)\,(k - x)^+\,dk + \int_{x_0}^{\infty} H''(k)\,(x - k)^+\,dk \end{aligned} \tag{1}$$

for all $(x, x_0) \in \mathbb{R}_+^2$ such that H' is continuous at x_0.

3. Suppose that the interest rate r is constant and that you can trade a risky asset with price S_t as well as *any* European option on that asset. Use (1) and the law of one price to show that the price at date 0 of a European derivative with terminal payoff $H(S_T)$ can be replicated by a *static* portfolio that holds

- n_0 units of the underlying asset
- An amount a_0 invested at the risk free rate
- $w(k)$ units of a put with strike k for all $k \leq x_0$
- $w(k)$ units of a call with strike k for all $k > x_0$

This result is known as the Carr–Madan formula. What happens if $x_0 = F_0(T)$ is the forward price of the underlying for delivery at date T?

4. Determine the static replication strategy (relative to an arbitrary reference point x_0) for a power derivative with payoff $H(x) = x^p$ for some $p \in \mathbb{R}$.

5. In practice, what are in the limits of the Carr–Madan result?

Part 2. The VIX index

Since 2003, the VIX index is defined by

$$\left(\frac{\mathrm{VIX}_t}{100}\right)^2 \eta = 2 \sum_i \left(\frac{\Delta K_i}{K_i^2}\right) e^{r(T-t)} \mathcal{O}_t(T, K_i) - \left(\frac{F_t(T)}{K_i} - 1\right)^2 \tag{2}$$

where $\eta = \frac{30}{252}$ expressed in years, $F_t(T)$ is the forward price of the S&P500 (SPX) for delivery at date $T = t+\eta$, K_0 is the largest strike below $F_t(T)$ among existing thirty days SPX options, $K_{\pm i}$ is the ith strike above (below) the reference strike K_0, and $\mathcal{O}_t(T, K_i)$ is the price of an OTM SPX option with maturity T and strike K_i.[1]

In its marketing material the CBOE claims that VIX provides a "*measure of the market's expectation of future volatility*." To try and understand this claim, assume that markets are complete and that the level of the SPX evolves according to

$$\frac{dS_t}{S_t} = (r - \delta)\, dt + \sqrt{V_t} dB_t^{\mathbb{Q}} \tag{3}$$

where r denotes the constant interest rate, δ denotes the constant dividend yield on the index, $B_t^{\mathbb{Q}}$ is a Brownian motion under the risk neutral measure $\mathbb{Q}$, and $V_t \geq 0$ is a process that models the squared volatility of the index and which may depend on other sources of risk than just $B_t^{\mathbb{Q}}$

1. Show that the *realized variance*

$$\overline{V}_{t,T} := \int_t^T V_s ds$$

over the interval $[t, T]$ satisfies

$$x\overline{V}_{t,T} = \int_t^T \frac{dS_u}{S_u} - \log\left(\frac{S_T}{S_t}\right)$$

[1] In practice there may not always exist options with exactly thirty days to maturity. CBOE circumvents this difficulty by computing the right handside of (2) for the closest maturities before (near term) and after (next term) thirty days and then taking a maturity weighted average of the results to define the VIX index. See the CBOE's white paper for complete implementation details.

for some $x > 0$ to be determined.

2. Show that $E_t^{\mathbb{Q}}\left[\int_t^T \frac{dS_u}{S_u}\right] = \alpha\,(T-t)$ for some α to be determined.

3. Show that

$$E_t^{\mathbb{Q}}\left[\log\frac{S_T}{S_t}\right] = \log\frac{K_0}{S_t} - \left(1 - \frac{F_t(T)}{K_0}\right) - P_t(T, K_0)$$

where

$$P_t(T, K_0) := \int_0^{K_0} e^{r(T-t)}\mathrm{Put}_t\,(T,k)\,\frac{dk}{k^2} + \int_{K_0}^{\infty} e^{r(T-t)}\mathrm{Call}_t\,(T,k)\,\frac{dk}{k^2}$$

and $\mathrm{Put}_t/\mathrm{Call}_t(T,k)$ denotes the price at time t of a European put/call with maturity T and strike k written on SPX.

4. Show that

$$xE_t^{\mathbb{Q}}\left[\overline{V}_{t,T}\right] = P_t(T, K_0) + \log\frac{F_t(T)}{K_0} + \left(1 - \frac{F_t(T)}{K_0}\right)$$

where x is the constant from question 1 and combine this expression with a second order Taylor expansion of the log to conclude that

$$xE_t^{\mathbb{Q}}\left[\overline{V}_{t,T}\right] \approx P_t(T, K_0) - \left(1 - \frac{F_t(T)}{K_0}\right)^2 \text{(1/2)}$$

Explain how one can make sense of CBOE's claim regarding the VIX index starting from this expression. What are the main advantages of this measure relative to the realized variance over the last 30 days and to a Black-Scholes implied volatility? What are the key assumptions on the evolution of S_t that are required for the validity of your above arguments?

Part 3. Futures pricing

The CBOE lists two types of futures related to variance or volatility: VIX futures and Variance futures. In a *VIX futures* with maturity date T the terminal value of the futures price is simply given by

$$f_T^{\mathrm{VIX}}(T) = \mathrm{VIX}_T.$$

By contrast, the terminal value of the futures price in a *Variance futures* is given by

$$f_T^{\text{VA}}(T) = \frac{10,000}{T - t_0} \sum_{i=1}^{N} \left(\log \frac{\text{SPX}_{t_i}}{\text{SPX}_{t_{i-1}}} \right)^2$$

where t_0 is initial listing date of the contract and N denotes the total number of sample points between the listing date t_0 and the maturity date $t_N = T$.

To facilitate the analysis of these contracts, consider from now on an *idealized world* where i) the exchange effectively offers SPX options of all strikes and all maturities so that the VIX satisfies

$$\left(\frac{\text{VIX}_t}{100} \right)^2 = \frac{1}{\eta} E_t^{\mathbb{Q}} \left[\int_t^{t+\eta} V_u du \right]$$

and ii) the number of sample points $N \to \infty$ so that

$$f_T^{\text{VA}}(T) = \frac{10,000}{T - t_0} \int_{t_0}^{T} V_u du$$

gives the terminal settlement price of the variance futures. Assume further that the squared volatility of the SPX evolves according to

$$dV_t = \lambda (\theta - V_t)\, dt + \xi \rho \sqrt{V_t} dB_t^{\mathbb{Q}} + \xi \sqrt{|1 - \rho^2|\, V_t} dZ_t^{\mathbb{Q}}$$

where $\lambda . \theta, \xi > 0$ and $\rho \in [-1, 1]$ are constants and $W_t^{\mathbb{Q}}$ is a $\mathbb{Q}-$Brownian motion that is independent from the Brownian motion in (3).

1. Show that

$$\left(\frac{\text{VIX}_T}{100} \right)^2 = \frac{1}{\eta} (a + bV_T)$$

for some (a, b) to be determined in terms of η and $(\rho, \lambda, \theta, \xi)$.

2. Show that

$$-\log E_t^{\mathbb{Q}} \left[e^{-sV_T} \right] = c(T - t; s) + d(T - t; s) V_t, \qquad s > 0,$$

for some $c, d : \mathbb{R}_+^2 \to \mathbb{R}$ to be determined in terms of $(\rho, \lambda, \theta, \xi)$.

Hint. Use the fact that $E_t^{\mathbb{Q}} \left[e^{-sV_T} \right] = f(t, V_t)$ is a martingale to derive a PDE for $f(t, v)$. then look for a solution of the required form, and finally use separation of variables to derive ODEs for c and d. To solve these ODEs you may find it useful to use Mathematica (or Wolfram α).

3. Show that the variance futures price and the VIX futures price are respectively given by

$$f_t^{\mathrm{VA}}(T) = \frac{10,000}{T - t_0}\left(\int_{t_0}^{t} V_u du + a^*(T-t) + b^*(T-t)V_t\right)$$

and

$$f_t^{\mathrm{VIX}}(T) = \frac{100}{\sqrt{\eta}} E_t^{\mathbb{Q}}\left[\sqrt{a' + b' V_T}\right] \tag{4}$$

for some $a^*, b^* : \mathbb{R}_+ \to \mathbb{R}$ and $a', b' \in \mathbb{R}$ to be determined.

4. Use the Laplace transform identity

$$\int_0^\infty e^{-sz}\frac{ds}{\sqrt{s}} = \sqrt{\frac{\pi}{z}}, \qquad z > 0$$

to show that

$$\sqrt{x} = \frac{1}{2\sqrt{\pi}}\int_0^\infty \left(1 - e^{-sx}\right)\frac{ds}{\sqrt{s^3}}.$$

Explain how this representation can be used to reduce the computation of the VIX futures price (4) to the numerical evaluation of

$$\frac{50}{\sqrt{\pi\eta}}\int_0^\infty \left(1 - e^{-\ell(s,T-t,V_t)}\right)\frac{ds}{\sqrt{s^3}}$$

for some $\ell : \mathbb{R}_+^3 \to \mathbb{R}$ to be determined in *closed form*. This clever observation was first made in Broadie and Jain (2008).

5. What relation must hold between the index VIX_t and the futures price $f_t^{\mathrm{VA}}(t+\eta)$ in a variance futures with time to maturity $T - t = \eta$? What would you do if you observed that this relation fails?

6. Use your preferred language to write a code that computes the futures prices in a VIX futures as a function of time to maturity, the current squared volatility V_t, and the parameters (λ, θ, ξ) of the squared volatility process. Analyze numerically how the result depends on the parameters (λ, θ, ξ).

7. Use your preferred language to write a code that computes the futures prices in Variance futures as a function of time to maturity $T - t$, the accrued variance $\int_{t_0}^{t}(100\sqrt{V_u})^2 du$, the current squared volatility V_t, and the parameters (λ, θ, ξ) of the squared volatility process. Analyze graphically how the result depends on the parameters (λ, θ, ξ).

8. Use the code you developed in the previous two questions together with the market data in the excel file `Fin404-2025-VIXnCo-Data` to calibrate the current squared volatility V_t and the parameters (λ, θ, ξ).

9. What additional data would you require to calibrate the correlation ρ between the increments of SPX and its squared volatility?

10. Suppose that the implied volatility surface for European SPX options at time t is explicitly given by

$$\sigma_t(T, K) = \alpha(t, T, V_t) + \beta(t, T, V_t)\left(\log \frac{K}{S_t}\right) + \gamma(t, T, V_t)\left(\log \frac{K}{S_t}\right)^2$$

for some functions $\alpha, \beta, \gamma : \mathbb{R}_+^3 \to \mathbb{R}$. Provide a formula for the number of SPX futures and the number of variance futures in the initial replicating portfolio of an SPX call with strike $K = S_0$ and maturity $T = 1$.

General guidelines

1. The project is to be done in groups of **at least 3 and no more than 4 students** *without* communications between the groups.

2. The name and sciper number of each member of the group should appear clearly on the final report.

3. The final report should be typeset on no more than 15 A4 pages in 12pt font with 1.5cm margins and standard linespacing.

4. The code should be ready to run in one or more standalone files. If there are multiple files please also provide a readme with clear instructions.

5. The code should be commented with explanations for each line/block preferably in the code itself.

6. The code should not use any external financial engineering libraries or functions. Please develop your own. It may however include built-in optimization, differential equation solving, and root finding functions.

7. Your code and the final report should be uploaded on the moodle.

8. Only one final report and one code archive per group cab be uploaded. No need to also send your work by email.

9. Make sure to not simply copy existing material and to cite your sources. Your work will **automatically** be scanned by an antiplagiarism software.