Formalizing Ionescu-Tulcea: Countable Kernel Compositional Products

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1. Measures

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Let $(\Omega, \mathcal{A})$ be a measurable space, i.e. $\mathcal{A} \subseteq \mathfrak{P}(\Omega)$, s.t.

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2. $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$,
3. $A_n \in \mathcal{A} \Rightarrow \bigcup_{n \in \mathbb{N}} A_n \in \mathcal{A}$.

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Definition 1.1 (probability measure): A probability measure is a function $\mu : \mathcal{A} \to \mathbb{R}_{\geq 0}$, s.t.

1. $\mu(\emptyset) = 0$
2. For $A_n \in \mathcal{A}$ pairwise disjoint, $\mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \sum_{n\in\mathbb{N}} \mu(A_n)$.
3. $\mu(\Omega) = 1$

2. Kernels

2.1 Definition

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Definition 2.1.1 (Markov kernel): A Markov kernel from $(\Omega_0, \mathcal{A}_0)$ to $(\Omega_1, \mathcal{A}_1)$ is a map $K : \Omega_0 \times \mathcal{A}_1 \to \mathbb{R}$, s.t.

1. $\forall \omega \in \Omega_0 : K(\omega, \cdot)$ is a probability measure.
2. $\forall A \in \mathcal{A}_1 : K(\cdot, A)$ is measurable.

2.2 Compositional Product

Definition 2.2.1 (compositional product): Let μ be a measure on $(\Omega_0, \mathcal{A}_0)$ and K a kernel from $(\Omega_0, \mathcal{A}_0)$ to $(\Omega_1, \mathcal{A}_1)$. We define a measure on $(\Omega_0 \times \Omega_1, \mathcal{A}_0 \otimes \mathcal{A}_1)$ by

$$(\mu \times K)(A) = \int_{\Omega_0} K(\omega_0, A(\omega_0)) \, \mathrm{d}\mu(\omega_0)$$

where $A \in \mathcal{A}_0 \otimes \mathcal{A}_1$ and $A(\omega_0) = \{\omega_1 \mid (\omega_0, \omega_1) \in A\}$.

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Then by induction, we can define a measure on $\bigotimes_{m \leq n} (\Omega_m, \mathcal{A}_m)$ by

$$\mu \times K_1 \times \ldots \times K_n = ((\mu \times K_1) \times \ldots \times K_{n-1}) \times K_n$$

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These fulfill the following consistency property:

Lemma 2.3.1:

$\mu \times \ldots \times K_n(A) = \mu \times \ldots \times K_{n+m}(A \times \bigtimes_{n<k \leq n+m} \Omega_k)$

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The goal is to show that we can define "$\mu \times K_1 \times K_2 \times \ldots$" if μ is a probability measure and K_n are Markov kernels.

2.4 Ionescu-Tulcea

Theorem 2.4.1 (Ionescu-Tulcea): Let μ be a prob.-measure on $(\Omega_0, \mathcal{A}_0)$ and K_n be Markov kernels from $\bigotimes_{m<n}(\Omega_m, \mathcal{A}_m)$ to $(\Omega_n, \mathcal{A}_n)$ There exists a unique probability measure $\mathbb{P}$ on $\bigotimes_{n\in\mathbb{N}}(\Omega_n, \mathcal{A}_n)$, s.t.

$\forall n \in \mathbb{N} : \forall A \in \bigotimes_{m\leq n} \mathcal{A}_m :$

$$\mathbb{P}(A \times \bigtimes_{k>n} \Omega_k) = (\mu \times K_1 \times ... \times K_n)(A).$$

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Definition 2.4.2: Cylinder sets are sets of the form $A \times \bigtimes_{k>n} \Omega_k$ where $n \in \mathbb{N}$ and $A \in \bigotimes_{k\leq n} \mathcal{A}_k$.

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Definition 2.4.2: Cylinder sets are sets of the form $A \times \bigtimes_{k>n} \Omega_k$ where $n \in \mathbb{N}$ and $A \in \bigotimes_{k\le n} \mathcal{A}_k$.

Lemma 2.4.3: Cylinder sets are a $\bigcap$-stable generator of $\bigotimes_{n\in\mathbb{N}} \mathcal{A}_n$.

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Definition 2.5.3 (sigma additivity): A content μ is sigma additive if for all pairwise disjoint sequences $A_n \in \mathcal{A}$ with $\bigcup_n A_n \in \mathcal{A}$, we have $\mu\left(\bigcup_n A_n\right) = \sum_{n\in\mathbb{N}} \mu(A_n)$.

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Theorem 2.5.4 (Caratheodory's extension theorem): Every σ-additive content on a set algebra $\mathcal{A}$ can be extended to a measure on $\sigma(\mathcal{A})$.

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$$\begin{aligned} P : \quad & C \to [0,1] \\ & A \times \bigtimes_{k>n} \Omega_k \mapsto \mu \times \ldots \times K_n(A) \end{aligned}$$

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is well defined and a content (i.e. additive). It is left to show that P is σ-additive.

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It is known that it suffices to show for $B_n \in C$ and $B_n \downarrow \emptyset$, that $P(B_n) \to 0$.

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$$Q_{n,m} = K_n \times \dots \times K_{n+m}$$

$$f_{n,m}(\omega_{1,\dots,n}) = Q_{n,m}(\omega_{1,\dots,n}, A_{n+m+1}(\omega_{1,\dots,n}))$$

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We then would strengthen our induction and show

$$\exists \omega : \forall n \in \mathbb{N} : \inf_m f_{n,m}(\omega_{1,\ldots,n}) > 0$$

If $f_{n,0}(\omega_{1,\ldots,n}) > 0$ then $A_{n+1}(\omega_{1,\ldots,n}) \neq \emptyset$ then $\omega_{1,\ldots,n} \in A_n$ ......