Thoughts – Other's and mine

Notation

Polynomial spaces

2. In two variables, we refer to

$$\mathcal{P}_t := \{u(x, y) = \sum_{\substack{i+k \le t \\ i,k \ge 0}} c_{ik} x^i y^k\} \tag{5.1}$$

as the set of *polynomials of degree* $\le t$. If all polynomials of degree $\le t$ are used, we call them finite elements with *complete polynomials*.

Figure 1: Breaess [1] p.60

The polynomial families $\mathcal{P}_t$ are not used on rectangular partitions of a domain. We can see why by looking at the simplest example, the bilinear element. Instead of using $\mathcal{P}_t$ as we did for triangles, on rectangular elements we use the polynomial family which contains *tensor products*:

$$\mathcal{Q}_t := \{u(x, y) = \sum_{0 \le i,k \le t} c_{ik} x^i y^k\}. \tag{5.5}$$

If more general quadrilateral elements are involved, we can use appropriately transformed families.

Figure 2: Breaess [1] p.68

Let K be any triangle. Let $\mathcal{P}_k$ denote the set of all polynomials in two variables of degree $\le k$. The following table gives the dimension of $\mathcal{P}_k$.

k	$\dim \mathcal{P}_k$
1	3
2	6
3	10
$\vdots$	$\vdots$
k	$\frac{1}{2}(k+1)(k+2)$

Figure 3: Brenner and Scott [2] p.71

In this section we consider finite elements defined on rectangles. Let $\mathcal{Q}_k = \{\sum_j c_j\, p_j(x)\, q_j(y) : p_j, q_j$ polynomials of degree $\le k\}$. One can show that

(3.5.1) $$\dim \mathcal{Q}_k = (\dim \mathcal{P}_k^1)^2,$$

where $\mathcal{P}_k^1$ denotes the space of polynomials of degree less than or equal to k in one variable (cf. exercise 3.x.6).

Figure 4: Brenner and Scott [2] p85

Definition of a finite element

We have not yet given a formal definition of what we mean by a "finite element". To fill this gap define a *finite element* to mean a triple (K, P_K, Σ), where

- K is a geometric object, for example a triangle,
- P_K is a finite-dimensional linear space of functions defined on K,
- Σ is a set of degrees of freedom,

such that a function $v \in P_K$ is uniquely determined by the degrees of freedom Σ. From Example 3.1 we have that (K, P_K, Σ), where

- K is a triangle,
- $P_K = P_1(K)$,
- Σ is the values at the vertices of K,

is a finite element. In Fig 3.7 below we have collected some of the most common finite elements (cf [Ci]). The various degrees of freedom are denoted as follows:

Figure 5: Johnson [3] p.79

We follow Ciarlet's definition of a finite element (Ciarlet 1978).

(3.1.1) Definition. *Let*

(i) $K \subseteq \mathbb{R}^n$ *be a bounded closed set with nonempty interior and piecewise smooth boundary (the* **element domain***),*

(ii) $\mathcal{P}$ *be a finite-dimensional space of functions on* K *(the space of* **shape functions***) and*

(iii) $\mathcal{N} = \{N_1, N_2, \ldots, N_k\}$ *be a basis for* $\mathcal{P}'$ *(the set of* **nodal variables***).*

Then $(K, \mathcal{P}, \mathcal{N})$ *is called a* **finite element**.

It is implicitly assumed that the nodal variables, N_i, lie in the dual space of some larger function space, e.g., a Sobolev space.

(3.1.2) Definition. *Let* $(K, \mathcal{P}, \mathcal{N})$ *be a finite element. The basis* $\{\phi_1, \phi_2, \ldots, \phi_k\}$ *of* $\mathcal{P}$ *dual to* $\mathcal{N}$ *(i.e.,* $N_i(\phi_j) = \delta_{ij}$*) is called the* **nodal basis** *of* $\mathcal{P}$.

Figure 6: Brenner and Scott [2] p.69

In order to define a finite element, following [146], we need to specify three things.

- The geometry: we choose a reference element $\hat{K}$, a change of variables $F(\hat{x})$ and we set $K = F(\hat{K})$.
- A set $\hat{P}$ of polynomials on $\hat{K}$. For $\hat{p} \in \hat{P}$ we define on K, $p = \hat{p} \circ F^{-1}$.
- A set of degrees of freedom $\hat{\Sigma}$, that is, a set of linear forms $\{\hat{\ell}_i\}_{1\leq i\leq \dim \hat{P}}$ on $\hat{P}$. We say that this set is unisolvent when these linear forms are linearly independent, i.e. the knowledge of $\hat{\ell}_i(\hat{p})$ for all i completely defines $\hat{p}$.

Figure 7: Boffi, Brezzi and Martin [4] p.67

5.8 Definition. A *finite element* is a triple (T, Π, Σ) with the following properties:

(i) T is a polyhedron in $\mathbb{R}^d$. (The parts of the surface ∂T lie on hyperplanes and are called *faces*.)

(ii) Π is a subspace of $C(T)$ with finite dimension s. (Functions in Π are called *shape functions* if they form a basis of Π.)

(iii) Σ is a set of s linearly independent functionals on Π. Every $p \in \Pi$ is uniquely defined by the values of the s functionals in Σ. – Since usually the functionals involve point evaluation of a function or its derivatives at points in T, we call these *(generalized) interpolation conditions*.

In (ii) s is the *number of local degrees of freedom* or *local dimension*.

Although generally Π consists of polynomials, it is not enough to look only at polynomial spaces, since otherwise we would exclude piecewise polynomial elements such as the Hsieh–Clough–Tocher element. In fact, there are even finite elements consisting of piecewise rational functions; see Wachspress [1971].

Figure 8: Braess [1] p.70

Misc

We have already encountered several examples of affine families. The families $\mathcal{M}_0^k$ and the rectangular elements considered so far are affine families. For example, $\mathcal{M}_0^k$ is defined by the triple $(\hat{T}, \mathcal{P}_k, \Sigma_k)$, where

$$\hat{T} := \{(\xi, \eta) \in \mathbb{R}^2;\ \xi \geq 0,\ \eta \geq 0,\ 1 - \xi - \eta \geq 0\} \tag{5.11}$$

is the unit triangle and $\Sigma_k := \{p(z_i);\ i = 1, 2, \ldots, s := k(k+1)/2\}$ is the set of nodal basis points z_i in Remark 5.6.

Figure 9: Reference Triangle, Breass [1] p.73

(3.4.1) Definition. *Let $(K, \mathcal{P}, \mathcal{N})$ be a finite element and let* $F(x) = \mathbf{A}x + \mathbf{b}$ *(*$\mathbf{A}$ *nonsingular) be an affine map. The finite element $(\widehat{K}, \widehat{\mathcal{P}}, \widehat{\mathcal{N}})$ is* **affine equivalent** *to $(K, \mathcal{P}, \mathcal{N})$ if*

(i) $F(K) = \hat{K}$
(ii) $F^*\hat{\mathcal{P}} = \mathcal{P}$ *and*
(iii) $F_*\mathcal{N} = \hat{\mathcal{N}}$.

We write $(K, \mathcal{P}, \mathcal{N}) \underset{F}{\simeq} (\widehat{K}, \widehat{\mathcal{P}}, \widehat{\mathcal{N}})$ if they are affine equivalent.

(3.4.2) *Remark.* Recall that the *pull-back* F^* is defined by $F^*(\hat{f}) := \hat{f} \circ F$ and the *push-forward* F_* is defined by $(F_*N)(\hat{f}) := N\big(F^*(\hat{f})\big)$.

Figure 10: Affine equivalence, Brenner and Scott [2] p.82

Bibliography

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