FEM for 2D problems

Mathematical foundations (some)

- Functions $\mathbb{R}^2 \to \mathbb{R}$ and gradient
- Vector fields $\mathbb{R}^2 \to \mathbb{R}^2$ and divergence
- Divergence theorem and integration by parts

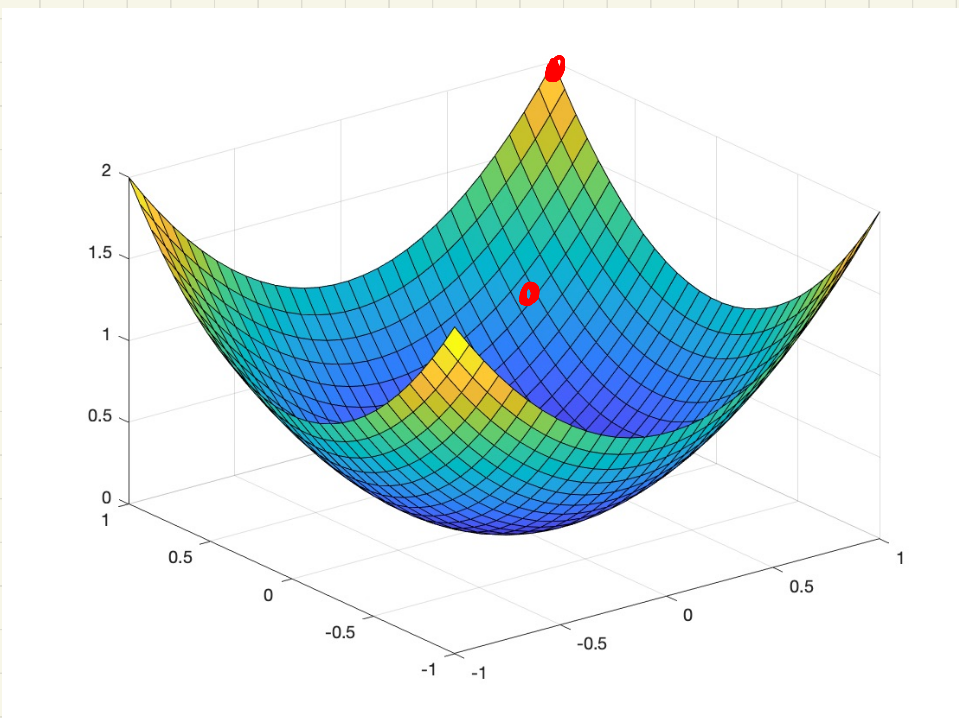
$\rightsquigarrow$ Informal introduction, no rigorous math

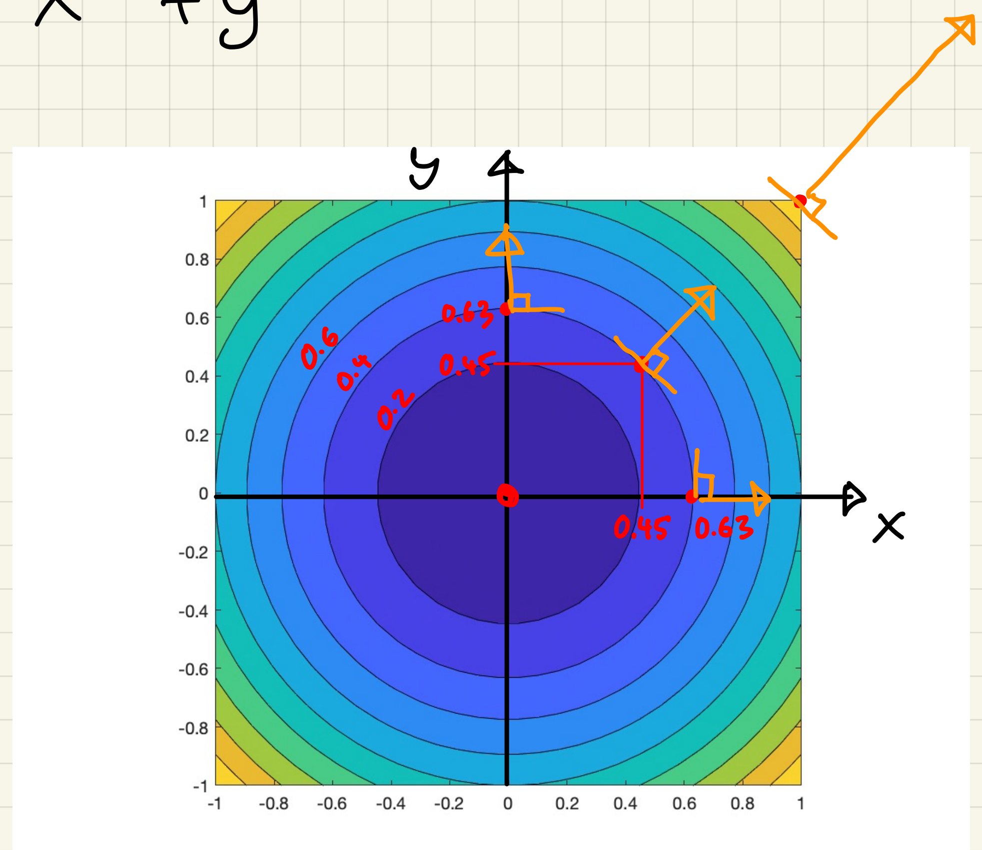
Functions $\mathbb{R}^2 \to \mathbb{R}$ and gradient

Example

$$f : \mathbb{R}^2 \to \mathbb{R}, \quad f(x,y) = x^2 + y^2$$

$$\sqrt{2 \cdot 0.9^2} = 1.27$$





$$\nabla f(0.45, 0.45) = \begin{pmatrix} 0.9 \\ 0.9 \end{pmatrix}$$

$$\nabla f(0.63, 0) = \begin{pmatrix} 1.26 \\ 0 \end{pmatrix}$$

$$\nabla f(0, 0.63) = \begin{pmatrix} 0 \\ 1.26 \end{pmatrix}$$

$$\nabla f(0,0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\nabla f(1,1) = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

$$f_x(x,y) = 2x$$

$$f_y(x,y) = 2y$$

$$\rightsquigarrow \nabla f(x,y) = 2 \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

$\rightsquigarrow$ Gradient points in the direction of steepest ascend!
Length of vector indicates slope.

Partial derivatives

$$\frac{\partial}{\partial x} f(x,y) = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}, \qquad f_x(x,y) = \frac{\partial}{\partial x} f(x,y)$$

$$\frac{\partial}{\partial y} f(x,y) = \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h}, \qquad f_y(x,y) = \frac{\partial}{\partial y} f(x,y)$$

Gradient

$\nabla f : \mathbb{R}^2 \to \mathbb{R}^2$

$$\boxed{\operatorname{grad} f(x,y) = \nabla f(x,y) = \begin{pmatrix} f_x(x,y) \\ f_y(x,y) \end{pmatrix}}$$

$$\begin{tikzcd} (x,y) \arrow[d] \\ \boxed{\nabla f} \arrow[d] \\ \begin{pmatrix} f_x(x,y) \\ f_y(x,y) \end{pmatrix} \end{tikzcd}$$

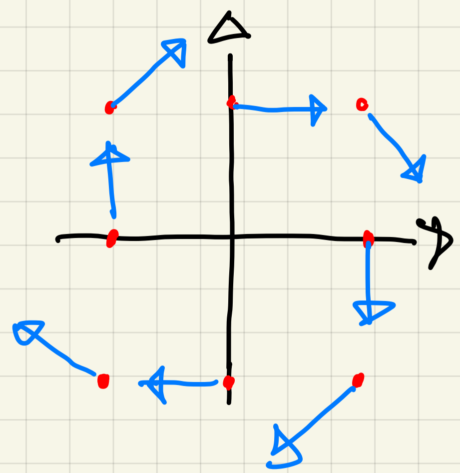
Vector fields $\mathbb{R}^2 \to \mathbb{R}^2$ and divergence

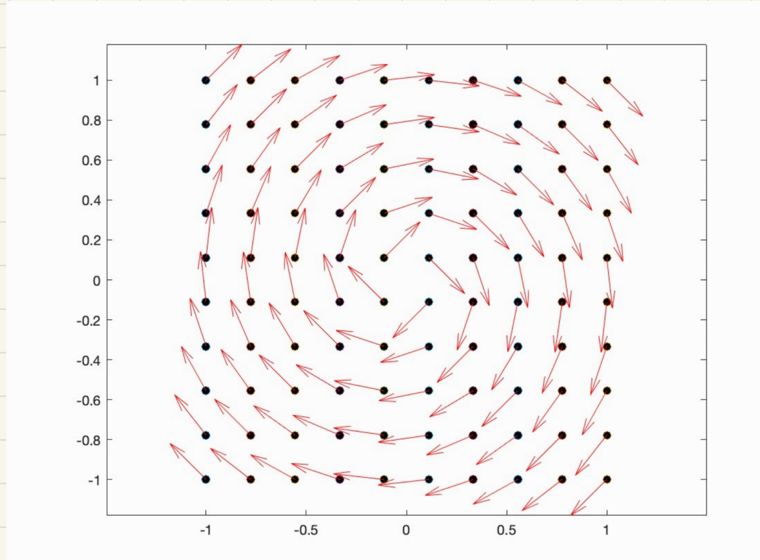
Example

$$\underline{v}: \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}^2, \quad \underline{v}(x,y) = \frac{1}{\sqrt{x^2+y^2}} \cdot \begin{pmatrix} y \\ -x \end{pmatrix}$$

$$v_x(x,y) = \frac{1}{\sqrt{x^2+y^2}} \cdot y = (x^2+y^2)^{-1/2} \cdot y$$

$$v_y(x,y) = -\frac{1}{\sqrt{x^2+y^2}} \cdot x = -(x^2+y^2)^{-1/2} \cdot x$$





$$v_{x,x}(x,y) = 2 \cdot x \cdot \left(-\tfrac{1}{2}\right) \cdot (x^2+y^2)^{-3/2} \cdot y = -(x^2+y^2)^{-3/2} \cdot x \cdot y$$

$$v_{y,y}(x,y) = -2 \cdot y \cdot \left(-\tfrac{1}{2}\right) \cdot (x^2+y^2)^{-3/2} \cdot x = (x^2+y^2)^{-3/2} \cdot x \cdot y$$

$$\operatorname{div} \underline{v}(x,y) = 0 \qquad (\underline{v} \text{ is divergence-free})$$

Divergence

$$\underline{v} : \mathbb{R}^2 \to \mathbb{R}^2, \quad \underline{v}(x,y) = \begin{pmatrix} v_x(x,y) \\ v_y(x,y) \end{pmatrix}, \quad v_x, v_y : \mathbb{R}^2 \to \mathbb{R}$$

$$v_{x,x}(x,y) = \frac{\partial}{\partial x} v_x(x,y), \quad v_{x,y}(x,y) = \frac{\partial}{\partial y} v_x(x,y)$$

$$v_{y,x}(x,y) = \frac{\partial}{\partial x} v_y(x,y), \quad v_{y,y}(x,y) = \frac{\partial}{\partial y} v_y(x,y)$$

For a vector field $\underline{v} : \mathbb{R}^2 \to \mathbb{R}^2$ the divergence $\operatorname{div} \underline{v}$ is the function $\operatorname{div} \underline{v} : \mathbb{R}^2 \to \mathbb{R}$ with

$$\operatorname{div} \underline{v}(x,y) = v_{x,x}(x,y) + v_{y,y}(x,y)$$

Example

With $f: \mathbb{R}^2 \to \mathbb{R}$ and $\underline{u}: \mathbb{R}^2 \to \mathbb{R}^2$, we define $\underline{v}: \mathbb{R}^2 \to \mathbb{R}^2$

$$\underline{v}(x,y) = f(x,y) \cdot \underline{u}(x,y) = \begin{pmatrix} f(x,y) \cdot u_x(x,y) \\ f(x,y) \cdot u_y(x,y) \end{pmatrix}$$

What is $\operatorname{div} \underline{v}$?

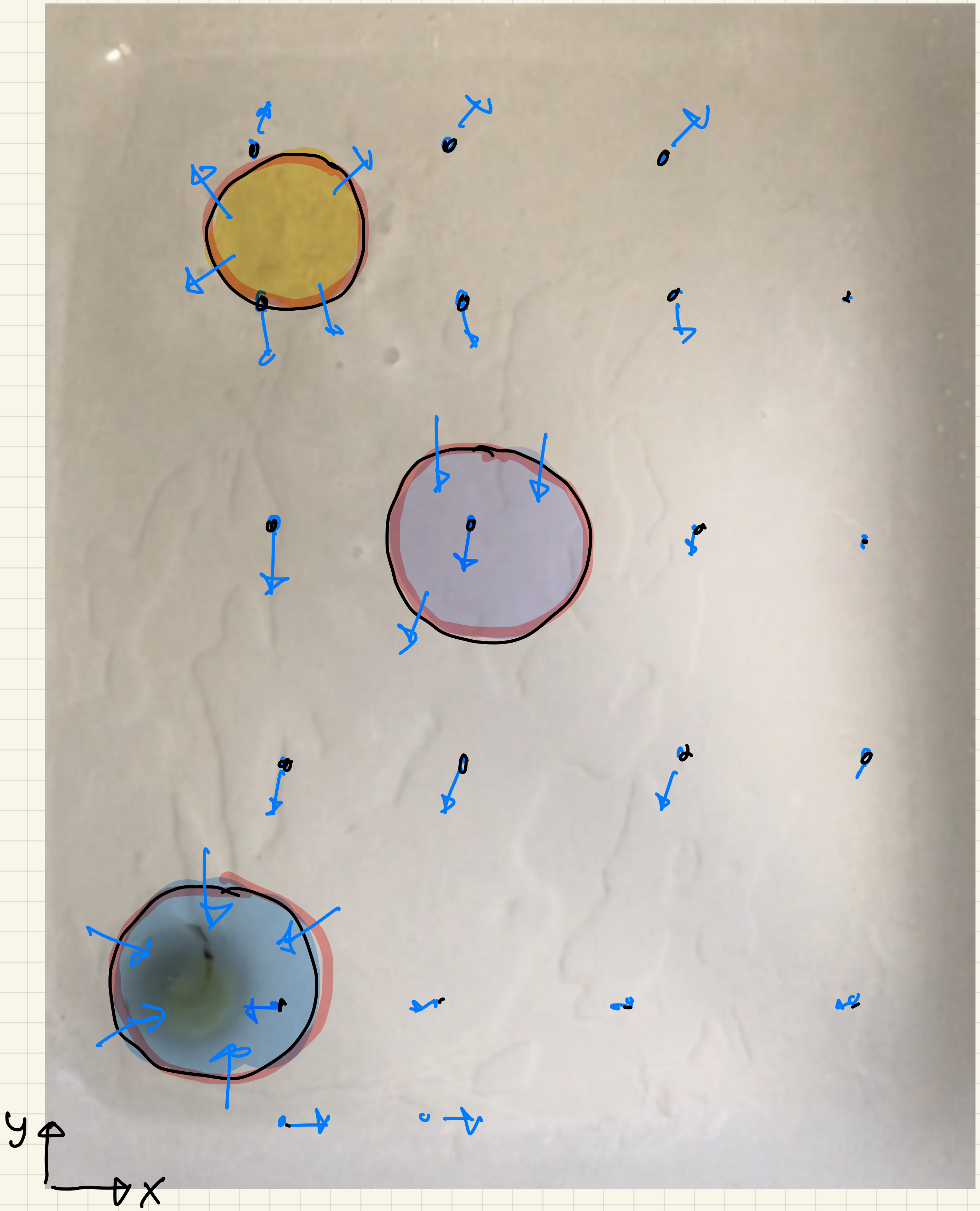
$$\operatorname{div} \underline{v}(x,y) = \frac{\partial}{\partial x}\left(f(x,y) \cdot u_x(x,y)\right) + \frac{\partial}{\partial y}\left(f(x,y) \cdot u_y(x,y)\right)$$

$$= f_x(x,y) \cdot u_x(x,y) + f(x,y) \cdot u_{x,x}(x,y) + f_y(x,y) \cdot u_y(x,y) + f(x,y) \cdot u_{y,y}(x,y)$$

$$\boxed{\operatorname{div} \underline{v}(x,y) = \nabla f(x,y) \cdot \underline{u}(x,y) + f(x,y) \cdot \operatorname{div} \underline{u}(x,y)}$$

$$\nabla f = \begin{pmatrix} f_x \\ f_y \end{pmatrix} \qquad \underline{u} = \begin{pmatrix} u_x \\ u_y \end{pmatrix}$$

Example: Fluid flow in 2D



Mathematical description

$\underline{v} : \mathbb{R}^2 \to \mathbb{R}^2$

→ Velocity vector

Regions

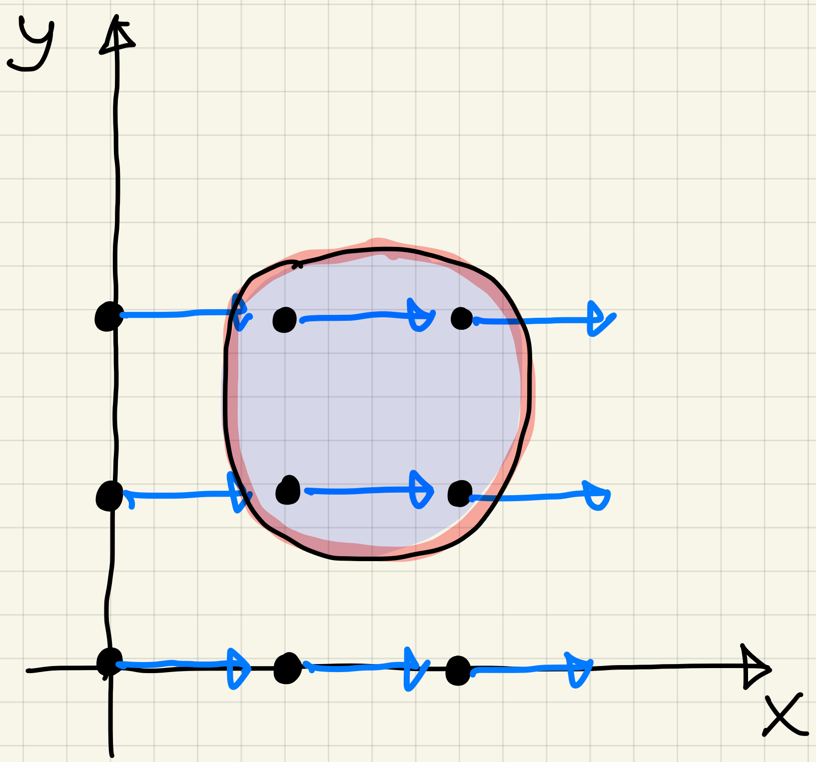
What goes in comes out

More out than in

More in than out

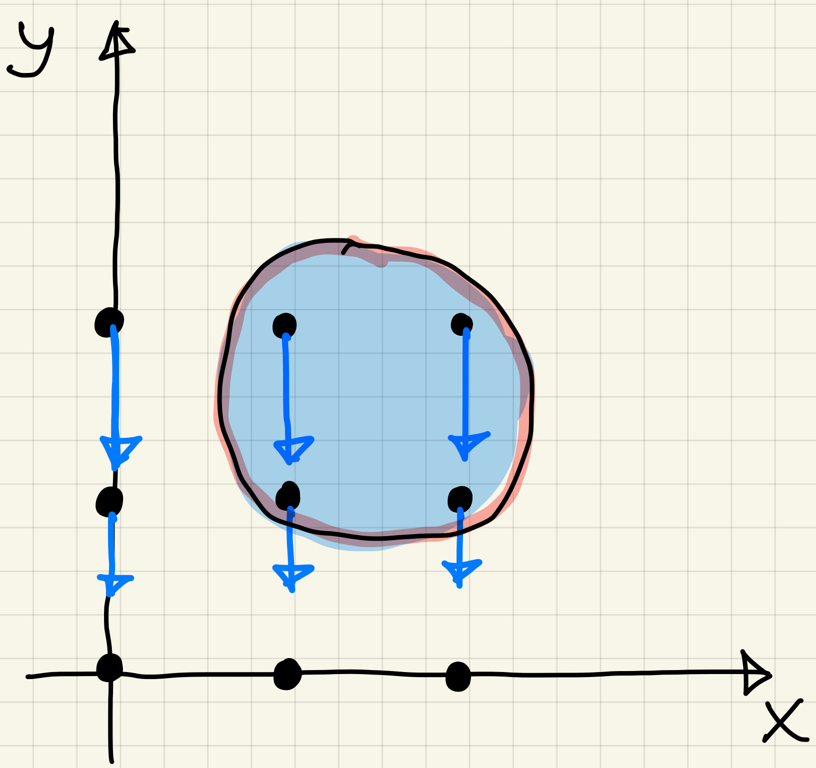
Flow over the boundary

Sources and sinks



$$\underline{v}(x,y) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

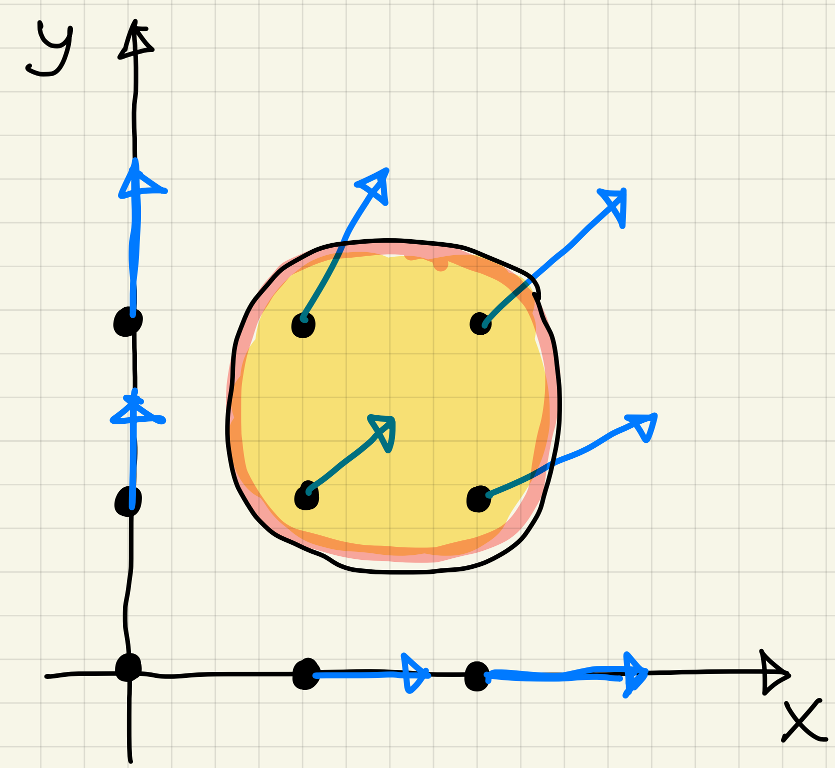
$$\operatorname{div} \underline{v}(x,y) = 0$$



$$\underline{v}(x,y) = \begin{pmatrix} 0 \\ -y \end{pmatrix}$$

$$\operatorname{div} \underline{v}(x,y) = -1$$

$\rightarrow$ Sink



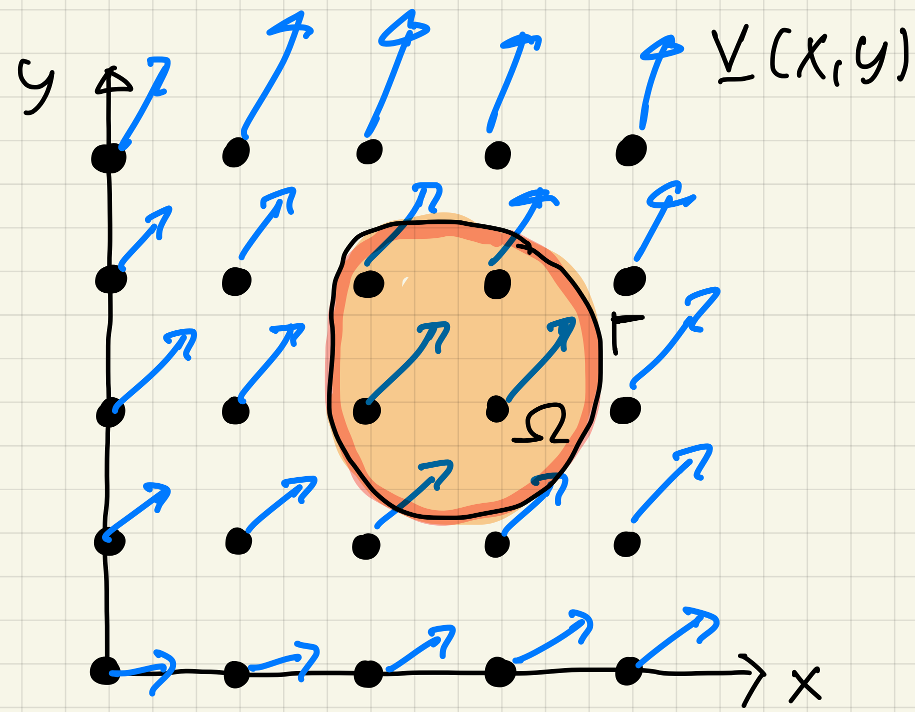
$$\underline{v}(x,y) = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\operatorname{div} \underline{v}(x,y) = 2$$

$\rightarrow$ Source

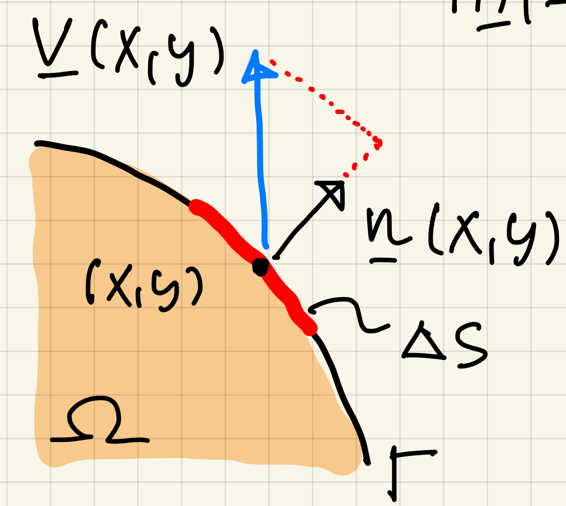
Divergence theorem (Integralsatz von Gauß)

Intuition (fluid flow)



$\underline{n}$: Outward unit vector orthogonal to Γ

$|\underline{n}| = 1$



$$\Delta f = \underline{v}(x,y) \cdot \underline{n}(x,y) \cdot \Delta s$$

Production in Ω

$$P = \int_\Omega \operatorname{div} \underline{v}(x,y)\, dA$$

Flow over boundary Γ

$$f = \int_\Gamma \underline{v}(x,y) \cdot \underline{n}(x,y)\, ds$$

Intuitively we state

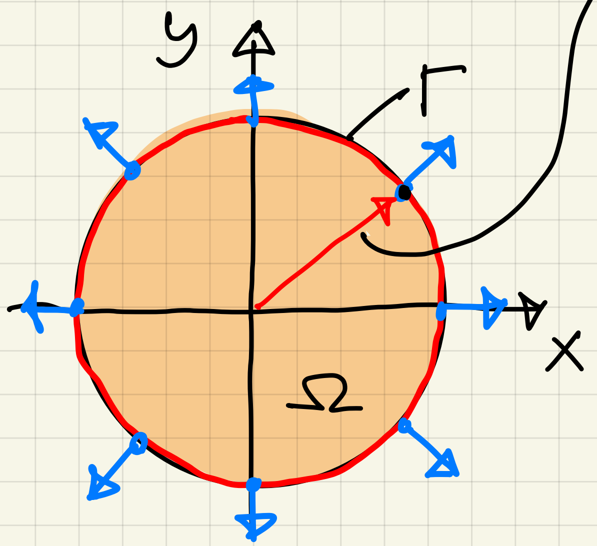
$$P = f$$

Mathematical formulation

$$\boxed{\int_\Omega \operatorname{div} \underline{v}(x,y)\, dA = \int_\Gamma \underline{v}(x,y) \cdot \underline{n}(x,y) \cdot ds}$$

Proof skipped!

Example



$$\begin{pmatrix} x \\ y \end{pmatrix}, \quad \sqrt{x^2+y^2} = 1$$

$$\Omega = \{ \underline{x} \in \mathbb{R}^2, \ |\underline{x}| \le 1 \}$$

$$\Gamma = \{ \underline{x} \in \mathbb{R}^2, \ |\underline{x}| = 1 \}$$

$$\underline{v}(x,y) = \begin{pmatrix} x \\ y \end{pmatrix} \qquad \operatorname{div} \underline{v}(x,y) = 2$$

Task: Verify that divergence theorem holds in this case

$$\int_\Omega \operatorname{div} \underline{v} \, dA = \int_\Omega 2 \, dA = 2 \cdot \pi \cdot 1^2 = 2\pi$$

$$\int_\Gamma \underline{v}(x,y) \cdot \underline{n}(x,y) \, ds = \int_\Gamma \underbrace{\begin{pmatrix} x \\ y \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}}_{1} ds = \int_\Gamma 1 \, ds = 1 \cdot 2 \cdot \pi \cdot r = 2\pi \qquad \checkmark$$

2D Integration by parts formula

Function $f: \mathbb{R}^2 \to \mathbb{R}$ and vector field $\underline{u}: \mathbb{R}^2 \to \mathbb{R}^2$

$$\int_\Omega \operatorname{div}(f \cdot \underline{u})\, dA = \int_\Gamma f \cdot \underline{u} \cdot \underline{n}\, ds$$

$$\int_\Omega (\nabla f \cdot \underline{u} + f \cdot \operatorname{div} \underline{u})\, dA = \int_\Gamma f \cdot \underline{u} \cdot \underline{n}\, ds$$

$$\boxed{\int_\Omega f \cdot \operatorname{div} \underline{u}\, dA = \int_\Gamma f \cdot \underline{u} \cdot \underline{n}\, ds - \int_\Omega \nabla f \cdot \underline{u}\, dA}$$

Integration by parts

$$\int_a^b u' \cdot v\, dx = [u \cdot v]_a^b - \int_a^b u \cdot v'\, dx$$