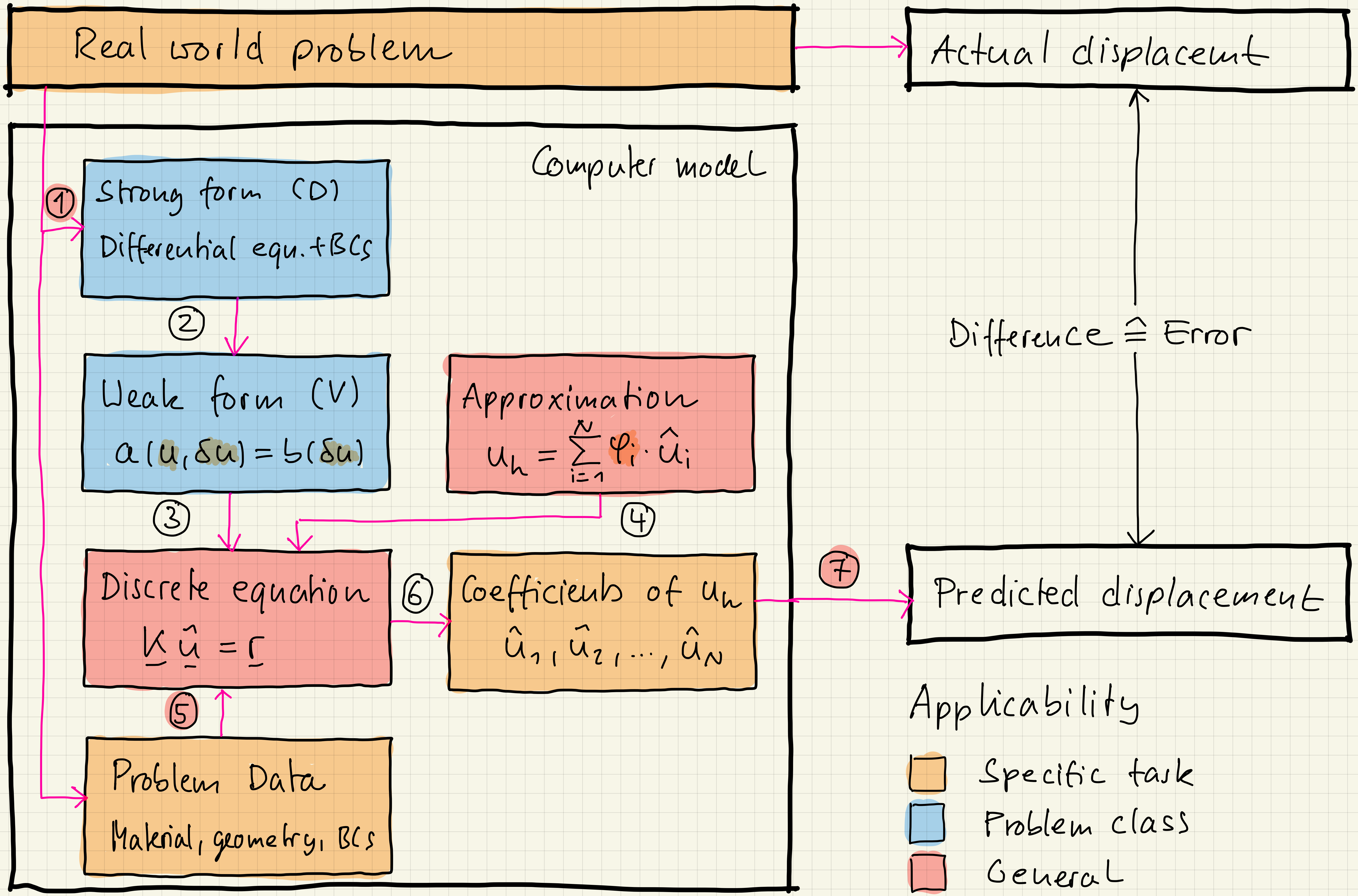
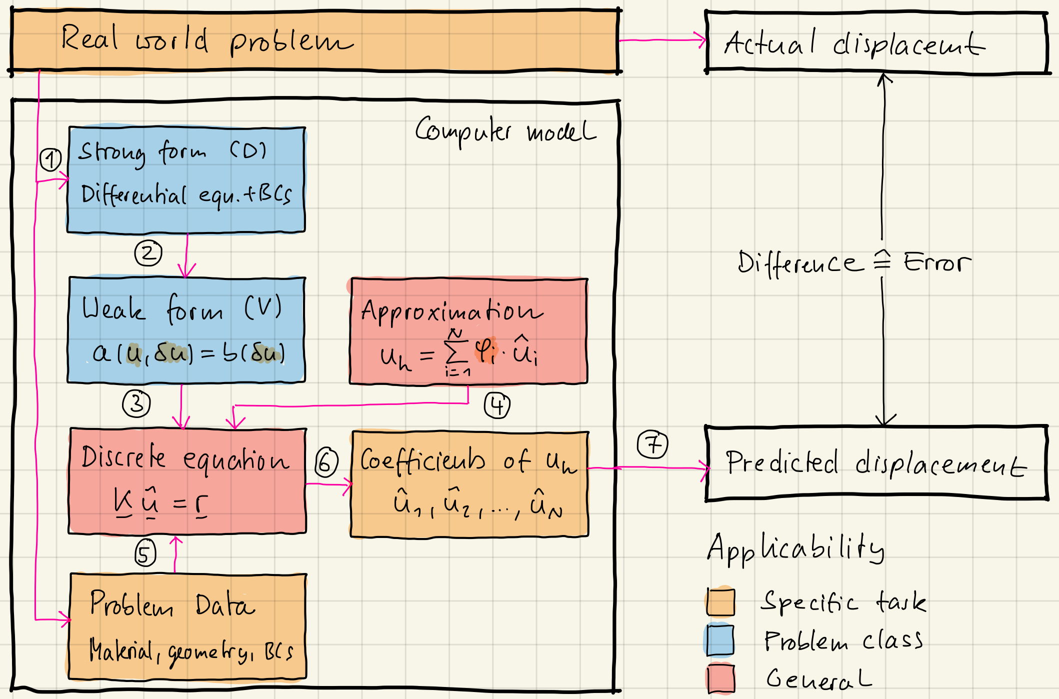
FEM for 1D problems

Summary and discussion of results


Real world problem
Actual displacement
Computer model
1
Strong form (D)
Differential equ.+BCs
2
Weak form (V)
$a(u,\delta u)=b(\delta u)$
Approximation
$u_h=\sum_{i=1}^{N}\varphi_i\cdot\hat{u}_i$
3
4
Discrete equation
$\underline{K}\,\hat{\underline{u}}=\underline{r}$
6
Coefficients of u_h
$\hat{u}_1,\hat{u}_2,\dots,\hat{u}_N$
7
Predicted displacement
Difference $\widehat{=}$ Error
5
Problem Data
Material, geometry, BCs
Applicability
Specific task
Problem class
General

Errors introduced in each step



1. **Modelling error**
   - Nonlinear material behaviour of soil
   - Material properties vary

2. None here

3. None here

4. **Discretization error** $e = u - u_h$

5. **Data error** GIGO (Garbage in, garbage out)

6. Roundoff errors
   - Most of the time not a problem

7. **Interpration error**
   - Draw the right conclusions!

Validation and verification

Validation – Do the equations in (D) capture all relevant effects?

"Do I solve the right equation?"

Verification – Does my numerical solution solve (D)?

"Do I solve the equation right?"

Discretization error

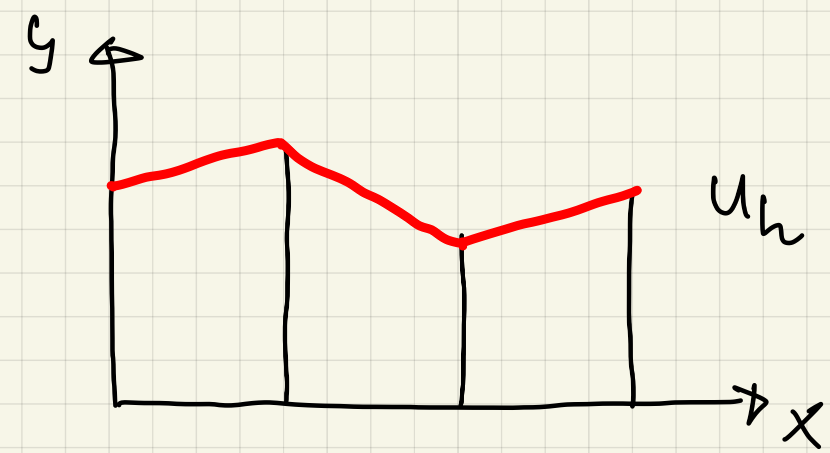
Error function

$$e = u - u_h$$

How to make sure the discretization error is sufficiently small

- Exact solution, but only available in simple cases
- Theory of error estimators (mathematicians)
- Rules of thumb and engineering expertise
  - Inspect stresses and jumps
  - Study results for different meshes

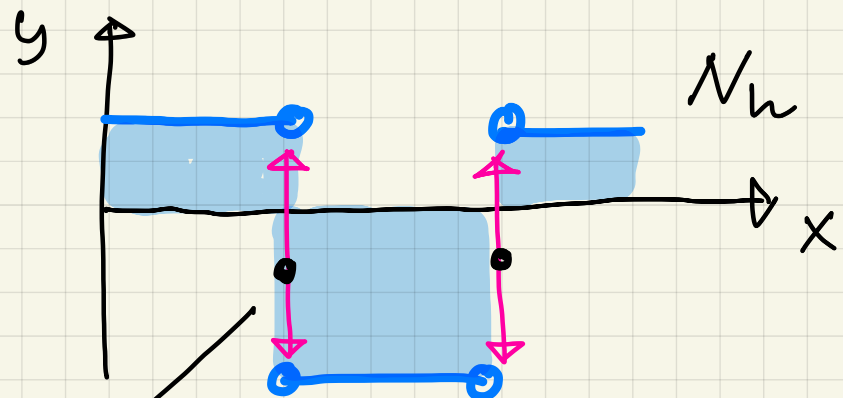
Axial force



u_h is piecewise linear

N_h is piecewise constant

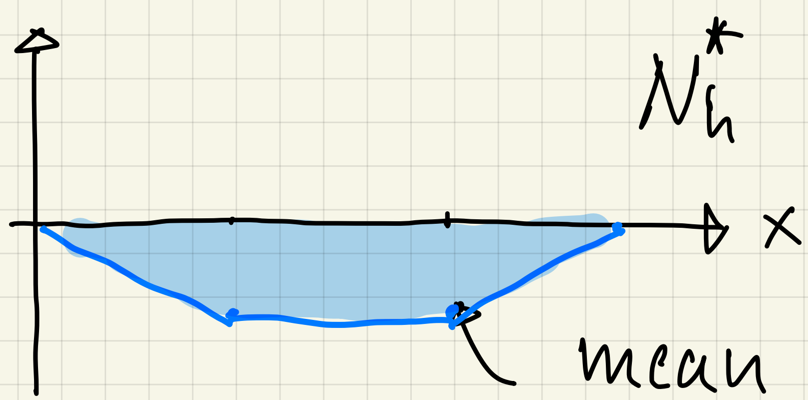
$$N(x) = EA\,u'(x)$$



In FE-programs: Element solution } Use that to assess the quality of your results

Jumps physically not possible

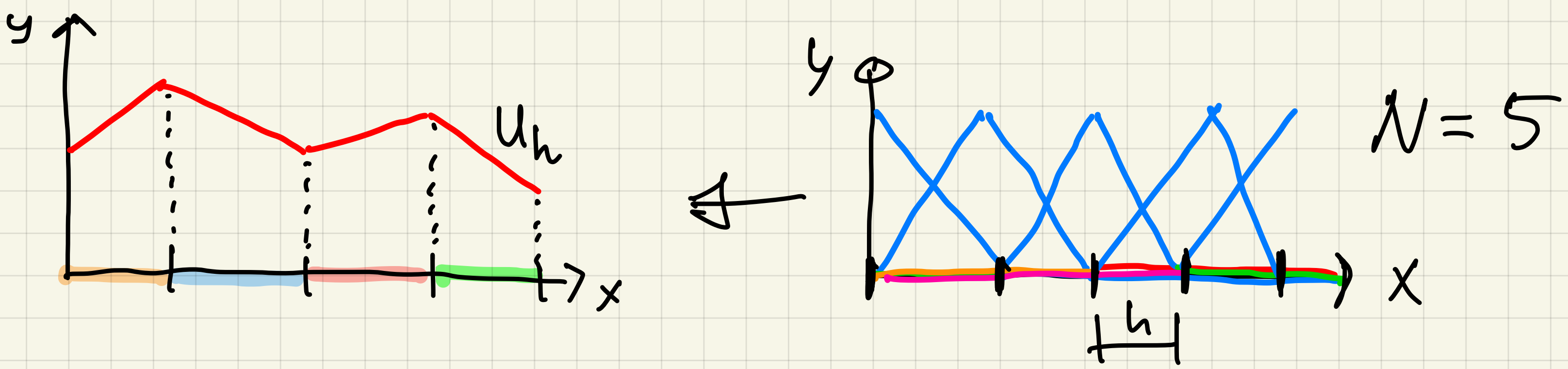
"Improved" plot



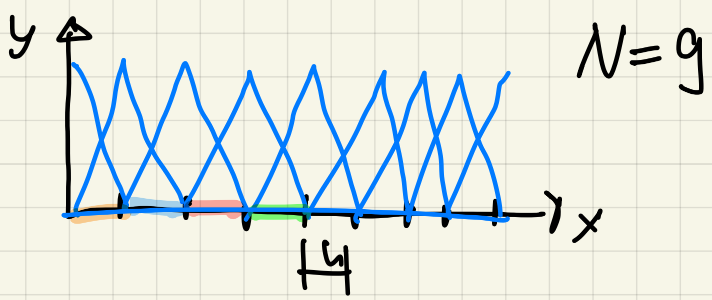
mean value of both elements

Nodal solution in FE-software } Show this to your clients

How to improve the approximate solution u_h

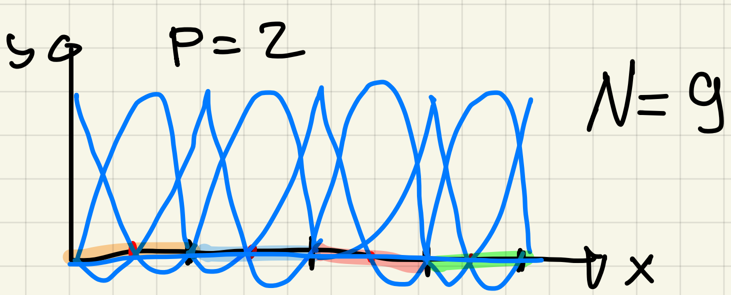


Reduce element size



h-version of finite element method

Increase polynomial degree



p-version of FEM

up to $p \cong 15$

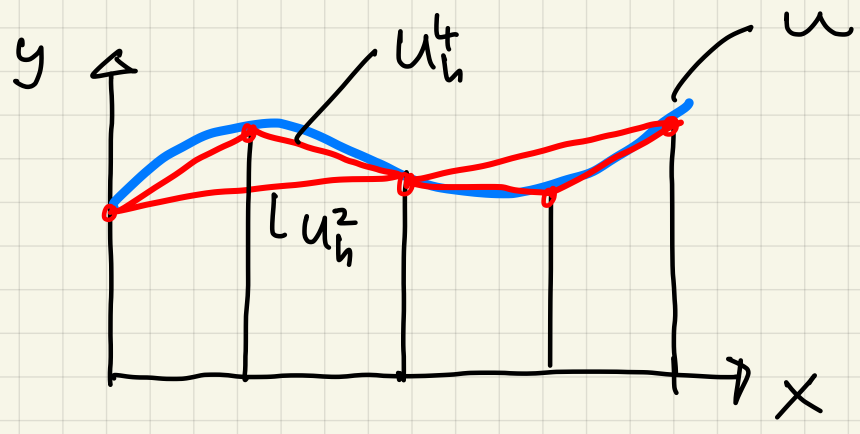
Combination

hp-version of FEM

Converges faster to exact solution than h-version but difficult to program

→ Better solution with same N

Convergence of solution



Sequence of functions u_h^k with

$$\lim_{k \to \infty} (u_h^k - u) = 0$$

Proof is difficult (we leave that to mathematicians)

In our computation, we used the supremum-norm

$$\| f \|_\infty = \sup \{ f(x) \mid 0 \leq x \leq L \}$$

(gleichmäßige Konvergenz)

More common: Energy-norm

$$\| f \|_a = \frac{1}{2} a(f, f)$$

$$a(u, \delta u) = EA \int_0^l u' \delta u' \, dx + c \int_0^l u \, \delta u \, dx + S \cdot u(l) \cdot \delta u(l)$$

→ Computes energy associated with the error

Convergence plots

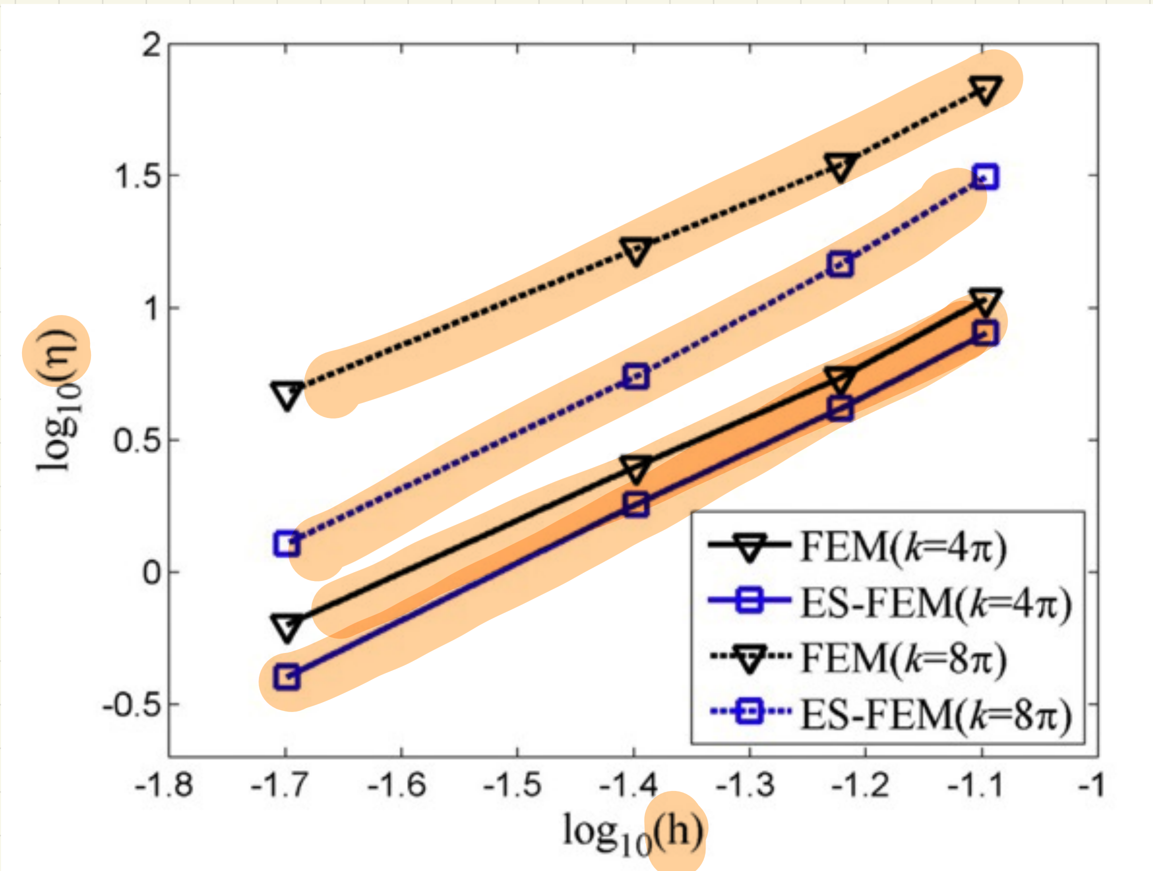
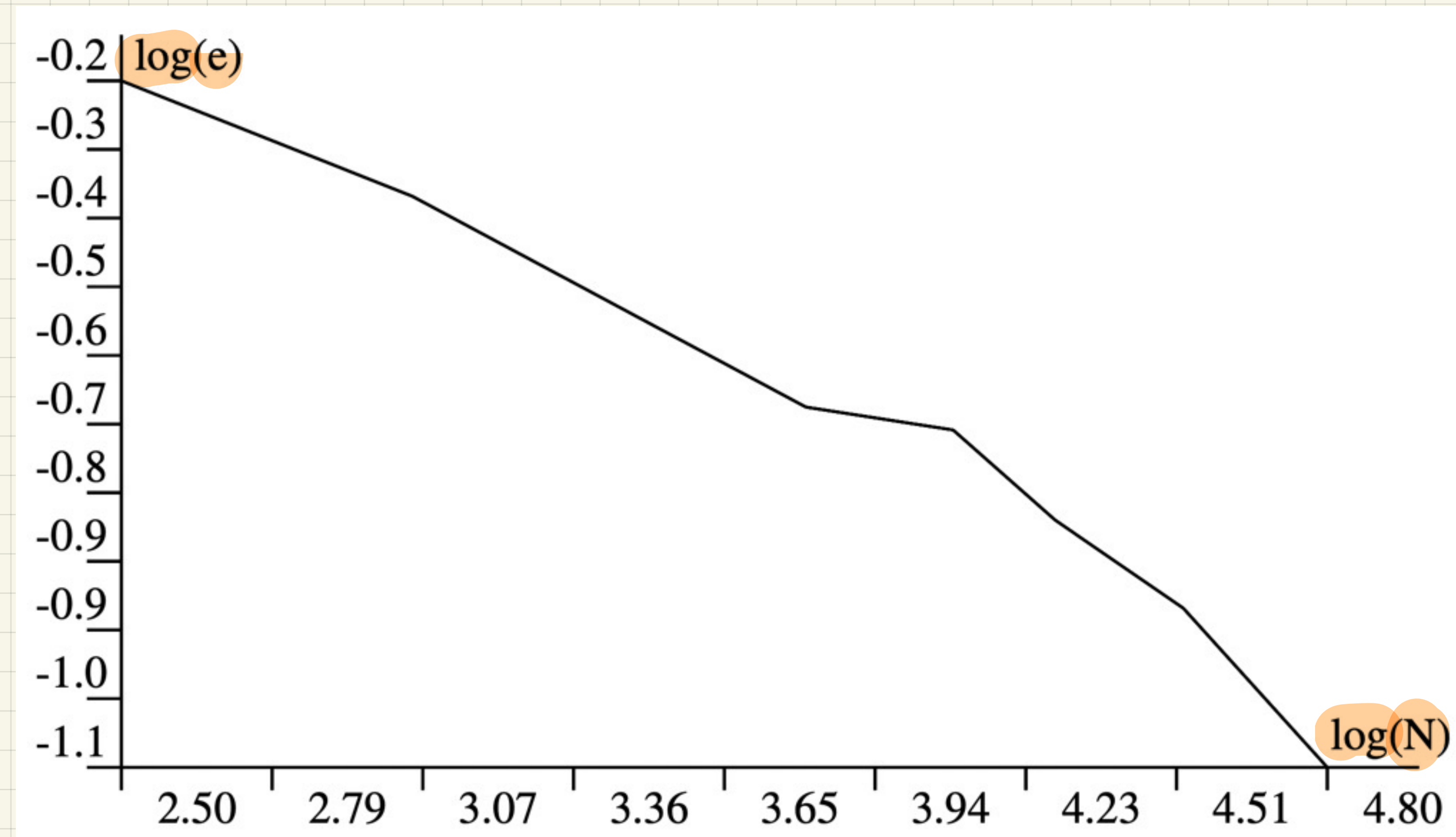


Fig. 9. The convergence rate results for FEM and ES-FEM.

Chai et al. (2018)



Rachowicz et al. (2006)