FEM for 1D problems

Numerical solution
- Pile foundation problem

Variational problem (V): Find function $u: [0,L] \to \mathbb{R}$ such that

$$EA \cdot \int_0^L u' \cdot \delta u' \, dx + C \cdot \int_0^L u \cdot \delta u \, dx + S \cdot u(L) \cdot \delta u(L) = n \cdot \int_0^L \delta u \, dx + F \cdot \delta u(0)$$
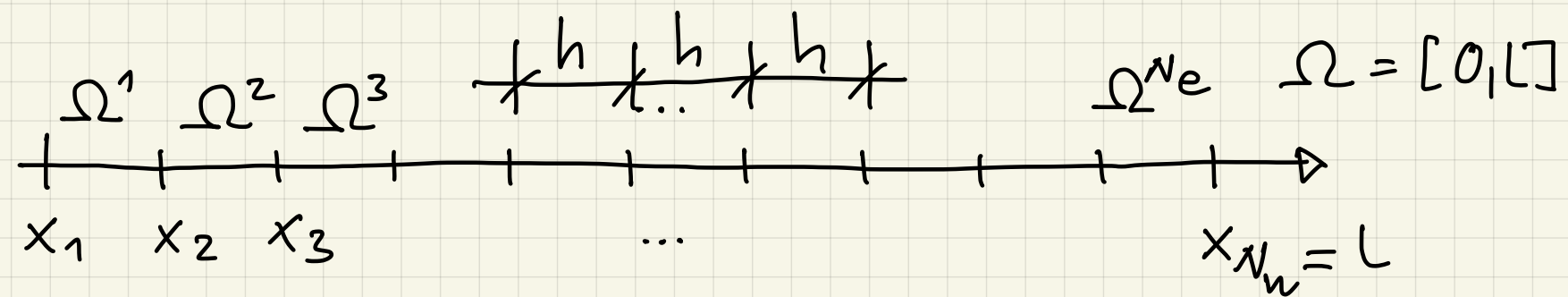
for all (admissible) test functions δu.
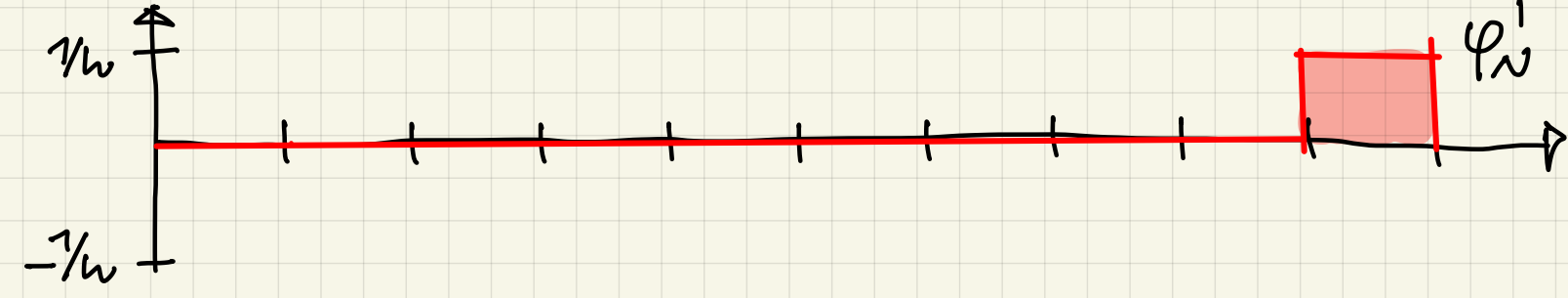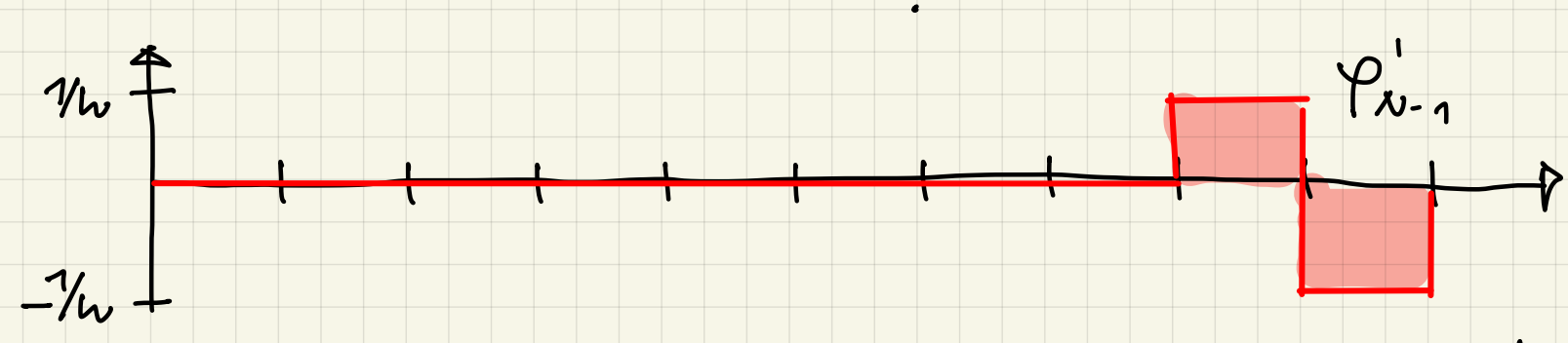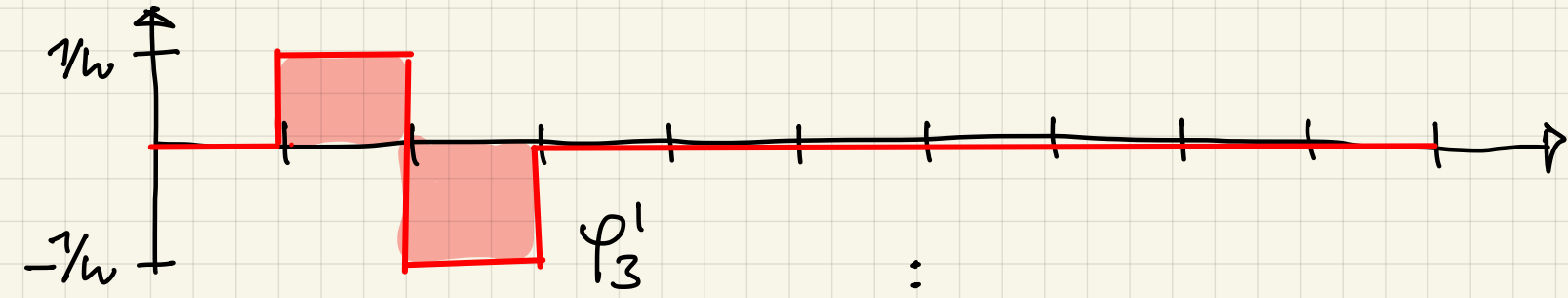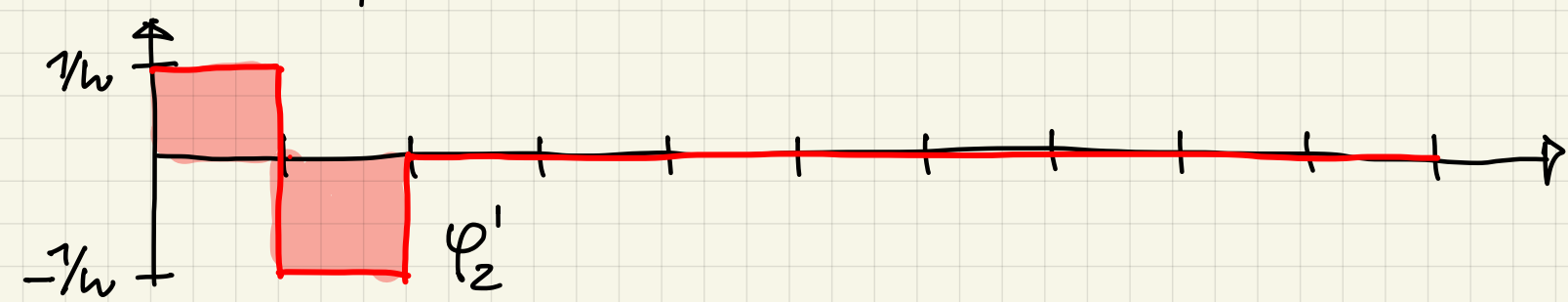
We have in this case

$$a(u, \delta u) = EA \cdot \int_0^L u' \cdot \delta u' \, dx + C \cdot \int_0^L u \cdot \delta u \, dx + S \cdot u(L) \cdot \delta u(L)$$

$$b(\delta u) \quad = n \cdot \int_0^L \delta u \, dx + F \cdot \delta u(0)$$

Choice of basis functions

N_n – Number of nodes
N_e – Number of elements
N – Number of basis functions
h – Element size

Ω^1 Ω^2 Ω^3 $\;h\;h\;h$ Ω^{N_e} $\Omega = [0, L]$

x_1 x_2 x_3 $\quad\cdots\quad$ $x_{N_n} = L$
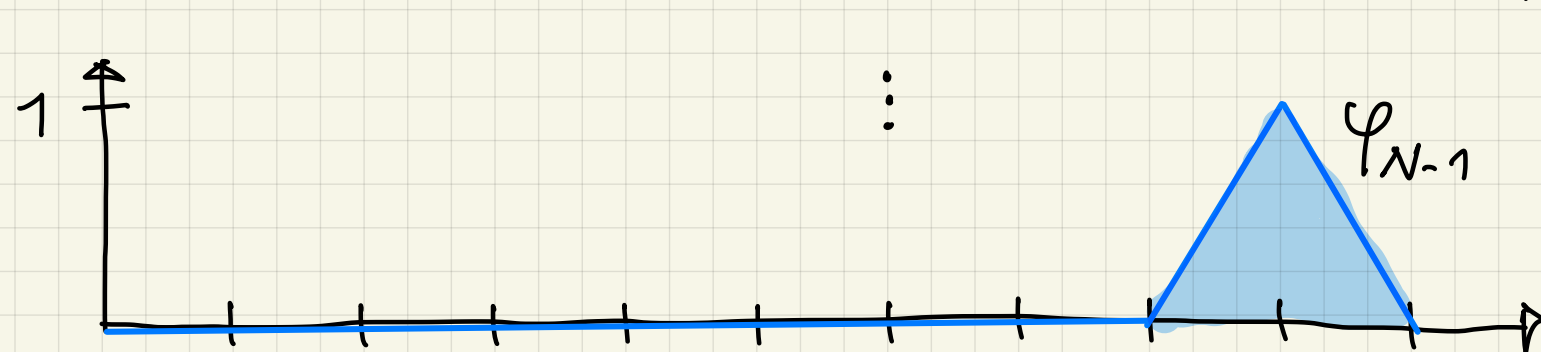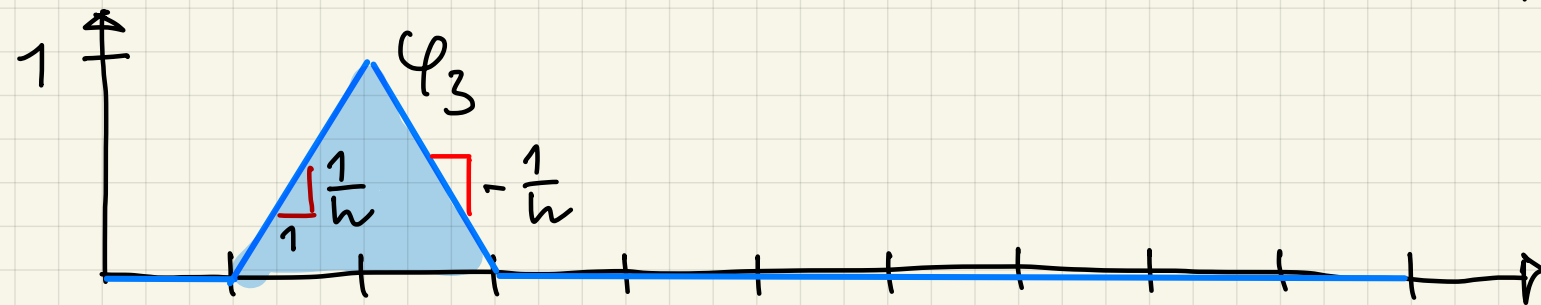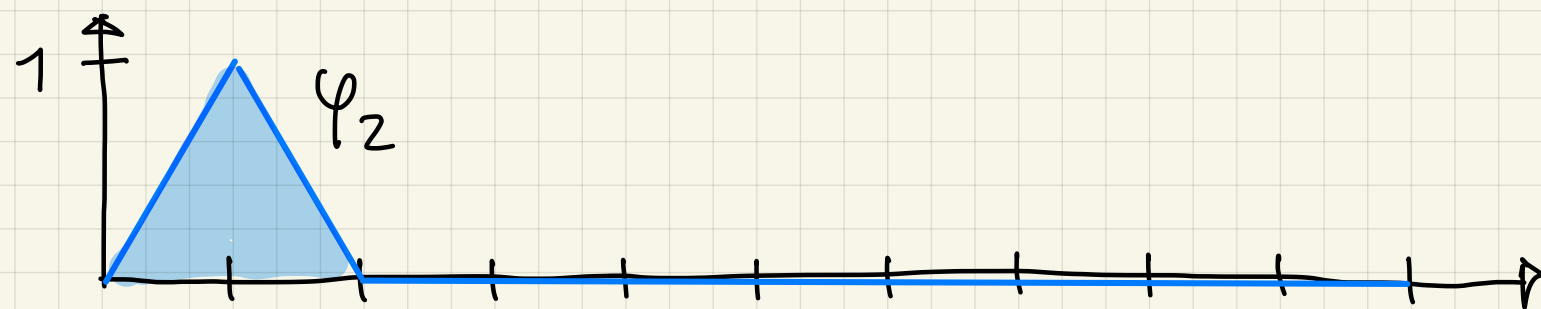
1 $\quad \varphi_1$

$\frac{1}{h}$ $\quad \varphi_1'$

1 $\quad \varphi_2$

$\frac{1}{h}$ $\quad -\frac{1}{h}$ $\quad \varphi_2'$

1 $\quad \varphi_3$

$\frac{1}{h}$ $\quad -\frac{1}{h}$ $\quad \varphi_3'$

1 $\quad \varphi_{N-1}$

$\frac{1}{h}$ $\quad -\frac{1}{h}$ $\quad \varphi_{N-1}'$

1 $\quad \varphi_N$

$\frac{1}{h}$ $\quad -\frac{1}{h}$ $\quad \varphi_N'$
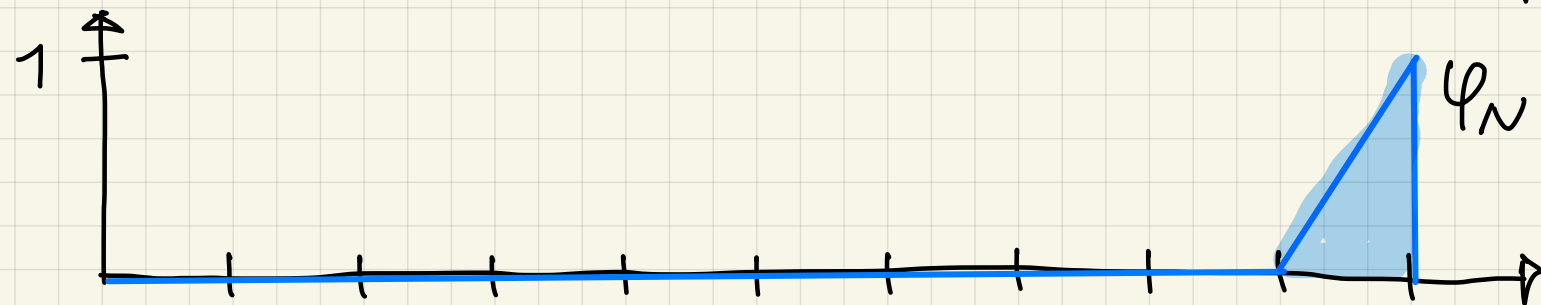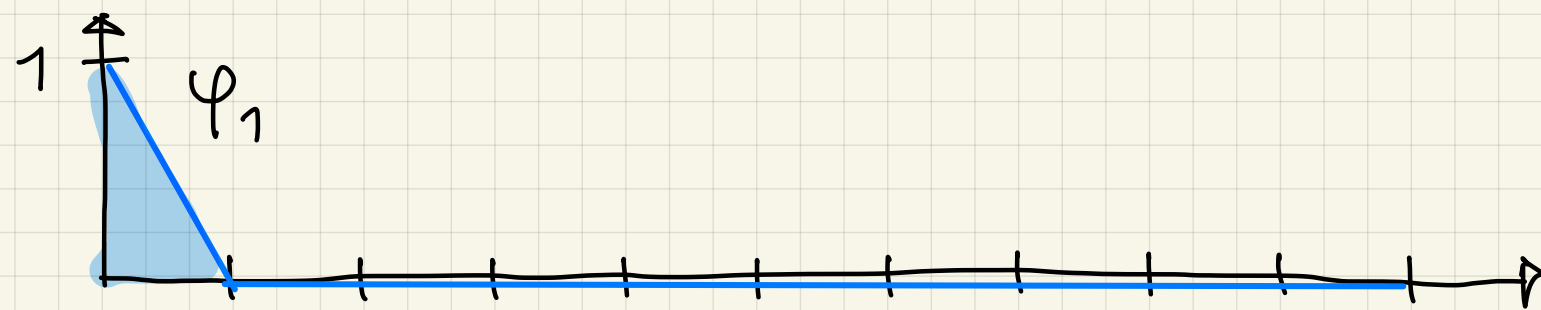
Split contributions to matrix $\underline{K}$

$$K_{ij} = a(\varphi_i, \varphi_j) = EA \cdot \int_0^L \varphi_i' \cdot \varphi_j' \, dx + C \cdot \int_0^l \varphi_i \cdot \varphi_j \, dx + S \cdot \varphi_i(l) \cdot \varphi_j(l)$$

$$= \quad K_{ij}^{EA} \quad + \quad K_{ij}^{C} \quad + \quad K_{ij}^{S}$$

$$a^{EA}(u, \delta u) = EA \cdot \int_0^l u' \cdot \delta u' \, dx \quad \rightsquigarrow \quad K_{ij}^{EA} = a^{EA}(\varphi_i, \varphi_j) = EA \cdot \int_0^l \varphi_i' \cdot \varphi_j' \, dx$$

$$a^{C}(u, \delta u) = C \cdot \int_0^l u \cdot \delta u \, dx \quad \rightsquigarrow \quad K_{ij}^{C} = a^{C}(\varphi_i, \varphi_j) = C \cdot \int_0^l \varphi_i \cdot \varphi_j \, dx$$

$$a^{S}(u, \delta u) = S \cdot u(l) \cdot \delta u(l) \quad \rightsquigarrow \quad K_{ij}^{S} = a^{S}(\varphi_i, \varphi_j) = S \cdot \varphi_i(l) \cdot \varphi_j(l)$$

Contribution of EA

$$K_{ij}^{EA} = a^{EA}(\varphi_i, \varphi_j) = EA \cdot \int_0^l \varphi_i' \cdot \varphi_j' \, dx$$

$$k_{11}^{EA} = EA \cdot \int_0^l \varphi_1' \cdot \varphi_1' \, dx = EA \cdot h \cdot \frac{1}{h^2} = \frac{EA}{h}$$

$$k_{12}^{EA} = EA \cdot \int_0^l \varphi_1' \cdot \varphi_2' \, dx = EA \cdot h \cdot \frac{-1}{h^2} = -\frac{EA}{h}$$

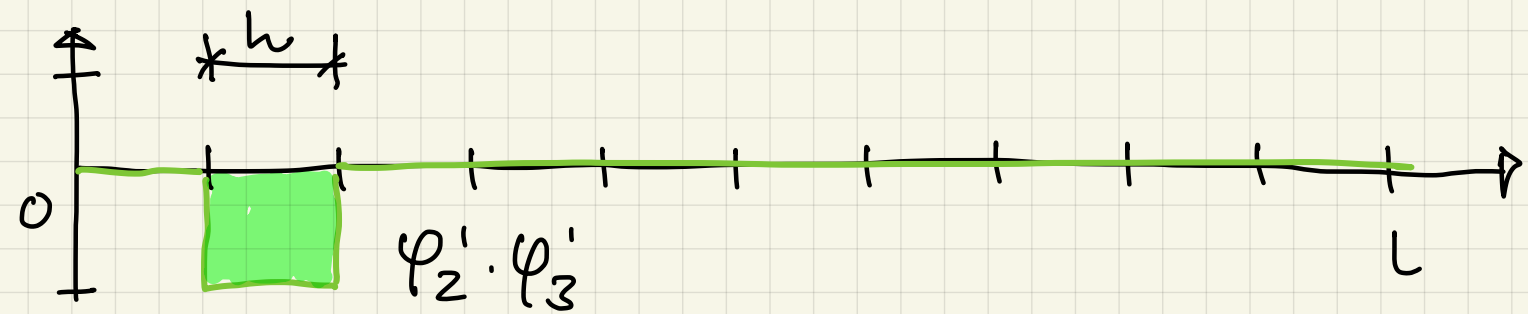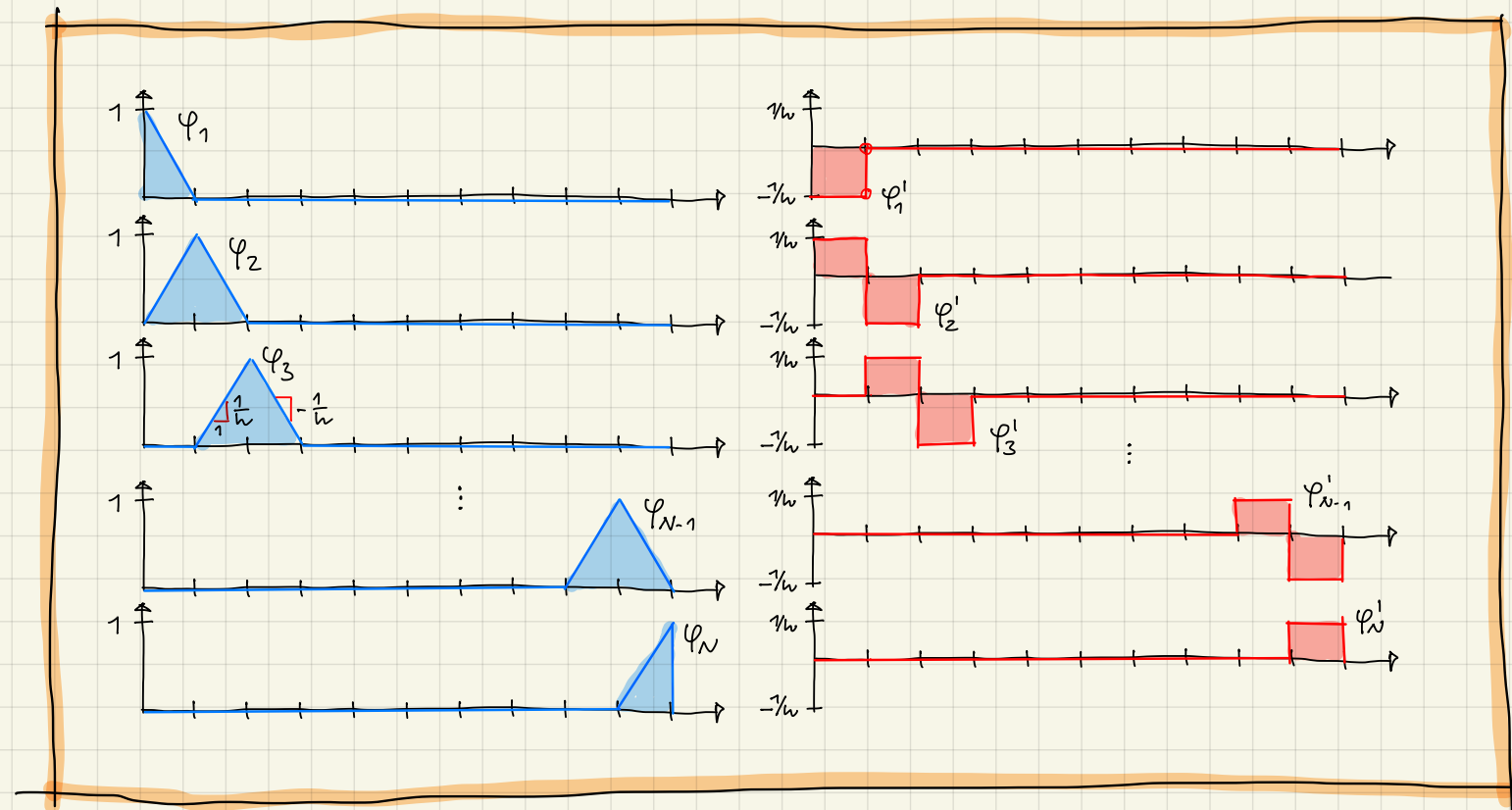$$k_{13}^{EA} = EA \cdot \int_0^l \varphi_1' \cdot \varphi_3' \, dx = 0$$

$$k_{14}^{EA} = \quad \ldots \quad = K_{1N}^{EA} = 0$$

$$k_{21}^{EA} = \qquad k_{12}^{EA} \qquad = -\frac{EA}{h}$$

$$k_{22}^{EA} = EA \cdot \int_0^l \varphi_2' \cdot \varphi_2' \, dx = EA \cdot 2 \cdot h \cdot \frac{1}{h^2} = \frac{2EA}{h}$$

$$K_{23}^{EA} = EA \cdot \int_0^l \varphi_2' \cdot \varphi_3' \, dx = EA \cdot h \cdot \frac{-1}{h^2} = -\frac{EA}{h} \quad , \quad k_{34}^{EA} = \ldots = K_{3N}^{EA} = 0$$

$$k_{31}^{EA} = 0 \; , \; k_{32}^{EA} = -\frac{EA}{h} \; , \; k_{33}^{EA} = \frac{2EA}{h} \; , \; k_{34}^{EA} = -\frac{EA}{h}$$

$$\underline{K}^{EA} = \frac{EA}{h} \begin{bmatrix} 1 & -1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 1 \end{bmatrix}$$

Sparse and symmetric