FEM for 2D problems

Heat conduction: Weak formulation

Differential equation

$$-\lambda \operatorname{div} \nabla \theta = w$$

Multiplication with test function $\delta\theta \in V_0$

$$-\lambda \cdot \operatorname{div} \nabla \theta \cdot \delta\theta = w \cdot \delta\theta$$

where V_0 is the set of all "nice" functions $u : \Omega \to \mathbb{R}$ with $u(x,y) = 0$ for $(x,y) \in \Gamma_D$.

Integrate over Ω

$$-\int_\Omega \lambda \cdot \operatorname{div} \nabla \theta \cdot \delta\theta \, dA = \int_\Omega w \cdot \delta\theta \, dA$$

Ok!

Not symmetric & second derivatives
→ Integration by parts

Integration by parts formula

$f : \Omega \to \mathbb{R}$, $\underline{u} : \Omega \to \mathbb{R}^2$

$$\int_\Omega f \cdot \operatorname{div} \underline{u} \, dA = \int_\Gamma f \, \underline{u} \cdot \underline{n} \, ds - \int_\Omega \nabla f \cdot \underline{u} \, dA$$

$$\Leftrightarrow \int_\Omega \operatorname{div} \underline{u} \cdot f \, dA = \int_\Gamma \underline{u} \cdot \underline{n} \cdot f \, ds - \int_\Omega \underline{u} \cdot \nabla f \, dA$$

Integration by parts

($\nabla\theta$ takes the role of $\underline{u}$, $\delta\theta$ of f)

Boundary conditions!

$$\int_\Omega \lambda \cdot \operatorname{div} \nabla\theta \cdot \delta\theta \, dA = \int_\Gamma \lambda \cdot \overbrace{\nabla\theta \cdot \underline{n}}^{=0 \text{ on } \Gamma_N} \cdot \overbrace{\delta\theta}^{=0 \text{ on } \Gamma_D} \, ds - \boxed{\int_\Omega \lambda \cdot \nabla\theta \cdot \nabla\delta\theta \, dA}$$

Boundary term only for Γ_R

$$-\int_\Omega \lambda \cdot \operatorname{div} \nabla\theta \cdot \delta\theta \, dA = -\int_{\Gamma_R} \lambda \cdot \nabla\theta \cdot \underline{n} \cdot \delta\theta \, ds + \int_\Omega \lambda \cdot \nabla\theta \cdot \nabla\delta\theta \, dA$$

flip sign!

Boundary condition on Γ_R

$$-\lambda \cdot \nabla\theta \cdot \underline{n} = h \cdot (\theta - \theta^*)$$

Insert

$$-\int_\Omega \lambda \cdot \operatorname{div} \nabla\theta \cdot \delta\theta \, dA = -\int_{\Gamma_R} \lambda \cdot \nabla\theta \cdot \underline{n} \cdot \delta\theta \, ds + \int_\Omega \lambda \cdot \nabla\theta \cdot \nabla\delta\theta \, dA$$

$$= \int_{\Gamma_R} h \cdot (\theta - \theta^*) \cdot \delta\theta \, ds + \int_\Omega \lambda \cdot \nabla\theta \cdot \nabla\delta\theta \, dA$$

$$= \int_{\Gamma_R} h \cdot \theta \cdot \delta\theta \, ds - \int_{\Gamma_R} h \cdot \theta^* \cdot \delta\theta \, ds + \int_\Omega \lambda \cdot \nabla\theta \cdot \nabla\delta\theta \, dA$$

Finally

$$\underbrace{\int_\Omega \lambda \cdot \nabla\theta \cdot \nabla\delta\theta \, dA + \int_{\Gamma_R} h \cdot \theta \cdot \delta\theta \, ds}_{\text{Bilinear form}} = \underbrace{\int_\Omega w \cdot \delta\theta \, dA + \int_{\Gamma_R} h \cdot \theta^* \cdot \delta\theta \, ds}_{\text{linear form}}$$

Variational problem (heat 2D): Find $\theta \in V_0$ such that

$$\int_\Omega \lambda \cdot \nabla\theta \cdot \nabla\delta\theta \, dA + \int_{\Gamma_R} h \cdot \theta \cdot \delta\theta \, ds = \int_\Omega w \cdot \delta\theta \, dA + \int_{\Gamma_R} h \cdot \theta^* \cdot \delta\theta \, ds$$

for all test functions $\delta\theta \in V_0$. The space (set) V_0 contains all "nice" functions $u: \Omega \to \mathbb{R}$ with $u(x,y) = 0$ for $(x,y) \in \Gamma_D$.

Remarks:

- "nice" means that we can compute the integrals
- Obviously we have

  $a: V_0 \times V_0 \to \mathbb{R}$ with $a(\theta, \delta\theta) = \int_\Omega \lambda \cdot \nabla\theta \cdot \nabla\delta\theta \, dA + \int_{\Gamma_R} h \cdot \theta \cdot \delta\theta \, ds$

  $b: V_0 \to \mathbb{R}$ with $b(\delta\theta) = \int_\Omega w \cdot \delta\theta \, dA + \int_{\Gamma_R} h \cdot \theta^* \cdot \delta\theta \, ds$

  where a is a scalar product and b a linear form. As before!
- Natural Neumann BCs are automatically satisfied!