FEM for 2D problems

Robin boundary conditions

Variational problem (heat 2D): Find $\theta \in V_0$ such that

$$\int_\Omega \lambda \cdot \nabla\theta \cdot \nabla\delta\theta \, dA + \int_{\Gamma_R} h \cdot \theta \cdot \delta\theta \, ds = \int_\Omega w \cdot \delta\theta \, dA + \int_{\Gamma_R} h \cdot \theta^* \cdot \delta\theta \, ds$$

for all test functions $\delta\theta \in V_0$. The space (set) V_0 contains all "nice" functions $u : \Omega \to \mathbb{R}$ with $u(x,y) = 0$ for $(x,y) \in \Gamma_D$.

We define

$$a_{int}(\theta, \delta\theta) = \int_\Omega \lambda \cdot \nabla\theta \cdot \nabla\delta\theta \, dA \qquad b_{int}(\delta\theta) = \int_\Omega w \cdot \delta\theta \, dA$$

$$a_R(\theta, \delta\theta) = \int_{\Gamma_R} h \cdot \theta \cdot \delta\theta \, ds \qquad b_R(\delta\theta) = \int_{\Gamma_R} h \cdot \theta^* \cdot \delta\theta \, ds$$

$$a(\theta, \delta\theta) = a_{int}(\theta, \delta\theta) + a_R(\theta, \delta\theta) \qquad b(\delta\theta) = b_{int}(\delta\theta) + b_R(\delta\theta)$$

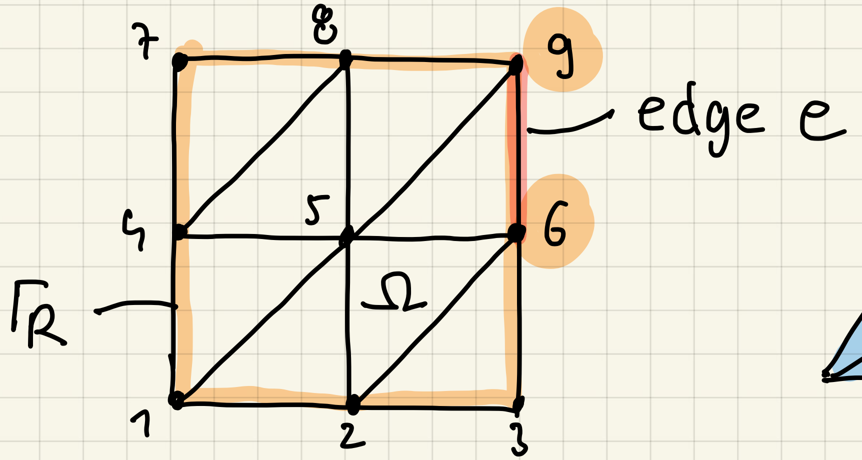
Linear System with Robin BCs

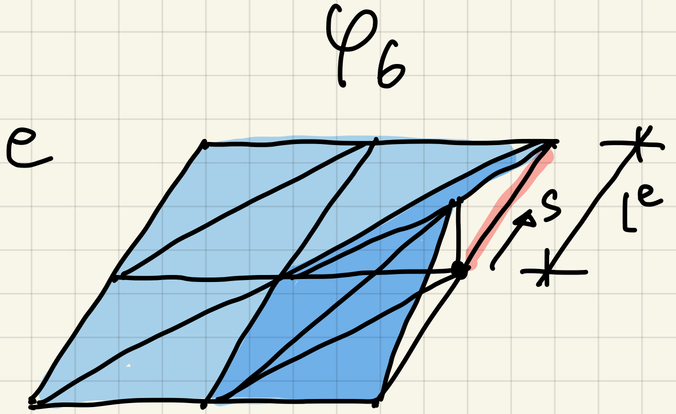
$$\left(\int_0^L \triangleright\cdot\triangleright \, ds = \frac{1}{3}\cdot L \,,\ \int_0^L \triangleright\triangleleft \, ds = \frac{1}{6}\cdot L\right)$$

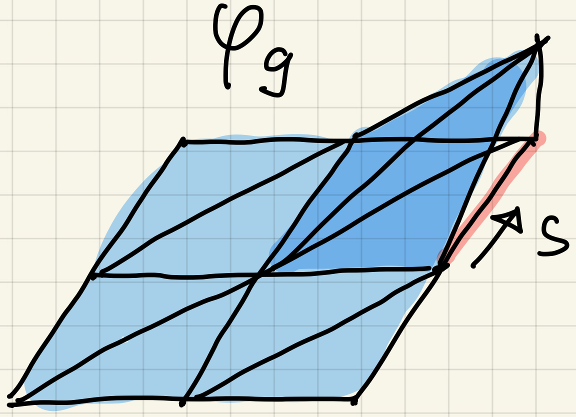
$\underline{K} = \underline{K}^{int} + \underline{K}^{R}$ where $K^{int}_{ij} = a_{int}(\varphi_i,\varphi_j)$ and $K^{R}_{ij} = a_R(\varphi_i,\varphi_j) = \int_{\Gamma_R} h\cdot\varphi_i\cdot\varphi_j \, ds$

$\underline{r} = \underline{r}^{int} + \underline{r}^{R}$ where $r^{int}_{i} = b_{int}(\varphi_i)$ and $r^{R}_{i} = b_R(\varphi_i) = \int_{\Gamma_R} h\cdot\theta^*\cdot\varphi_i \, ds$

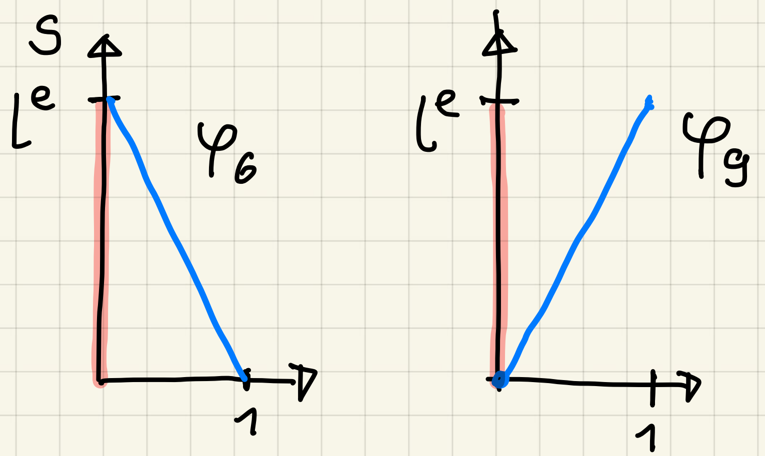
Example







On edge e



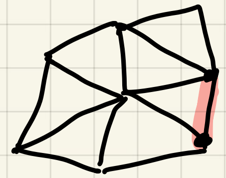
$$K^R_{66} = \int_0^{l^e} h\cdot\left(1-\frac{s}{l^e}\right)\cdot\left(1-\frac{s}{l^e}\right) ds = \frac{1}{3}\cdot h\cdot l^e$$

$$r^R_{6} = \int_0^{l^e} h\cdot\theta^*\cdot\left(1-\frac{s}{l^e}\right) ds = \frac{1}{2}\cdot h\cdot\theta^*\cdot l^e$$

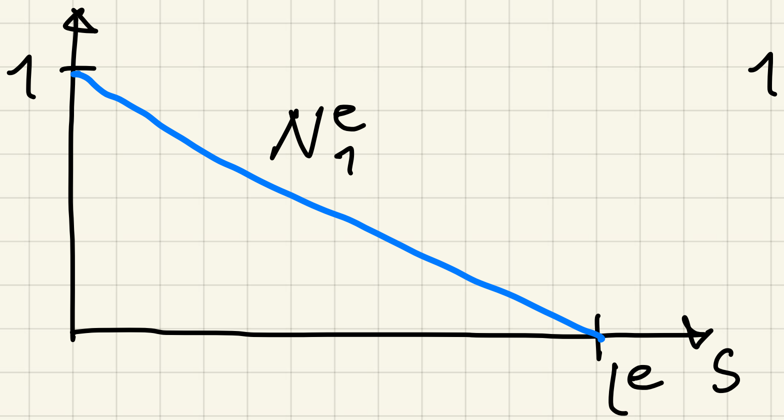
$$K^R_{69} = \int_0^{l^e} h\cdot\left(1-\frac{s}{l^e}\right)\cdot\frac{s}{l^e} \, ds = \frac{1}{6}\cdot h\cdot l^e$$

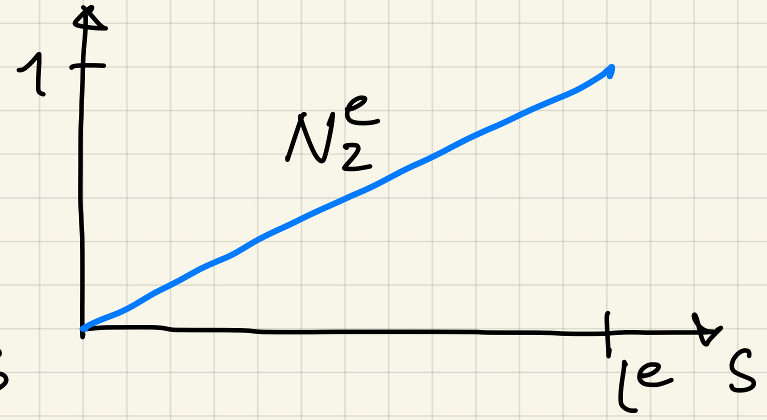
$$r^R_{9} = \quad \dots \quad = \frac{1}{2} h\cdot\theta^*\cdot l^e$$

Edge wise computation



Functions on edge e





Edge forms

$$a_R^e(\theta, \delta\theta) = \int_0^{l^e} h \cdot \theta \cdot \delta\theta \, ds$$

$$b_R^e(\delta\theta) = \int_0^{l^e} h \cdot \theta^* \cdot \delta\theta \, ds$$

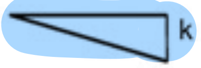
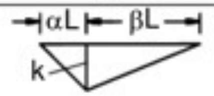
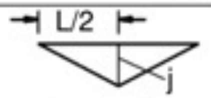
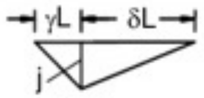
where $\theta, \delta\theta : [0, l^e] \to \mathbb{R}$

Contributions of edge e

$$\left.\begin{aligned} k^e_{R,ij} &= a^e_R(N^e_i, N^e_j) \\ r^e_{R,i} &= b^e_R(N^e_i) \end{aligned}\right\} \text{Assembly into global system as before}$$

→ See pile foundation

Integraltafel

Nr.		A	B	C	D	E	F	G
								$\int j^2 dx$
1		jk	$\frac{1}{2}jk$	$\frac{1}{2}j\,(k_1 + k_2)$	0	$\frac{1}{4}jk$	$\frac{1}{2}jk$	j^2
2		$\frac{1}{2}jk$	$\frac{1}{3}jk$	$\frac{1}{6}j\,(k_1 + 2k_2)$	$-\frac{1}{6}jk$	0	$\frac{1}{6}jk\,(1 + \alpha)$	$\frac{1}{3}j^2$
3		$\frac{1}{2}jk$	$\frac{1}{6}jk$	$\frac{1}{6}j\,(2k_1 + k_2)$	$\frac{1}{6}jk$	$\frac{1}{4}jk$	$\frac{1}{6}jk\,(1 + \beta)$	$\frac{1}{3}j^2$
4		$\frac{1}{2}k\,(j_1 + j_2)$	$\frac{1}{6}k\,(j_1 + 2j_2)$	$\frac{1}{6}[\,j_1\,(2k_1 + k_2) + j_2\,(k_1 + 2k_2)]$	$\frac{1}{6}k\,(_1 - j_2)$	$\frac{1}{4}j_1k$	$\frac{1}{6}k\,[\,j_1\,(1 + \beta) + j_2\,(1 + \alpha)]$	$\frac{1}{3}\,(j_1^2 + j_1 j_2 + j_2^2)$
5		0	$-\frac{1}{6}jk$	$\frac{1}{6}j\,(k_1 - k_2)$	$\frac{1}{3}jk$	$\frac{1}{4}jk$	$\frac{1}{6}jk\,(1 - 2\alpha)$	$\frac{1}{3}j^2$
6		$\frac{1}{4}jk$	0	$\frac{1}{4}jk_1$	$\frac{1}{4}jk$	$\frac{1}{4}jk$	$\frac{1}{4}jk\beta$	$\frac{1}{4}j^2$
7		$\frac{1}{4}jk$	$\frac{1}{4}jk$	$\frac{1}{4}jk_2$	$-\frac{1}{4}jk$	$-\frac{1}{8}jk$	$\frac{1}{4}jk\alpha$	$\frac{1}{4}j^2$
8		$\frac{1}{2}jk$	$\frac{1}{4}jk$	$\frac{1}{4}j\,(k_1 + k_2)$	0	$\frac{1}{8}jk$	$\frac{jk}{12\beta}\,(3 - 4\alpha^2)$	$\frac{1}{3}j^2$
9		$\frac{1}{2}jk$	$\frac{1}{6}jk\,(1 + \gamma)$	$\frac{1}{6}j\,[\,k_1\,(1 + \delta) + k_2\,(1 + \gamma)\,]$	$\frac{1}{6}jk\,(1 - 2\gamma)$	$\frac{1}{4}jk\delta$	$\frac{jk}{6\beta\gamma}\,(2\gamma - \gamma^2 - \alpha^2)$ $\gamma \geq \alpha$	$\frac{1}{3}j^2$
10		$\frac{2}{3}jk$	$\frac{1}{3}jk$	$\frac{1}{3}j\,(k_1 + k_2)$	0	$\frac{1}{6}jk$	$\frac{1}{3}jk\,(1 + \alpha\beta)$	$\frac{8}{15}j^2$
11		$\frac{1}{3}jk$	$\frac{1}{6}jk$	$\frac{1}{6}j\,(k_1 + k_2)$	0	$\frac{1}{12}jk$	$\frac{1}{6}jk\,(1 - 2\alpha\beta)$	$\frac{1}{5}j^2$
12		$\frac{2}{3}jk$	$\frac{1}{4}jk$	$\frac{1}{12}j\,(5k_1 + 3k_2)$	$\frac{1}{6}jk$	$\frac{7}{24}jk$	$\frac{1}{12}jk\,(5 - \alpha - \alpha^2)$	$\frac{8}{15}j^2$