# FEM for 1D problems

Numerical solution

# Ingredients of finite element solution

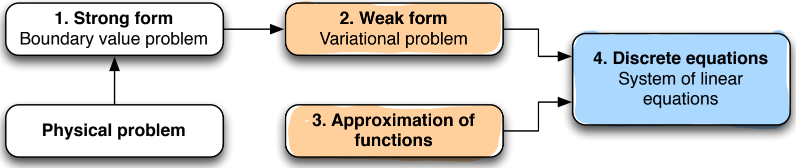



Diagram according to Fish and Belytschko, 2007

**Strong form:** Mathematical model of real world process, differential equation and boundary conditions

**Weak form:** Basis for finite element solution

**Approximation of functions:** Construct approximate solution by combining predefined functions

**Discrete equations:** Inserting predefined functions into weak form yields linear system of equations

It's only math once the boundary value problem has been formulated!

# Previous results

$a: V \times V \to \mathbb{R}$ - Bilinear form

$b: V \to \mathbb{R}$ - linear form

**Abstract variational problem (AV):** Find function $u \in V$ such that

$$a(u, \delta u) = b(\delta u)$$

for all test functions $\delta u \in V$.

## Approximate solution

$\dim(V) = \infty$

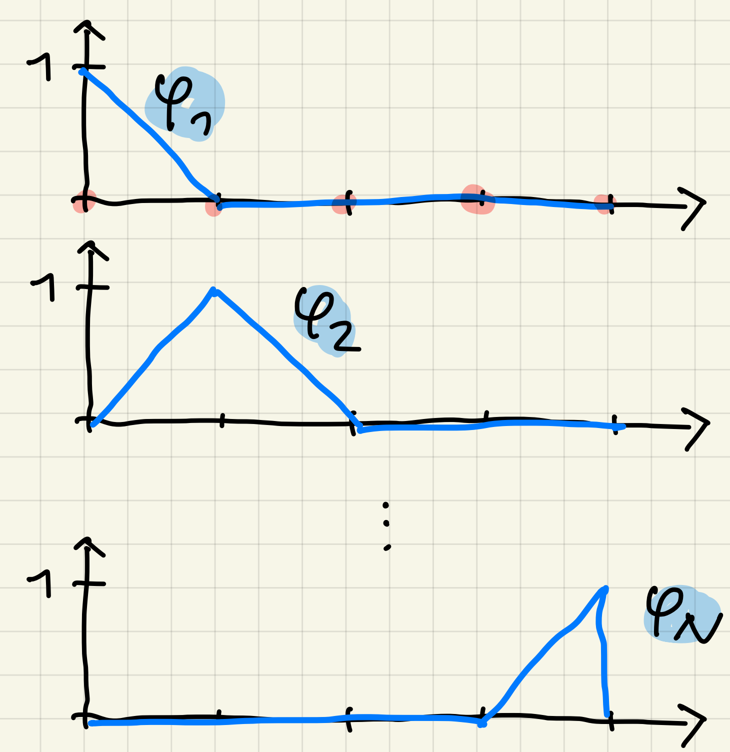
$\dim(V_h) = N$

$$u_h = \sum_{i=1}^{N} \varphi_i \cdot \hat{u}_i$$

$V$ = 'All (nice) functions on $[0, L]$'

$V_h = \{ \varphi_i \cdot \hat{u}_i \mid \hat{u}_i \in \mathbb{R}, i = 1, \dots, N \}$

= 'All functions we can construct from $\varphi_i$'




$u_h \in V_h \subset V$

$V_h$ is a finite dimensional subspace of $V$

**Abstract discrete variational problem (ADV):**

Find function $u_h \in V_h$ such that

$$a(u_h, \delta u_h) = b(\delta u_h)$$

for all test functions $\delta u_h \in V_h$.

**Step 1:** Replace one equation which has to hold for all $\delta u_h \in V_h$ by $N$ equations containing only $u_h$ as unknown.

**Step 2:** Insert $u_h = \sum_{i=1}^{N} \varphi_i \cdot \hat{u}_i$ and compute $\hat{u}_i$, $i = 1, \ldots, N$.

**Step 1:** Replace one equation which has to hold for all $\delta u_h \in V_h$ by $N$ equations containing only $u_h$ as unknown. Insert $\delta u_h = \sum_{i=1}^{N} \varphi_i \cdot \delta \hat{u}_i$ into $a(u, \delta u) = b(\delta u)$.

The approximate solution $u_h$ then has to satisfy

$$a\left(u_h, \sum_{i=1}^{N} \varphi_i \cdot \delta \hat{u}_i\right) = b\left(\sum_{i=1}^{N} \varphi_i \cdot \delta \hat{u}_i\right) \quad \text{for all} \quad \delta \hat{u}_i \in \mathbb{R}$$

Expand sum

$$\begin{aligned} b\left(\sum_{i=1}^{N} \varphi_i \cdot \delta \hat{u}_i\right) &= b(\varphi_1 \cdot \delta \hat{u}_1 + \varphi_2 \cdot \delta \hat{u}_2 + \ldots + \varphi_N \cdot \delta \hat{u}_N) \\ &= b(\varphi_1 \cdot \delta \hat{u}_1) + b(\varphi_2 \cdot \delta \hat{u}_2) + \ldots + b(\varphi_N \cdot \delta \hat{u}_N) \\ &= \delta \hat{u}_1 \cdot b(\varphi_1) + \delta \hat{u}_2 \cdot b(\varphi_2) + \ldots + \delta \hat{u}_N \cdot b(\varphi_N) \\ &= \sum_{i=1}^{N} \delta \hat{u}_i \cdot b(\varphi_i) \end{aligned}$$

$$a\left(u_h, \sum_{i=1}^{N} \varphi_i \cdot \delta \hat{u}_i\right) = \sum_{i=1}^{N} \delta \hat{u}_i \cdot a(u_h, \varphi_i)$$

We obtain

$$\sum_{i=1}^{N} \delta \hat{u}_i \cdot a(u_h, \varphi_i) = \sum_{i=1}^{N} \underbrace{\delta \hat{u}_i}_{\in \mathbb{R}} \cdot b(\varphi_i) \iff \sum_{i=1}^{N} \delta \hat{u}_i \overbrace{\left(a(u_h, \varphi_i) - b(\varphi_i)\right)}^{\text{Has to be } 0 \text{ if relation holds for all } \delta \hat{u}_i} = 0$$

for all $i = 1, \ldots, N$

↳ Next page

We found that the propositions

$$a(u_h, \delta u_h) = b(\delta u_h) \quad \text{for all } \delta u_h \in V_h \tag{1}$$

$$a\left(u_h, \sum_{i=1}^{N} \varphi_i \cdot \delta \hat{u}_i\right) = b\left(\sum_{i=1}^{N} \varphi_i \cdot \delta \hat{u}_i\right) \quad \text{for all } \delta \hat{u}_i \in \mathbb{R} \tag{2}$$

$$\boxed{a(u_h, \varphi_i) = b(\varphi_i) \quad , \quad i = 1, \dots, N} \tag{3}$$

are equivalent!

$N$ equations!

**Step 2:** Insert $u_h = \sum_{j=1}^{N} \varphi_j \cdot \hat{u}_j$

$$\boxed{a(u_h, \varphi_i) = b(\varphi_i) \quad , \quad i = 1, \dots, N}$$

We obtain

$$a\left(\sum_{j=1}^{N} \varphi_j \cdot \hat{u}_j , \varphi_i\right) = b(\varphi_i)$$

$$\sum_{j=1}^{N} \underbrace{a(\varphi_j, \varphi_i)} \cdot \hat{u}_j = \underbrace{b(\varphi_i)}$$

$$\sum_{j=1}^{N} K_{ij} \cdot \hat{u}_j = r_i$$

Zur Erinnerung

$$\underline{y} = \underline{A}\,\underline{x} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} \cdot x_1 + a_{12} \cdot x_2 + \dots + a_{1m} \cdot x_m \\ a_{21} \cdot x_1 + a_{22} \cdot x_2 + \dots + a_{2m} \cdot x_m \\ \vdots \\ a_{n1} \cdot x_1 + a_{n2} \cdot x_2 + \dots + a_{nm} \cdot x_m \end{bmatrix}$$

$$y_i = \sum_{j=1}^{m} a_{ij} \cdot x_j$$

Result: We can compute $\hat{u}_j$ by solving the linear system

$$\underline{K}\,\underline{\hat{u}} = \underline{r}$$

where

$$K_{ij} = a(\varphi_i, \varphi_j) \quad , \quad r_i = b(\varphi_i)$$

$\underline{K}$ – Stiffness matrix

$\underline{r}$ – Load vector

$$u_h = \sum_{j=1}^{N} \varphi_j \cdot \hat{u}_j$$