

Report Assignment 4: Control of a pendulum

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Contents

1	Modelling and identification of the system	1
1.1	Theoretical continuous model	1
1.2	State-space equation for the nonlinear model	2
1.3	Linearization of the model	2
1.4	Determination of the pendulum length	3
1.5	Discretization	4

List of Figures

1	Depiction of the pendulum placed on the cart [1]	2
2	Free body diagram of the pendulum [1]	2
3	Measured values for the experiment to determine ζ	4

List of Tables

1 Modelling and identification of the system

1.1 Theoretical continuous model

The configuration of the pendulum placed on the cart is depicted in Figure 1, along with the definition of the position of the cart x and the angle of the pendulum θ . In Figure 2, the free body diagram of the pendulum is shown. The real pendulum is depicted as a mass m , placed in the center of gravity of the the real pendulum. This mass is connected to the rotation point via a massless bar with length L . This way, the moment of inertia J can easily be computed:

$$J = m * L^2 \quad (1)$$

To derive the continuous model of the velocity-steered cart with pendulum, Newton's second law of motion for rotation is used. There are three contributions to the total moment working on the pendulum system. These are caused by gravity, damping and an inertial force because the pendulum is placed in an accelerating reference frame. These can be filled in into Newton's second law:

$$J\ddot{\theta} = -mgL\sin(\theta) - m\ddot{x}L\cos(\theta) - c\dot{\theta} \quad (2)$$

Using Equation 1, this can be simplified to:

$$L\ddot{\theta} + \ddot{x}\cos(\theta) = -g\sin(\theta) - \frac{c}{mL}\dot{\theta} \quad (3)$$

The parameters of the theoretical continuous model of the velocity-steered cart are x and θ . The input is the cart velocity \dot{x} which has units $[m/s]$. The outputs of the model are the cart position x with units $[m]$ and the pendulum angle θ with units $[rad]$.

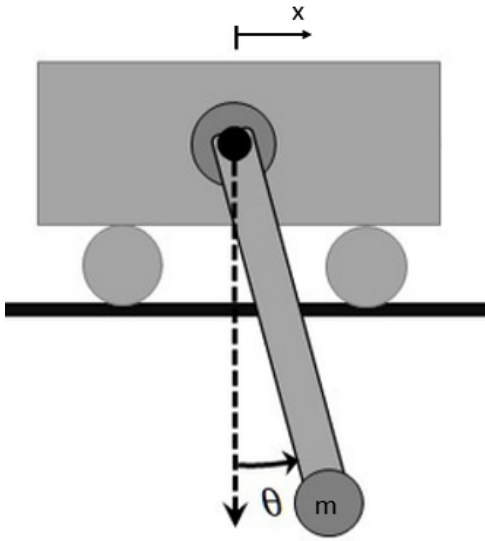


Figure 1: Depiction of the pendulum placed on the cart [1]

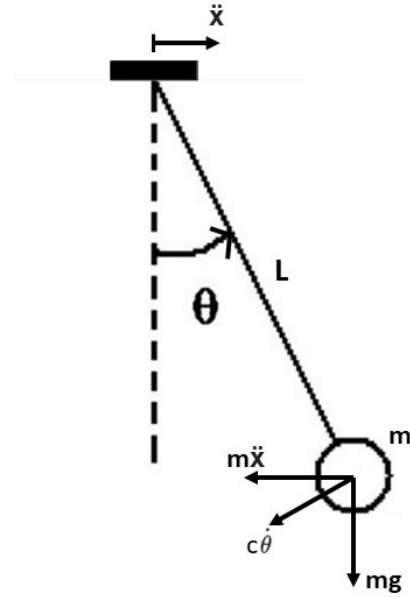


Figure 2: Free body diagram of the pendulum [1]

1.2 State-space equation for the nonlinear model

To derive a state-space equation for the non-linear model, following states are used:

$$\xi = [x, \theta, L\dot{\theta} + \dot{x}\cos(\theta)]^T \quad (4)$$

In this state vector, x stands for the position of the cart and θ stands for the pendulum angle. In the third state, $L\dot{\theta} + \dot{x}\cos(\theta)$ the tangential velocity of the centre of gravity of the pendulum with relation to a standstill reference frame can be recognized. For this reason, we will call the third state v_{tan} from now on. The state vector then becomes:

$$\xi = [x, \theta, v_{tan}]^T \quad (5)$$

Using Equation 3 and the definition of ξ , the nonlinear state-space equation can be derived. The input is the velocity of the cart and is denoted as v_c .

$$\dot{\xi} = \mathbf{f}(\xi, v_c) = \begin{bmatrix} \dot{x} \\ \dot{\theta} \\ L\ddot{\theta} + \ddot{x}\cos(\theta) - \dot{x}\sin(\theta)\dot{\theta} \end{bmatrix} = \begin{bmatrix} v_c \\ \frac{v_{tan} - v_c \cos(\theta)}{L} \\ -g\sin(\theta) - \left(\frac{v_c}{mL} + v_c \sin(\theta)\right)\left(\frac{v_{tan} - v_c \cos(\theta)}{L}\right) \end{bmatrix} \quad (6)$$

$$\mathbf{y} = \begin{bmatrix} x \\ \theta \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \xi + 0 * v_c \quad (7)$$

1.3 Linearization of the model

The nonlinear model can be linearized around the equilibrium state $\xi_0 = [0, 0, 0]^T$. Both θ and v_{tan} have to be zero in the equilibrium state. The value of x doesn't really matter, but is taken equal to zero for convenience. The equilibrium input v_{c0} is also chosen to be equal to zero. Calculating the jacobian matrices \mathbf{F} and \mathbf{G} , we can rewrite the model as:

$$\delta \dot{\xi} = \mathbf{F} \delta \xi + \mathbf{G} \delta v_c \quad (8)$$

Where $\delta\xi$ and δv_c are defined as follows:

$$\begin{aligned}\delta\xi &= \xi - \xi_0 = \xi \\ \delta v_c &= v_c - v_{c0} = v_c\end{aligned}\tag{9}$$

Using nonlinear Equation 6, \mathbf{F} and \mathbf{G} are defined as:

$$\begin{aligned}\mathbf{F} &= \left. \frac{\partial \mathbf{f}}{\partial \xi} \right|_{\xi_0, v_{c0}} \\ \mathbf{G} &= \left. \frac{\partial \mathbf{f}}{\partial v_c} \right|_{\xi_0, v_{c0}}\end{aligned}\tag{10}$$

From Equation 10, the following matrices arise:

$$\begin{aligned}\mathbf{F} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{L} \\ 0 & -g & \frac{-c}{mL^2} \end{bmatrix} \\ \mathbf{G} &= \begin{bmatrix} 1 \\ \frac{-1}{L} \\ \frac{c}{mL^2} \end{bmatrix}\end{aligned}\tag{11}$$

As the measurement equation was already linear, we find the following linear state-space model:

$$\delta\dot{\xi} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{L} \\ 0 & -g & \frac{-c}{mL^2} \end{bmatrix} \delta\xi + \begin{bmatrix} 1 \\ \frac{-1}{L} \\ \frac{c}{mL^2} \end{bmatrix} \delta v_c\tag{12}$$

$$\mathbf{y} = \begin{bmatrix} x \\ \theta \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \xi + 0 * v_c\tag{13}$$

By applying the laplacetransformation to Equation 12, following transferfunctions can be found.

$$\begin{aligned}\frac{d\delta x}{dt} &= v_c \rightarrow H_x(s) = \frac{\delta X(s)}{\delta V_c(s)} = \frac{1}{s} \\ \frac{d\delta\theta}{dt} &= \frac{v_{tan} - v_c}{L} \rightarrow H_\theta(s) = \frac{\delta\Theta(s)}{\delta V_c(s)} = \frac{-1}{sL} \\ \frac{d\delta v_{tan}}{dt} &= -g\delta\theta - \frac{c}{mL^2}\delta v_{tan} + \frac{c}{mL^2}\delta v_c \rightarrow H_{v_{tan}}(s) = \frac{\delta V_{tan}(s)}{\delta V_c(s)} = \frac{1}{\frac{mL^2}{c}s + 1}\end{aligned}\tag{14}$$

1.4 Determination of the pendulum length

The damping ratio ζ can be determined, using the logarithmic decrement. In this experiment, the maximal amplitude of the angle θ for two consecutive periods are measured. This is shown in Figure 3, where T stands for the period of the oscillation. The logarithmic decrement δ is then defined as [2]:

$$\delta = \ln\left(\frac{\theta(t)}{\theta(t+T)}\right) = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}\tag{15}$$

Where $\theta(t)$ and $\theta(t+T)$ are both the values of the maximal angle in their period. From Equation 15, ζ can be derived:

$$\zeta = \frac{\delta}{\sqrt{(2\pi)^2 + \delta^2}}\tag{16}$$

To derive the length of the pendulum, the period of the oscillations is measured. From the measured period, we can derive the measured (damped) pulsation, which is an estimate for the real pulsation.

$$\omega_d = \frac{2\pi}{T} = \omega_n \sqrt{1-\zeta^2}\tag{17}$$

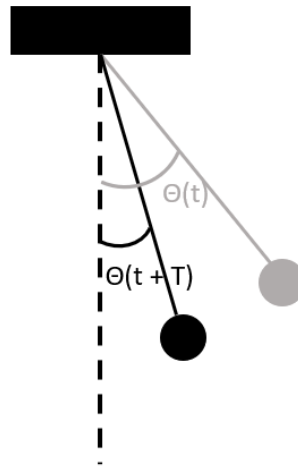


Figure 3: Measured values for the experiment to determine ζ

From Equation 17, the natural pulsation ω_n can be derived. As $\omega_n = \sqrt{\frac{g}{L}}$, L can be found as:

$$L = \frac{g}{\omega_n^2} = \frac{g(1 - \zeta^2)}{\omega_d^2} \quad (18)$$

The numerical values are: $L = 0.1261m$, $\zeta = 0.0327$

1.5 Discretization

References

- [1] Aurelio Arenas. The use of a nintendo wii remote control in physics experiments. https://www.researchgate.net/publication/258272199_The_use_of_a_Nintendo_Wii_remote_control_in_physics_experiments, 2013. Accessed December 17th 2021.
- [2] W. Desmet. *Mechanische trillingen - Slides*. Cursusdienst VTK, 2021.