Report Assignment 4: Control of a pendulum

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Matthias Derez, Toon Servaes

Contents

List of Figures

List of Tables

1 Modellation and identification of the system

1.1 Theoretical continuous model

The configuration of the pendulum placed on the cart is depicted in Figure 1, along with the definition of the position of the cart x and the angle of the pendulum θ. In Figure 2, the free body diagram of the pendulum is shown. The real pendulum is depicted as a mass m, placed in the center of gravity of the the real pendulum. This mass is connected to the rotation point via a massless bar with length L. This way, the moment of inertia J can easily be computed:

$$J = m * L^2 \tag{1}$$

To derive the continuous model of the velocity-steered cart with pendulum, Newton's second law of motion for rotation is used. There are three contributions to the total moment working on the pendulum system. These are caused by gravity, damping and an inertial force because the pendulum is placed in an accelerating reference frame. These can be filled in into Newton's second law:

$$J\ddot{\theta} = -mgLsin(\theta) - m\ddot{x}Lcos(\theta) - c\dot{\theta} \tag{2}$$

Using Equation 1, this can be simplified to:

$$L\ddot{\theta} + \ddot{x}cos(\theta) = -gsin(\theta) - \frac{c}{mL}\dot{\theta} \tag{3}$$

The parameters of the theoretical continuous model of the velocity-steered cart are x and θ. The input is the cart velocity $\dot{x}$ which has units $[m/s]$. The outputs of the model are the cart position x with units $[m]$ and the pendulum angle θ with units $[rad]$.
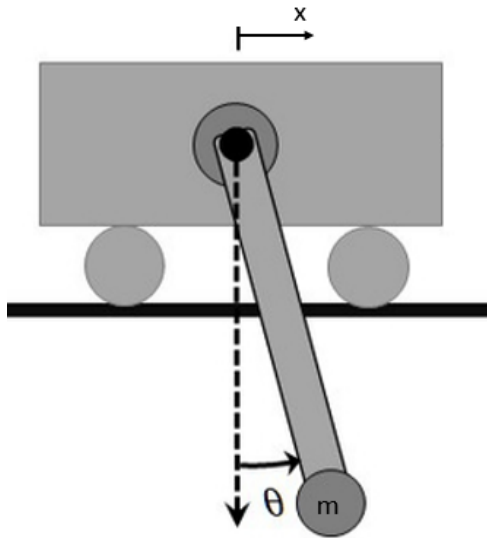

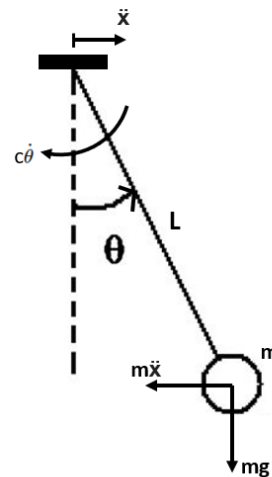Figure 1: Depiction of the pendulum placed on the cart [3]

Figure 2: Free body diagram of the pendulum [1]

1.2 State-space equation for the nonlinear model

To derive a state-space equation for the non-linear model, following states are used:

$$\xi = [x, \theta, L\dot{\theta} + \dot{x}cos(\theta)]^T \tag{4}$$

In this state vector, x stands for the position of the cart and θ stands for the pendulum angle. In the third state, $L\dot{\theta} + \dot{x}cos(\theta)]^T$ the tangential velocity of the centre of gravity of the pendulum with relation

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to a standstill reference frame can be recognized. For this reason, we will call the third state v_{tan} from now on. The state vector then becomes:

$$\xi = [x, \theta, v_{tan}]^T \tag{5}$$

Using Equation 3 and the definition of ξ, the nonlinear state-space equation can be derived. The input is the velocity of the cart and is denoted as v_c.

$$\dot{\xi} = \mathbf{f}(\xi, v_c) = \begin{bmatrix} \dot{x} \\ \dot{\theta} \\ L\ddot{\theta} + \ddot{x}cos(\theta) - \dot{x}sin(\theta)\dot{\theta} \end{bmatrix} = \begin{bmatrix} v_c \\ \frac{v_{tan} - v_c cos(\theta)}{L} \\ -gsin(\theta) - (\frac{c}{mL} + v_c sin(\theta))(\frac{v_{tan} - v_c cos(\theta)}{L}) \end{bmatrix} \tag{6}$$

$$\mathbf{y} = \begin{bmatrix} x \\ \theta \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \xi + 0 * v_c \tag{7}$$

1.3 Linearization of the model

The nonlinear model can be linearized around the equilibrium state $\xi_0 = [0, 0, 0]^T$. Both θ and v_{tan} have to be zero in the equilibrium state. The value of x doesn't really matter, but is taken equal to zero for convenience. The equilibrium input v_{c0} is also chosen to be equal to zero. Calculating the jacobian matrices $\mathbf{F}$ and $\mathbf{G}$, we can rewrite the model as:

$$\delta\dot{\xi} = \mathbf{F}\delta\xi + \mathbf{G}\delta v_c \tag{8}$$

Where $\delta\xi$ and δv_c are defined as follows:

$$\begin{aligned} \delta\xi = \xi - \xi_0 = \xi \\ \delta v_c = v_c - v_{c0} = v_c \end{aligned} \tag{9}$$

Using nonlinear Equation 6, $\mathbf{F}$ and $\mathbf{G}$ are defined as:

$$\begin{aligned} \mathbf{F} = \left.\frac{\partial \mathbf{f}}{\partial \xi}\right|_{\xi_0, v_{c0}} \\ \mathbf{G} = \left.\frac{\partial \mathbf{f}}{\partial v_c}\right|_{\xi_0, v_{c0}} \end{aligned} \tag{10}$$

From Equation 10, the following matrices arise:

$$\begin{aligned} \mathbf{F} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{L} \\ 0 & -g & \frac{-c}{mL^2} \end{bmatrix} \\ \mathbf{G} = \begin{bmatrix} 1 \\ \frac{-1}{L} \\ \frac{c}{mL^2} \end{bmatrix} \end{aligned} \tag{11}$$

As the measurement equation was already linear, we find the following linear state-space model:

$$\delta\dot{\xi} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{L} \\ 0 & -g & \frac{-c}{mL^2} \end{bmatrix} \delta\xi + \begin{bmatrix} 1 \\ \frac{-1}{L} \\ \frac{c}{mL^2} \end{bmatrix} \delta v_c \tag{12}$$

$$\mathbf{y} = \begin{bmatrix} x \\ \theta \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \xi + 0 * v_c \tag{13}$$

By applying the laplacetransformation to Equation 12, following transferfunctions can be found.

$$\begin{aligned} \frac{d\delta x}{dt} = v_c \rightarrow H_x(s) = \frac{\delta X(s)}{\delta V_c(s)} = \frac{1}{s} \\ \frac{d\delta\theta}{dt} = \frac{v_{tan} - v_c}{L} \rightarrow H_\theta(s) = \frac{\delta\Theta(s)}{\delta V_c(s)} = \frac{-1}{sL} \\ \frac{d\delta v_{tan}}{dt} = -g\delta\theta - \frac{c}{mL^2}\delta v_{tan} + \frac{c}{mL^2}\delta v_c \rightarrow H_{v_{tan}}(s) = \frac{\delta V_{tan}(s)}{\delta V_c(s)} = \frac{1}{\frac{mL^2}{c}s + 1} \end{aligned} \tag{14}$$

1.4 Determination of the pendulum length

The damping ratio ζ can be determined, using the logarithmic decrement. In this experiment, the maximal amplitude of the angle θ for two consecutive periods are measured. This is shown in Figure 3, where T stands for the period of the oscillation. The logarithmic decrement δ is then defined as [2]:

$$\delta = ln\left(\frac{\theta(t)}{\theta(t+T)}\right) = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} \tag{15}$$

Where $\theta(t)$ and $\theta(t+T)$ are both the values of the maximal angle in their period. From Equation 15, ζ can be derived:

$$\zeta = \frac{\delta}{\sqrt{(2\pi)^2 + \delta^2}} \tag{16}$$

By measuring the period of the oscillations, the distance L from the rotating point to the centre of gravity can be calculated. Since the pendulum has two <u>BOUTEN IN HET ENGELS</u> detached at the end of it, the centre of gravity will be locatied close to the end, but the distributed mass of the pendulum will cause the measured length L_m to be a bit larger than the calculated length$L$.

From the measured period, we can derive the measured (damped) pulsation, which is an estimate for the real pulsation.

$$\omega_d = \frac{2\pi}{T} = \omega_n\sqrt{1-\zeta^2} \tag{17}$$

From Equation 17, the natural pulastion ω_n can be derived. As $\omega_n = \sqrt{\frac{g}{L}}$, L can be found as:

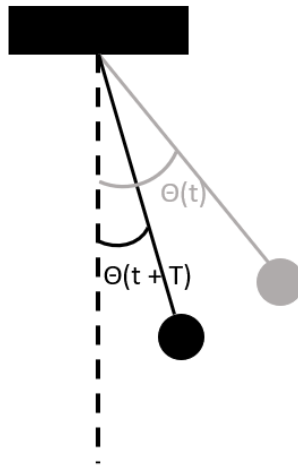$$L = \frac{g}{\omega_n^2} = \frac{g(1-\zeta^2)}{\omega_d^2} \tag{18}$$

Figure 3: Measured values for the experiment to determine ζ

The numerical values are: $L = 0.1261m$, $\zeta = 0.0327$ <u>ZETA DIMENSIELOOS?</u>. As expected, the measured length $L_m = $ <u>MEASURED LENGTH</u> is a larger than de calculated length L. If the damping is neglected, the measured pulsation ω_d is equal to the natural pulsation ω_n and the length can be calculated as $L' = \frac{g}{\omega_d^2} = 1.1262m$.

The damping coefficient c is related to the damping ratio ζ via Equation 19. As the pendulum mass is unknown, the moment of inertia J is unknown and c can not be calculated.

$$\zeta = \frac{c}{c_{crit}} = \frac{c}{2J\omega_n} \tag{19}$$

For the reminder of this report, the damping will be neglected.

1.5   Discretization

To discretize the linear model, the forwart Euler method is used:

$$\delta\dot{\xi} = \frac{\delta\xi[k+1] - \delta\xi[k]}{T_s} \tag{20}$$

Filling in Equation 20 in Equation 8, following equation can be derived:

$$\delta\xi[k+1] = (\mathbf{I} + T_s\mathbf{F})\delta\xi[k] + T_s\mathbf{G}\delta v_c[k] \tag{21}$$

Which can be translated into the following discretized, linear state-space model (where the damping is neglected, $c = 0$):

$$\delta\xi[k+1] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{T_s}{L} \\ 0 & -gT_s & 1 \end{bmatrix} \delta\xi[k] + \begin{bmatrix} T_s \\ \frac{-T_s}{L} \\ 0 \end{bmatrix} v_c[k]$$

$$\begin{bmatrix} x[k] \\ \theta[k] \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \xi[k] + 0 * v_c[k] \tag{22}$$

To discretize the nonlinear model, the forward Euler method is used as well:

$$\dot{\xi} = \frac{\xi[k+1] - \xi[k]}{T_s} \tag{23}$$

By filling in Equation 23 in Equation 6 the following equation can be derived:

$$\xi[k+1] = T_s\mathbf{f}(\xi[k], v_c[k]) + \xi[k] \tag{24}$$

By neglecting the damping of the system, following discretized, nonlinear state-space model can be found.

$$\xi[k+1] = \begin{bmatrix} T_s v_c[k] + x[k] \\ T_s * \frac{v_{tan}[k] - v_c[k]cos(\theta[k])}{L} + \theta[k] \\ T_s\left(-gsin(\theta[k]) - v_c[k]sin(\theta[k])\frac{v_{tan}[k] - v_c[k]cos(\theta[k])}{L} \right) + v_{tan}[k] \end{bmatrix} \tag{25}$$

References

[1] Aurelio Arenas.   The   use   of   a   nintendo   wii   remote   control   in   physics   experiments. `https://www.researchgate.net/publication/258272199_The_use_of_a_Nintendo_Wii_remote_control_in_physics_experiments`, 2013. Accessed December 17th 2021.

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[3] Michel Levis. Common and not zo common pendulum configurations. `https://www.quanser.com/blog/common-and-not-so-common-pendulum-configurations/`, 2019.   Accessed December 17th 2021.