Faculty of Engineering Science
Department of Mechanical Engineering
Control theory [H04X3a]

Report Assignment 1: identification of the motors

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1 Model Structure

1.1 Discrete-time Model Structure

In order to select the discrete-time model structure for the DC motors, the equivalent electric circuit in Figure 1 is used. By using Newton's second law and Kirchhoff's voltage law, following equations can be derived:

$$\begin{aligned} J\ddot{\theta} + b\dot{\theta} &= K_t i \\ L\frac{di}{dt} + Ri &= V - K_e\dot{\theta} \end{aligned} \tag{1}$$

The meaning of each parameter is:

- J = moment of inertia of the rotor [$\mathrm{kg \cdot m^2}$]
- θ = angular position [rad]
- b = motor viscous friction constant [$\mathrm{N \cdot m \cdot s}$]
- K_e = electromotive force constant [$\frac{\mathrm{V}}{\mathrm{rad \cdot s}}$] = K_t = motor torque constant [$\frac{\mathrm{N \cdot m}}{\mathrm{A}}$] = K
- R = electric resistance [Ω]
- L = electric inductance [H]

Continuous-time Transfer Function

Applying the Laplace transform to both equations in (1) leads to:

$$\begin{aligned} Js^2\Theta(s) + bs\Theta(s) &= KI(s) \\ LsI(s) + RI(s) &= V(s) - Ks\Theta(s) \end{aligned} \tag{2}$$

assuming $\theta(0) = 0$, $\dot{\theta}(0) = 0$ and $i(0) = 0$. After some calculations, the continuous-time transfer function describing the behaviour of this system can be derived:

$$H(s) = \frac{\dot{\Theta}(s)}{V(s)} = \frac{s\Theta(s)}{V(s)} = \frac{K}{(Js+b)(Ls+R)+K^2} \tag{3}$$

where the input is the voltage source $V(s)$ applied to the motor's armature and the output is the rotational velocity of the wheel $\dot{\Theta}(s)$. One can rewrite this transfer function in order to obtain a more general and well-known form:

$$\begin{aligned} H(s) &= \frac{K}{(Js+b)(Ls+R)+K^2} \\ &= \frac{K}{LJs^2 + (Lb+JR)s + K^2 + bR} \\ &= \frac{\frac{K}{LJ}}{s^2 + \frac{Lb+JR}{LJ}s + \frac{K^2+bR}{LJ}} \\ &= C\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \end{aligned} \tag{4}$$

where $C = \frac{K}{K^2+bR}$, $\omega_n^2 = \frac{K^2+bR}{LJ}$ and $\zeta = \frac{Lb+JR}{2\sqrt{LJ(K^2+bR)}}$.

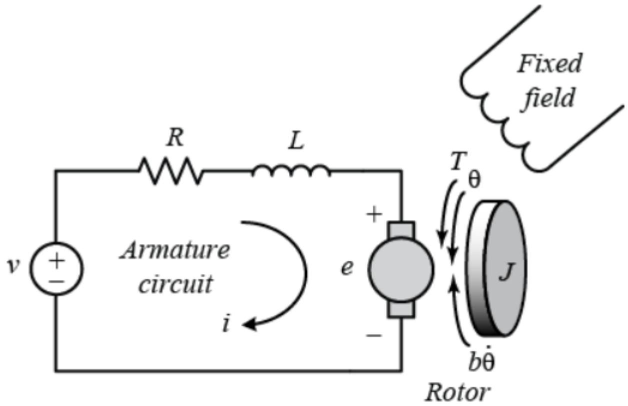

Figure 1: Electric equivalent circuit of the armature and the free-body diagram of the rotor.

Discrete-time Transfer Function

The discrete-time transfer function is derived from the continuous-time one by using a zero-order hold sampling process. To this extent, one has to compute $H(z) = (1 - z^{-1})\mathcal{Z}(\mathcal{L}^{-1}\{\frac{H(s)}{s}\} \times \delta_T(t))$. The software running on the Arduino, called 'MicroOS', samples at 100 Hz. Consequently, the sampling period $T_s = 0.01$ s.

Firstly, Equation (4) can be rewritten as follows:

$$H(s) = C\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = C\frac{a^2 + b^2}{(s+a)^2 + b^2} \tag{5}$$

where $a = \zeta\omega_n$ and $b = \sqrt{\omega_n^2(1-\zeta^2)}$. Next, using transform pair no. 22 on page 4 of the course formulary leads to:

$$\mathcal{Z}\left(\mathcal{L}^{-1}\left\{\frac{H(s)}{s}\right\} \times \delta_T(t)\right) = C\frac{z(Az+B)}{(z-1)\left[z^2 - 2e^{-aT_s}\cos(bT_s)z + e^{-2aT_s}\right]} \tag{6}$$

where $A = 1 - e^{-aT_s}\cos(bT_s) - \frac{a}{b}e^{-aT_s}\sin(bT_s)$ and $B = e^{-2aT_s}\cos(bT_s) + \frac{a}{b}e^{-aT_s}\sin(bT_s) - e^{-aT_s}\cos(bT_s)$. Finally, the zero-order hold equivalent is determined:

$$H(z) = (1 - z^{-1})\mathcal{Z}\left(\mathcal{L}^{-1}\left\{\frac{H(s)}{s}\right\} \times \delta_T(t)\right) = C\frac{(Az+B)}{\left[z^2 - 2e^{-aT_s}\cos(bT_s)z + e^{-2aT_s}\right]} \tag{7}$$

More generally:

$$H(z) = \frac{b_0 z + b_1}{z^2 + a_0 z + a_1} \tag{8}$$

However, the MicroOS software on the Arduino stores the control command calculated during discrete-time interval k in a memory buffer until discrete-time instance $k+1$. In this way, the delay between the measurement of the output and sending out of the control command is increased by one sampling period T_s. In the z-domain, this is equivalent to a multiplication of the transfer function by z^{-1}, leading to the final, generalised discrete-time transfer function of the system:

$$H(z) = \frac{b_0 z + b_1}{z^3 + a_0 z^2 + a_1 z} \tag{9}$$

The order of the numerator is 1, while the order of the denominator is 3. This is in line with the strict causality condition, which demands that the order of the numerator is strictly smaller than the denominator.

Number of delays blablabla.

1.2 Simplified Model

Another, simplified model is also examined. The reason for this is that equally accurate results may be achieved using simpler equations. This simplified model is achieved by neglecting the inductance term in Equation 3, because its effect is several times smaller in comparison with the mechanical terms. This leads to:

$$\begin{aligned} H_{\text{simple}}(s) &= \frac{K}{JRs + (bR + K^2)} \\ &= \frac{\frac{K}{JR}}{s + \frac{bR+K^2}{JR}} \\ &= \frac{1}{C}\frac{a}{s+a} \end{aligned} \tag{10}$$

where $C = \frac{K}{K^2+bR}$, which is the same as in Equation 4. Now, transform pair no. 12 leads to the zero-order hold equivalent of this simplified model:

$$H_{\text{simple}}(z) = \frac{1}{C}\frac{1 - e^{-aT_s}}{z - e^{-aT_s}} \tag{11}$$

More generally:

$$H_{\text{simple}}(z) = \frac{b_0}{z + a_0} \tag{12}$$

Again, the delay is taken into account, resulting in the final, simplified, discrete-time transfer function of the system:

$$H_{\text{simple}}(z) = \frac{b_0}{z^2 + a_0 z} \tag{13}$$

Order, number of delays blablabla.

Both the transfer functions of the complex model (9) and the simple model (13) are used in the rest of the assignment. The unknown parameters b_0, b_1, a_0 and a_1 are determined in Section X using the least-squares method. This is done for each motor separately. Then, depending on which model gives the best results, either the complex or the simple model is chosen.

1.3 Input and Output

As previously mentioned, the input of the model is the voltage source $V(s)$ applied to the motor's armature. This voltage is controlled by the Arduino and is expressed in Volts [V]. The output is the rotational velocity of the wheel $\dot{\Theta}(s)$, expressed in [rad/s].

2 Identification of each 'DC Motor + Wheel'

2.1 Excitation Signal

In order to identify the system 'DC motor + wheel', it is necessary to perform a dedicated experiment that actively excites the system, while the *persistency of excitation* condition is satisfied. This means that the excitation must be rich enough, i.e. so that all frequency modes of the system are excited and observable in the output. To this extent, a square wave is chosen, as displayed in Figure 2 on the left. Note that the input voltage also remains 0 for some time, instead of jumping to -6 or 6 instantaneously.

This step-up step-down pattern is repeated four times in order to appreciate the noise.

average out

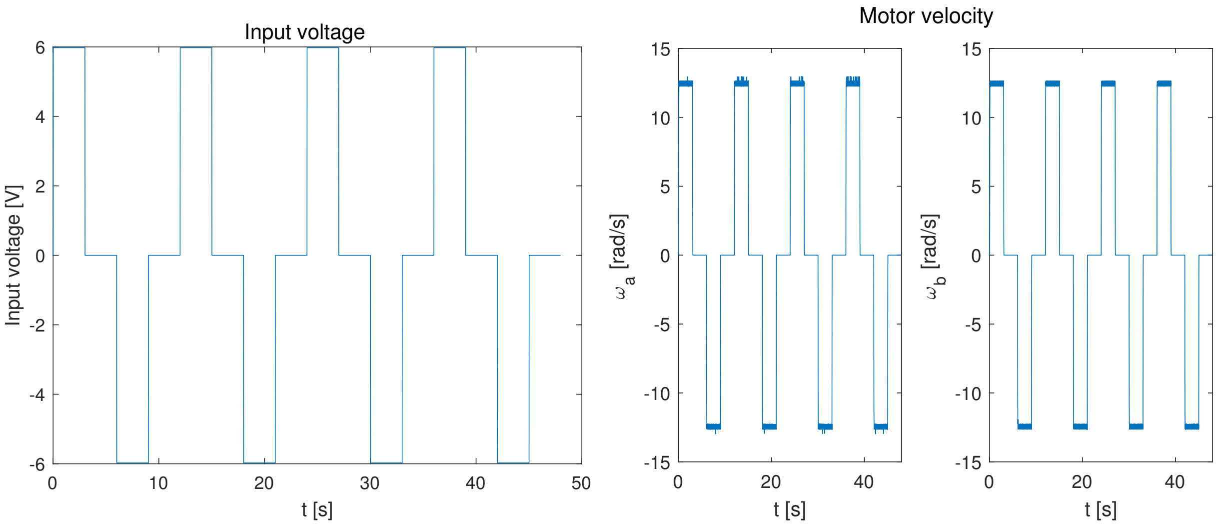

Figure 2: **Left**: Input voltage [V]. **Right**: Motor velocities, for motor A and B respectively.

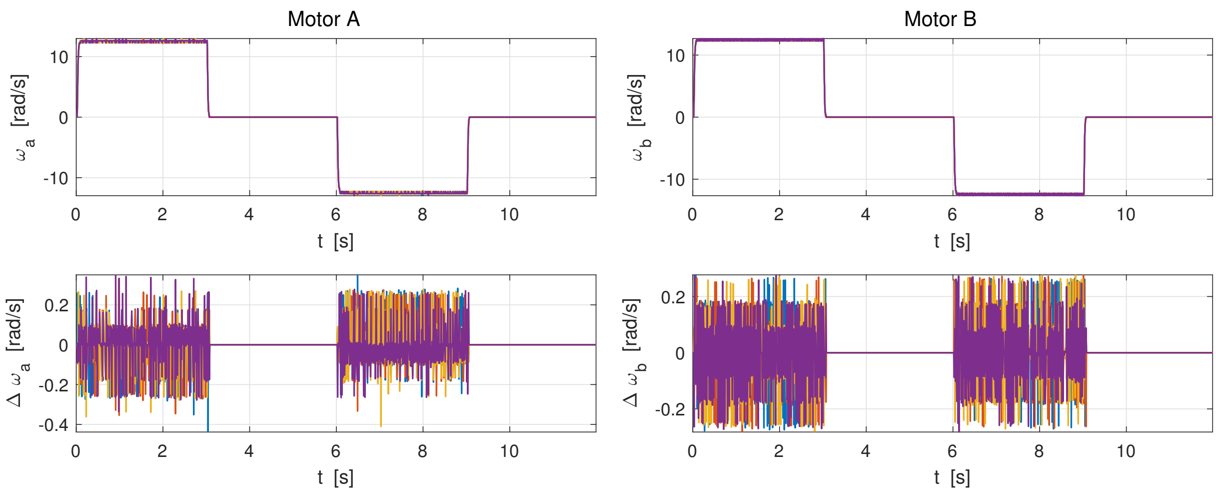

Figure 3: **Top**: Motor velocities averaged out to one period of the square wave excitation. **Bottom**: Deviation of the motor velocities with regard to the averaged out velocities, for each of the periods.

2.2 System Parameters

2.3 Filtering

2.4 Experimental Model Validation

3 Identification of the Cart on the Ground

3.1 Model Validation

3.2 Re-identification

Conclusion

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