

Report Assignment 3: State Feedback and State Estimation

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1 Design of the State Feedback Controller

In this Section, a state feedback controller is designed in order to control the position of the cart while driving on a straight line. This position control loop is then added on top of the velocity controllers designed in Assignment 2.

Input & Output

Figure 1 visualizes the discrete time LTI system that is examined in this assignment. The input and output are given by:

$$\begin{aligned} \text{input } u(t) &= v(t) = R\omega(t) = \dot{x}(t) \\ \text{output } y(t) &= -x(t) \end{aligned} \quad (1)$$

where $v(t)$ is the velocity of the cart, R the radius of the wheels and ω the rotational velocity of the wheels. The minus sign in the output equation is due to choice of the coordinate system, so that $-x$ represents a positive value.

State Space Model

There is only one state, the position x , so the matrices in the state equation (and measurement equation) are scalars.

Continuous state space model:

$$\begin{cases} \dot{x}(t) = v(t) = u(t) & \text{state equation} \\ y(t) = -x(t) & \text{measurement equation} \end{cases} \quad (2)$$

Discretization using Forward-Euler scheme:

$$\dot{x}[k] = \frac{x[k+1] - x[k]}{T_s} + O(T_s^2) \quad (3)$$

leads to:

$$\begin{cases} \dot{x}[t+1] = x[k] + T_s u[k] & \text{state equation} \\ y[t] = -x[t] & \text{measurement equation} \end{cases} \quad (4)$$

with T_s the sampling time of 0.01 s. From the general discretized state space form, one can deduce that $A = 1$, $B = T_s$, $C = -1$ and $D = 0$.

Close Loop Transfer Function

Assuming full state feedback, the closed loop transfer function is:

$$\begin{aligned} H(z) &= \frac{Y(z)}{R(z)} = (C - DK)(z - A - BK)^{-1}B + D \\ &= -(z - 1 + T_s K)^{-1}T_s \\ &= \frac{-T_s}{z - 1 + T_s K} \end{aligned} \quad (5)$$

with K the state feedback gain. Subsequently, the closed loop system has one pole, which is located at

$$p_d = 1 - T_s K \quad (6)$$

This equation indicates that the poles move to the left for increasing K , or inversely, move to the right for decreasing K , as seen in Figure 2. In the discrete time domain, the system is stable if the poles are within the unit circle. For this particular system, this means $0 < K < 200$ Hz.

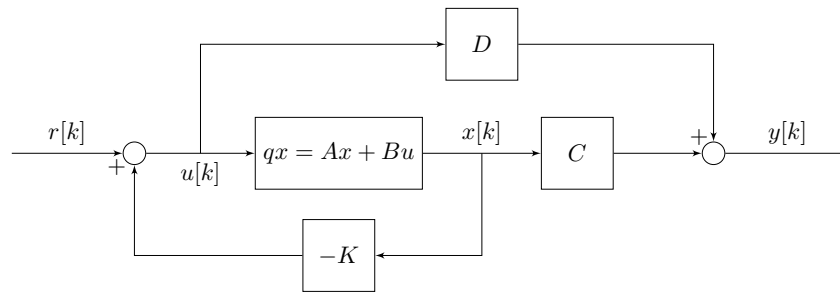


Figure 1: Flow diagram of the closed loop system. [1]

Furthermore, a pole that is close to the unit circle, e.g. for $K = 10$ or $K = 190$, leads to a slower responding system. Equivalently, a pole that is closer to the origin leads to a faster response. These conclusions are demonstrated by Figure 3. The limit cases are when, on the one hand, the pole is located on the unit circle itself, meaning that the system is on the edge of stability. On the other hand, a pole which lies at the origin corresponds to an infinitely fast system, i.e. a pure time delay. This would however mean that the control signal will be infinitely large, thus saturating the actuators. So there is a trade-off between response time and cost in terms of required actuation signal. Both of these limit cases are depicted in Figure 3.

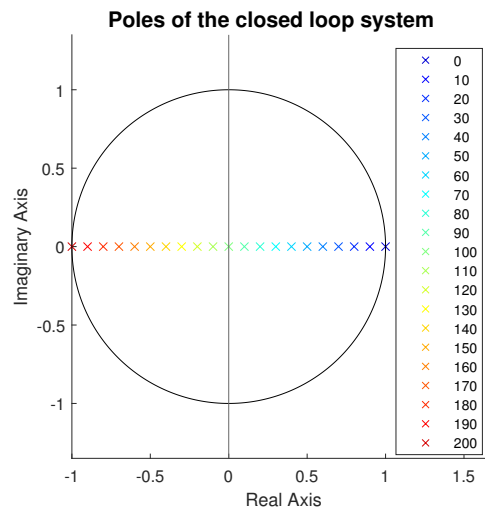


Figure 2: Pole locations of the closed-loop system for varying values of the state feedback gain.

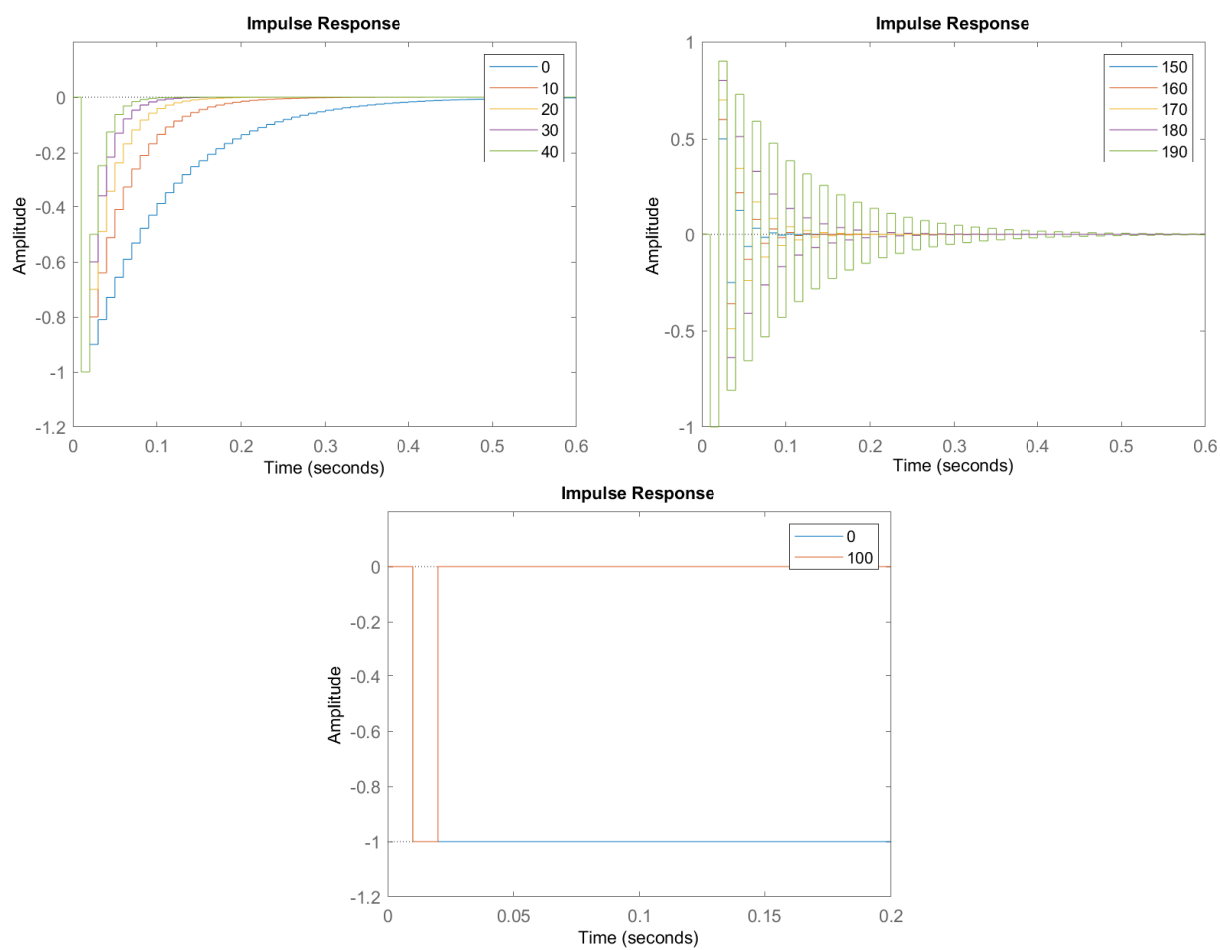


Figure 3: Impulse response of the closed-loop system for varying values of the state feedback gain K .

2 Kalman Filter

As previously elaborated, the measurement equation is equal to

$$y[t] = -x[t] \quad (7)$$

so the C and D matrices of the state-space model are scalar values, respectively -1 and 0 . In this Section, these values are used to validate the principles of the Kalman filter on this system. To this extent, some mathematical derivations are done.

Kalman gain

Firstly, an expression for the time-varying Kalman gain L_{k+1} is derived as a function of the state estimate covariance $\hat{P}_{k|k}$, the process noise covariance Q and measurement noise covariance R . This starts from the following equations in Chapter 10 of *Control Theory - Handouts* [2]:

$$\mathbf{L}_{k+1} = \hat{\mathbf{P}}_{k+1|k} \mathbf{C}^T S_{k+1}^{-1} \quad (8)$$

$$S_{k+1} = \mathbf{C} \hat{\mathbf{P}}_{k+1|k} \mathbf{C}^T + R_{k+1} \quad (9)$$

$$\hat{\mathbf{P}}_{k+1|k} = \mathbf{A} \hat{\mathbf{P}}_{k|k} \mathbf{A}^T + \mathbf{Q}_k \quad (10)$$

All matrices and vectors are scalar values in this specific system, so from now on, the boldface notation is left out. Inserting Equation 8 and 9 in Equation 10 and using $A = 1$ and $C = -1$ yields:

$$\begin{aligned} L_{k+1} &= - \left(\hat{P}_{k|k} + Q_k \right) \left(\hat{P}_{k+1|k} + R_{k+1} \right)^{-1} \\ &= - \frac{\hat{P}_{k|k} + Q_k}{\hat{P}_{k+1|k} + R_{k+1}} \\ &= - \frac{\hat{P}_{k|k} + Q_k}{\hat{P}_{k|k} + Q_k + R_{k+1}} \end{aligned} \quad (11)$$

where Equation 10 was used again in the last step.

Taking the limit for $Q_k \rightarrow \infty$ results in $L_{k+1} = -1$, which in turn leads to $\hat{x}_{k+1|k+1} = -y_{k+1}$. This is literally the measurement equation of the state-space model (7). One can explain this as follows: the larger Q , the greater the variation of the process noise, the less confidence in the model. Or equivalently, the more confidence in the measurement. In the extreme case, i.e. $Q_k \rightarrow \infty$, one has so little confidence in the model that the state equation is entirely neglected.

Next, taking the limit for $R_{k+1} \rightarrow \infty$ prompts $L_{k+1} = 0$, which results in $\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k}$. The a priori state estimate is equal to the a posteriori state estimate, meaning that the correction step in the Kalman filter process is neglected. This is yet again easily explainable: the larger R , the greater the variation of the measurement noise, the less confidence in the measurement. Or equivalently, the more confidence in the model. The extreme case $R_{k+1} \rightarrow \infty$ has so little confidence in the measurement that the innovation residual is completely not taken into account, thus eliminating the correction step.

Next state estimate covariance

Secondly, an expression for the next state estimate covariance $\hat{P}_{k+1|k+1}$ is derived as a function of the previous state estimate covariance $\hat{P}_{k|k}$, the process noise covariance Q and measurement noise covariance R . Again, starting from an equation given in the handouts:

$$\hat{P}_{k+1|k+1} = (1 - L_{k+1}C) \hat{P}_{k+1|k} = (1 + L_{k+1}) \hat{P}_{k+1|k} \quad (12)$$

Using Equation 10 and 11 yields:

$$\begin{aligned}\hat{P}_{k+1|k+1} &= \left(1 - \frac{\hat{P}_{k|k} + Q_k}{\hat{P}_{k|k} + Q_k + R_{k+1}}\right) \hat{P}_{k+1|k} \\ &= \frac{R_{k+1}}{\hat{P}_{k|k} + Q_k + R_{k+1}} (\hat{P}_{k|k} + Q_k)\end{aligned}\tag{13}$$

Taking the limit for $Q_k \rightarrow \infty$ results in $\hat{P}_{k+1|k+1} = R_{k+1}$. Again, $Q_k \rightarrow \infty$ indicates that there is zero confidence in the model, so only the measurements get used to adapt the state estimate. This way, the reliability of the estimator is purely based on the reliability of the measurement. In other words, the uncertainty on the state estimate evolves according to the measurement noise covariance. Whereas this uncertainty evolves depending on both the model and the measurements in normal operation of the filter.

Further, the limit for $R_{k+1} \rightarrow \infty$ prompts $\hat{P}_{k+1|k+1} = \hat{P}_{k|k} + Q_k = \hat{P}_{k+1|k}$. Once more, $R_{k+1} \rightarrow \infty$ indicates zero confidence in the measurements. The a priori covariance matrix of the estimation error is equal to the a posteriori covariance matrix, so the correction step is neglected. In other words, the reliability of the estimator is purely based on the reliability of the model. In normal operation of the filter, $\hat{P}_{k+1|k+1} \preceq \hat{P}_{k+1|k}$, meaning that the uncertainty of the state estimate decreases, whereas in this limit case, the uncertainty does not decrease after the correction step, but it stays the same.

Steady state covariance

Now,

References

- [1] Siang Lim. Control block diagram with tikz. <https://www.overleaf.com/latex/examples/control-block-diagram-with-tikz/jfdcfxhgjmtz>, 2017. Last accessed December 8th 2021.
- [2] Swevers J. Pipeleers G. *Control Theory - Handouts*. Cursusdienst VTK, 2021.