Orbitopes

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Joint work with Marc Pfetsch (ZIB)

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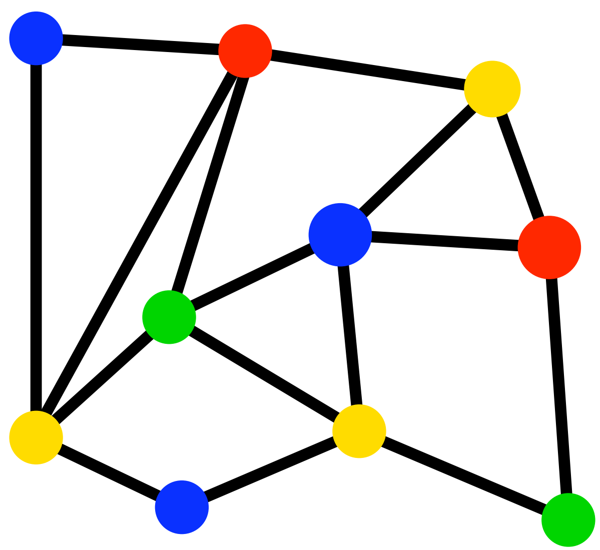
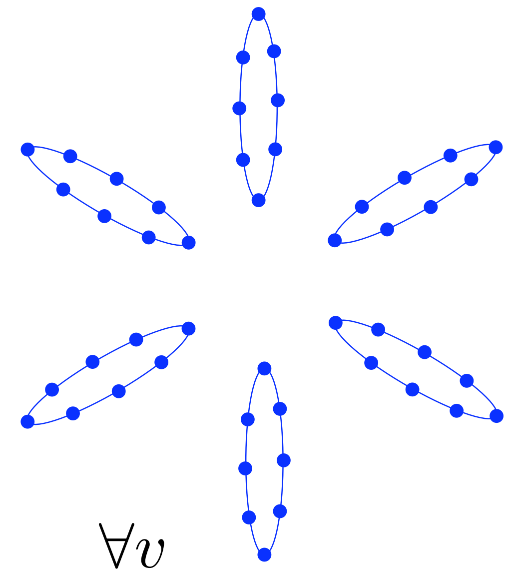
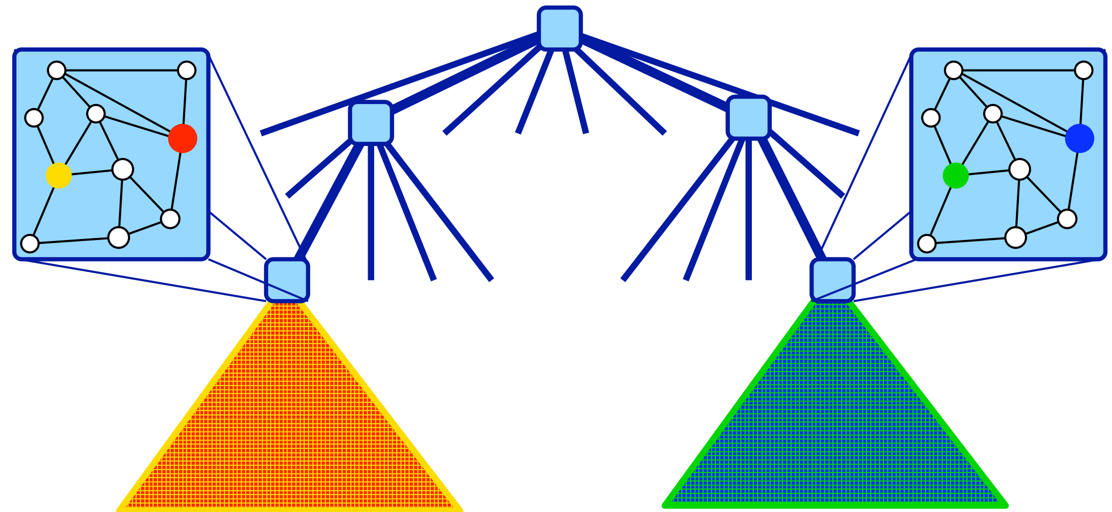


$$
\begin{array}{llcll}
\min & \displaystyle\sum_{c=1}^{C_{\max}} y_c & & & \\
\text{s.t.} & \displaystyle\sum_{c} x_{v,c} & = & 1 & \forall v \\
 & x_{v,c} + x_{w,c} & \leq & 1 & \forall c \; \forall \{v,w\} \in E \\
 & x_{v,c} & \leq & y_c & \forall v \; \forall c \\
 & x_{v,c} & \in & \{0,1\} & \forall v \; \forall c
\end{array}
$$

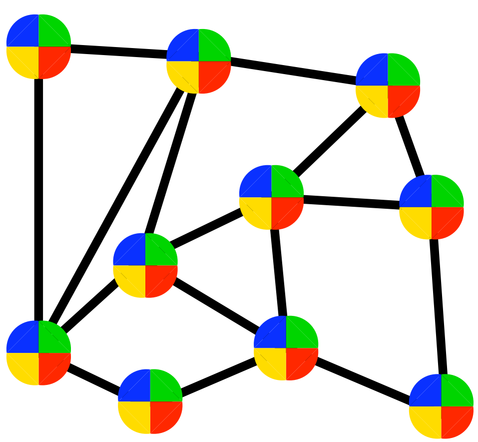


Symmetric group operates with constant objective function value on orbits.

Two problems



Unnecessarily large search space (Branch-and-cut)



Useless LP-bounds

$$
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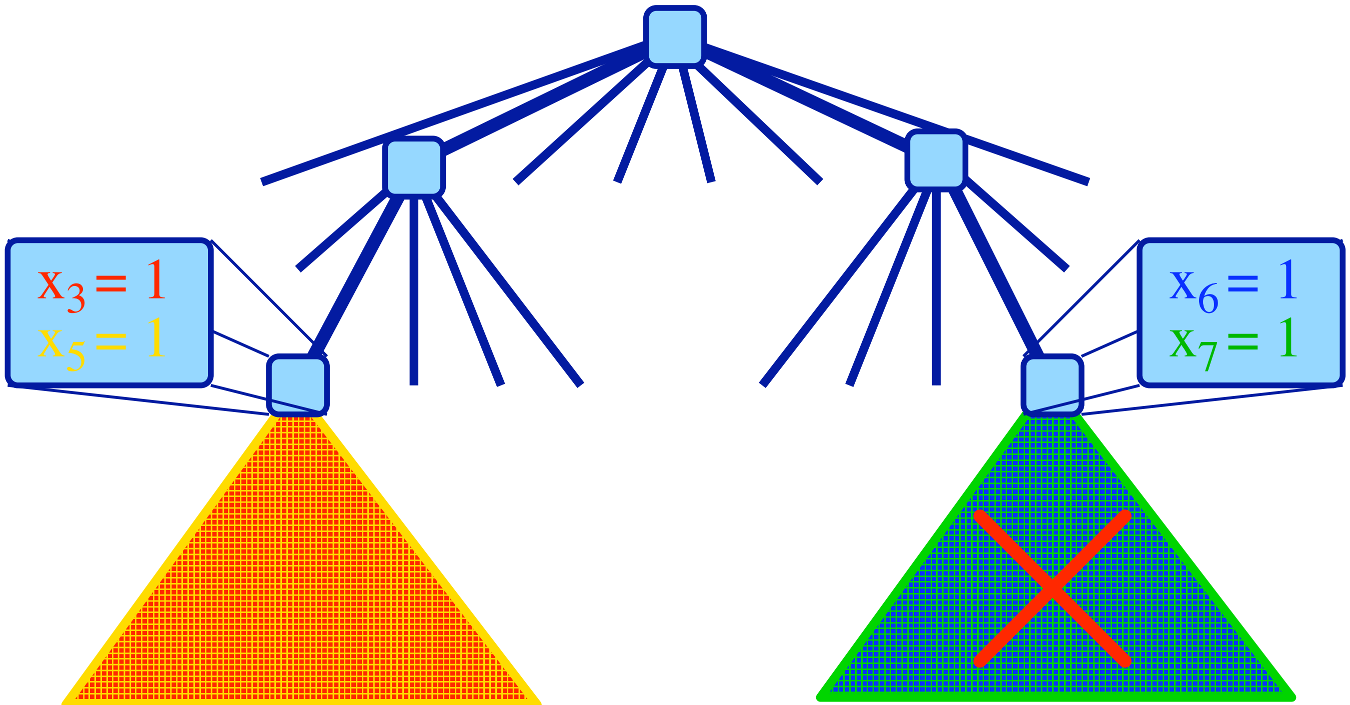
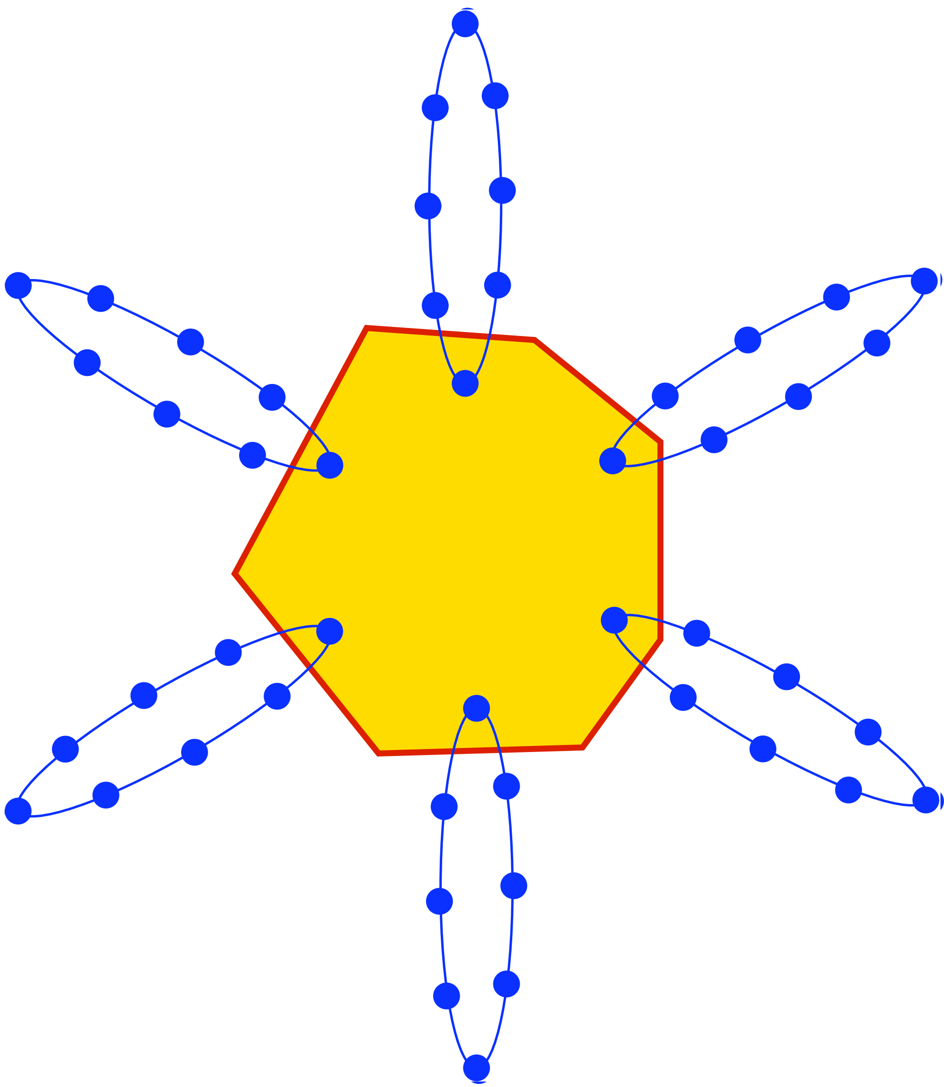
\end{array}
$$



- Criterion: {3,5} is lexicographically minimal in its orbit
- Decision uses a suitable generating system of the symmetry group (Schreier-Sims table)
- [Margot 2002+]



This talk:

An optimal way to cut off unnecessary parts of the orbits (for certain types of IPs).

1	0	0	0	0	0	0
0	1	0	0	0	0	0
1	0	0	0	0	0	0
0	1	0	0	0	0	0
0	0	1	0	0	0	0
1	0	0	0	0	0	0
0	1	0	0	0	0	0
0	0	0	1	0	0	0
0	0	0	0	1	0	0
0	0	1	0	0	0	0

Orbitope $\mathrm{O}(n,r)$:

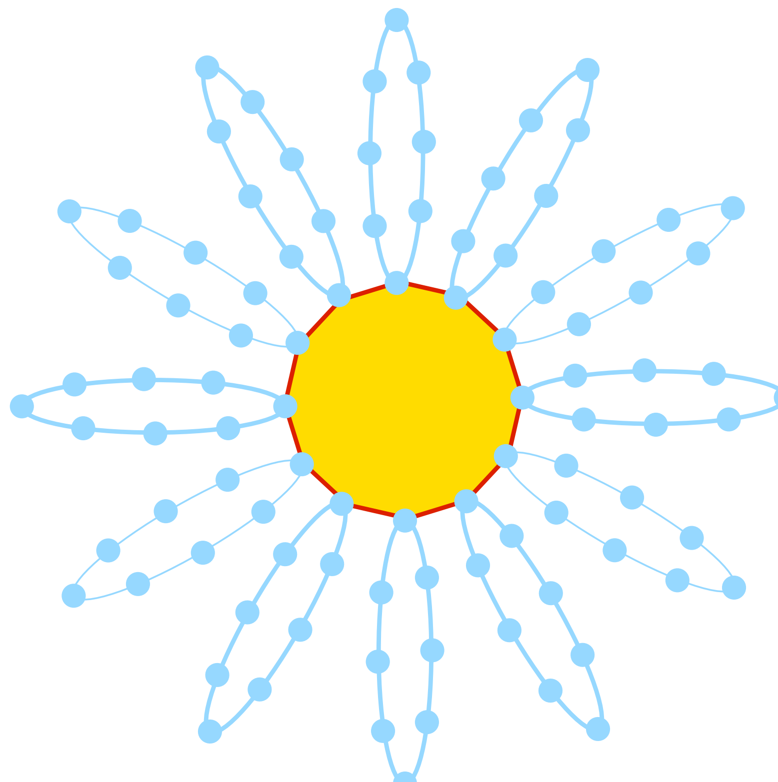
Framework: Assignment variables

Convex hull of all $n \times r$ 0/1-matrices with

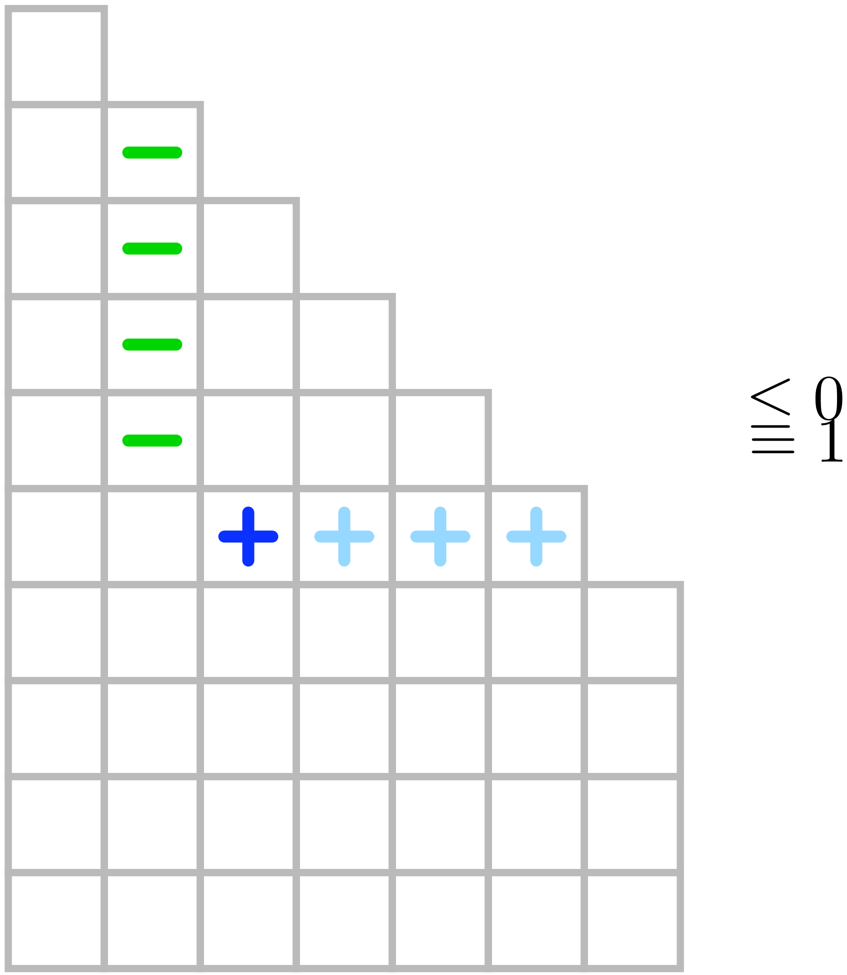
$x_{i,j} \in \{0,1\}$

(a) Exactly one 1 per row $\sum_j x_{i,j} = 1$

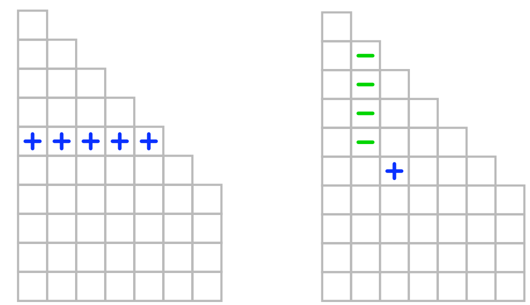
(b) Lex-decreasing-sorted columns



IP-formulation



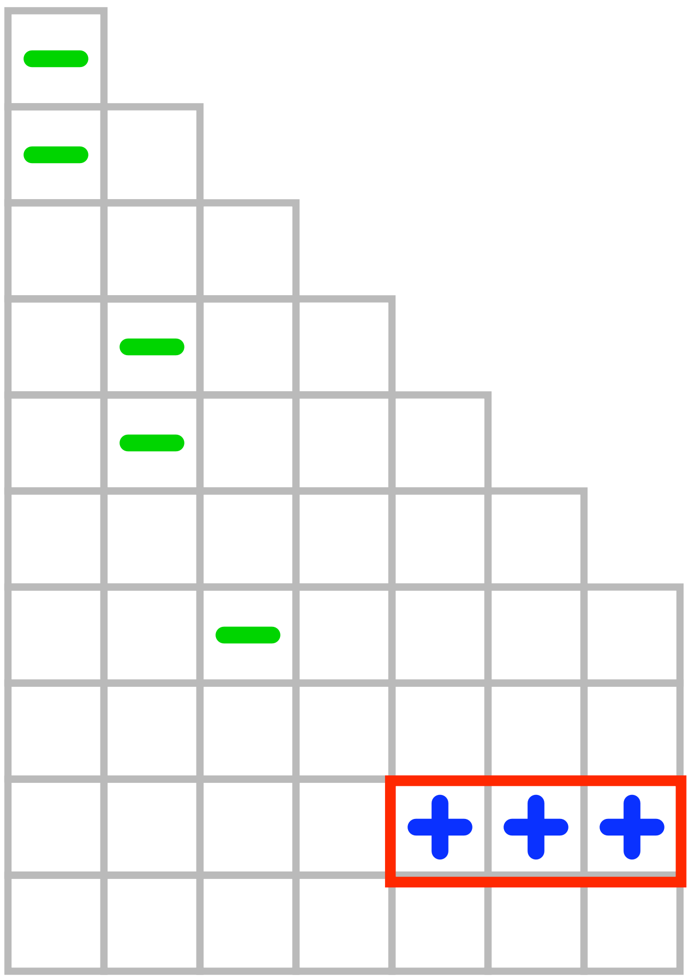
- The inequalities

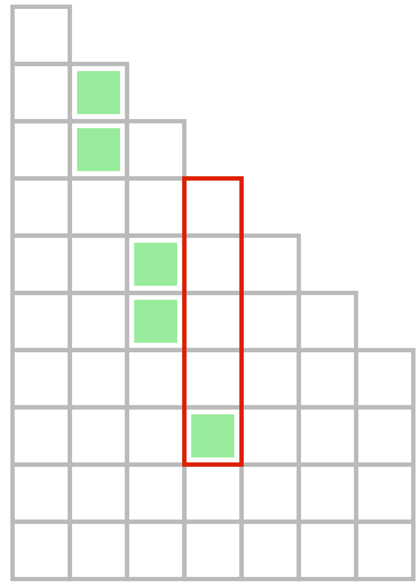


describe the 0/1-points in the orbitope (cf. Diaz et al. 01).

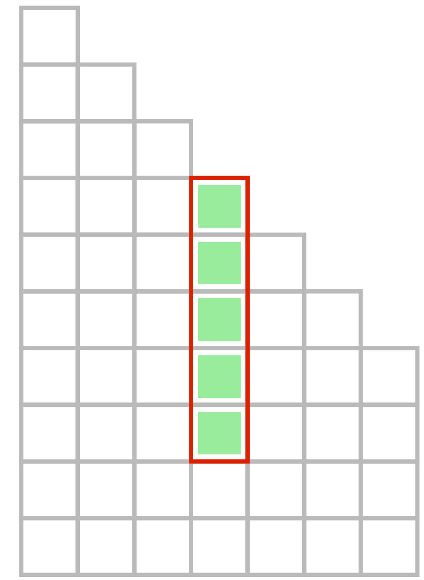
- But: They do not provide a complete description.

Shifted-column inequalities





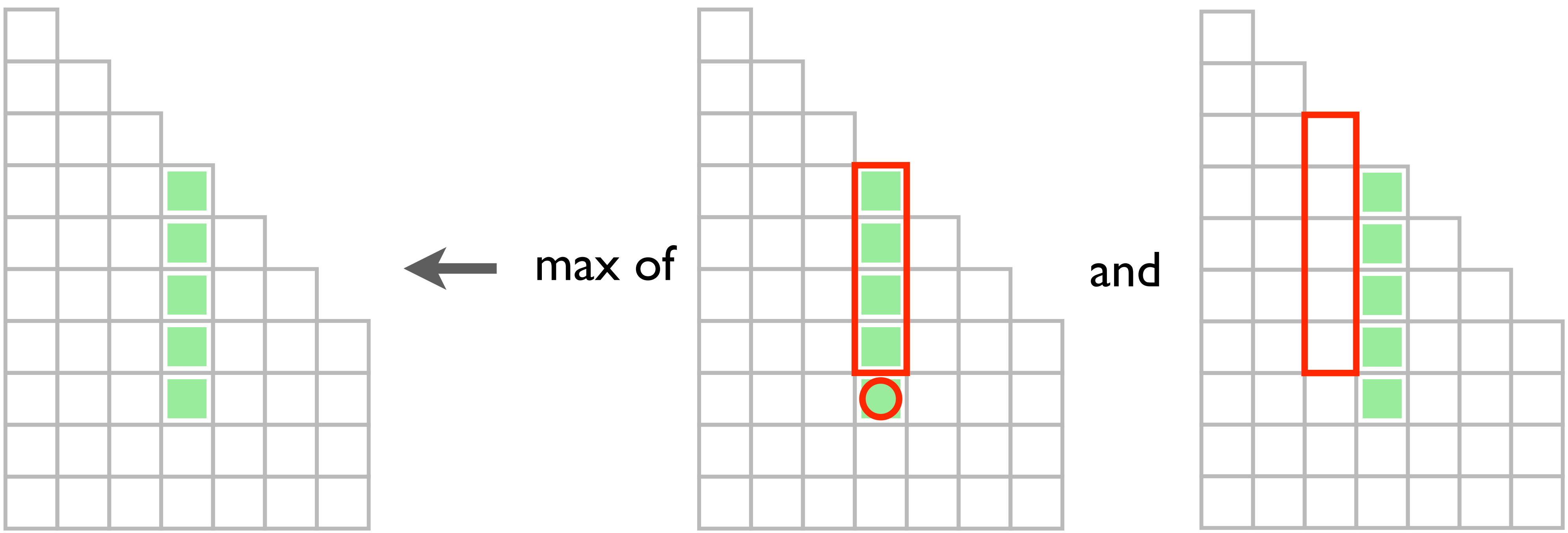
shifting of



(exponentially many)
shifted columns inequalities (SCI)

"Basis"

Crucial step: For a point $X^\star$ we can compute in $\mathrm{O}(nr)$ steps $X^\star$-maximum shiftings of all columns.



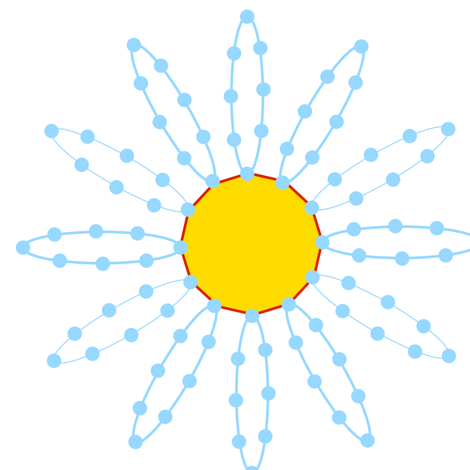
The separation problem for SCIs can be solved in linear time.

Ideal description of orbitopes

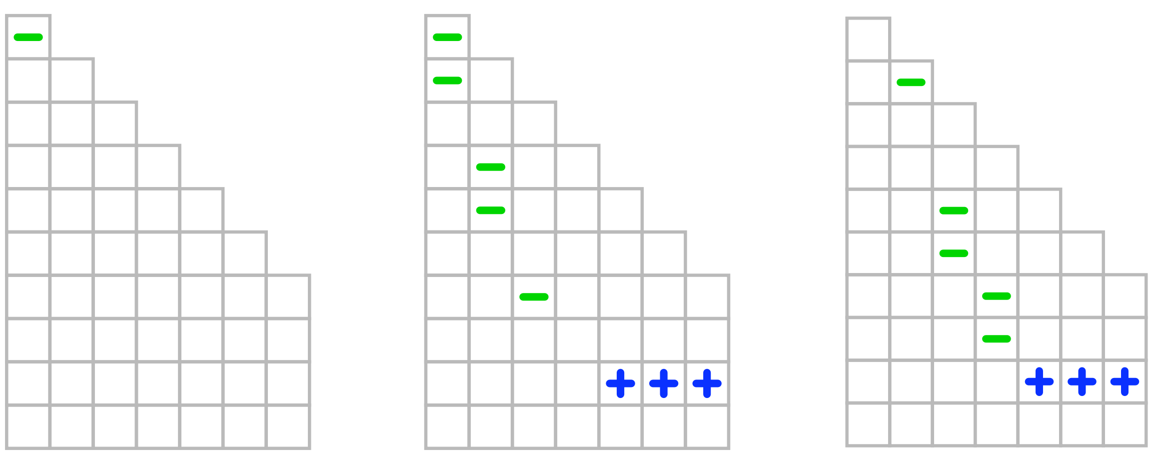
Theorem (K & Pfetsch, 2005)

The orbitope $\mathrm{O}(n,r)$ equals the polytope $Q(n,r)$ that is described by the following constraints:

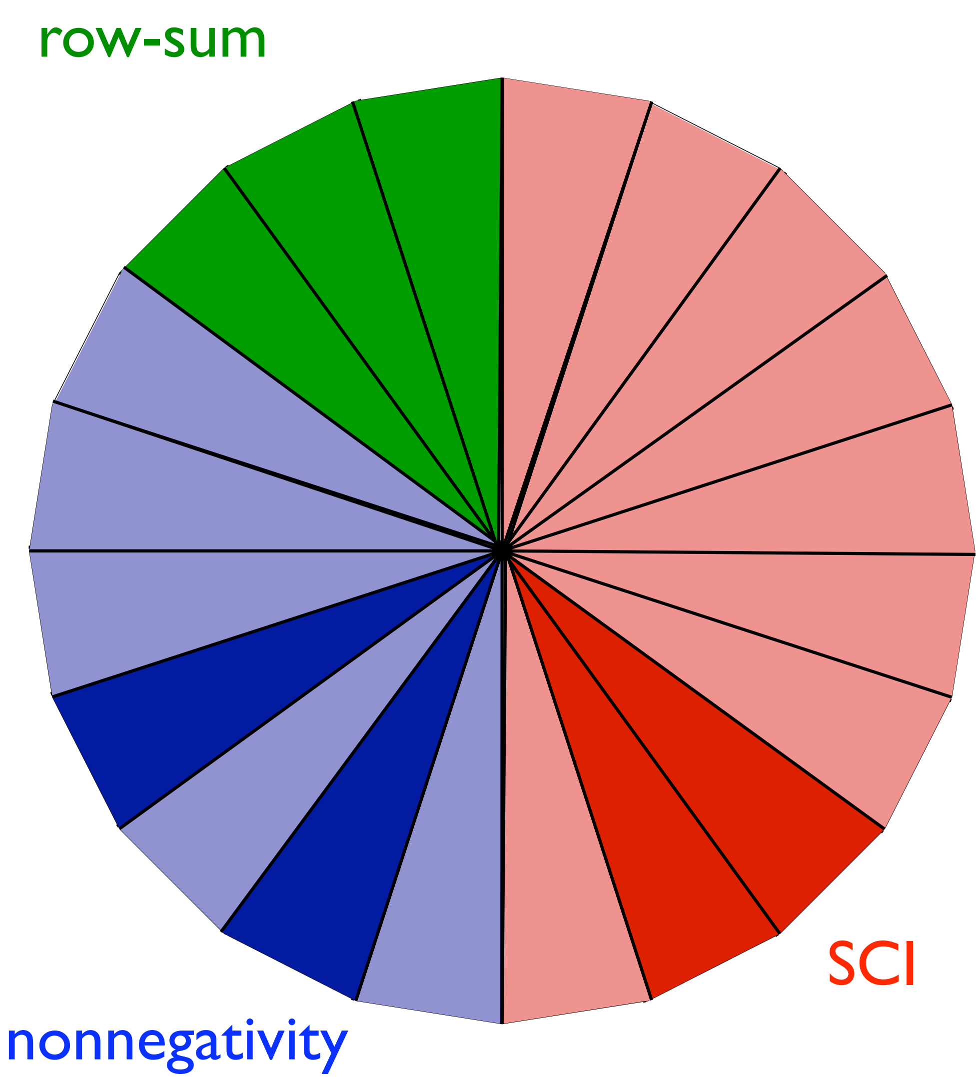
(i) nonnegativity constraints
(ii) row-sum equations
(iii) shifted-column inequalities



The description is irredundant up to:



Proof strategy



Let $X^\star$ be a vertex of $Q(n,r)$.

Let $\mathcal{B}^\star$ be a basis of $X^\star$ with all row-sum and as many as possible nonnegativity constraints.

If no SCI: $X^\star$ is a 0/1-vector

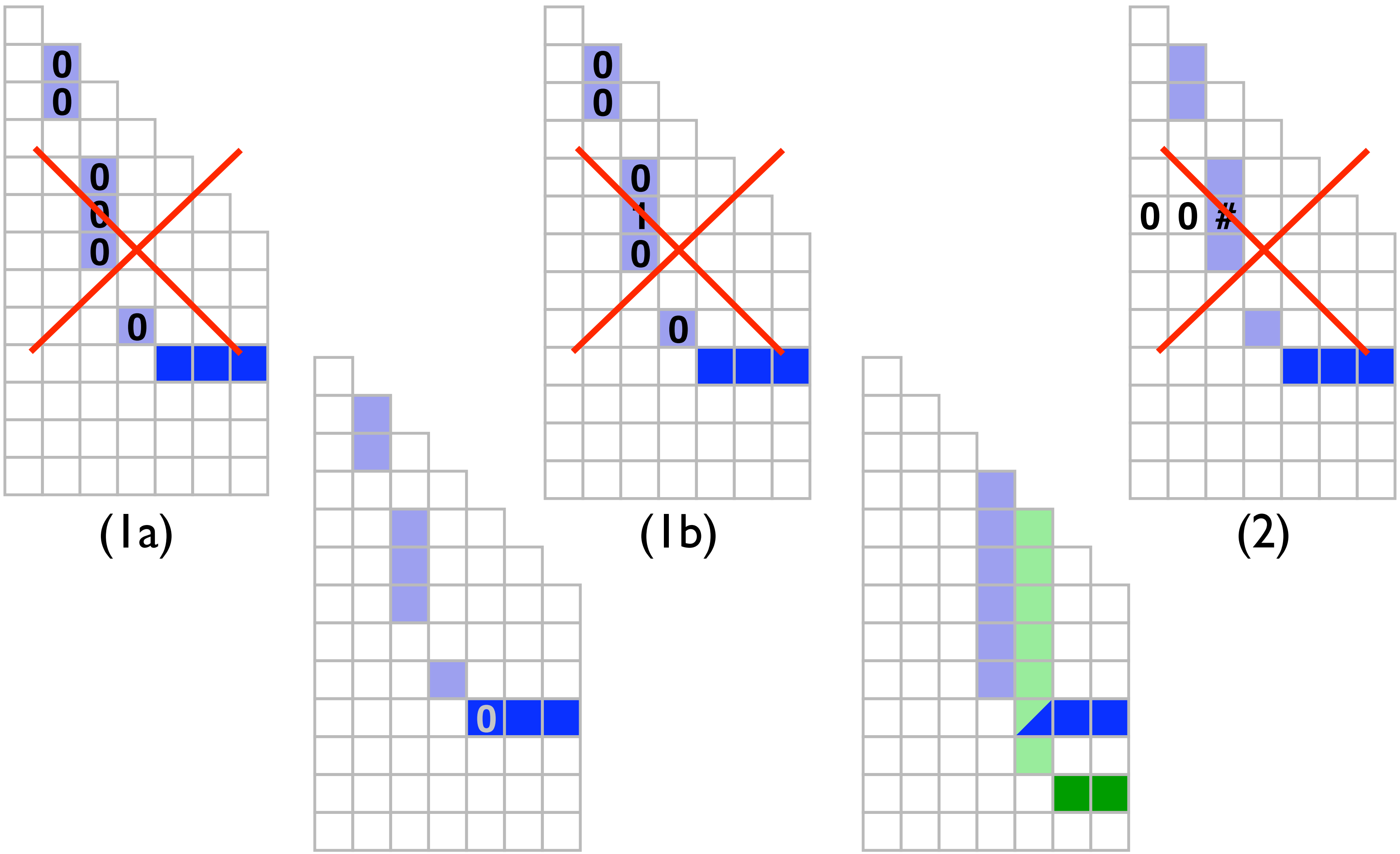
Otherwise:

Modify $\mathcal{B}^\star$ and show that there is another solution.

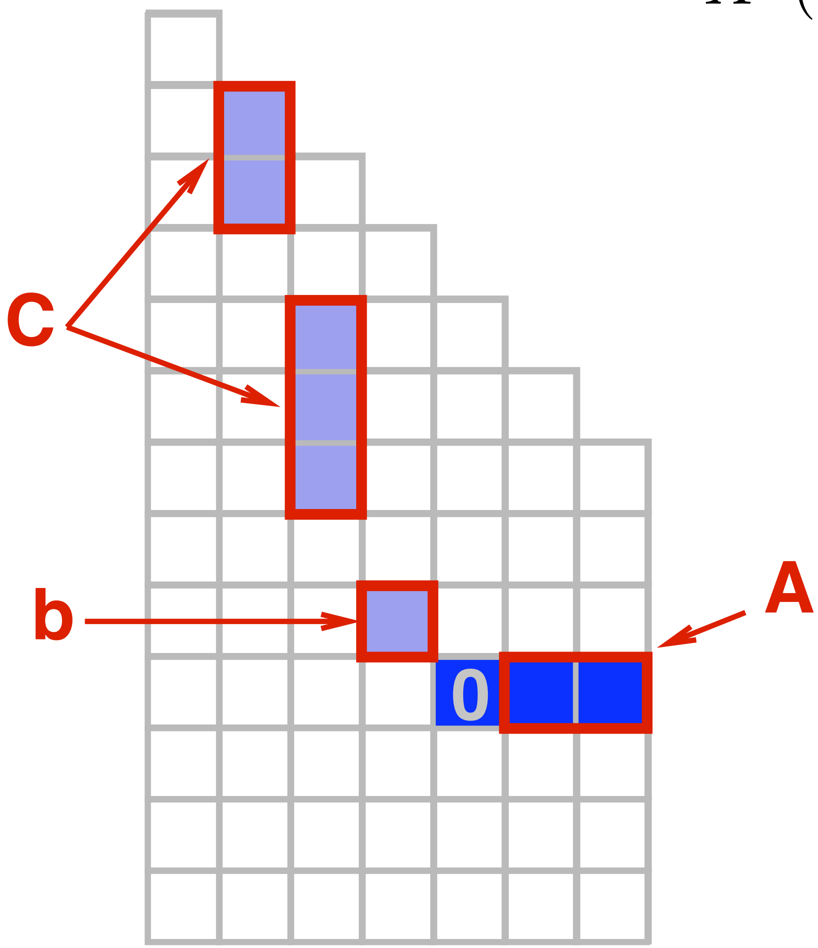
Contradiction

Modification of the basis

Configurations to exclude:



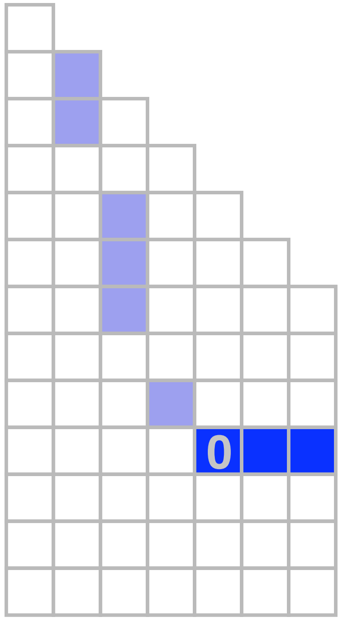
(1a) (1b) (2) (3) (4)



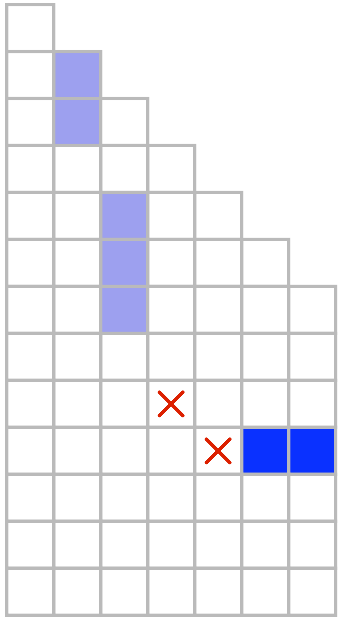
$$X^\star(A) \leq X^\star(C) \leq X^\star(C) + X^\star(b) = X^\star(A)$$

Thus: $X^\star(b) \;=\; 0$ and $X^\star(A) = X^\star(C)$

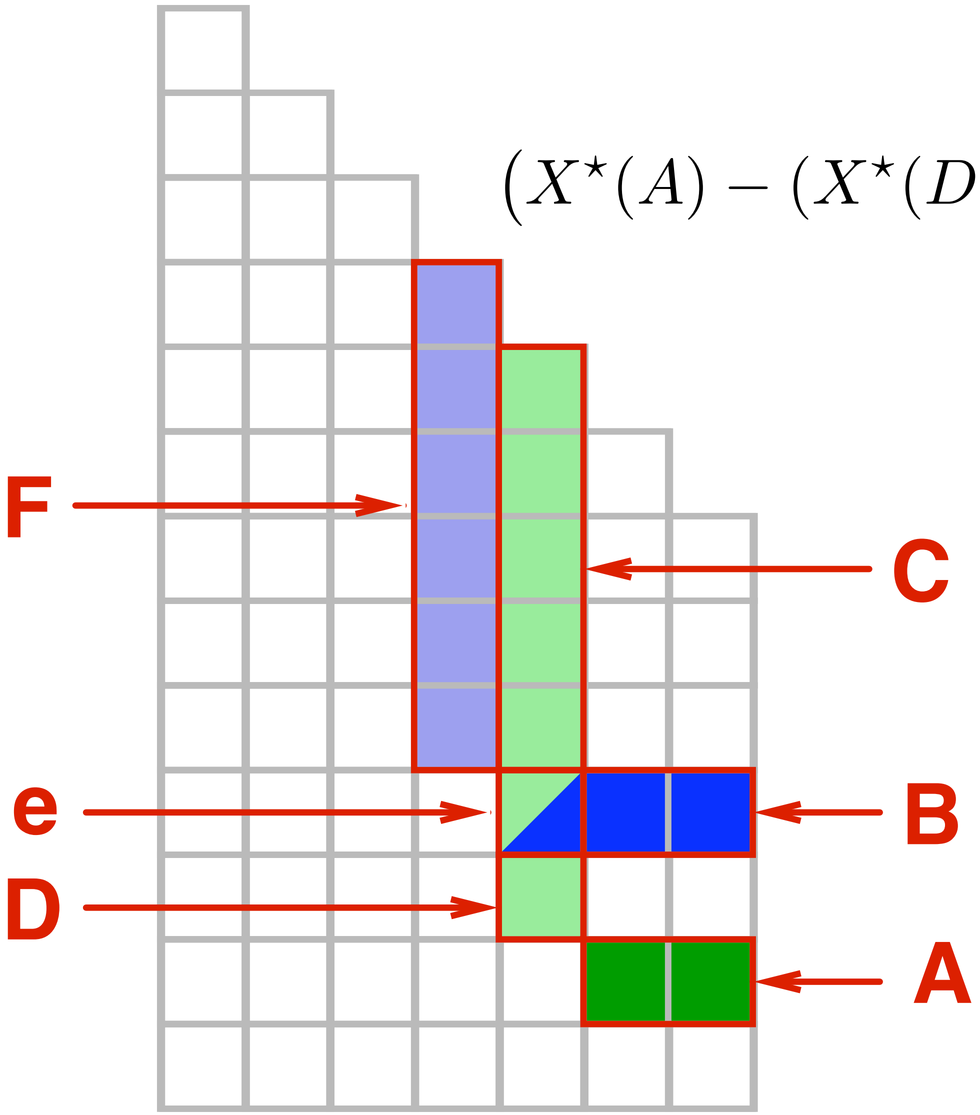
Replace:



by



$$\begin{aligned} X^\star(A) - (X^\star(D) + X^\star(e) + X^\star C) &= 0 \\ X^\star(B) + X^\star(e) - X^\star(F) &= 0 \\ \big(X^\star(A) - (X^\star(D) + X^\star(F))\big) + \big(X^\star(B) - (X^\star(C))\big) &= 0 \end{aligned}$$

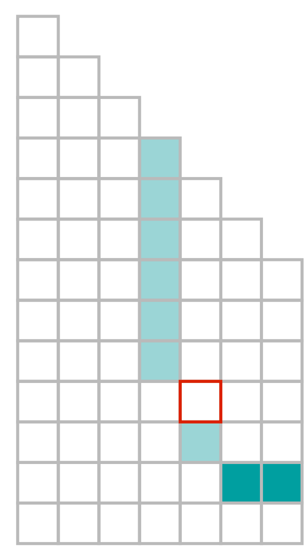


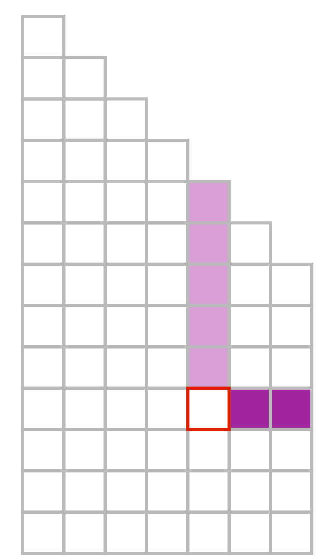
Thus (since $X^\star$ satisfies all SCIs):

$$X^\star(A) - (X^\star(D) + X^\star(F)) = 0$$

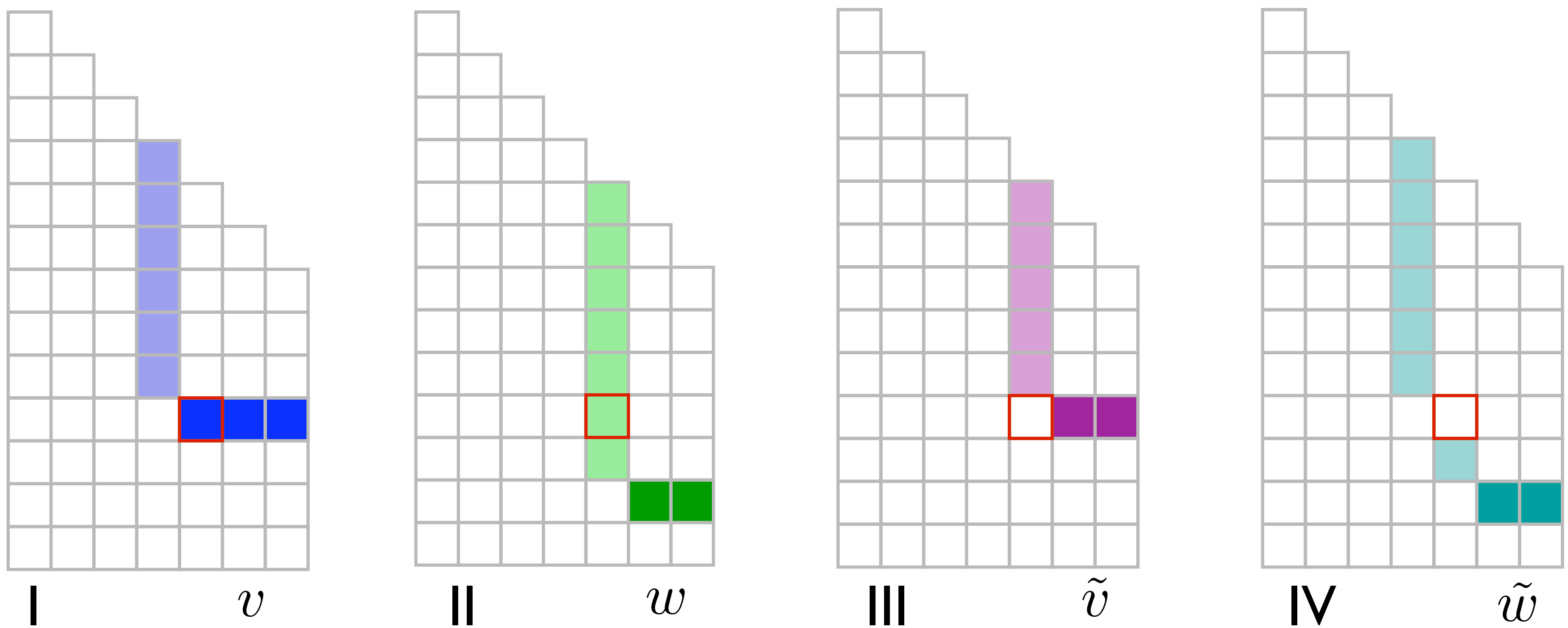
and

$$X^\star(B) - X^\star(C) = 0$$





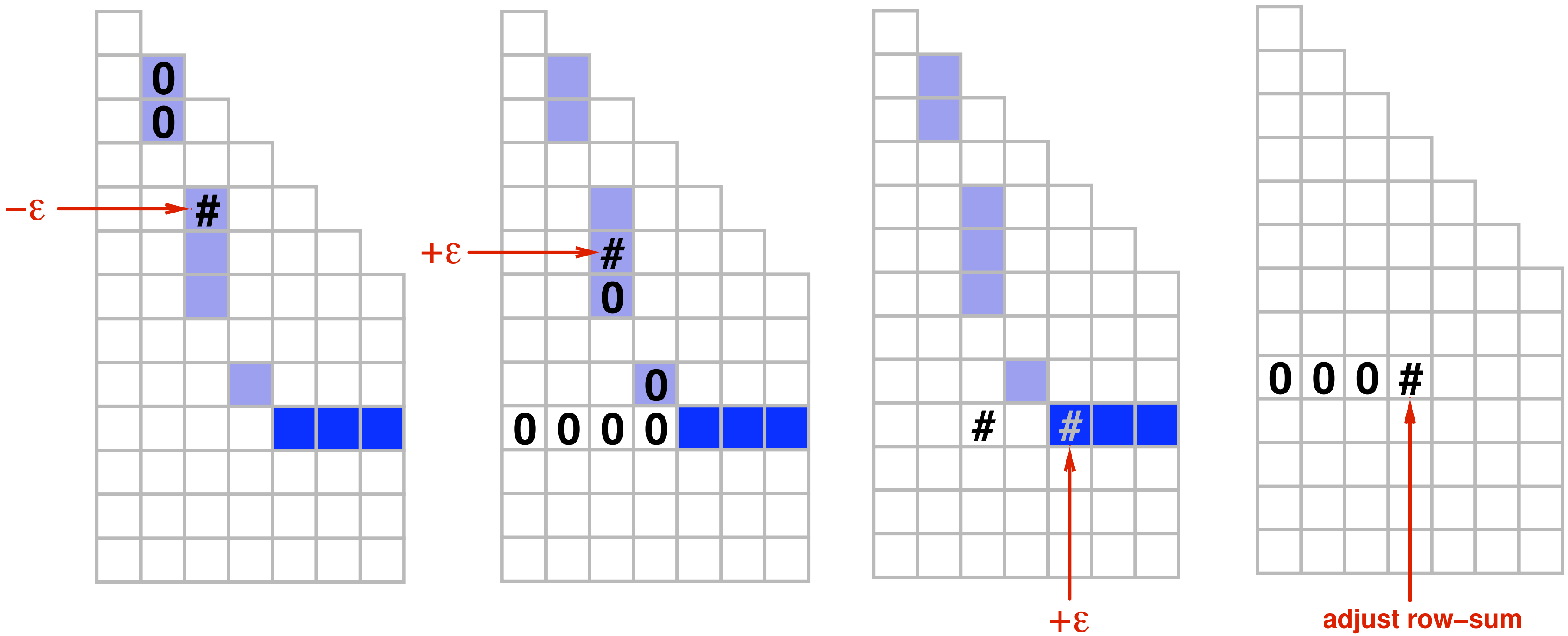
Ensuring (4) ctd.



Lemma: If $\operatorname{lin}(\mathcal{L} \cup \{v, w\}) = \mathbb{R}^q$ and $v + w = \tilde{v} + \tilde{w}$ (thus $v = \tilde{v} + \tilde{w} - w$), then there are $u_1, u_2 \in \{w, \tilde{v}, \tilde{w}\}$ such that $\operatorname{lin}(\mathcal{L} \cup \{u_1, u_2\}) = \mathbb{R}^q$ holds.

Thus there are two inequalities among II, III, and IV, by which we can replace I and II.

1. Modify $X^\star$ to $\tilde{X}$:



2. Using (1a), (1b), (2), (3), (4): $\tilde{X}$ satisfies the equation system corresponding to $\tilde{\mathcal{B}}$.

3. BUT: If any modification in 1. was done (i.e., $\tilde{\mathcal{B}}$ contains some SCI), then $\tilde{X} \neq X^\star$: contradiction.

Remarks

- Full-, packing, and covering-orbitopes
- Other group operations (e.g., cyclic groups)
- Interplay of orbitopes with special polyhedra
- “Academic applications” (Coloring, k-partitioning, etc.)
- Real-world IP-models
- Combination with Margot’s method

Thank you for your attention.