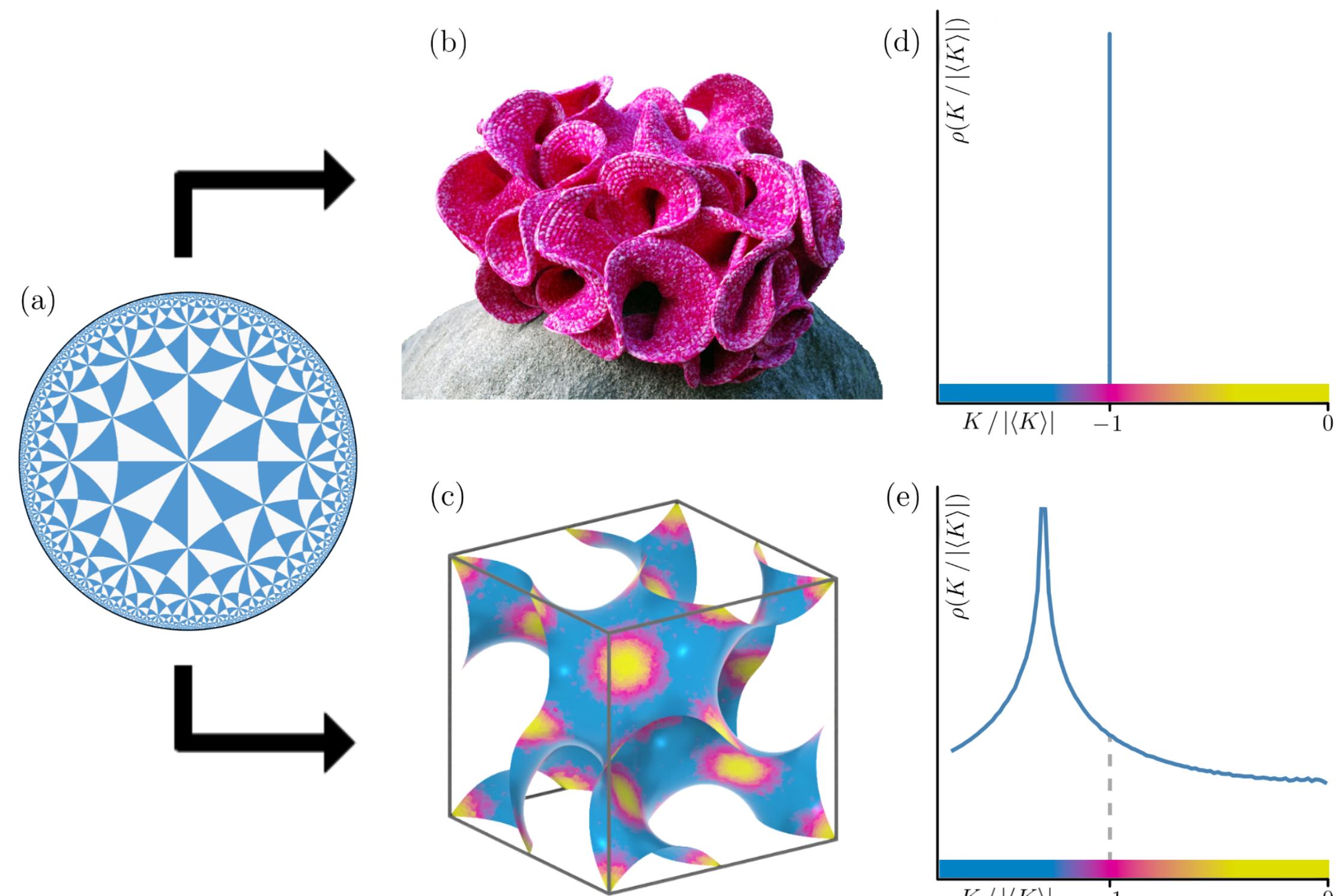


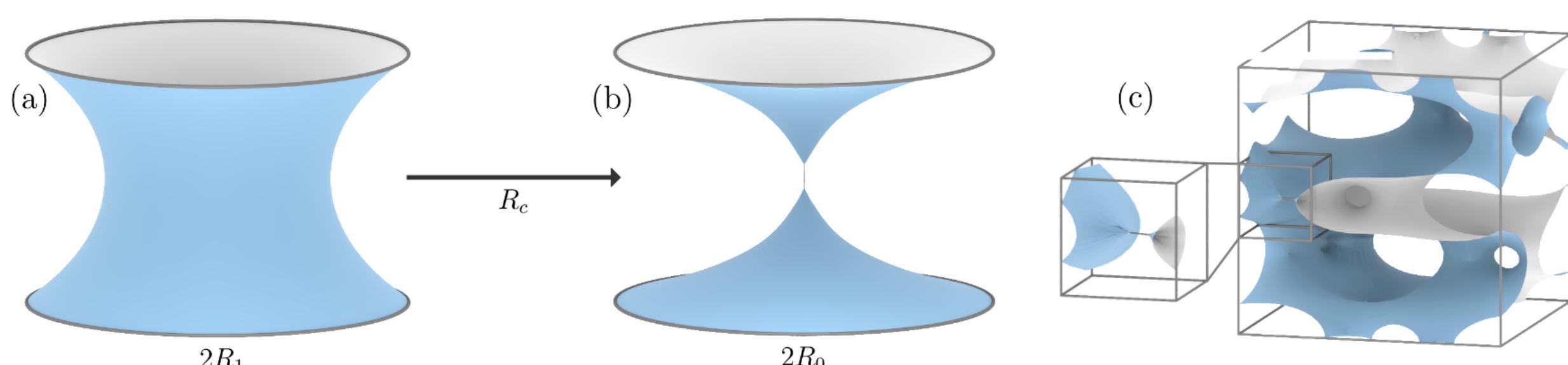
Exploring the Homogeneity of Disordered Minimal Surfaces[†]

1. Hilbert's Embedding Theorem



Hilbert's Embedding Theorem states that no complete regular isometric immersion of the hyperbolic plane \mathbb{H}^2 (a) into \mathbb{R}^3 exists. Daina Taimina's crocheted pseudosphere¹ (b) and the Gyroid (c) weaken this theorem's assumption by containing geometric singularities or Gaussian curvature fluctuations.

4. Topology-Stabilized Curvature Optimization



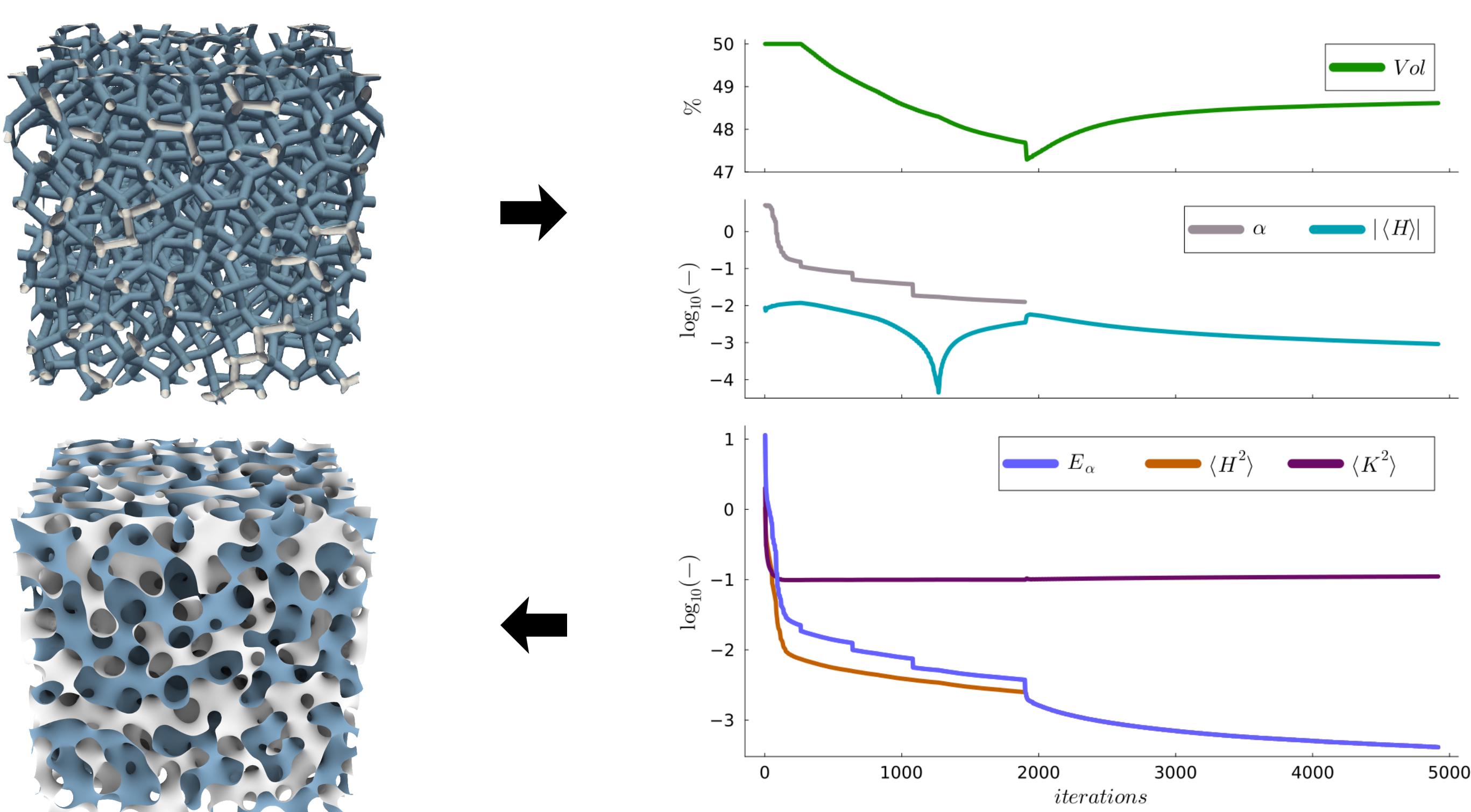
There exists a critical wireframe radius R_c below which the catenoid "pinches off" (a-b), wanting to attain the Goldschmidt solution. We observed this behavior in our experiments as well (c).

To prevent channel collapse and to avoid local minima, we propose a new objective function for use in the Surface Evolver⁶:

$$E_\alpha(\mathcal{M}) = \int_{\mathcal{M}} H^2 dA + \alpha \cdot \int_{\mathcal{M}} K^2 dA$$

with $\alpha = \min \left\{ \alpha', w \cdot \alpha' \cdot \frac{\int_{\mathcal{M}} H^2 dA}{\int_{\mathcal{M}} K^2 dA} \right\}$

for the previous elastic modulus α' and damping factor $w > 0$.



2. Homogeneity Measures for Surfaces

For a surface \mathcal{M} , denote the average Gaussian curvature by $\langle K \rangle$ and the Gaussian curvature variance by $\sigma = \sqrt{\langle K^2 \rangle - \langle K \rangle^2}$. Then, the *fluctuation of Gaussian curvature*² is given by

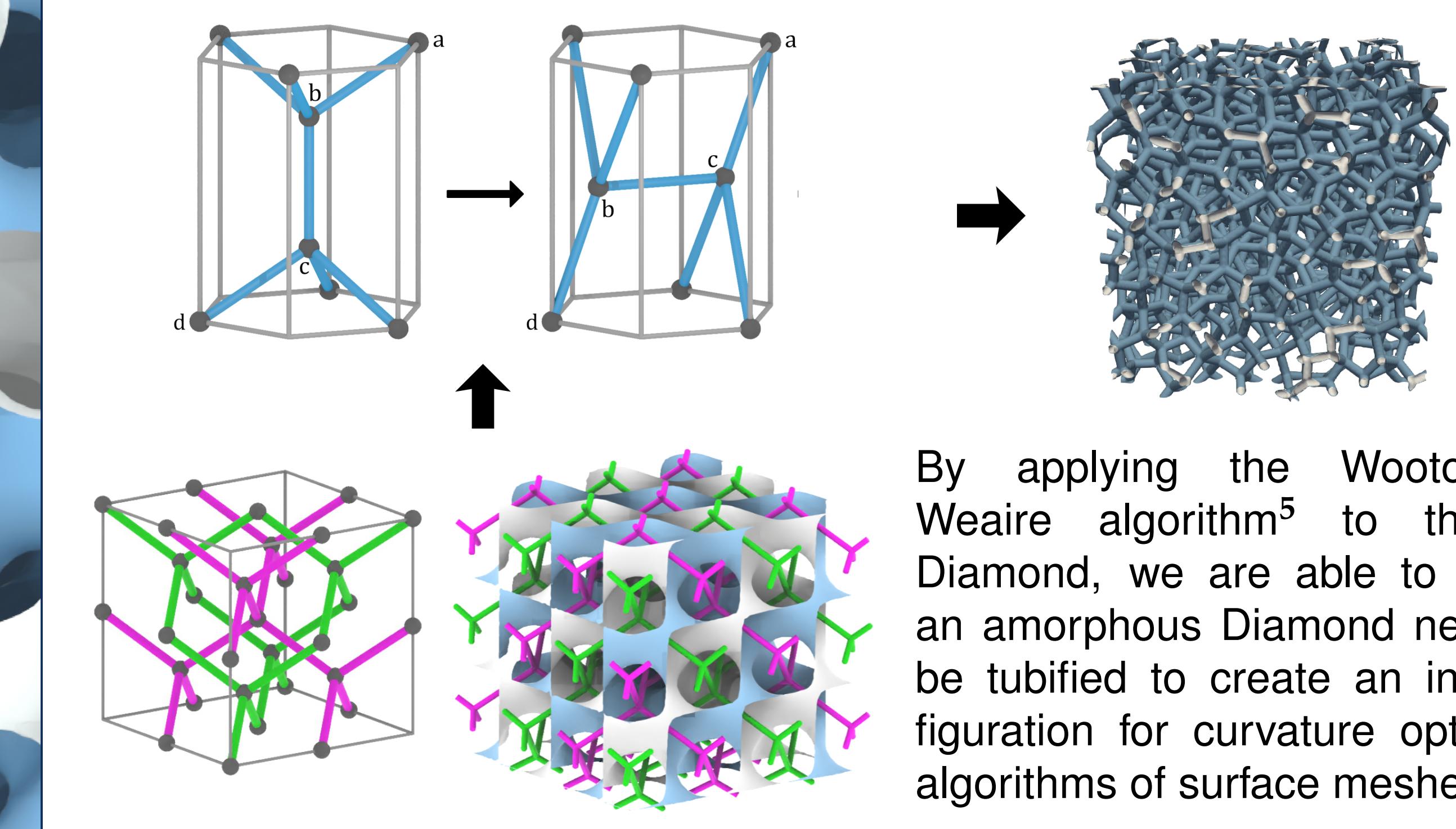
$$\frac{\sigma^2}{\langle K \rangle^2} = \frac{\langle K^2 \rangle - \langle K \rangle^2}{\langle K \rangle^2} = A(\mathcal{M}) \cdot \frac{\int_{\mathcal{M}} K^2 dA}{(\int_{\mathcal{M}} K dA)^2} - 1.$$

Conversely, we define the *isotropy index*³ of the surface, measuring the variations of the unit normal field \mathbf{n} , as

$$\beta_1^{0,2}(\mathcal{M}) = \left| \frac{\lambda_3(W_1^{0,2}(\mathcal{M}))}{\lambda_1(W_1^{0,2}(\mathcal{M}))} \right| \text{ for } W_1^{0,2}(\mathcal{M}) = \frac{1}{3} \int_{\mathcal{M}} \mathbf{n} \otimes \mathbf{n} dA$$

with the largest and smallest eigenvalues λ_1 and λ_3 , respectively. Both quantities are dimensionless⁴.

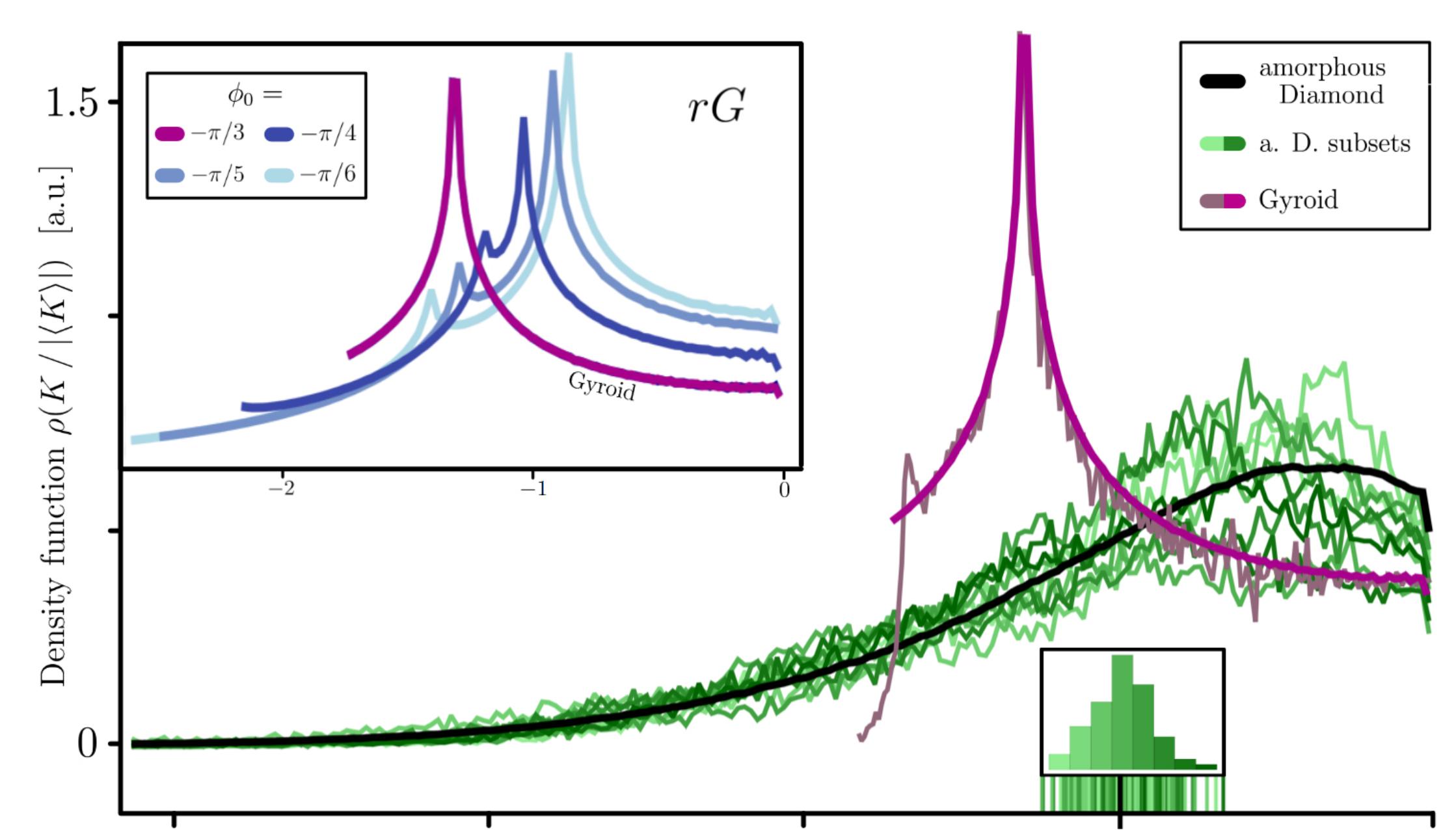
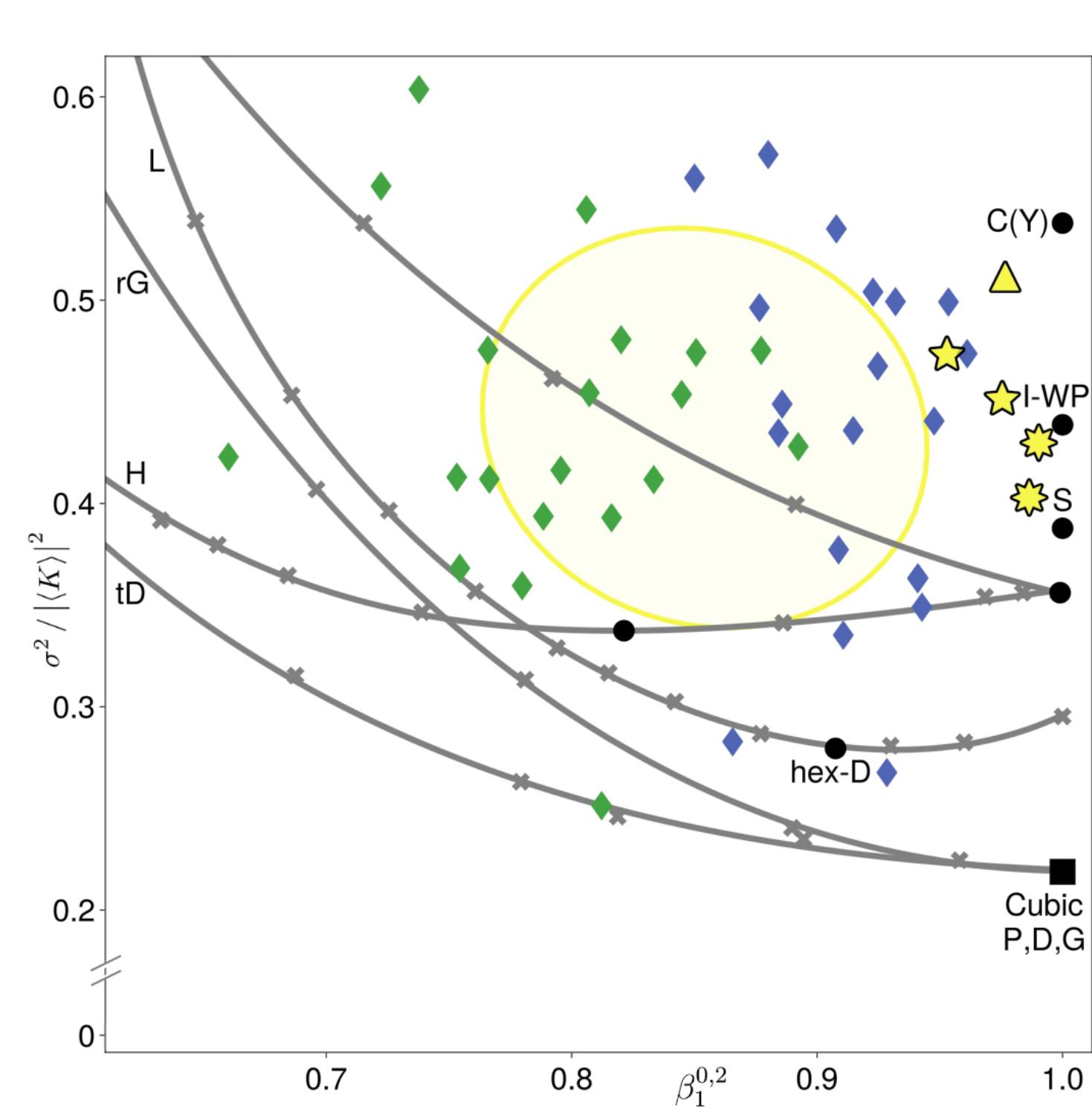
3. Amorphous Diamond Minimal Surface



By applying the Wootton-Winer-Weaire algorithm⁵ to the cubic Diamond, we are able to generate an amorphous Diamond net. It can be tubified to create an initial configuration for curvature optimization algorithms of surface meshes.

5. Superior Uniformity of the Gyroid

We sample 5 amorphous Diamond surfaces in the sizes $\chi = -432$ (triangle), $\chi = -2000$ (5-star) and $\chi = -8192$ (8-star), normalizing them to $\langle K \rangle = -1$. From these surfaces, we cut out random cubical subsamples with Euler characteristic $\chi = -16$ (green) and $\chi = 2^3 \cdot (-16)$ (blue). None of the amorphous minimal surfaces come close in curvature uniformity to the cubic Gyroid. In fact, we find that the Gyroid is particularly good at associating a minimal amount of its surface area to flat points and high-curvature regions.



[†]This project is joint work with M.C. Pedersen, M.E. Evans, M.A. Klatt, P.W.A. Schönhofer, G.E. Schröder-Turk.

^A Image from: A.-L. Jessop et al. In: Journal of the Royal Society, Interface **21** (2024).

^B Image from: H. Yin et al. In: PNAS **109**(27) (2012).

¹ Henderson and Taimina: Crocheting the Hyperbolic Plane. The Mathematical Intelligencer **23** (2009).

² Schröder-Turk, Fogden and Hyde: Bicontinuous Geometries and Mol. Self-Assembly. Euro. Phys. J. B **54** (2006).

³ Schröder-Turk et al.: Minkowski Tensors of Anisotropic Spatial Structure. New J. Phys. **15**.8 (2013).

⁴ Himmelmann: Optimization in Geometric Materials. PhD Thesis, University of Potsdam (2024).

⁵ Wootton, Winer and Weaire: Computer Gen. of Struct. Models of Amorphous Si and Ge. Phys. Rev. Lett. **54** (1985).

⁶ Barkema and Mousseau: High-Quality Continuous Random Networks. Physical Review B **62** (1999).

⁷ Brakke: The Surface Evolver. Experimental Mathematics **1**.2 (1992).