A Dirichlet Model for Interval Responses

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1 - Introduction

„I like being around other people"

Why Use Interval Responses?

Motivating Example:

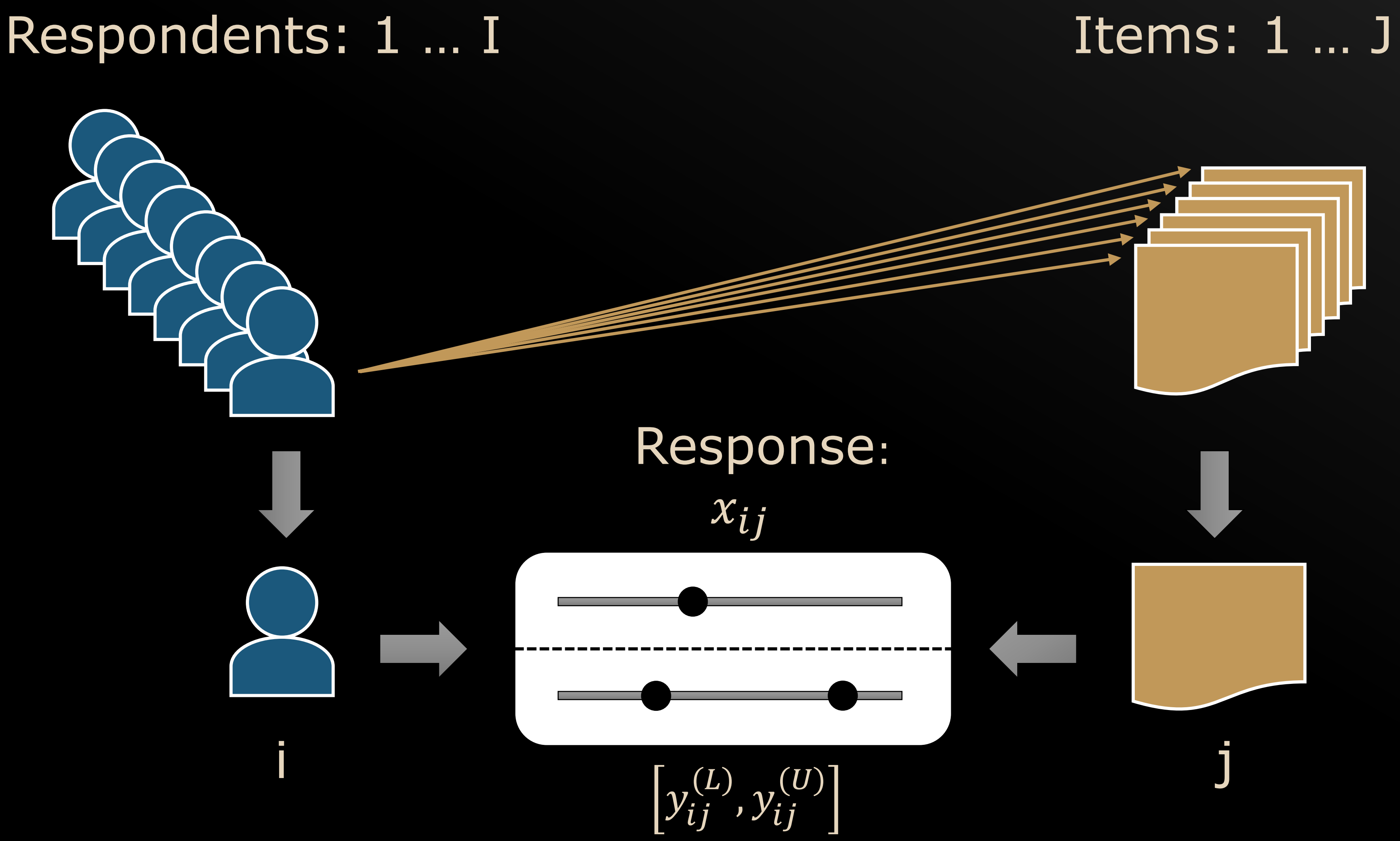
- Whole Trait Theory (Fleeson, 2001)
  - Trait: Distribution of states

➢ Accounting for variability

➢ Range of valid values

2 – IRT Models

Testing Scenario

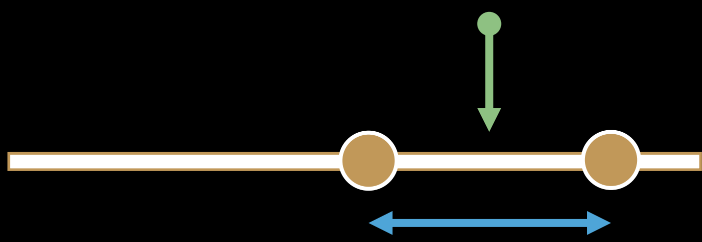


Interval Response

<u>Manifest Response:</u>

Interval Location (Midpoint):

$$\frac{y_{ij}^{(L)} + y_{ij}^{(U)}}{2}$$



Interval Width: $y_{ij}^{(U)} - y_{ij}^{(L)}$

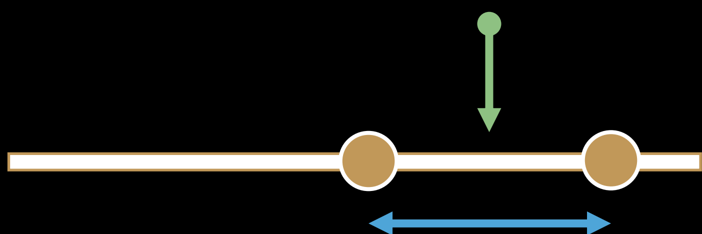
Interval Response

Manifest Response:

Latent Space:

Interval Location (Midpoint): $\frac{y_{ij}^{(L)} + y_{ij}^{(U)}}{2}$

Location Dimension

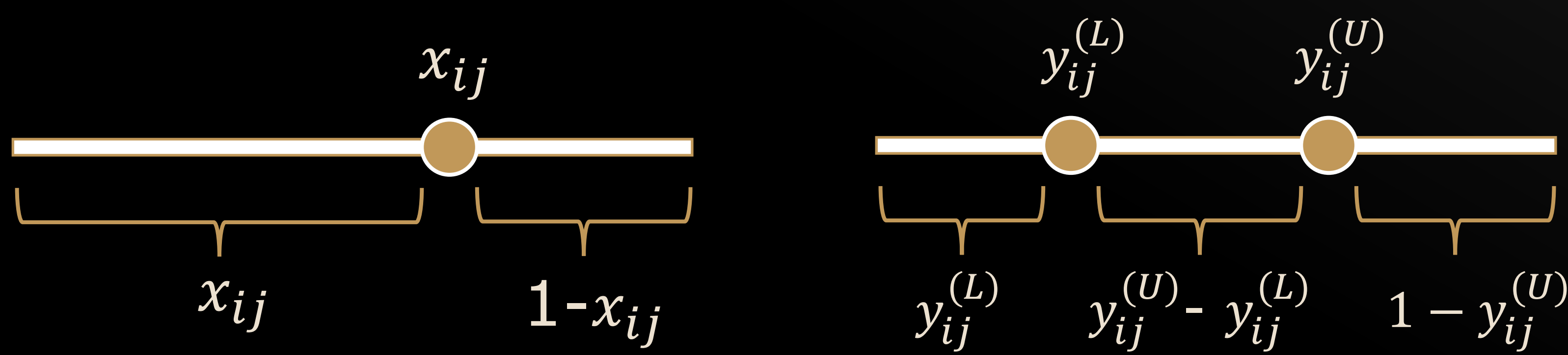


Interval Width: $y_{ij}^{(U)} - y_{ij}^{(L)}$

Expansion Dimension

Compositional Data

- Components must sum to one: simplex



Restrictions

No support for zero-components

- Single Response:

$$0 < x_{ij} < 1$$

- Interval Response:

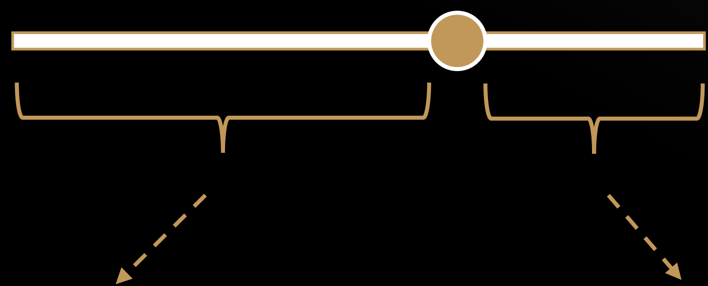
$$0 < y_{ij}^{(L)} < y_{ij}^{(U)} < 1$$

Beta Response Model (BRM)

Noel & Dauvier (2007)

$x_{ij} \sim \mathrm{Beta}\left(a_{ij}, d_{ij}\right);$

$$E\left(x_{ij}\right) = \frac{a_{ij}}{a_{ij}+d_{ij}}$$



$$a_{ij} = \exp\left[\alpha\left(\theta_i - \delta_j\right) + \tau_j\right] \qquad d_{ij} = \exp\left[-\alpha\left(\theta_i - \delta_j\right) + \tau_j\right]$$

Beta Response Model (BRM)

Parameters: Ability / Difficulty

$$a_{ij} = \exp\left[\alpha\underbrace{\left(\theta_i - \delta_j\right)} + \tau_j\right] \qquad d_{ij} = \exp\left[-\alpha\underbrace{\left(\theta_i - \delta_j\right)} + \tau_j\right]$$

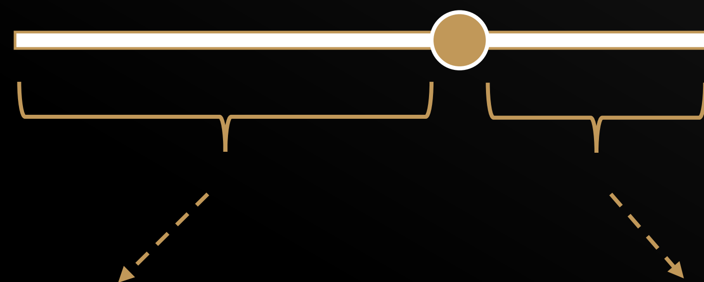
θ_i: Person ability

δ_j: Item difficulty

➢ Classic interpretation

Beta Response Model (BRM)

Parameters: Scaling



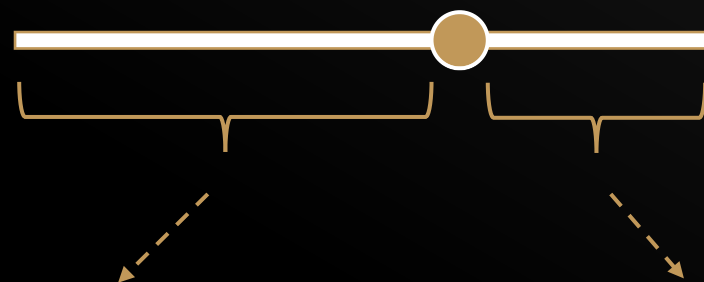
$$a_{ij} = \exp\left[\alpha\left(\theta_i - \delta_j\right) + \tau_j\right] \qquad d_{ij} = \exp\left[-\alpha\left(\theta_i - \delta_j\right) + \tau_j\right]$$

$\pm\alpha > 0$: Scaling

- Continuous model:
  Not a discrimination parameter!!

BETA RESPONSE MODEL (BRM)

Parameters: Precision

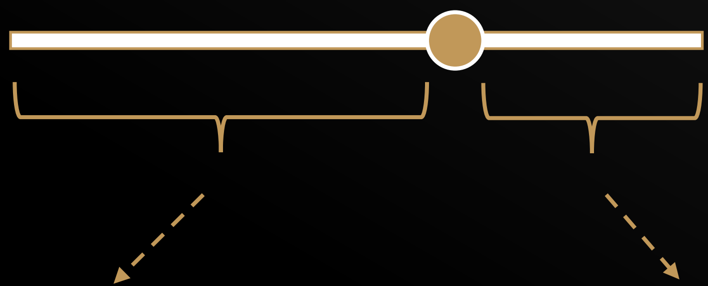


$$a_{ij} = \exp\left[\alpha\left(\theta_i - \delta_j\right) + \tau_j\right] \qquad d_{ij} = \exp\left[-\alpha\left(\theta_i - \delta_j\right) + \tau_j\right]$$

$\tau_j > 0$: Item precision (both additive!)

➢ Steeper density curves

Beta Response Model (BRM)
Parameters: Exponential Link



$$a_{ij} = \exp\left[\alpha(\theta_i - \delta_j) + \tau_j\right] \qquad d_{ij} = \exp\left[-\alpha(\theta_i - \delta_j) + \tau_j\right]$$

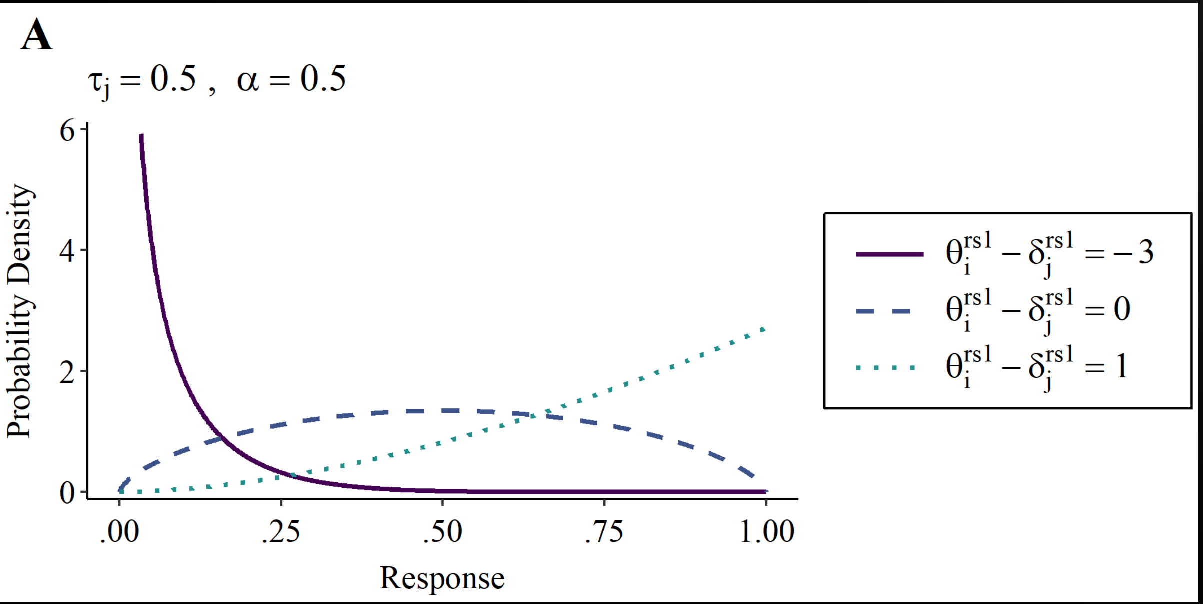
Example:

- $\theta_i - \delta_j = 0; \quad \tau_j = 0$

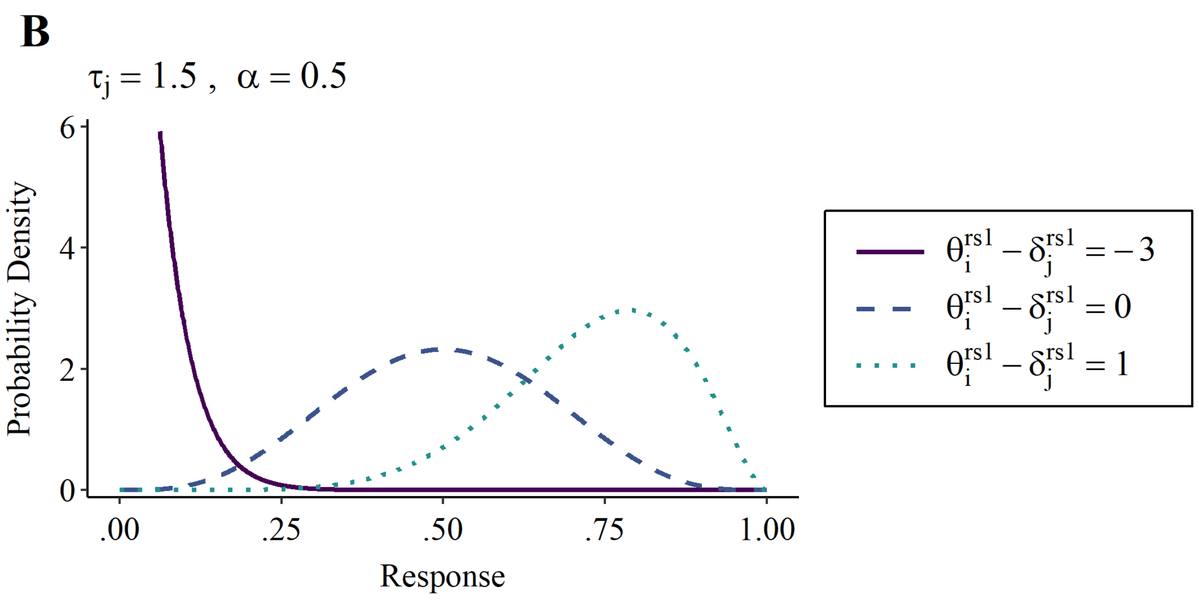
➢ $\exp(0) = 1$

➢ $\mathrm{Beta}(1, 1)$: uniform

BRM: Examples

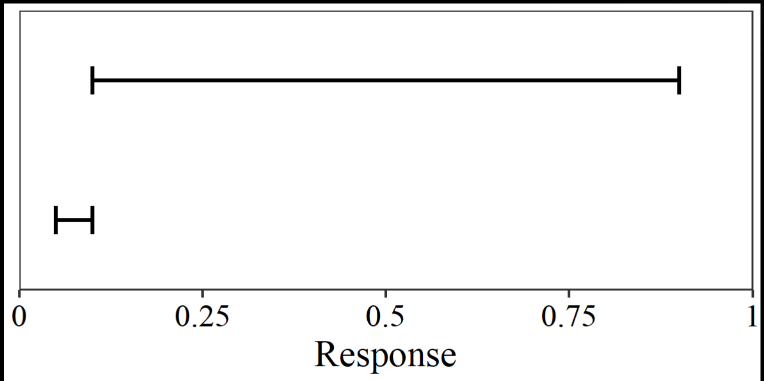


$\tau_j = 0.5$

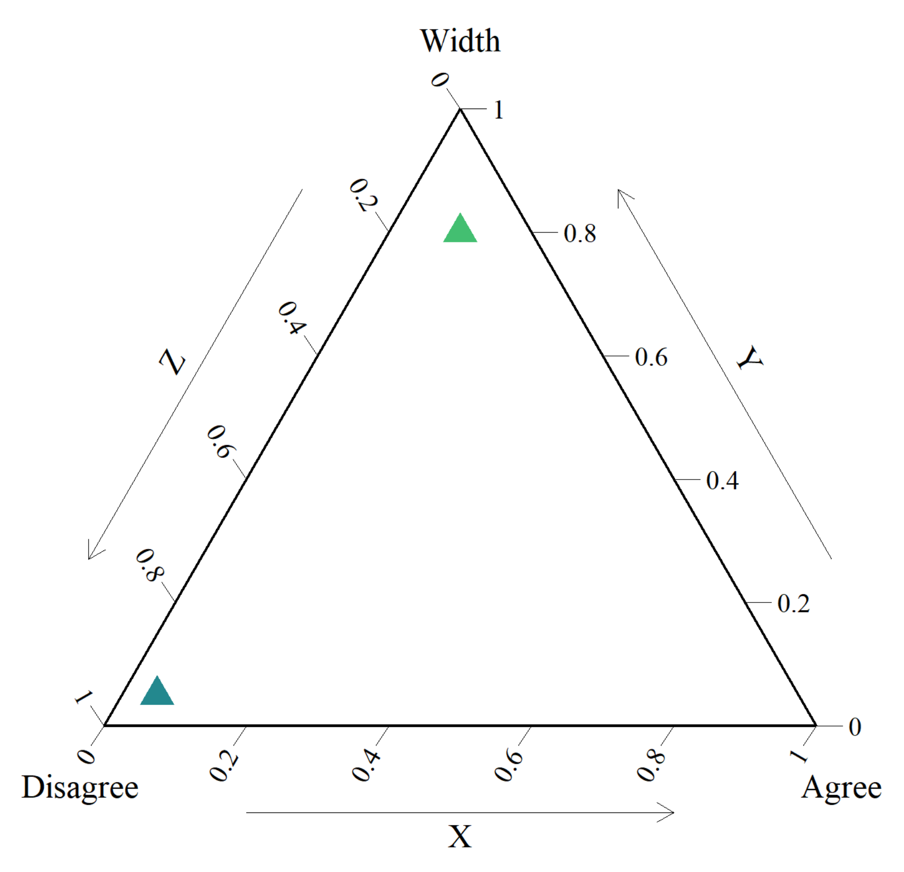


$\tau_j = 1.5$

Response Intervals - Ternary Space

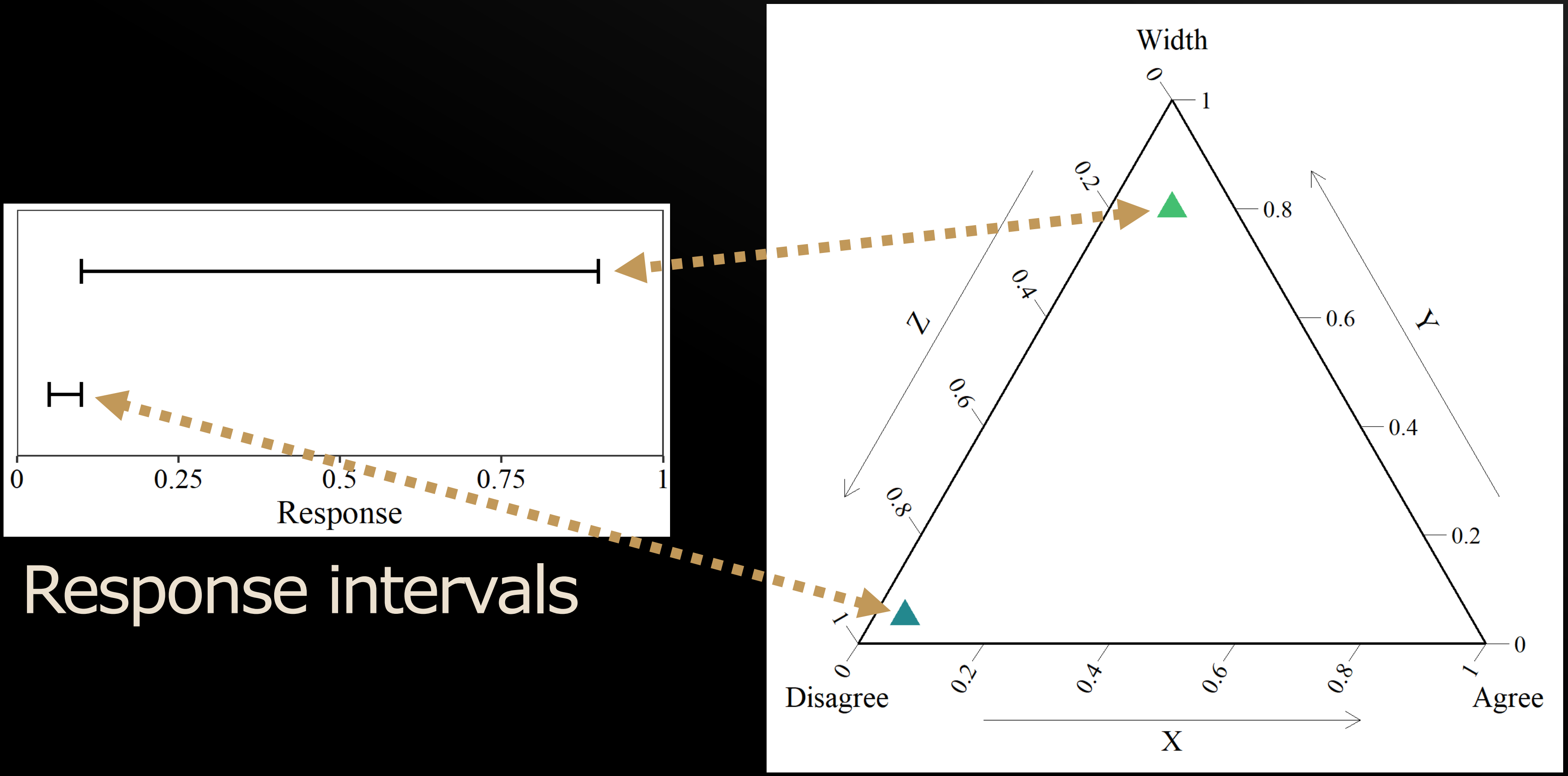


Response intervals



Location in ternary space

RESPONSE INTERVALS - TERNARY SPACE

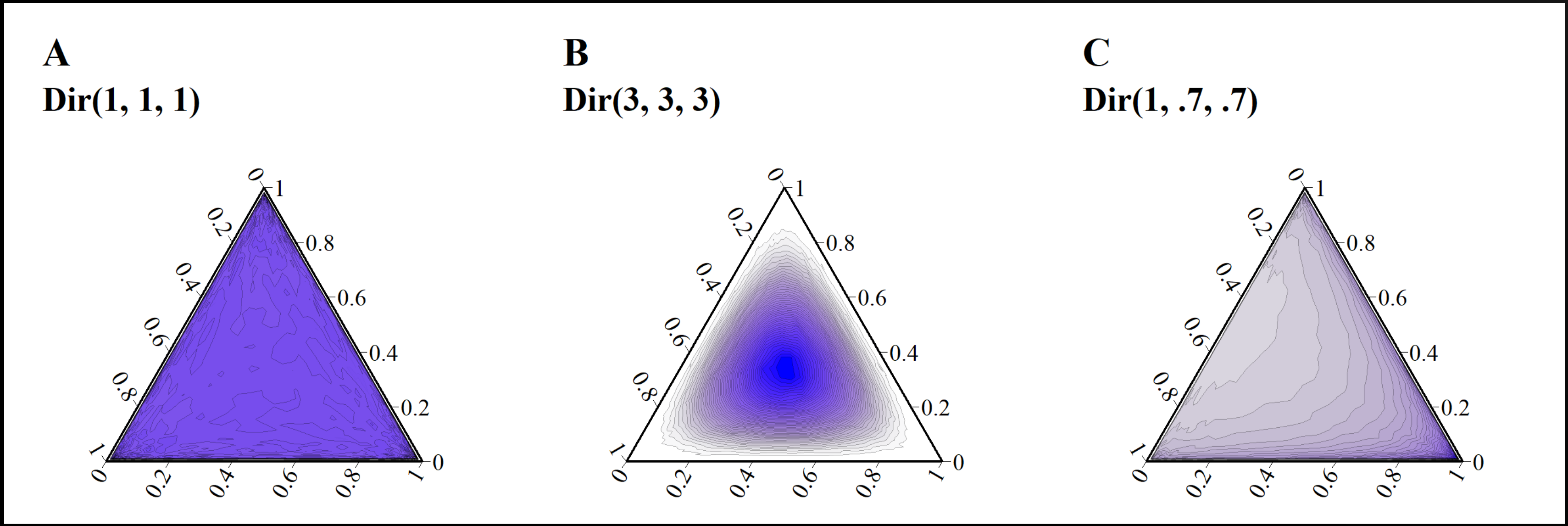


Response intervals

Location in ternary space

Dirichlet Distribution

The beta distribution generalizes to the Dirichlet distribution.

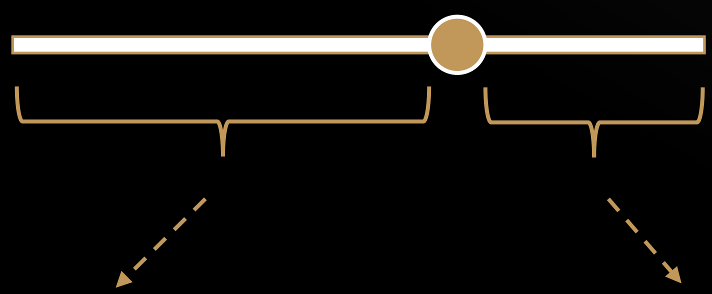


Uniform Unimodal Multimodal

Beta Response Model (BRM)
Noel & Dauvier (2007)

$x_{ij} \sim \text{Beta}\left(a_{ij}, d_{ij}\right);$

$$E\left(x_{ij}\right) = \frac{a_{ij}}{a_{ij}+d_{ij}}$$



$$a_{ij} = \exp\left[\alpha\left(\theta_i - \delta_j\right) + \tau_j\right] \qquad d_{ij} = \exp\left[-\alpha\left(\theta_i - \delta_j\right) + \tau_j\right]$$

Dirichlet Dual Response Model (DDRM)

Latent Parameterization

$$\boldsymbol{y}_{ij} \sim \text{Dirichlet}(a_{ij}, e_{ij}, d_{ij});$$

$$E\left(\boldsymbol{y}_{ij}\right) = \frac{a_{ij}}{a_{ij}+e_{ij}+d_{ij}}, \frac{e_{ij}}{a_{ij}+e_{ij}+d_{ij}}, \frac{d_{ij}}{a_{ij}+e_{ij}+d_{ij}}$$

$$a_{ij} = \exp\left[\alpha_\lambda\left(\theta_i - \delta_j\right) + \tau_j\right] \quad d_{ij} = \exp\left[-\alpha_\lambda\left(\theta_i - \delta_j\right) + \tau_j\right]$$

$$e_{ij} = \exp\left[\alpha_\epsilon\left(\eta_i + \gamma_j\right) + \tau_j\right]$$

Dirichlet Dual Response Model (DDRM)

Parameters: Precision

$$a_{ij} = \exp\left[\alpha_\lambda\left(\theta_i - \delta_j\right) + \tau_j\right] \qquad d_{ij} = \exp\left[-\alpha_\lambda\left(\theta_i - \delta_j\right) + \tau_j\right]$$

$$e_{ij} = \exp\left[\alpha_\epsilon\left(\eta_i + \gamma_j\right) + \tau_j\right]$$

- Location dimension: equivalent to the BRM
- Expansion dimension: controls the interval width
- Scaling $\alpha_\lambda / \alpha_\epsilon$ per dimension
- Precision τ_j across both dimensions

Dirichlet Dual Response Model (DDRM)

Parameters: Expansion Dimension

$$a_{ij} = \exp[\alpha_\lambda(\theta_i - \delta_j) + \tau_j] \qquad d_{ij} = \exp[-\alpha_\lambda(\theta_i - \delta_j) + \tau_j]$$

$$e_{ij} = \exp[\alpha_\epsilon(\eta_i + \gamma_j) + \tau_j]$$

η_i: Person expansion (preference for wider intervals)
γ_j: Item expansion (strength to evoke wider intervals)

- Higher values = wider response intervals

Dirichlet Dual Response Model (DDRM)

Exponential Link

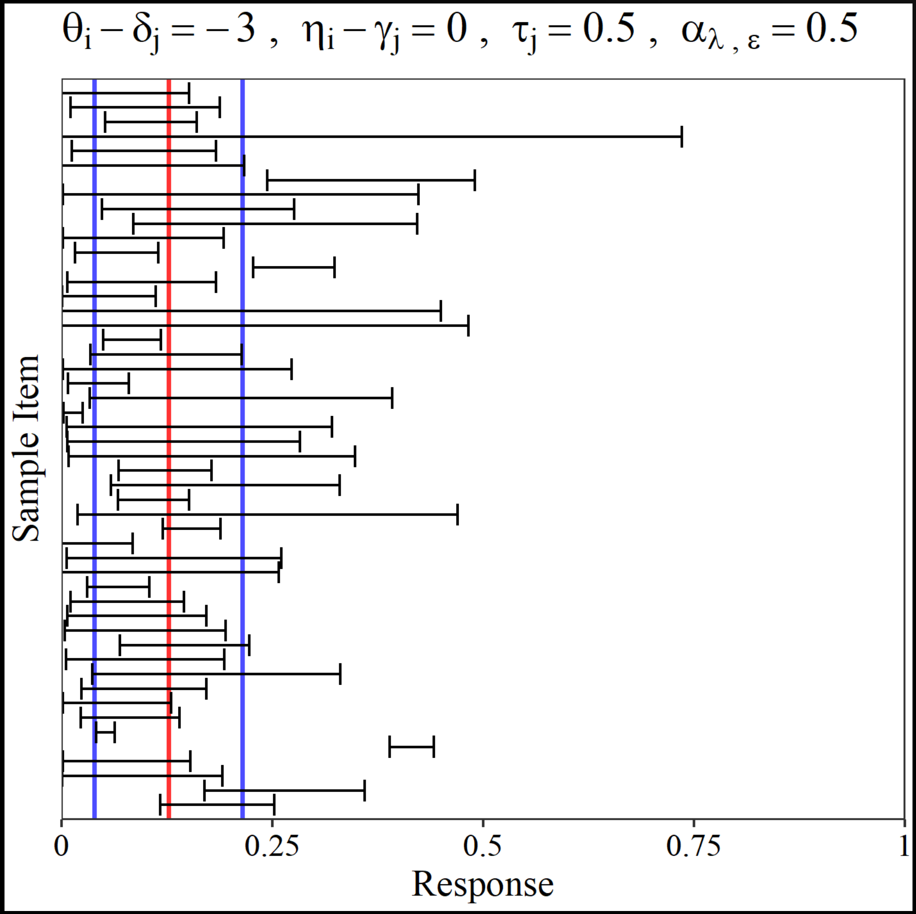
$$a_{ij} = \underbrace{\exp}\left[\alpha_\lambda\left(\theta_i - \delta_j\right) + \tau_j\right] \qquad d_{ij} = \underbrace{\exp}\left[-\alpha_\lambda\left(\theta_i - \delta_j\right) + \tau_j\right]$$

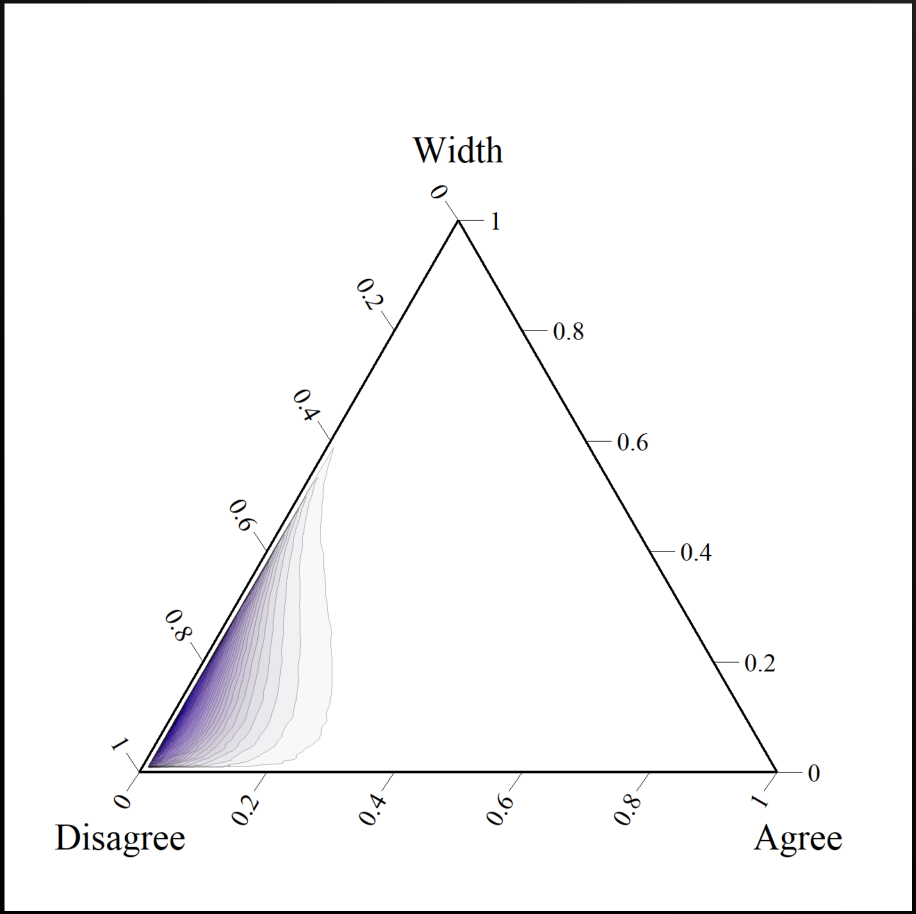
$$e_{ij} = \underbrace{\exp}\left[\alpha_\epsilon\left(\eta_i + \gamma_j\right) + \tau_j\right]$$

Example:

- $\theta_i - \delta_j = 0;$ $\eta_i + \gamma_j = 0;$ $\tau_j = 0$
- ➢ $\exp(0) = 1$
- ➢ $Dirichlet(1,1,1)$: uniform distribution over the simplex

DDRM Examples



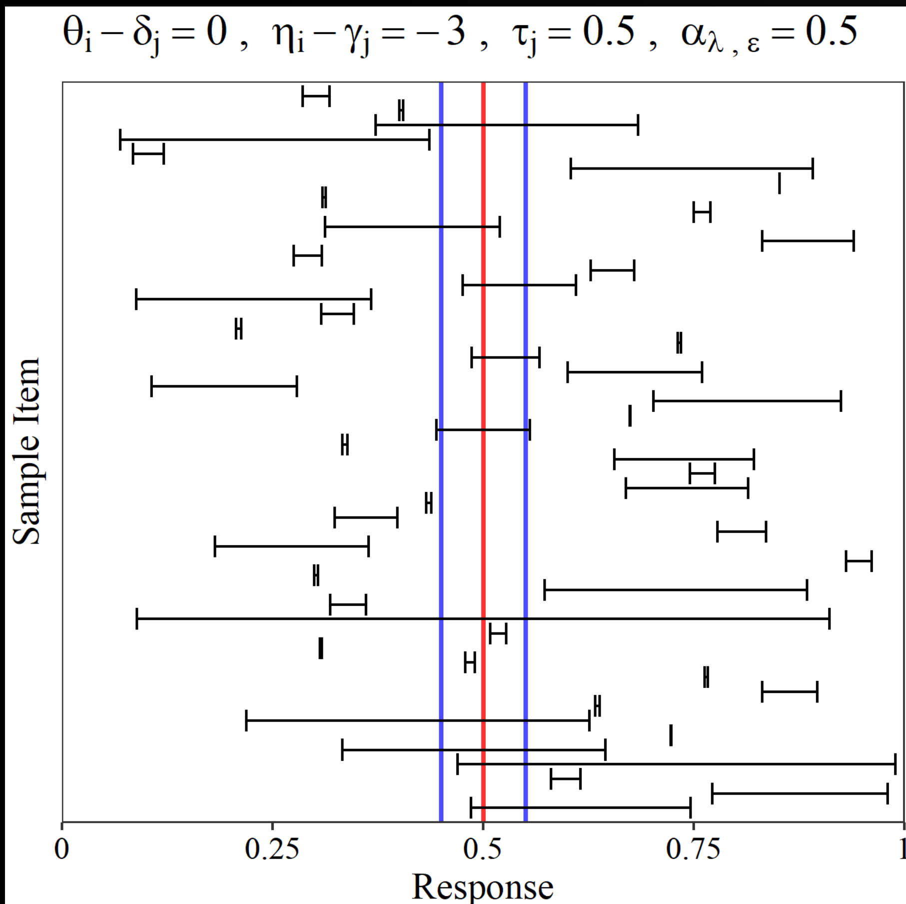


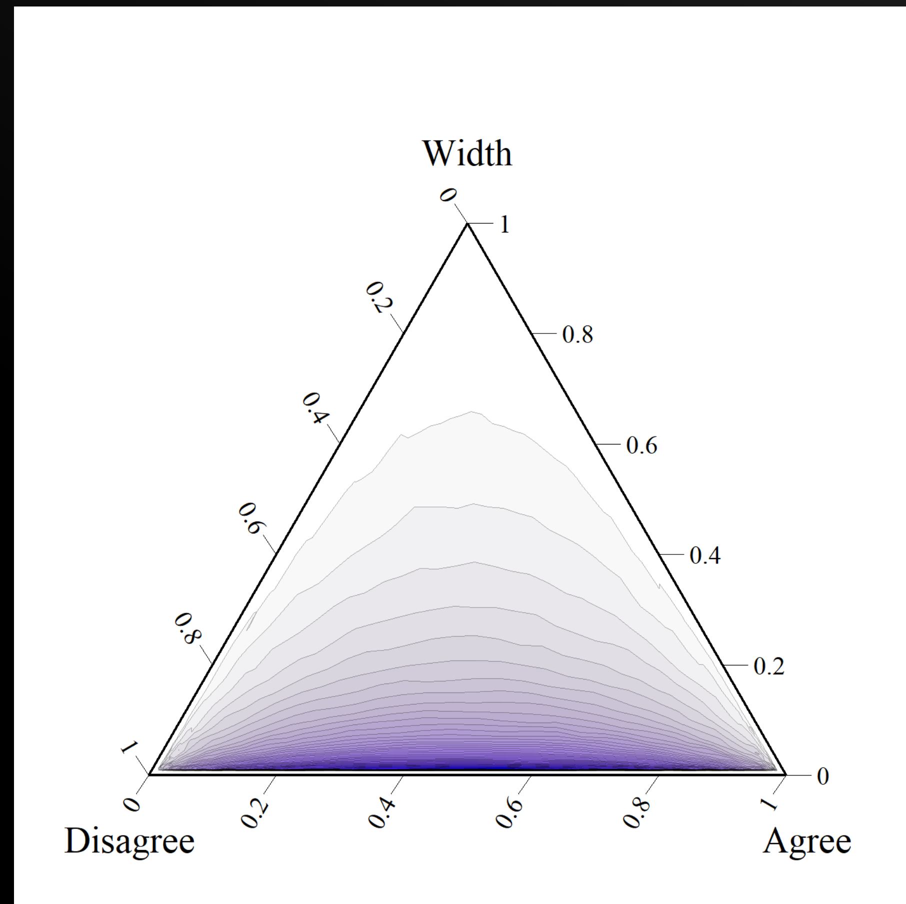
50 randomly drawn intervals

Response distribution density

- Red vertical line: expected interval location (midpoint)
- Blue vertical lines: expected lower and upper bound

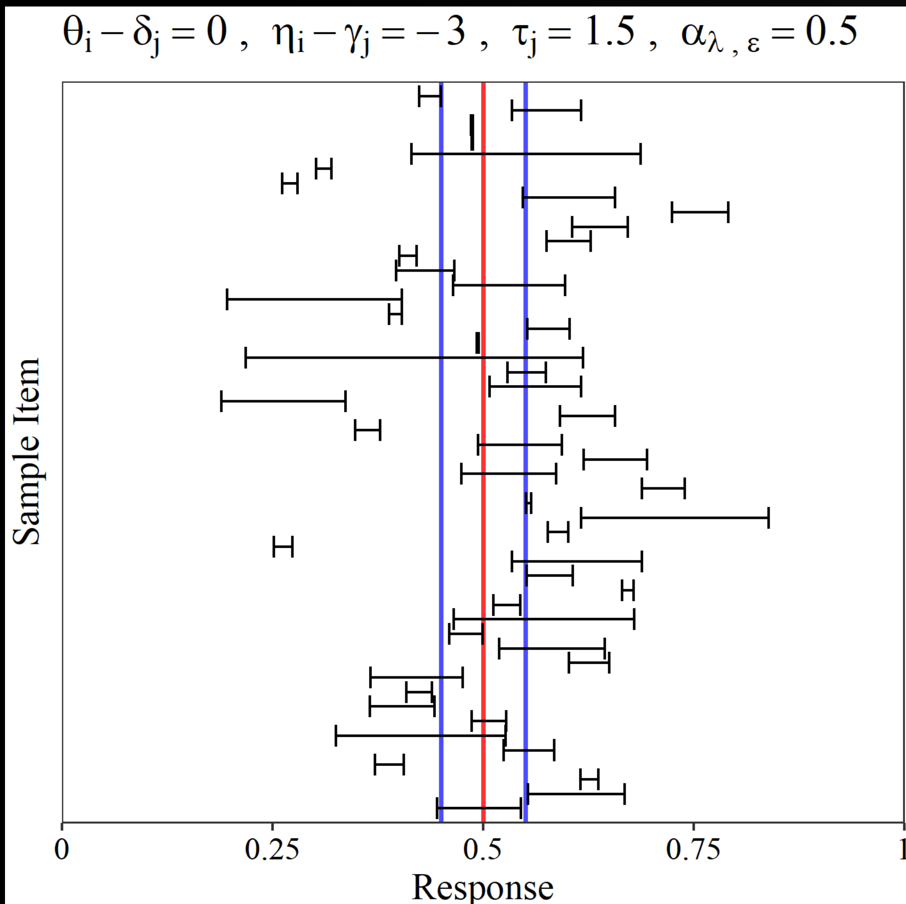
DDRM Examples
Comparison: Precision

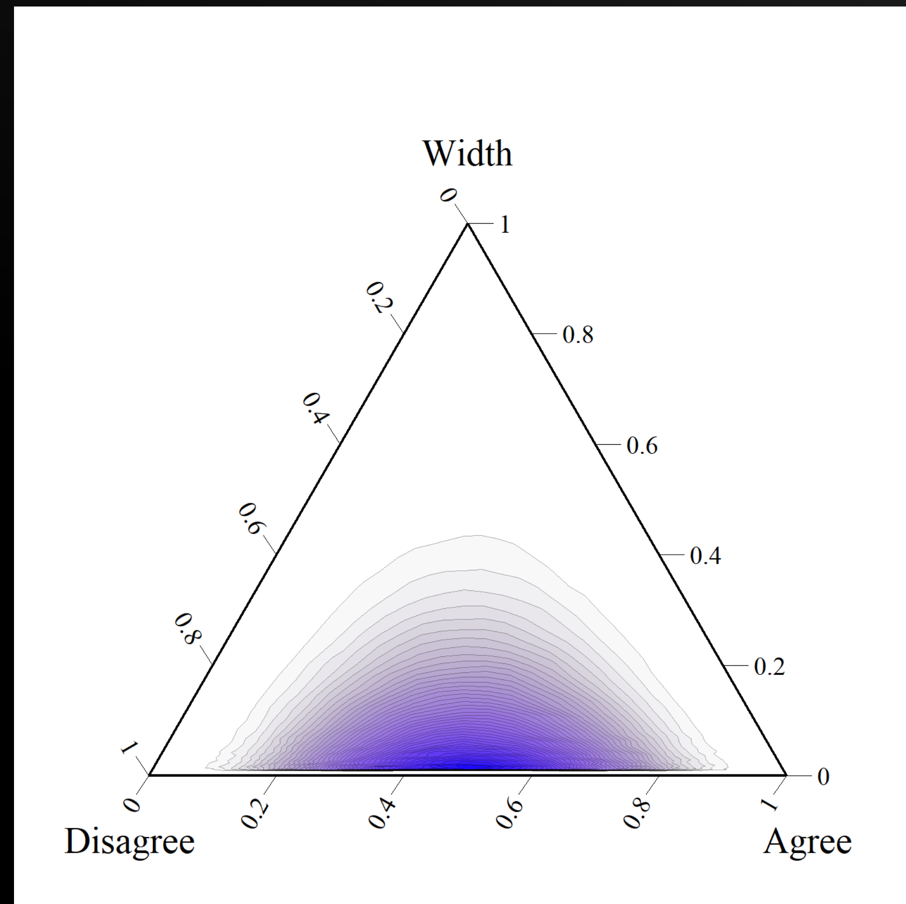




$$\tau_j = 0.5$$

DDRM Examples
Comparison: Precision





$$\tau_j = 1.5$$

3 – Empirical Example

EMPIRICAL EXAMPLE
Methods

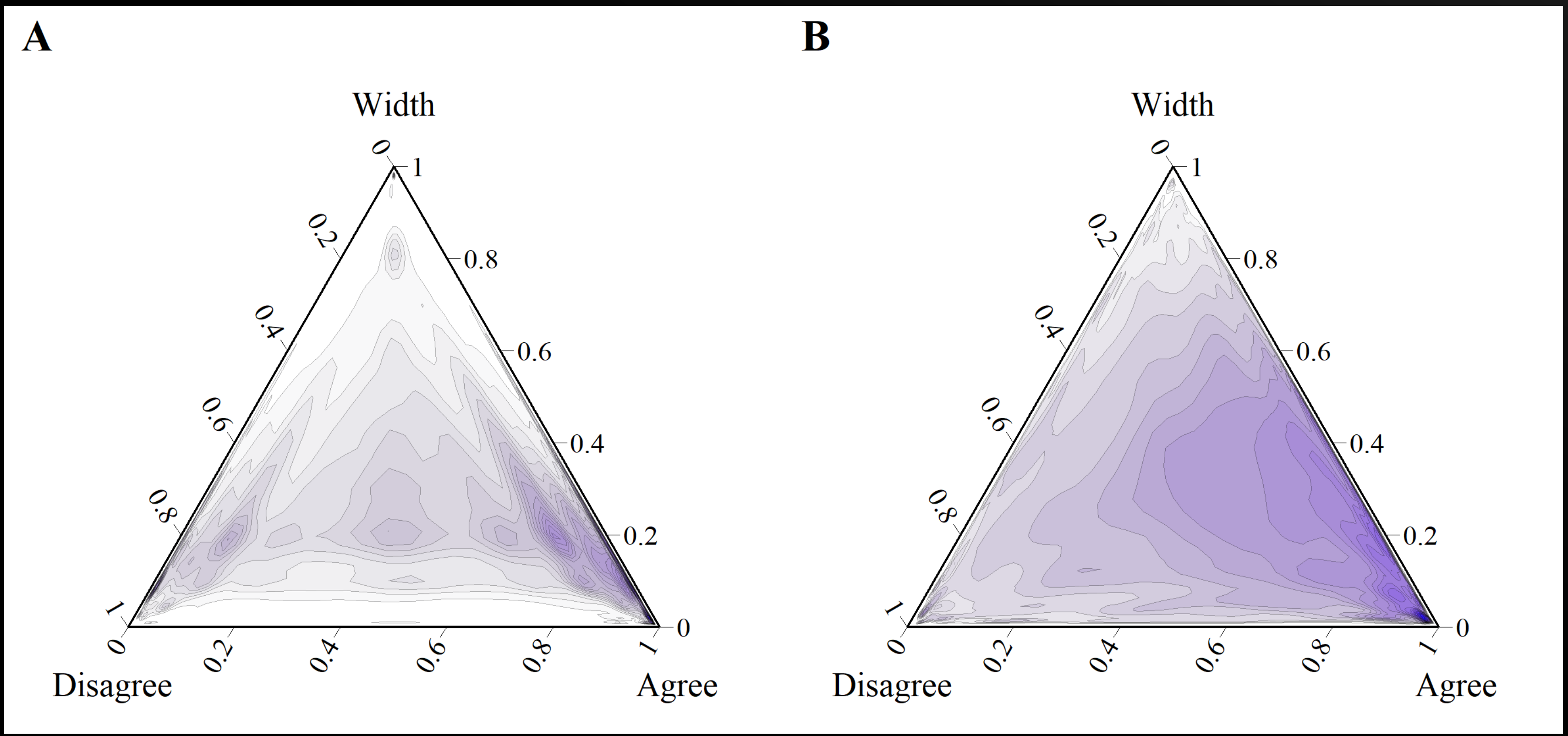
Two Extraversion scales:

- IPIP: 36 items (Interval Responses)
- BFI-2: 12 items (Single Responses)

Sample: n = 222 (f: 140 , m: 80, d: 2)

Posterior Predictive Checks

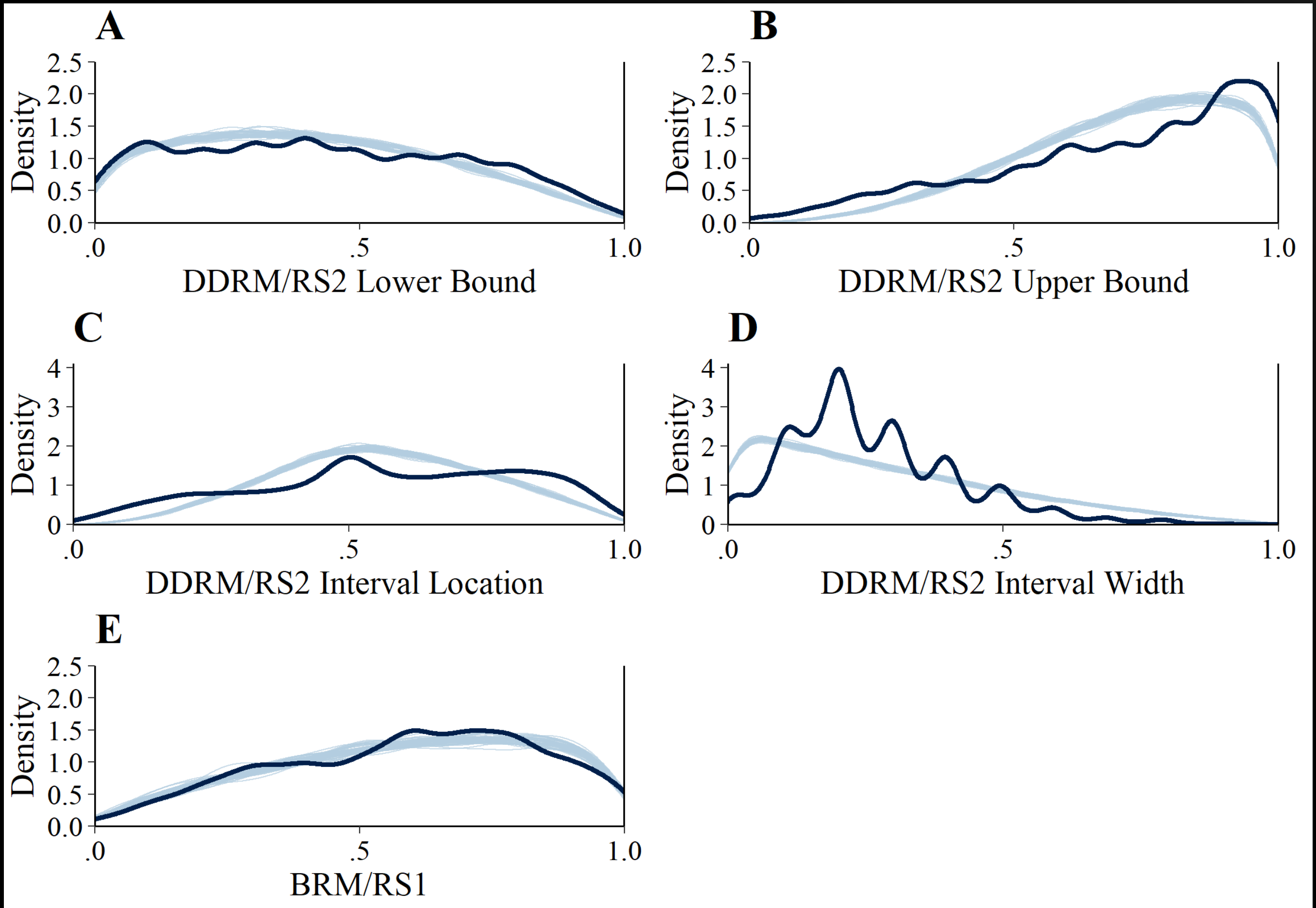
Ternary



Empirical

Replicated

Posterior Predictive Checks
Binary Marginal Densities

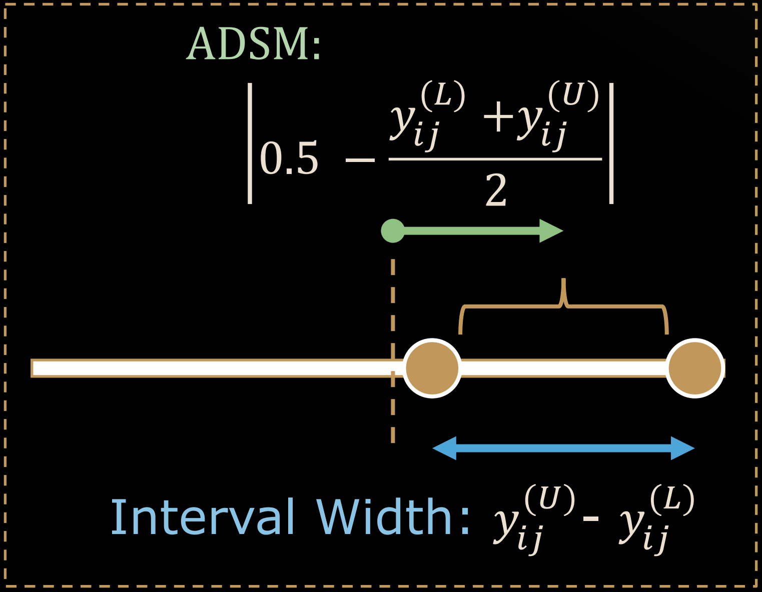


Dark lines: empirical; Lightlines: replicated

4 - Why Do We Need A Model?

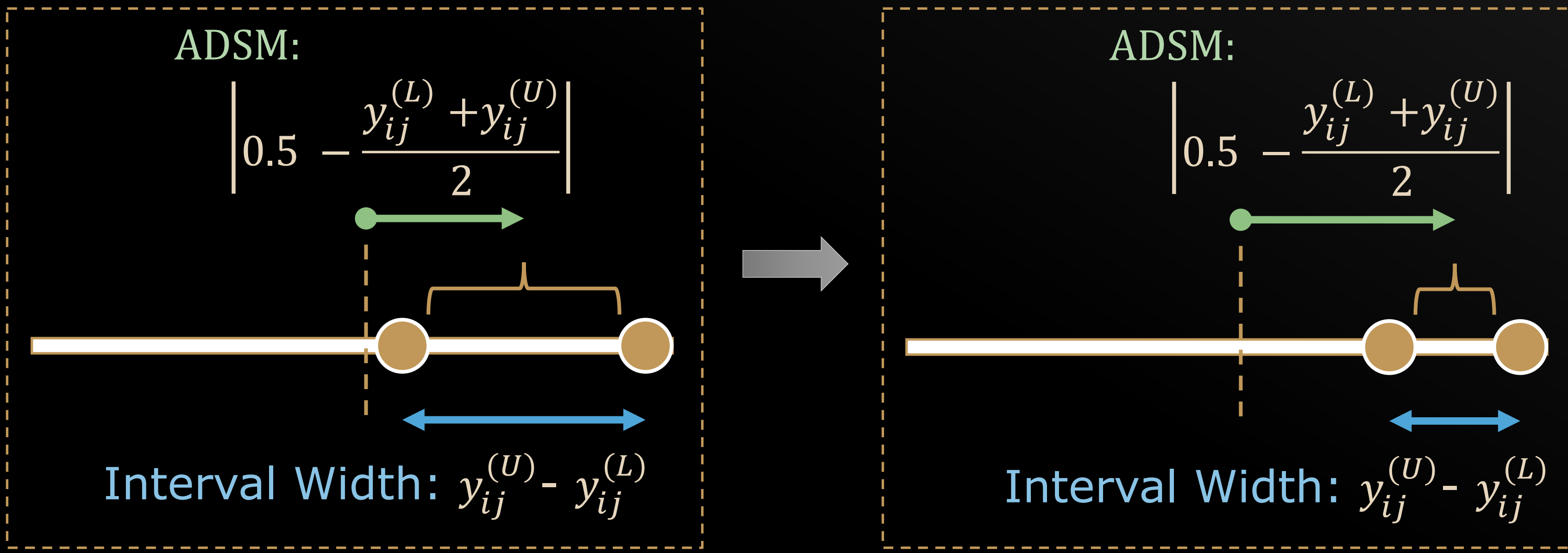
Boundedness
Scale-Inherent Correlation

- ADSM: Absolute Distance from Scale Midpoint



BOUNDEDNESS
Scale-Inherent Correlation

- ADSM: Absolute Distance from Scale Midpoint



- Negative correlation between ADMS and Interval Width

Boundedness
Comparison of Correlations

Manifest correlation: person mean scores

- ADSM
- Interval Width

Latent variable correlation: person parameters

- Absolute Location ($|\theta|$; remember: $M(\theta) = 0$)
- Expansion (η)

Boundedness
Comparison of Correlations

	Mean scores	Model parameters
Empirical:	$r = -.57$	$r = -.19$
Simulation:	$r = -.74$	$r = -.02$
True:		$(r = -.002)$

➢The model accounts for the scale-inherent correlation

Convergent Validity: Response Formats

Mean Scores vs. Estimates

Manifest correlation: person mean scores

- Single response
- Interval location (midpoint)

Latent variable correlation: person parameters

- Person location θ_{BRM}
- Person location θ_{DDRM}

Convergent Validity: Response Formats

Correlations in the Empirical Study:

Mean scores: $r = .81$

Model Parameters: $r = .87$

- ➢ Latent model improves convergence

Take Home Points

- High **convergent validity** of response formats
- Model **accounts** for **boundedness**
- **Additional information**: expansion dimension
  - Validity? What does it measure?
- Useful tool for analysis of interval responses

THANKS TO:

- Prof. Dr. Daniel W. Heck

- Prof. Dr. Andreas Voss

- Dr. Raphael Hartmann

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Slides: https://github.com/matthiaskloft/

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