A Dirichlet Model for Interval Responses

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1 - Introduction

„I like being around other people"

WHY USE INTERVAL RESPONSES?

Motivating Example:

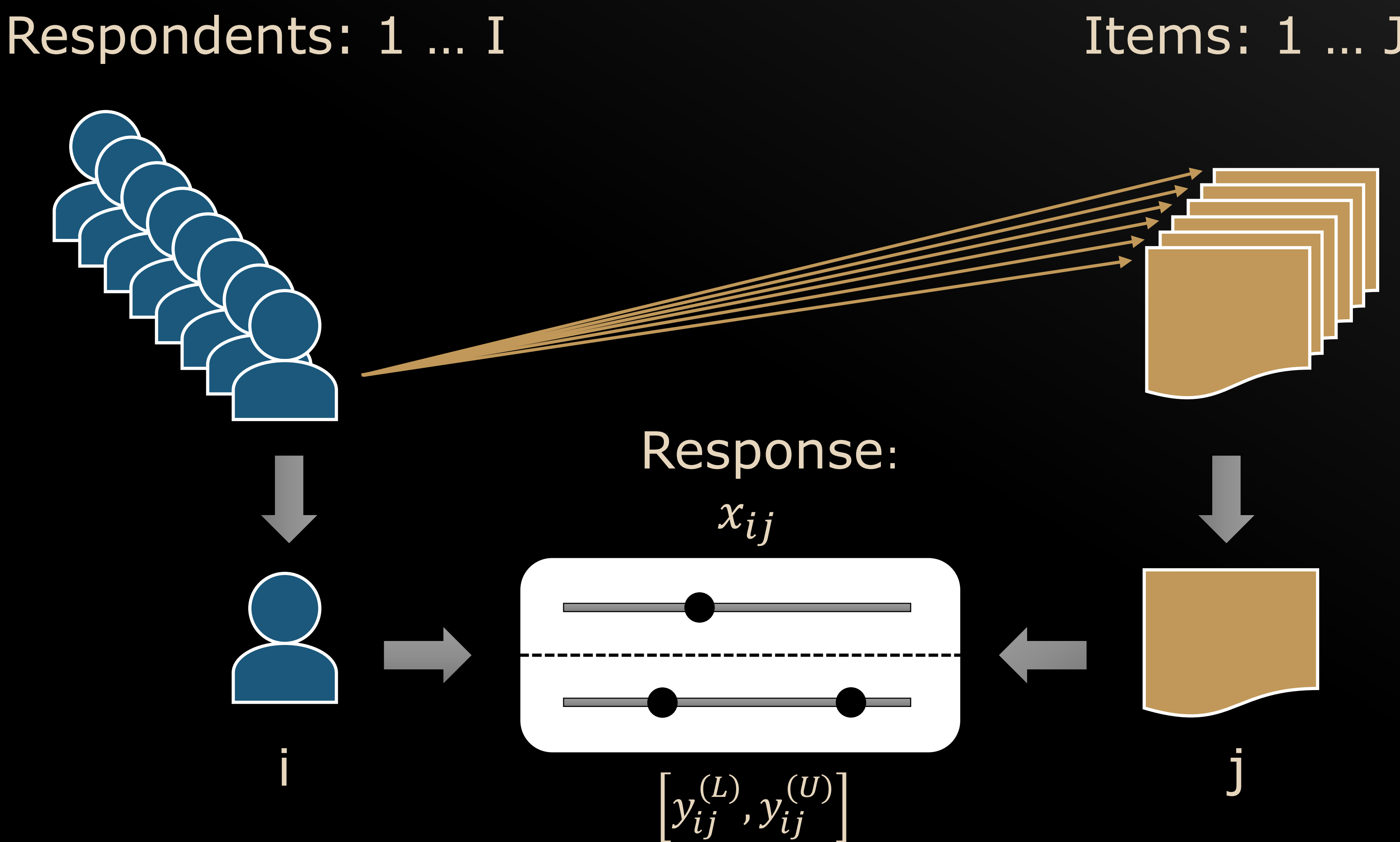
- Whole Trait Theory (Fleeson, 2001)
  - Trait: Distribution of states

➢ Accounting for variability

➢ Range of valid values

2 – IRT Models

Testing Scenario

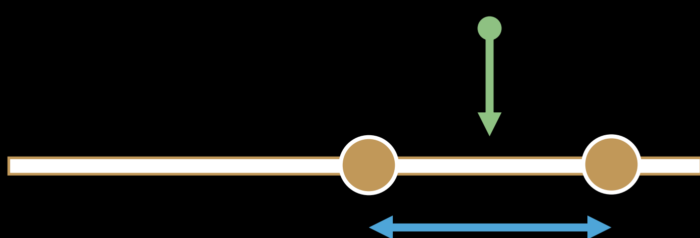


Interval Response

<u>Manifest Response:</u>

Interval Location (Midpoint):

$$\frac{y_{ij}^{(L)} + y_{ij}^{(U)}}{2}$$



Interval Width: $y_{ij}^{(U)} - y_{ij}^{(L)}$

Interval Response

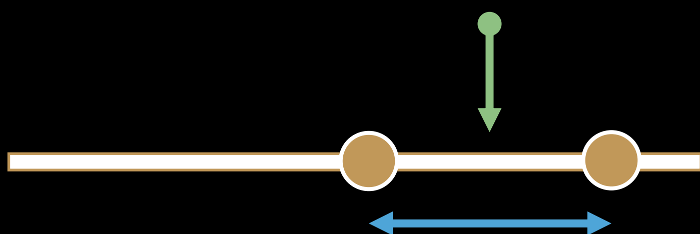
Manifest Response:

Latent Space:

Interval Location (Midpoint):

$$\frac{y_{ij}^{(L)} + y_{ij}^{(U)}}{2}$$

Location Dimension



Interval Width: $y_{ij}^{(U)} - y_{ij}^{(L)}$

Expansion Dimension

Compositional Data

- Components must sum to one: simplex

Restrictions

No support for zero-components

- Single Response:

$$0 < x_{ij} < 1$$

- Interval Response:

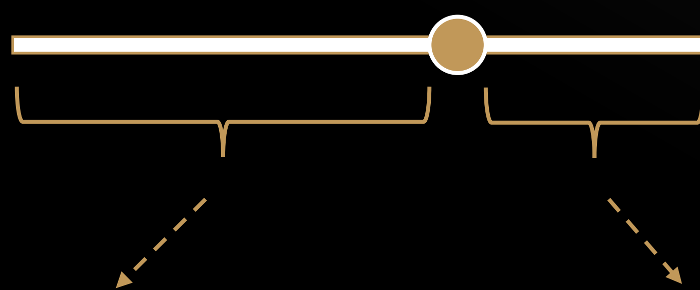
$$0 < y_{ij}^{(L)} < y_{ij}^{(U)} < 1$$

Beta Response Model (BRM)

Noel & Dauvier (2007)

$$x_{ij} \sim \text{Beta}\left(a_{ij}, d_{ij}\right);$$

$$E\left(x_{ij}\right) = \frac{a_{ij}}{a_{ij}+d_{ij}}$$

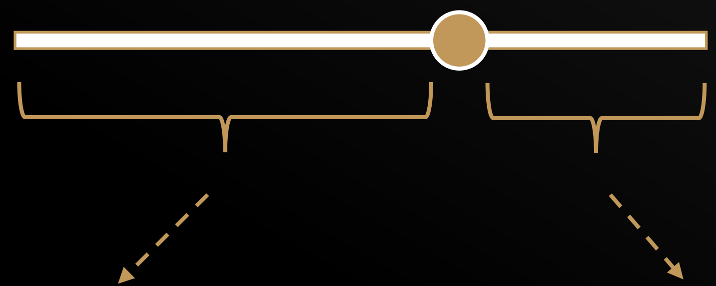


$$a_{ij} = \exp\left[\alpha\left(\theta_i - \delta_j\right) + \tau_j\right]$$

$$d_{ij} = \exp\left[-\alpha\left(\theta_i - \delta_j\right) + \tau_j\right]$$

Beta Response Model (BRM)

Parameters: Ability / Difficulty



$$a_{ij} = \exp\left[\alpha\left(\theta_i - \delta_j\right) + \tau_j\right] \qquad d_{ij} = \exp\left[-\alpha\left(\theta_i - \delta_j\right) + \tau_j\right]$$

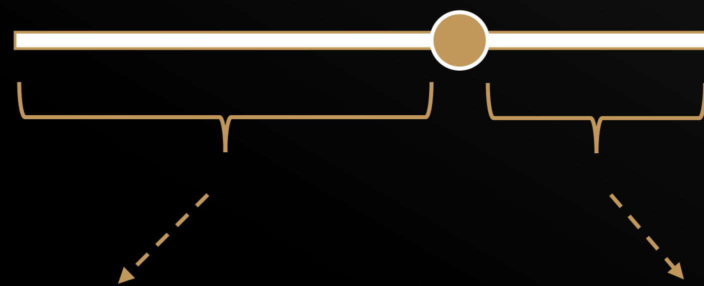
θ_i: Person ability

δ_j: Item difficulty

➢ Classic interpretation

Beta Response Model (BRM)

Parameters: Scaling

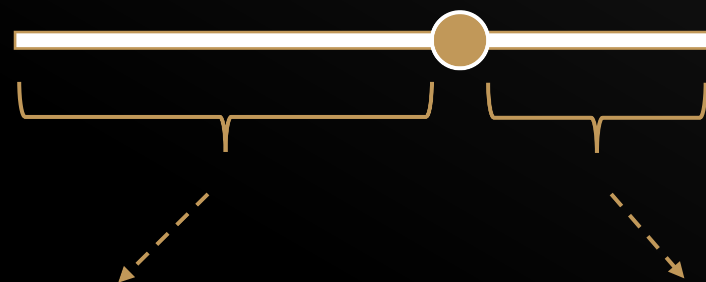


$$a_{ij} = \exp\left[\alpha\left(\theta_i - \delta_j\right) + \tau_j\right] \qquad d_{ij} = \exp\left[-\alpha\left(\theta_i - \delta_j\right) + \tau_j\right]$$

$\pm\alpha > 0$: Scaling

- Continuous model:
  Not a discrimination parameter!!

Beta Response Model (BRM)
Parameters: Precision



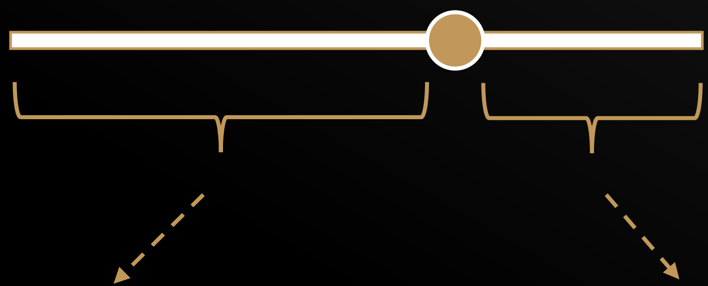
$$a_{ij} = \exp\left[\alpha(\theta_i - \delta_j) + \underbrace{\tau_j}\right] \qquad d_{ij} = \exp\left[-\alpha(\theta_i - \delta_j) + \underbrace{\tau_j}\right]$$

$\tau_j > 0$: Item precision (both additive!)

- ➢ Steeper density curves

Beta Response Model (BRM)

Parameters: Exponential Link



$$a_{ij} = \exp\left[\alpha\left(\theta_i - \delta_j\right) + \tau_j\right] \qquad d_{ij} = \exp\left[-\alpha\left(\theta_i - \delta_j\right) + \tau_j\right]$$

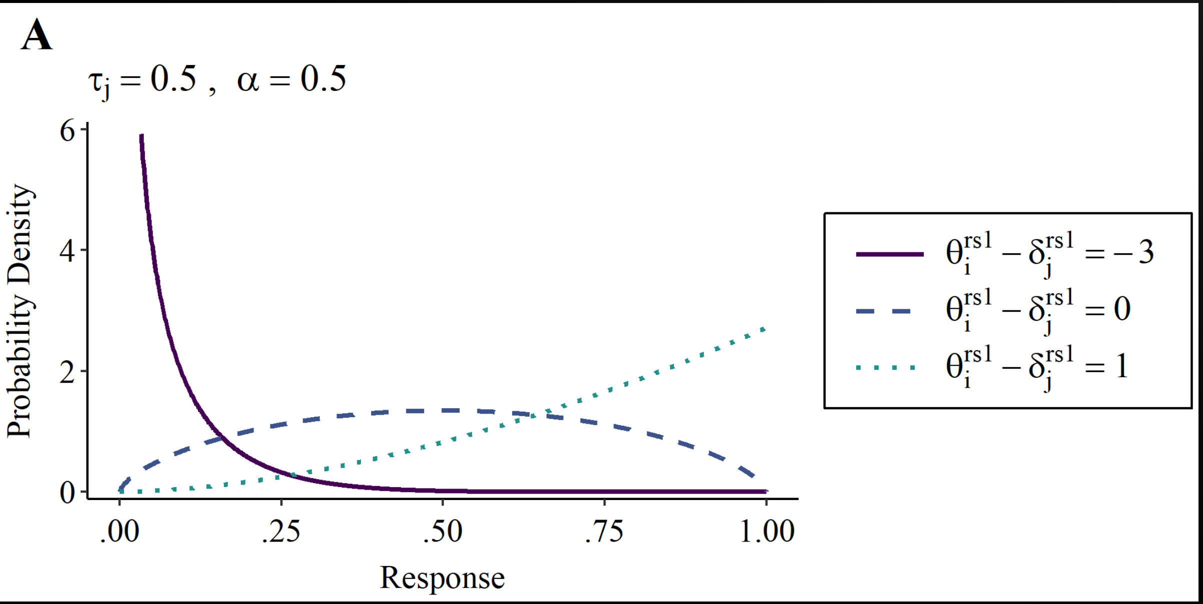
Example:

- $\theta_i - \delta_j = 0; \quad \tau_j = 0$

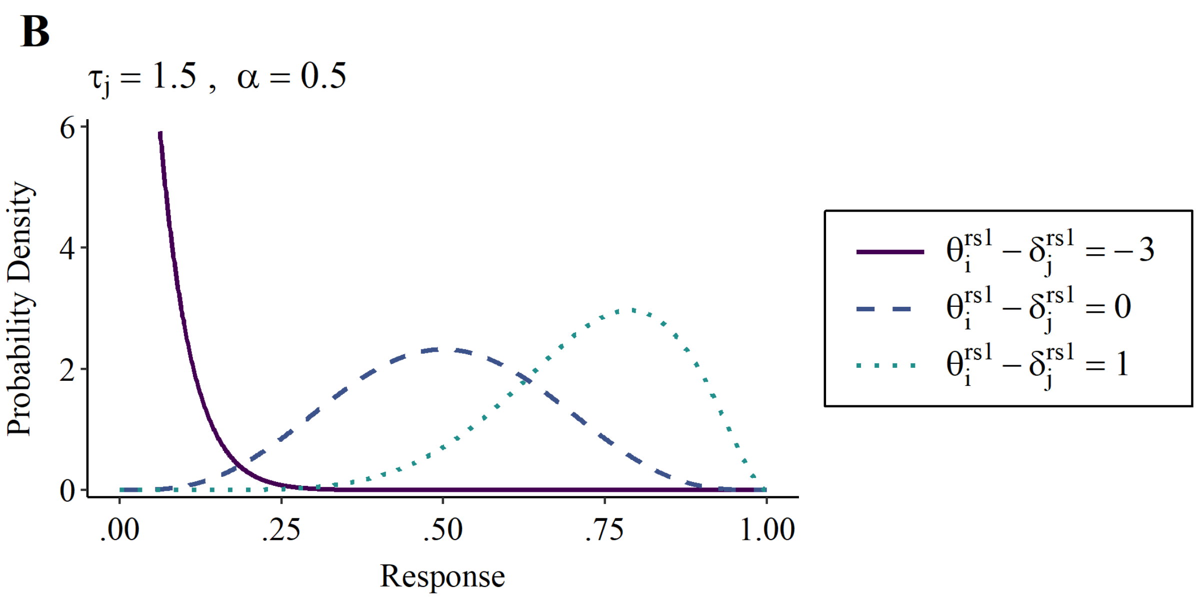
➢ $\exp(0) = 1$

➢ $\mathrm{Beta}(1, 1)$: uniform

BRM: Examples

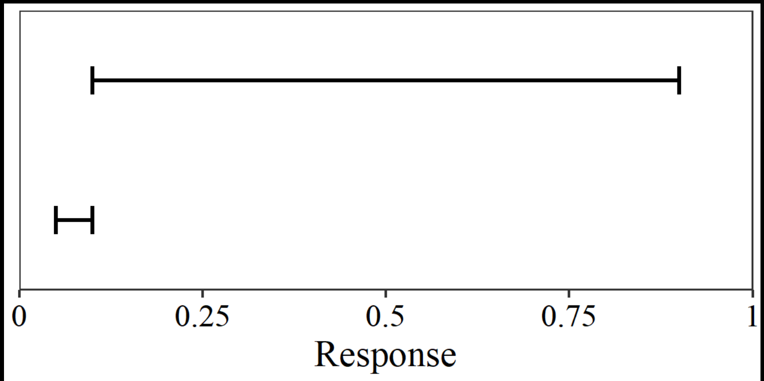


$\tau_j = 0.5$

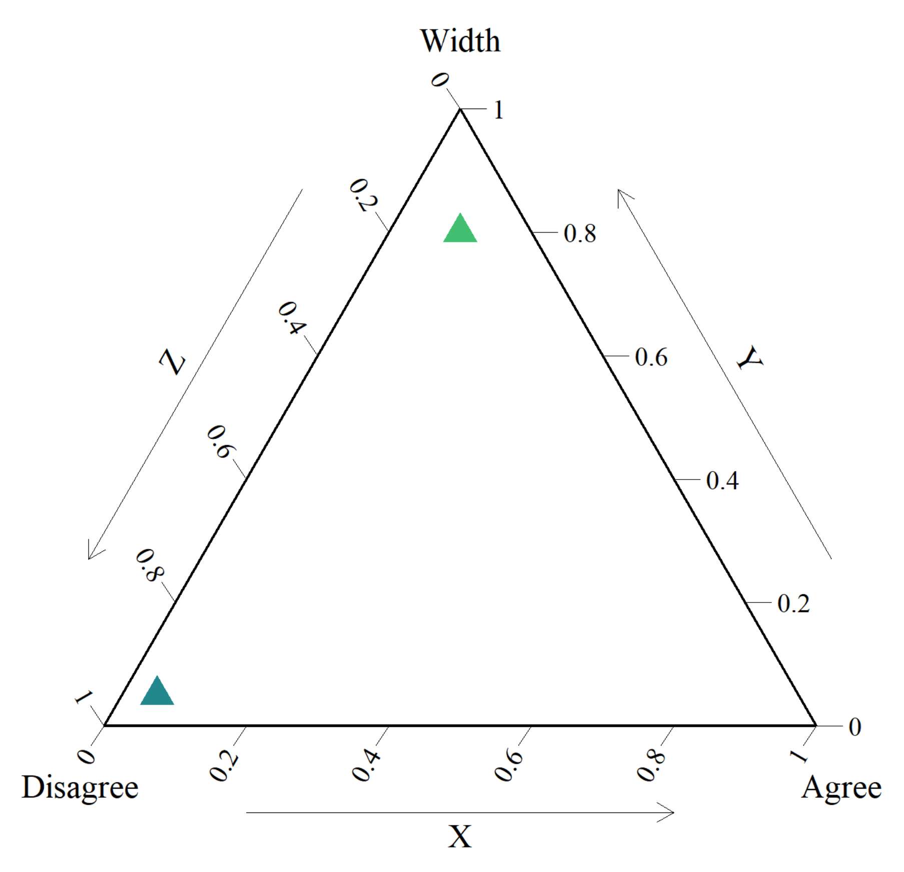


$\tau_j = 1.5$

Response Intervals - Ternary Space

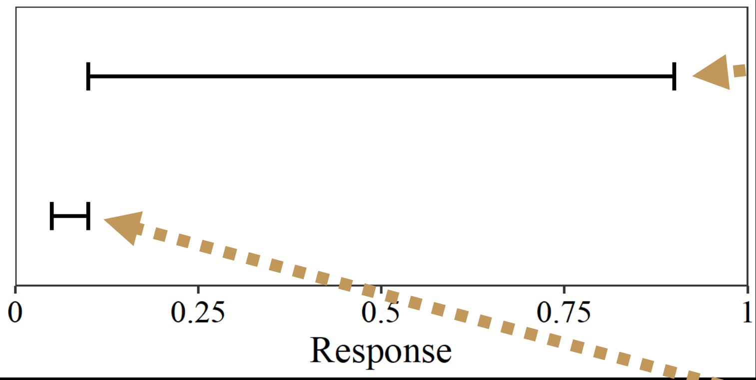


Response intervals

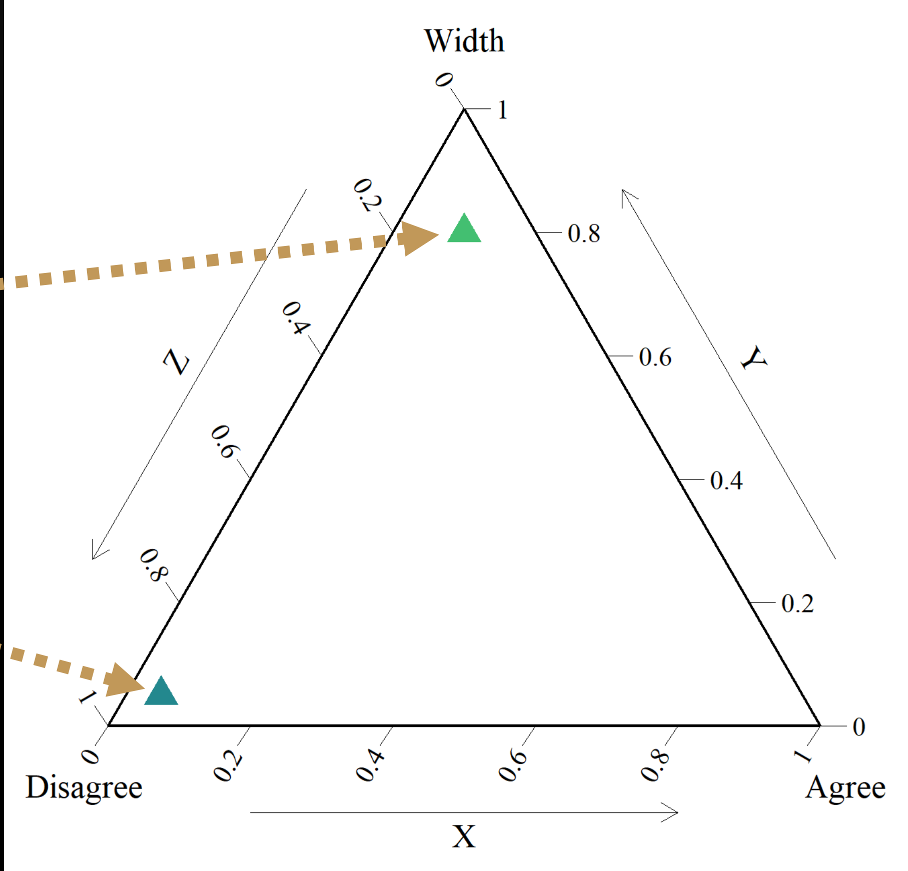


Location in ternary space

Response Intervals - Ternary Space



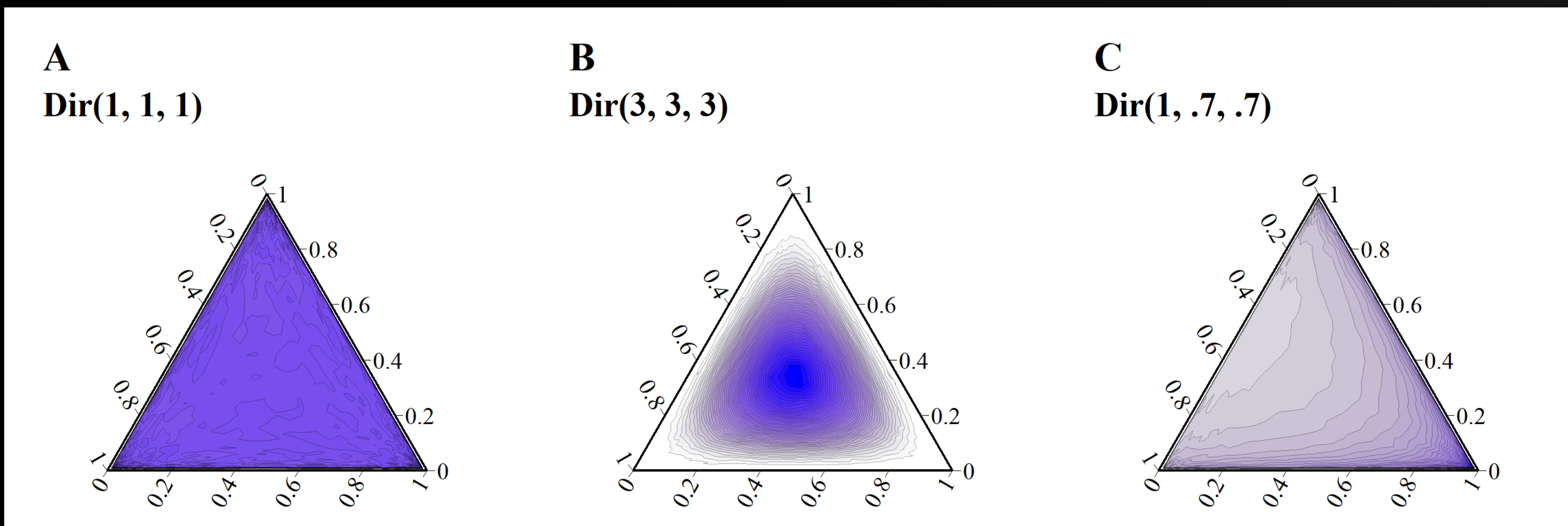
Response intervals



Location in ternary space

Dirichlet Distribution

The beta distribution generalizes to the Dirichlet distribution.

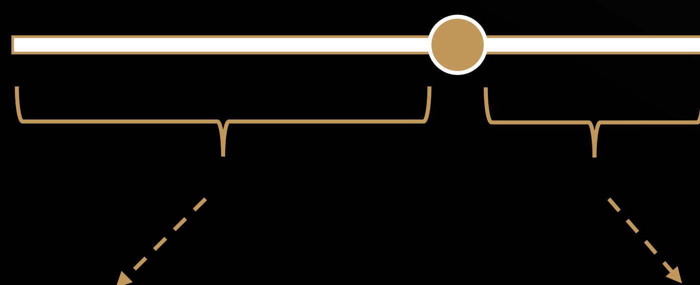


Uniform Unimodal Multimodal

Beta Response Model (BRM)
Noel & Dauvier (2007)

$$x_{ij} \sim \mathrm{Beta}(a_{ij}, d_{ij});$$

$$E(x_{ij}) = \frac{a_{ij}}{a_{ij} + d_{ij}}$$



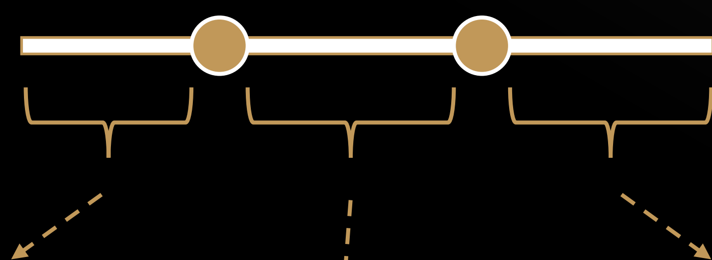
$$a_{ij} = \exp[\alpha(\theta_i - \delta_j) + \tau_j] \qquad d_{ij} = \exp[-\alpha(\theta_i - \delta_j) + \tau_j]$$

Dirichlet Dual Response Model (DDRM)

Latent Parameterization

$$\boldsymbol{y}_{ij} \sim \text{Dirichlet}(a_{ij}, e_{ij}, d_{ij});$$

$$E\left(\boldsymbol{y}_{ij}\right) = \frac{a_{ij}}{a_{ij}+e_{ij}+d_{ij}}, \frac{e_{ij}}{a_{ij}+e_{ij}+d_{ij}}, \frac{d_{ij}}{a_{ij}+e_{ij}+d_{ij}}$$



$$a_{ij} = \exp\left[\alpha_\lambda\left(\theta_i - \delta_j\right) + \tau_j\right] \qquad d_{ij} = \exp\left[-\alpha_\lambda\left(\theta_i - \delta_j\right) + \tau_j\right]$$

$$e_{ij} = \exp\left[\alpha_\epsilon\left(\eta_i + \gamma_j\right) + \tau_j\right]$$

Dirichlet Dual Response Model (DDRM)

Parameters: Precision

$$a_{ij} = \exp\left[\alpha_\lambda(\theta_i - \delta_j) + \tau_j\right] \qquad d_{ij} = \exp\left[-\alpha_\lambda(\theta_i - \delta_j) + \tau_j\right]$$

$$e_{ij} = \exp\left[\alpha_\epsilon(\eta_i + \gamma_j) + \tau_j\right]$$

- Location dimension: equivalent to the BRM
- Expansion dimension: controls the interval width
- Scaling $\alpha_\lambda/\alpha_\epsilon$ per dimension
- Precision τ_j across both dimensions

Dirichlet Dual Response Model (DDRM)

Parameters: Expansion Dimension

$$a_{ij} = \exp\left[\alpha_\lambda\left(\theta_i - \delta_j\right) + \tau_j\right] \qquad d_{ij} = \exp\left[-\alpha_\lambda\left(\theta_i - \delta_j\right) + \tau_j\right]$$

$$e_{ij} = \exp\left[\alpha_\epsilon\left(\eta_i + \gamma_j\right) + \tau_j\right]$$

η_i: Person expansion (preference for wider intervals)
γ_j: Item expansion (strength to evoke wider intervals)

- ➢ Higher values = wider response intervals

Dirichlet Dual Response Model (DDRM)

Exponential Link

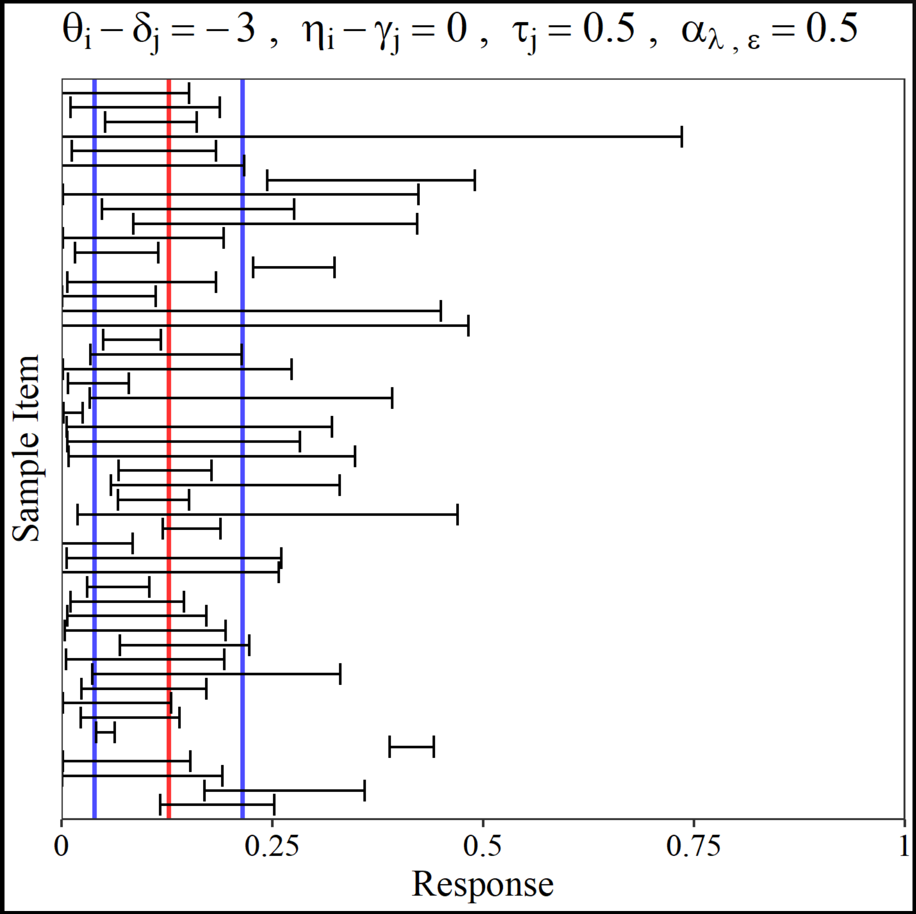
$$a_{ij} = \exp\left[\alpha_\lambda(\theta_i - \delta_j) + \tau_j\right] \qquad d_{ij} = \exp\left[-\alpha_\lambda(\theta_i - \delta_j) + \tau_j\right]$$

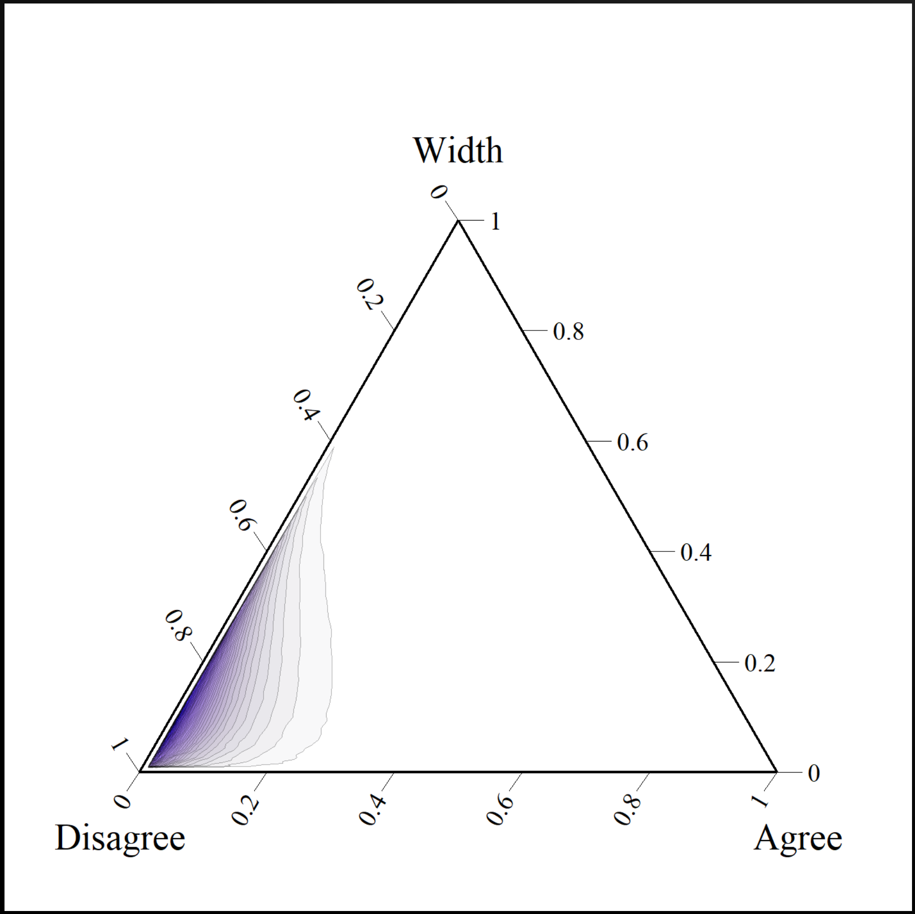
$$e_{ij} = \exp\left[\alpha_\epsilon(\eta_i + \gamma_j) + \tau_j\right]$$

Example:

- $\theta_i - \delta_j = 0;$ $\eta_i + \gamma_j = 0;$ $\tau_j = 0$
- ➢ $\exp(0) = 1$
- ➢ $Dirichlet(1,1,1)$: uniform distribution over the simplex

DDRM Examples



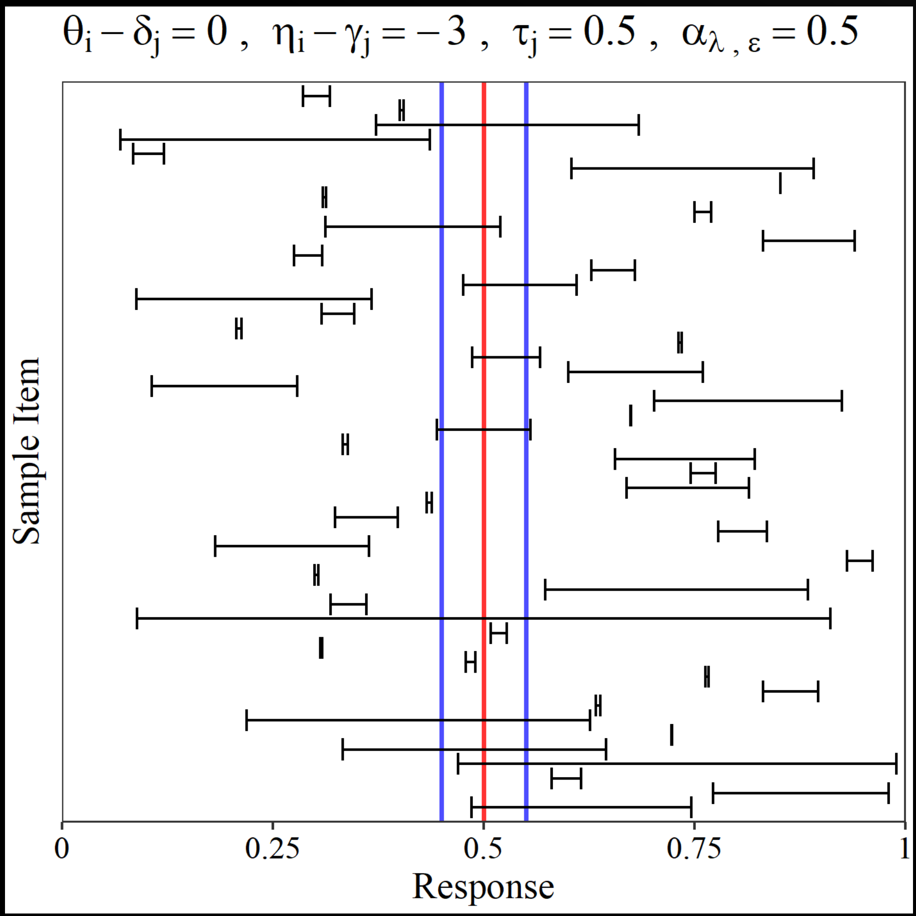


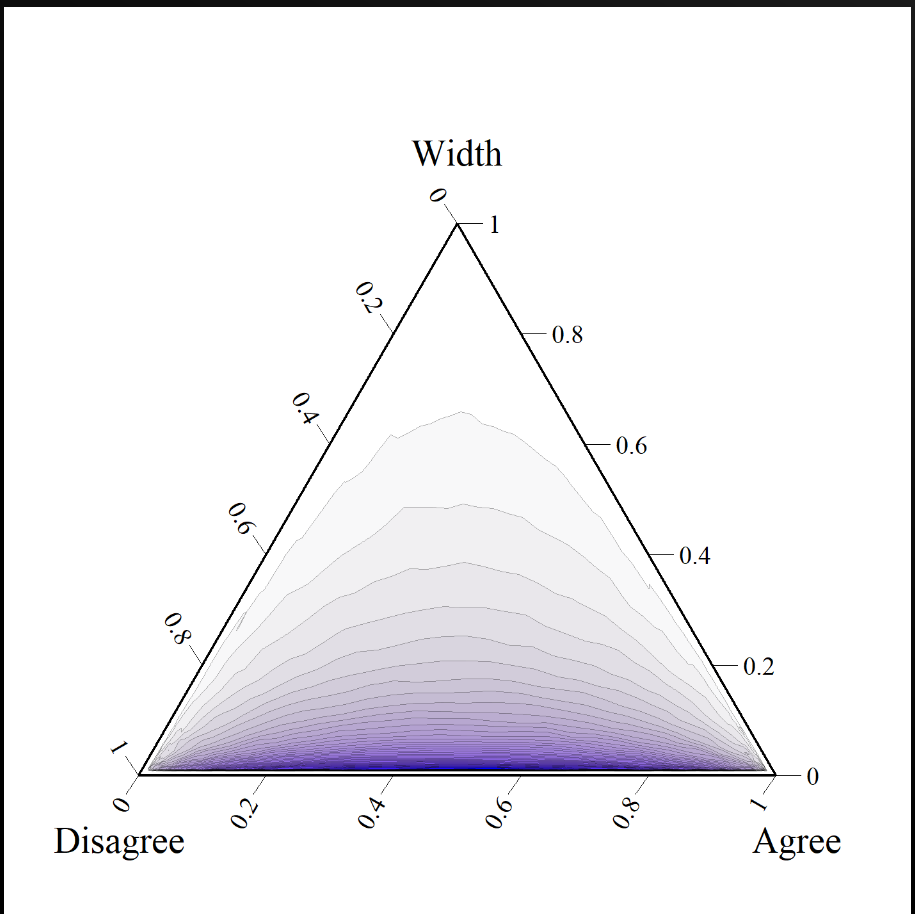
50 randomly drawn intervals

Response distribution density

- Red vertical line: expected interval location (midpoint)
- Blue vertical lines: expected lower and upper bound

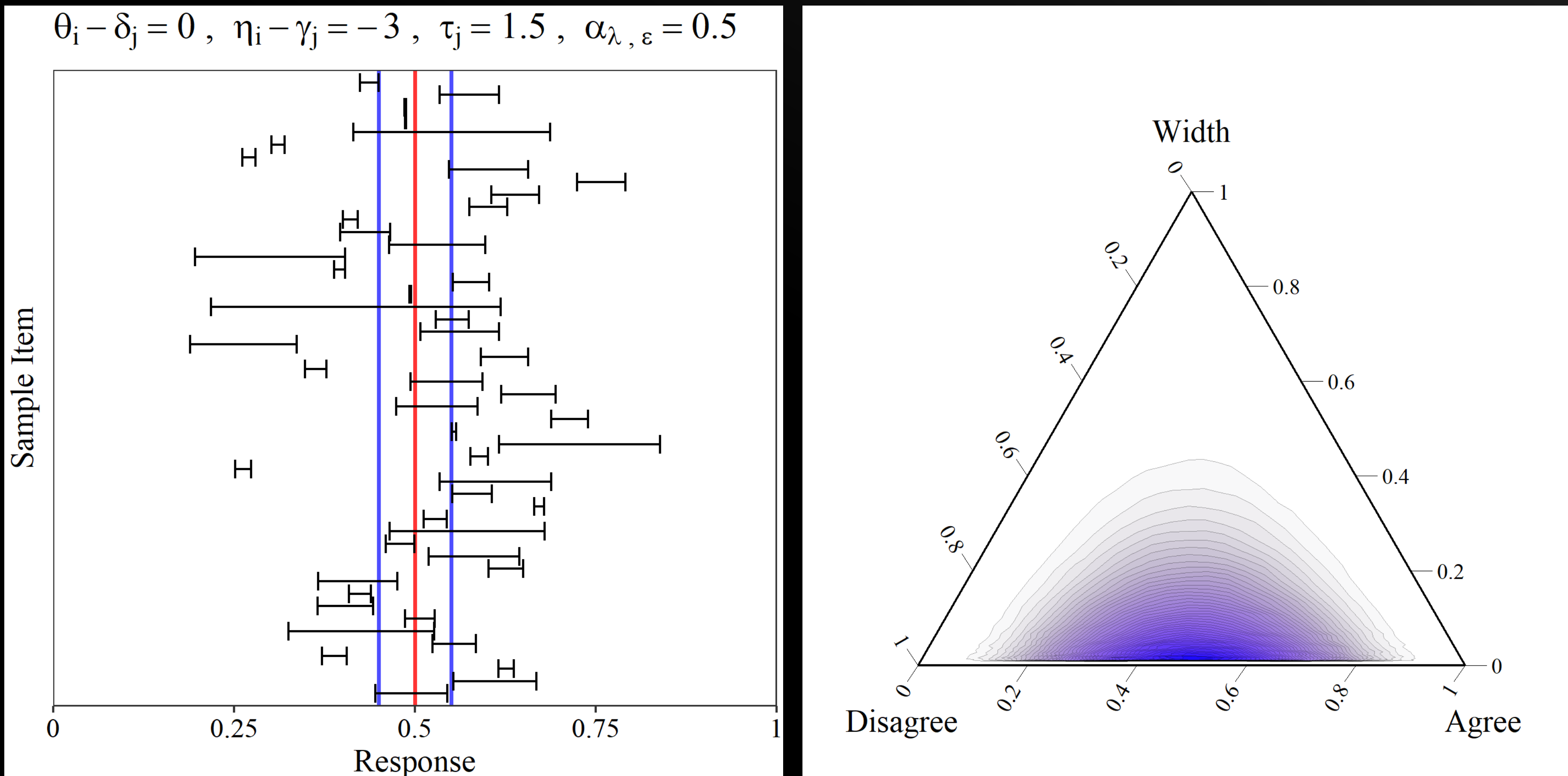
DDRM Examples
Comparison: Precision





$\tau_j = 0.5$

DDRM Examples
Comparison: Precision



$\tau_j = 1.5$

3 – Empirical Example

Empirical Example
Methods

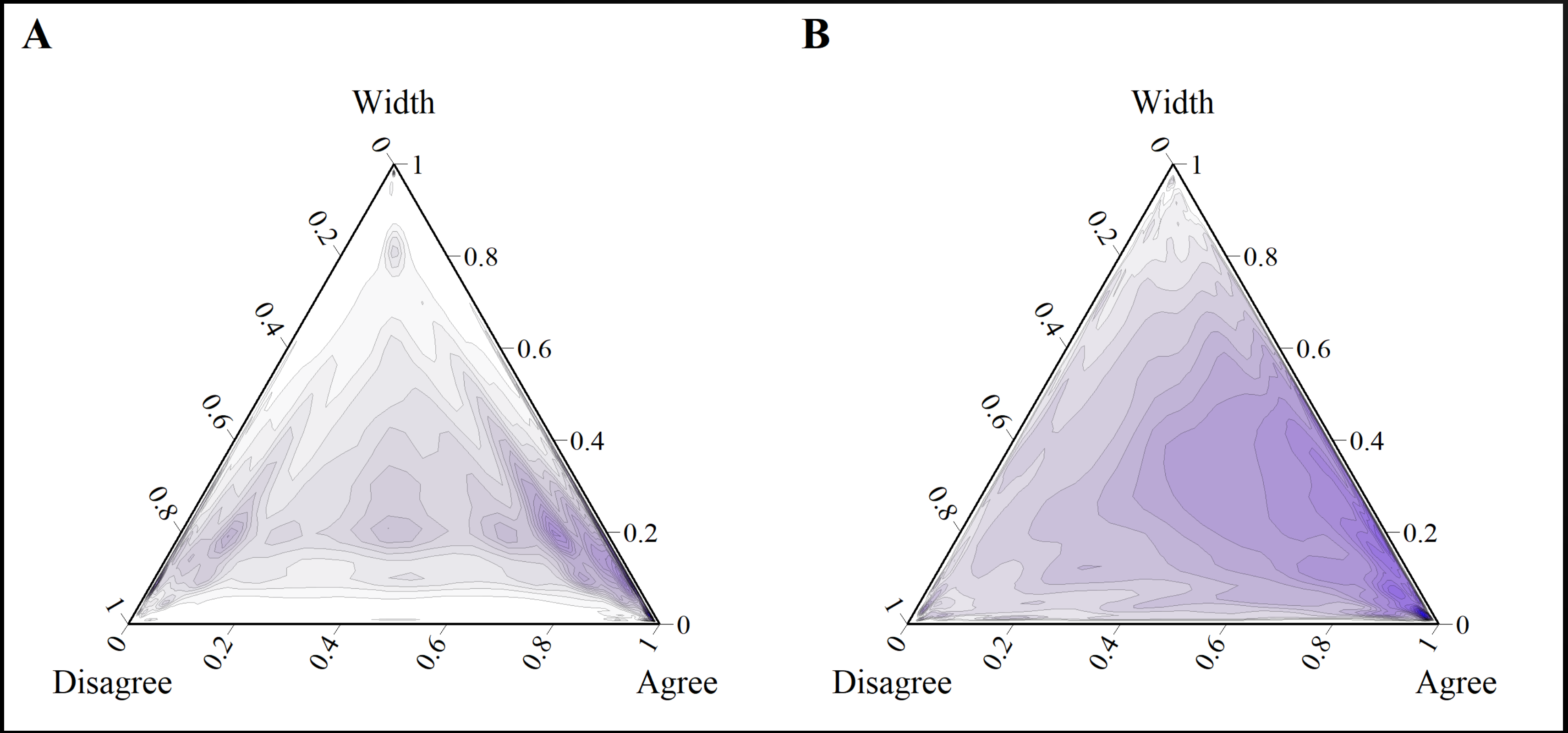
Two Extraversion scales:

- IPIP: 36 items (Interval Responses)
- BFI-2: 12 items (Single Responses)

Sample: n = 222 (f: 140 , m: 80, d: 2)

Posterior Predictive Checks

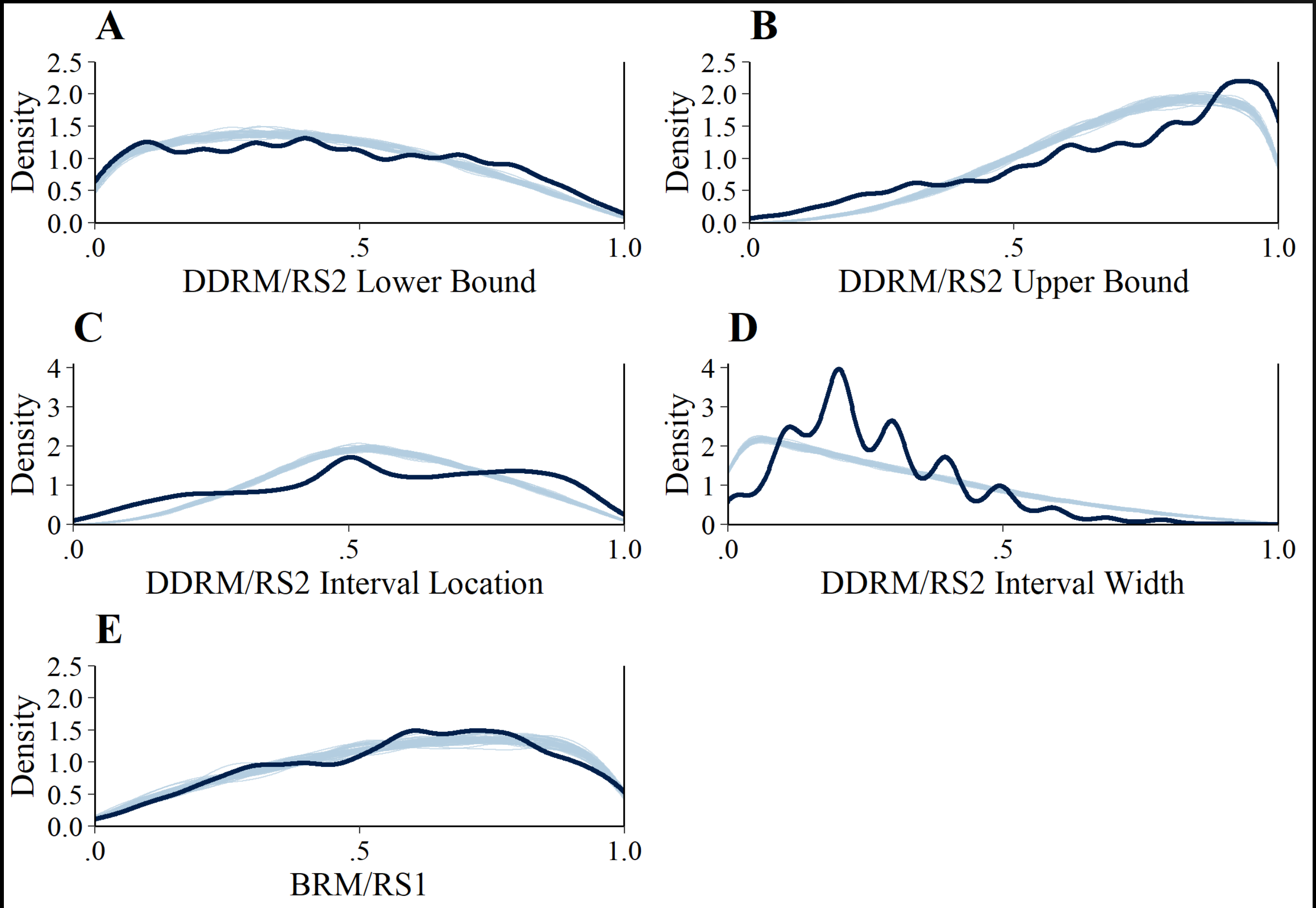
Ternary



Empirical

Replicated

Posterior Predictive Checks
Binary Marginal Densities



Dark lines: empirical; Lightlines: replicated

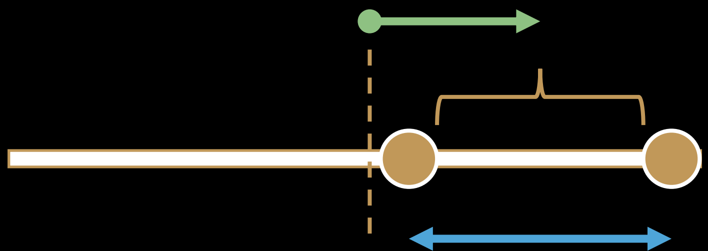
4 - Why Do We Need A Model?

Boundedness
Scale-Inherent Correlation

- ADSM: Absolute Distance from Scale Midpoint

ADSM:

$$\left|0.5 - \frac{y_{ij}^{(L)} + y_{ij}^{(U)}}{2}\right|$$

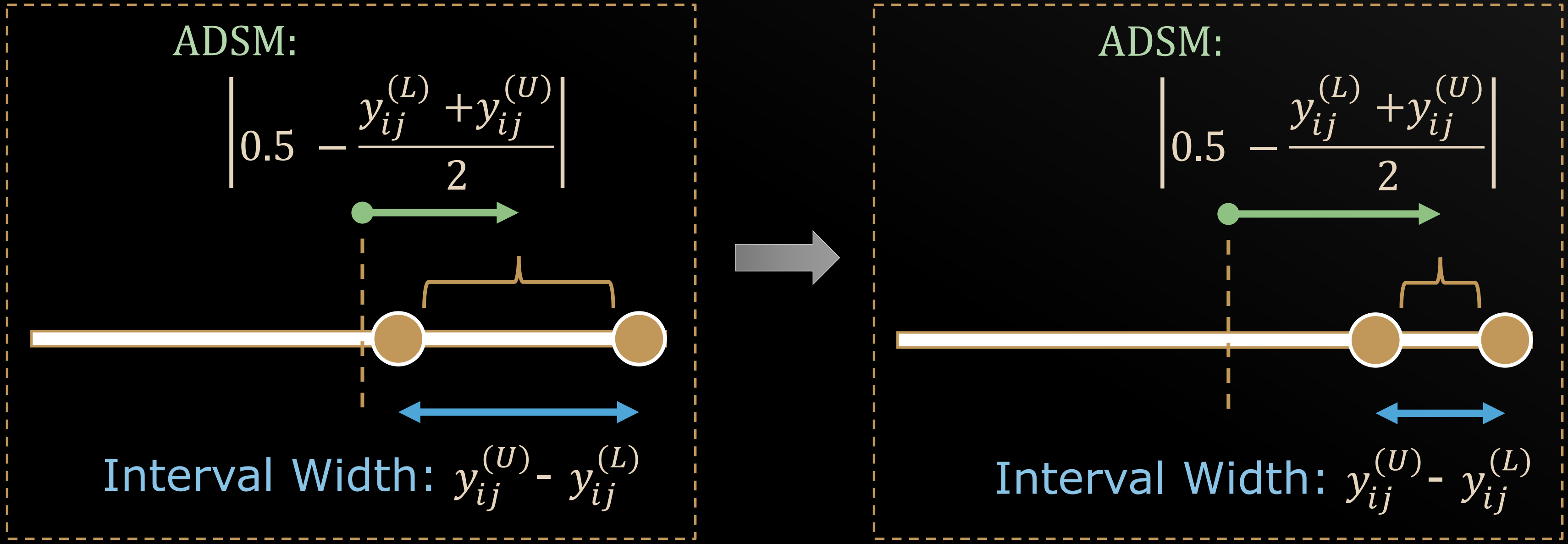


Interval Width: $y_{ij}^{(U)} - y_{ij}^{(L)}$

Boundedness

Scale-Inherent Correlation

- ADSM: Absolute Distance from Scale Midpoint



- Negative correlation between ADMS and Interval Width

Boundedness
Comparison of Correlations

Manifest correlation: person mean scores

- ADSM
- Interval Width

Latent variable correlation: person parameters

- Absolute Location ($|\theta|$; remember: $M(\theta) = 0$)
- Expansion (η)

Boundedness
Comparison of Correlations

	Mean scores	Model parameters
Empirical:	$r = -.57$	$r = -.19$
Simulation:	$r = -.74$	$r = -.02$
True:		$(r = -.002)$

- ➢The model accounts for the scale-inherent correlation

Convergent Validity: Response Formats

Mean Scores vs. Estimates

Manifest correlation: person mean scores

- Single response
- Interval location (midpoint)

Latent variable correlation: person parameters

- Person location θ_{BRM}
- Person location θ_{DDRM}

Convergent Validity: Response Formats

Correlations in the Empirical Study:

Mean scores: $r = .81$

Model Parameters: $r = .87$

- Latent model improves convergence

Take Home Points

- High **convergent validity** of response formats
- Model **accounts** for **boundedness**
- **Additional information**: expansion dimension
  - Validity? What does it measure?
- Useful tool for analysis of interval responses

Thanks to:

- Prof. Dr. Daniel W. Heck

- Prof. Dr. Andreas Voss

- Dr. Raphael Hartmann

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Slides: https://github.com/matthiaskloft/

References

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Additional Slides

Interval Types

I) Conjunctive Set:

- All valid values
- Conceptualization we used

II) Disjunctive Set:

- Only one valid value

Beta Response Model (BRM)

Noel & Dauvier, 2007

$$\underbrace{x_{ij}}_{\dfrac{A_{ij}}{A_{ij}+D_{ij}}} \qquad \underbrace{1\text{-}x_{ij}}_{\dfrac{D_{ij}}{A_{ij}+D_{ij}}}$$

- *A*: Agreement
- *D*: Disagreement

Beta Response Model (BRM)

Noel & Dauvier, 2007

x_{ij} $\quad$ $1\text{-}x_{ij}$

$\dfrac{A_{ij}}{A_{ij}+D_{ij}}$ $\quad$ $\dfrac{D_{ij}}{A_{ij}+D_{ij}}$

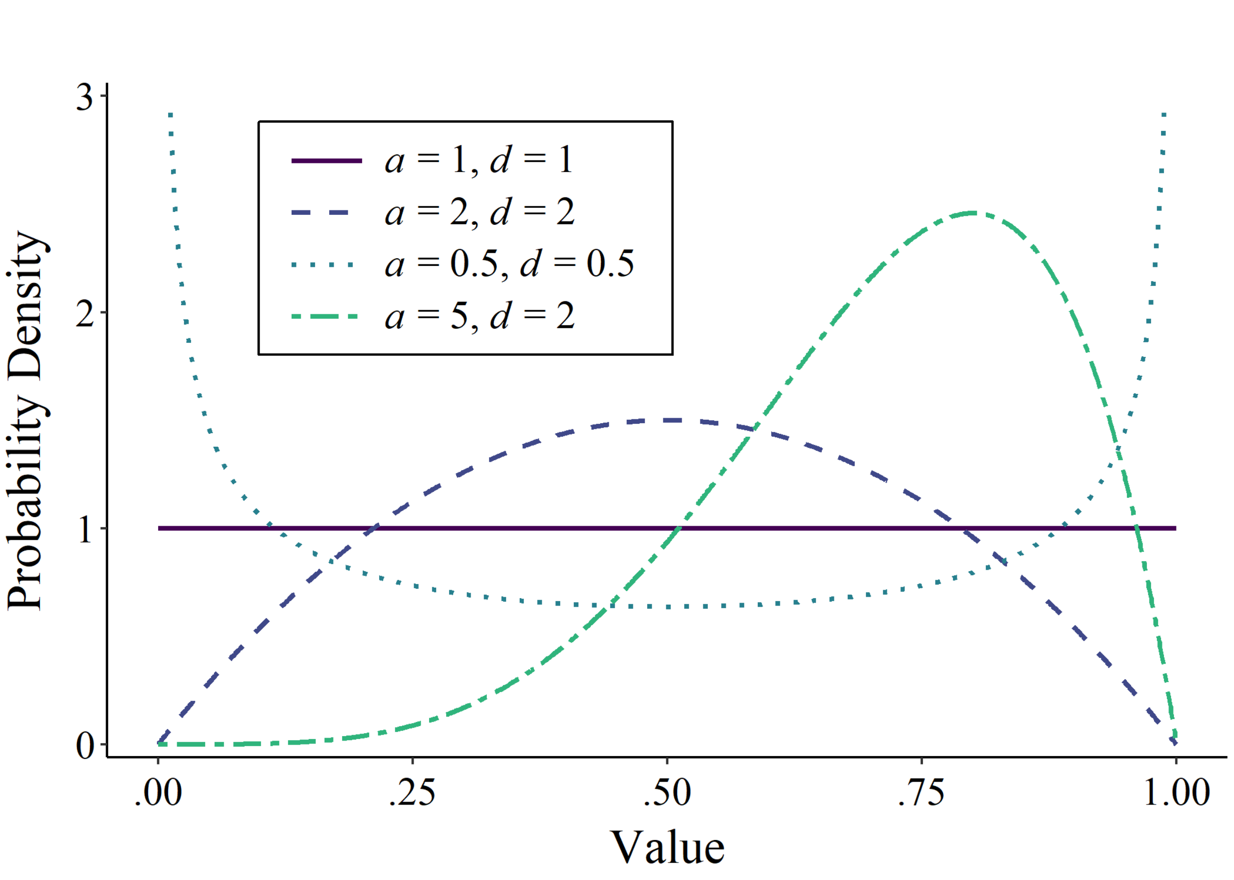
- *A*: Agreement
- *D*: Disagreement

$$A_{ij} \sim \Gamma(a_{ij}, s)$$

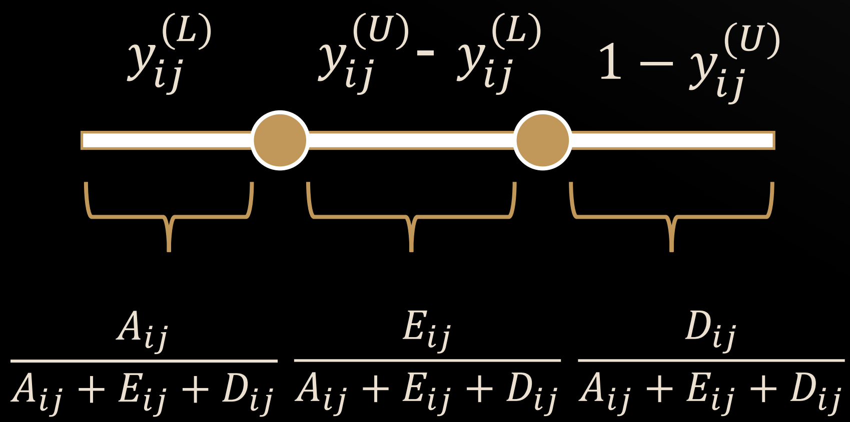
$$D_{ij} \sim \Gamma(d_{ij}, s)$$

➢ $x_{ij} \sim \text{Beta}(a_{ij}, d_{ij})$

Beta Distribution



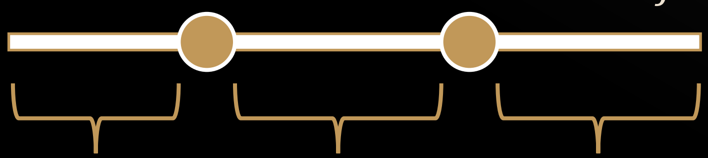
Dirichlet Dual Response Model (DDRM)



- *A*: Agreement
- *E*: Expansion
- *D*: Disagreement

Dirichlet Dual Response Model (DDRM)

$y_{ij}^{(L)}$ $\quad y_{ij}^{(U)} - y_{ij}^{(L)}$ $\quad 1 - y_{ij}^{(U)}$



$\frac{A_{ij}}{A_{ij} + E_{ij} + D_{ij}}$ $\quad \frac{E_{ij}}{A_{ij} + E_{ij} + D_{ij}}$ $\quad \frac{D_{ij}}{A_{ij} + E_{ij} + D_{ij}}$

- *A*: Agreement
- *E*: Expansion
- *D*: Disagreement

$$A_{ij} \sim \Gamma(a_{ij}, s)$$
$$E_{ij} \sim \Gamma(e_{ij}, s)$$
$$D_{ij} \sim \Gamma(d_{ij}, s)$$

➢ $x_{ij} \sim \text{Dirichlet}(a_{ij}, e_{ij}, d_{ij})$

Dirichlet Dual Response Model (DDRM)

Parameters: Ability / Difficulty

$$a_{ij} = \exp\left[\alpha_\lambda\left(\theta_i - \delta_j\right) + \tau_j\right] \quad d_{ij} = \exp\left[-\alpha_\lambda\left(\theta_i - \delta_j\right) + \tau_j\right]$$

$$e_{ij} = \exp\left[\alpha_\epsilon\left(\eta_i + \gamma_j\right) + \tau_j\right]$$

θ_i: Person location (ability)
δ_j: Item location (difficulty)

- ➢ Classic interpretation

Dirichlet Dual Response Model (DDRM)

Parameters: Scaling

$$a_{ij} = \exp\left[\alpha_\lambda\left(\theta_i - \delta_j\right) + \tau_j\right] \quad d_{ij} = \exp\left[-\alpha_\lambda\left(\theta_i - \delta_j\right) + \tau_j\right]$$

$$e_{ij} = \exp\left[\alpha_\epsilon\left(\eta_i + \gamma_j\right) + \tau_j\right]$$

$\pm\alpha_\lambda > 0$: Scaling, location dimension

$\pm\alpha_\epsilon > 0$: Scaling, expansion dimension

➢ Not a discrimination parameter!

Dirichlet Dual Response Model (DDRM)

Parameters: Precision

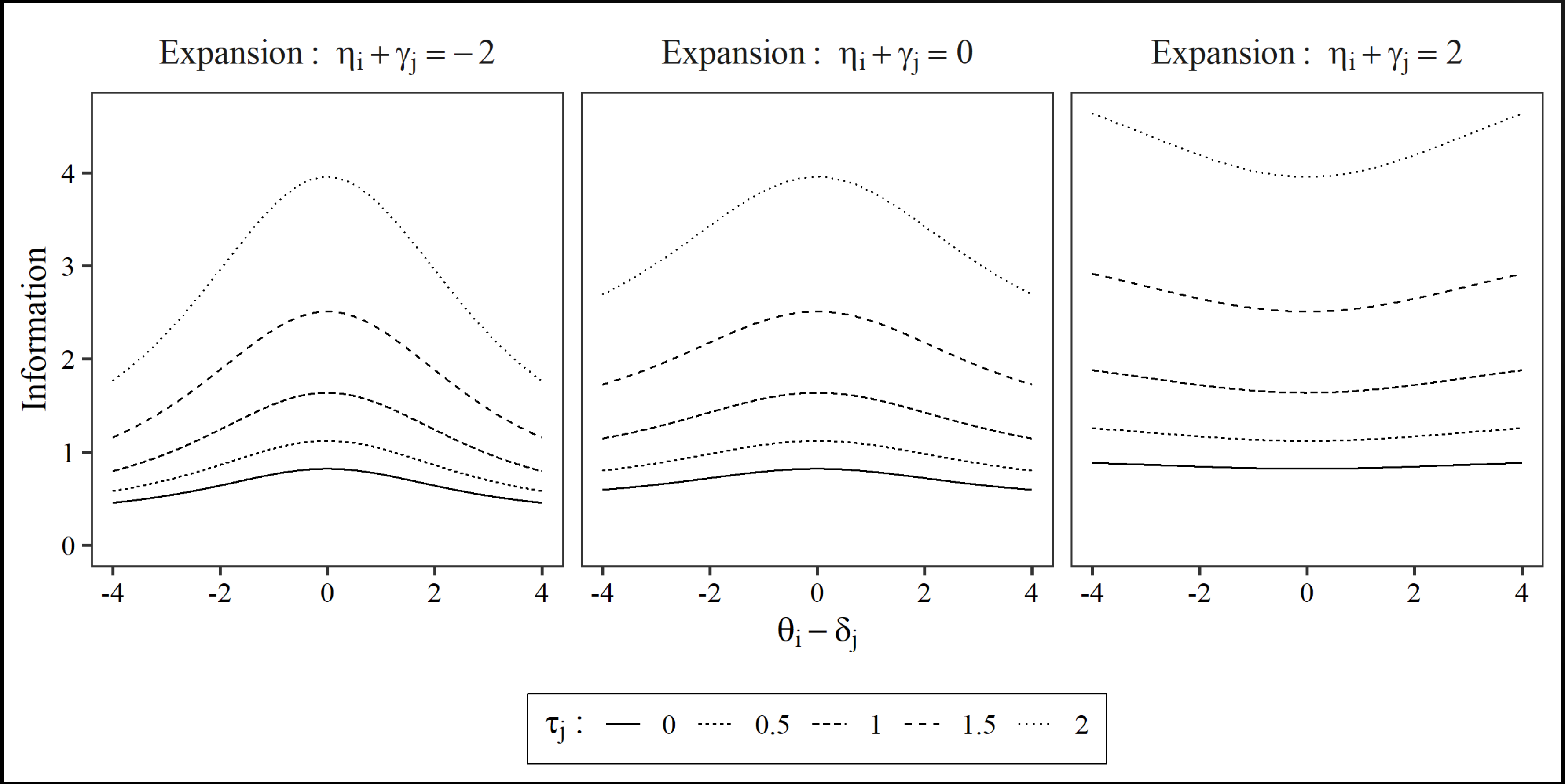
$$a_{ij} = \exp\left[\alpha_\lambda(\theta_i - \delta_j) + \tau_j\right] \qquad d_{ij} = \exp\left[-\alpha_\lambda(\theta_i - \delta_j) + \tau_j\right]$$

$$e_{ij} = \exp\left[\alpha_\epsilon(\eta_i + \gamma_j) + \tau_j\right]$$

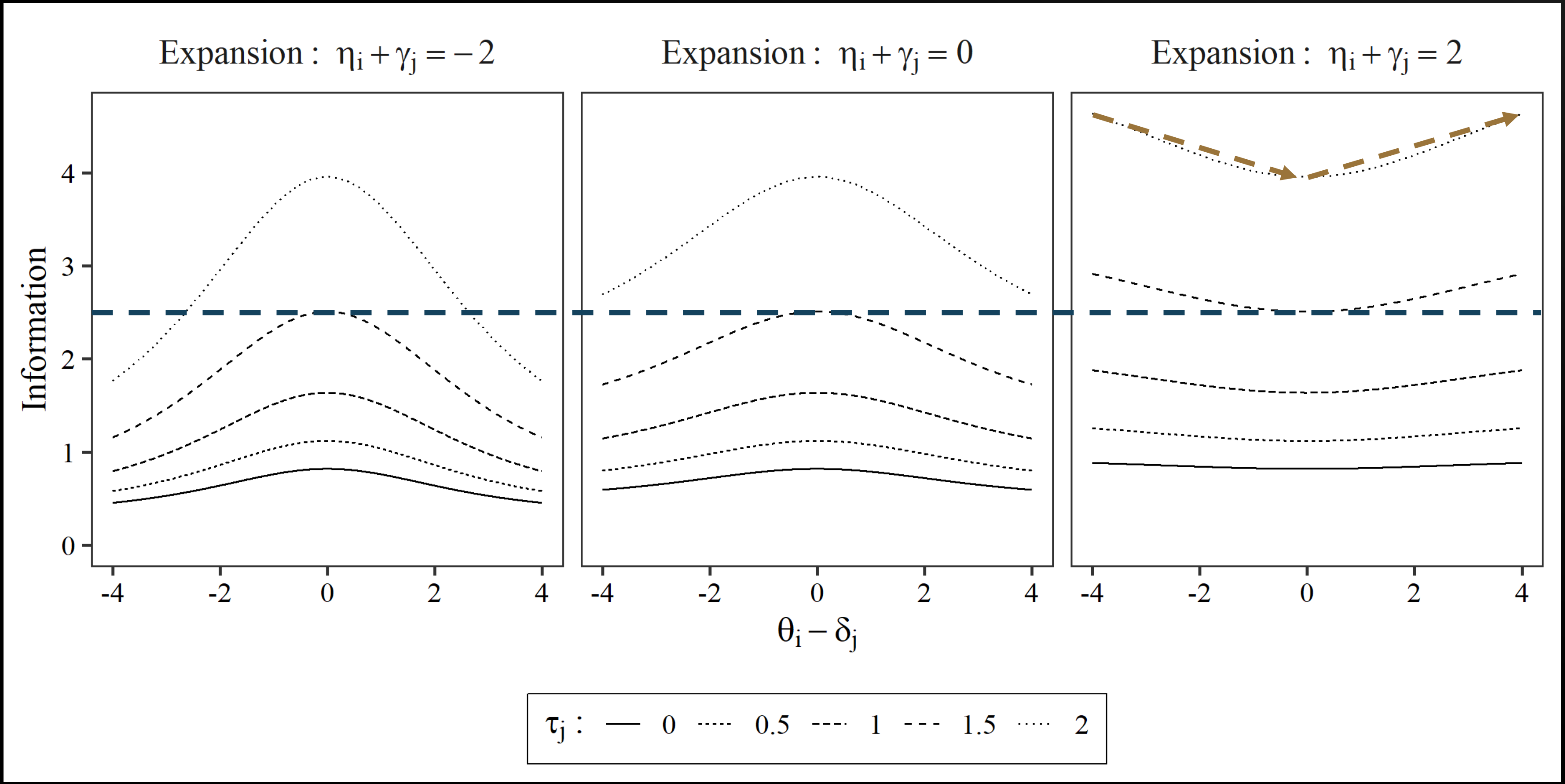
$\tau_j > 0$: Item precision (all three additive!)

- ➢ Steeper density curves

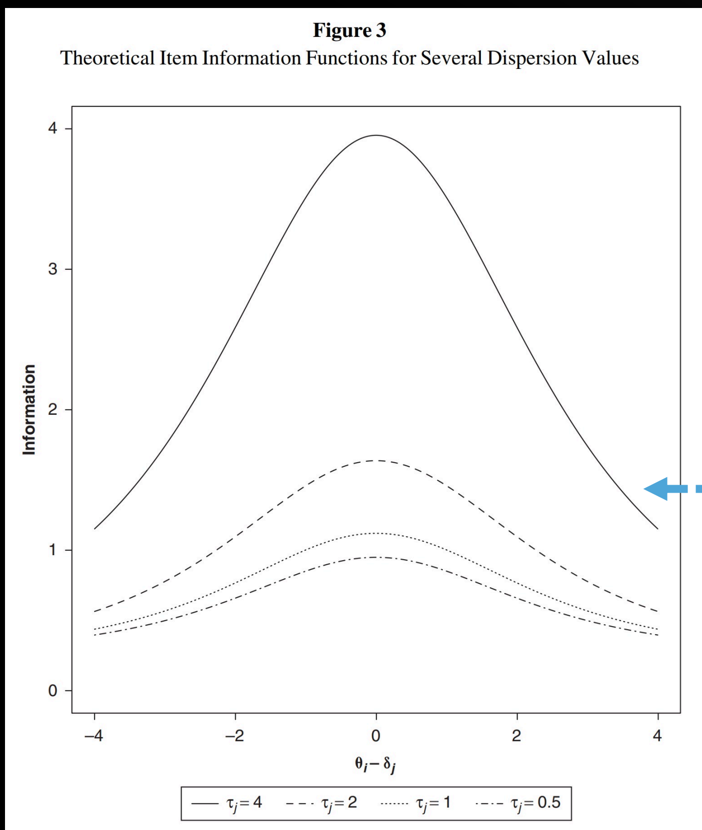
Item Information
Location Dimension



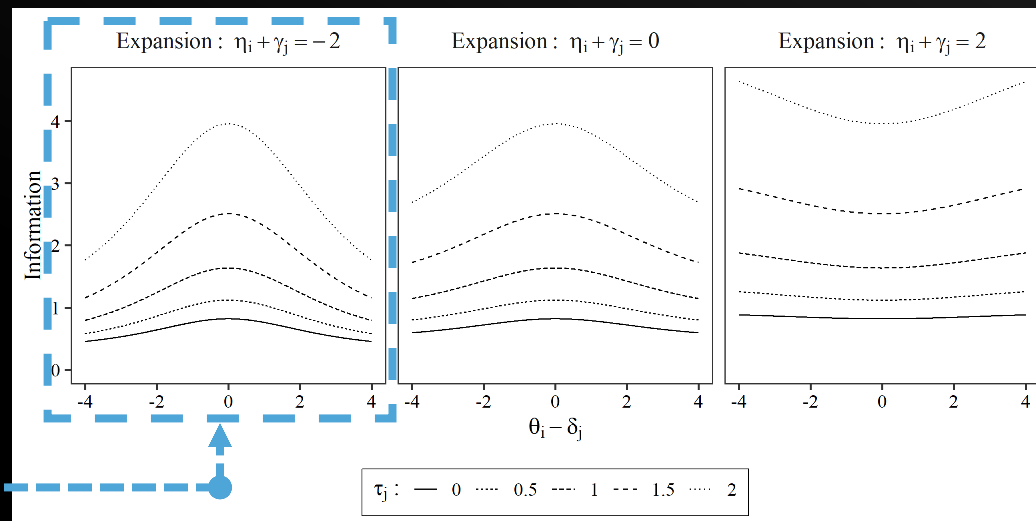
Item Information
Location Dimension



Item Information
Comparison with Beta Response Model

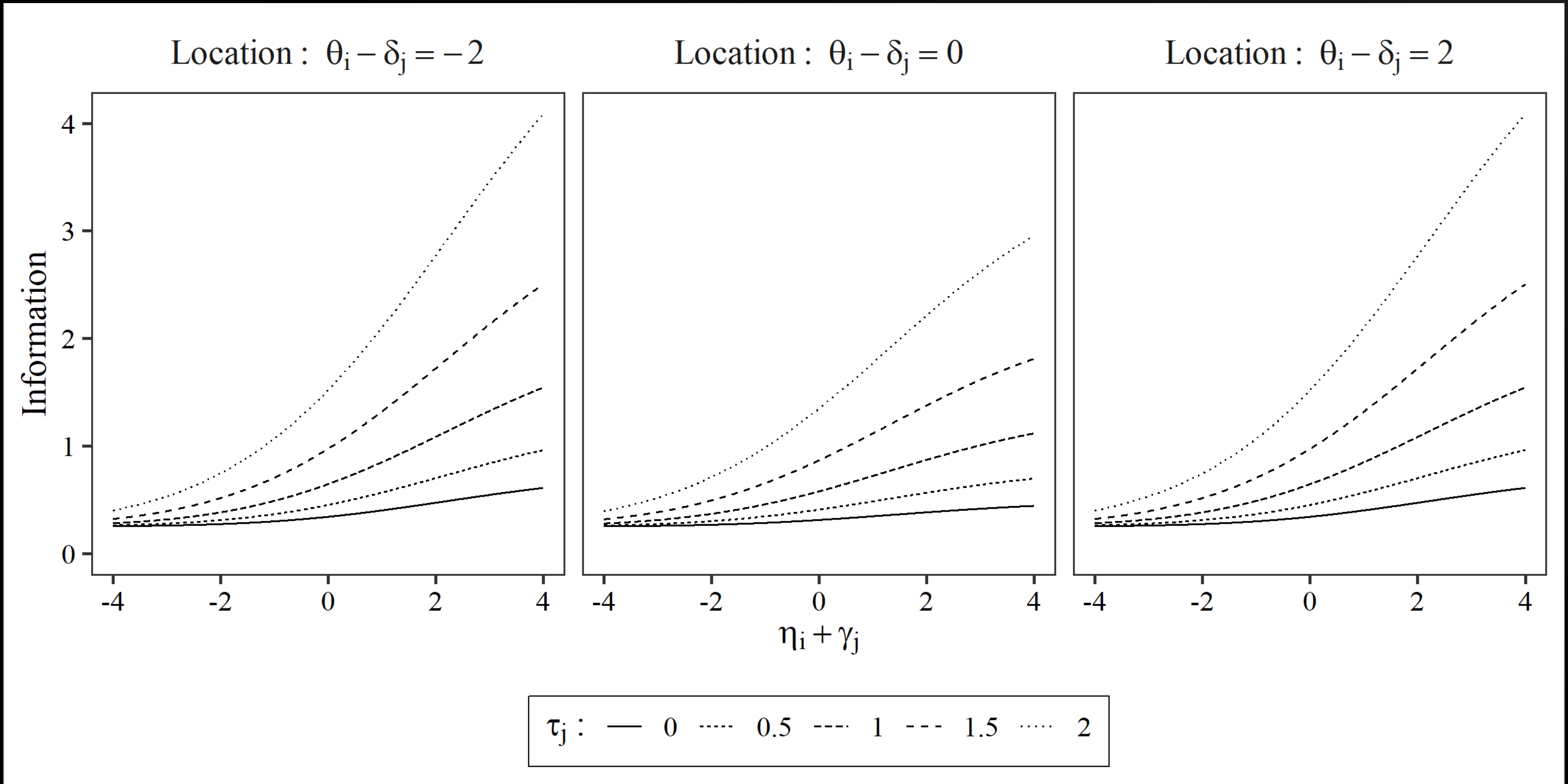


BRM (Noel & Dauvier, 2007)

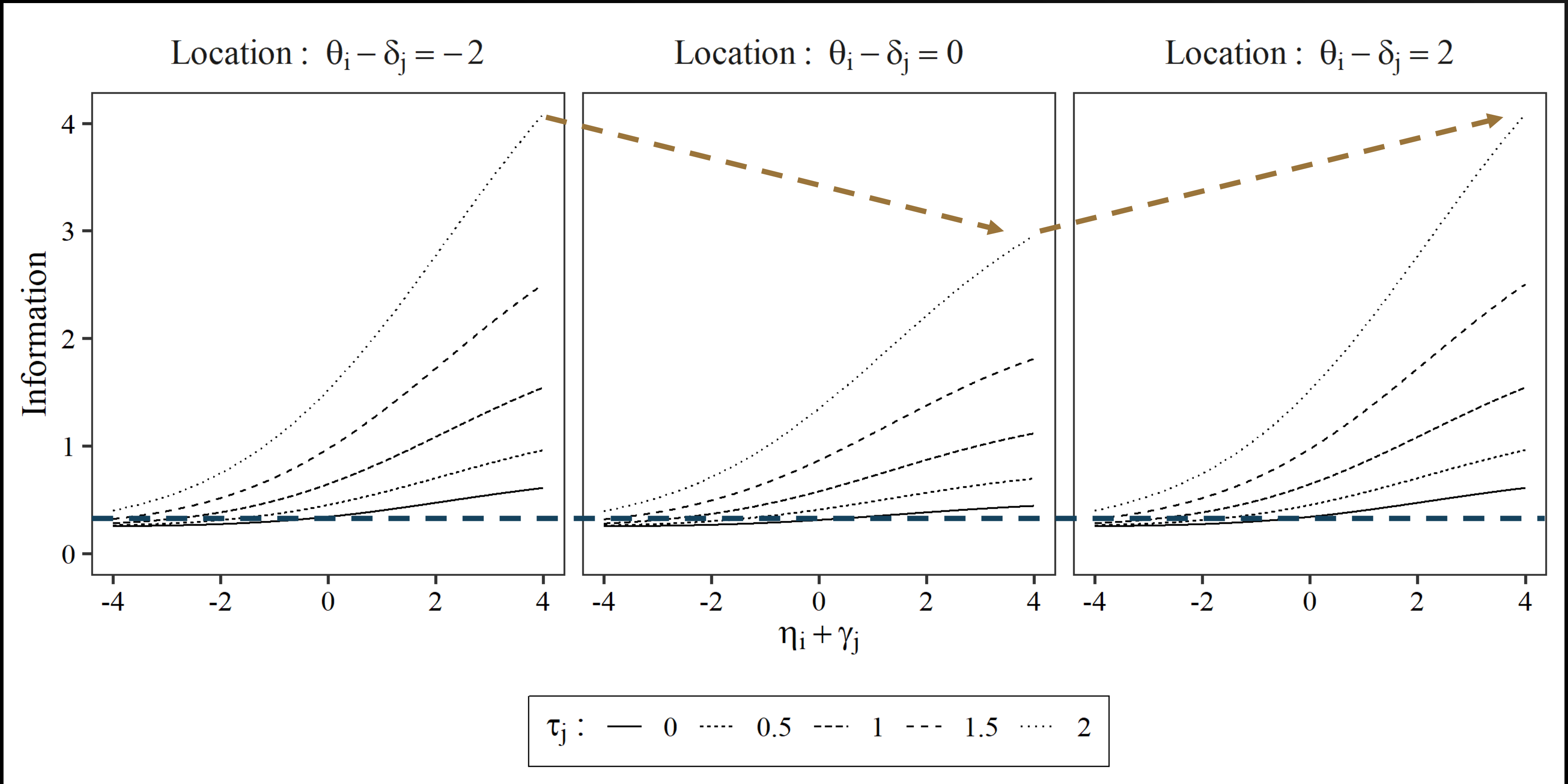


DDRM

Item Information
Expansion Dimension



ITEM INFORMATION
Expansion Dimension



Item Information

Conclusion

High sensitivity when:

- Location dimension = low / high (away from zero)
- Expansion dimension: high

➢ More information when response needs to be pushed towards the bounds of the response scale

3 - Simulation

for the DDRM

Setup

Numbers of

- Persons: 100, 250, 500
- Items: 10, 15, 20, 30
- Replications per condition: 200

Person Parameters:

- $\theta_i, \eta_i \sim N(0,1)$

Item Parameters:

- $\delta_j, \gamma_j \sim \text{sequence } [\text{-}2, 2] \text{ by } 4/n_{items}$
- $\tau_j \sim U(0,2)$

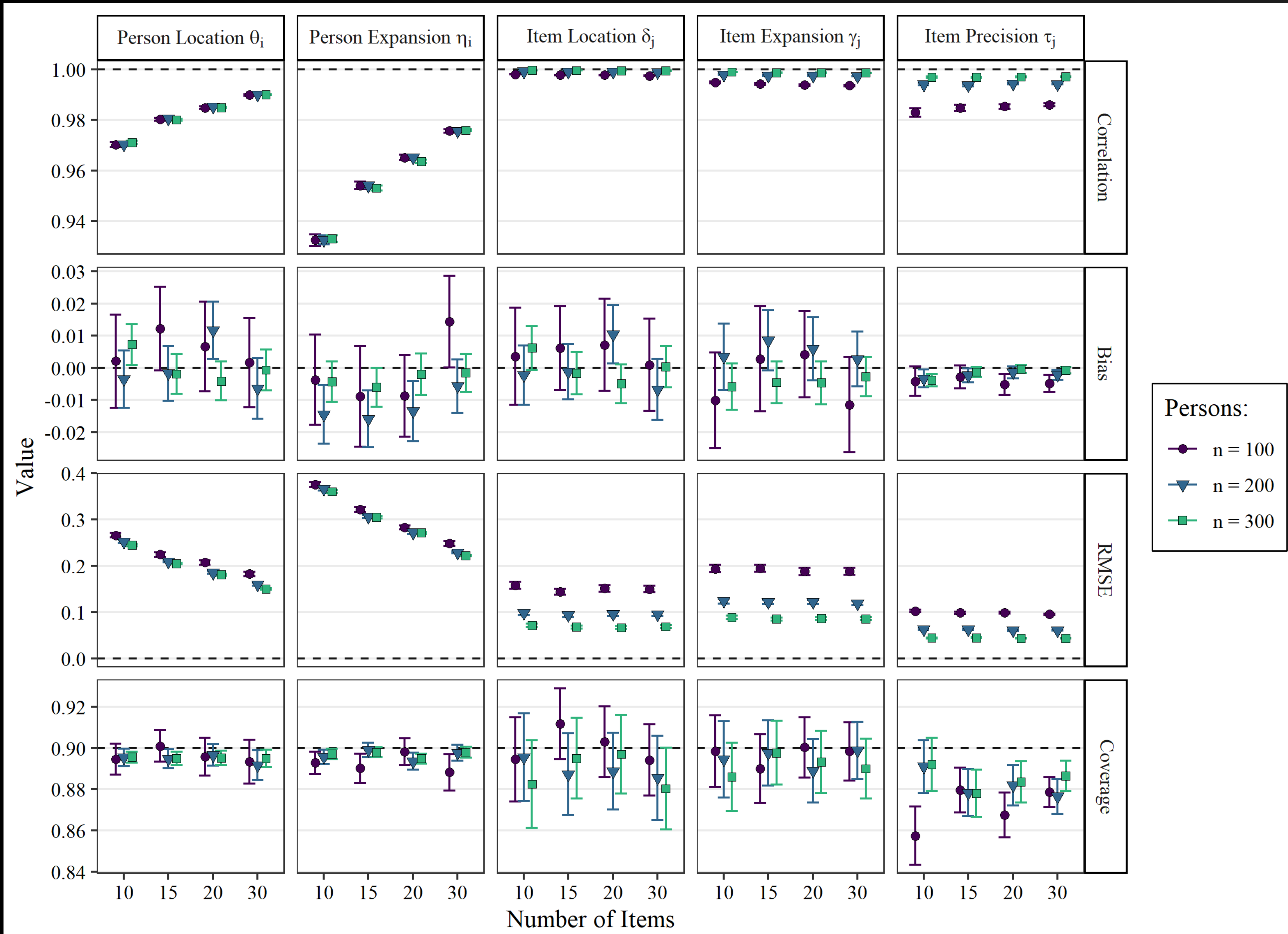
Scaling Parameters:

- $\alpha_\lambda, \alpha_\epsilon = 0.5$

Fit Measures

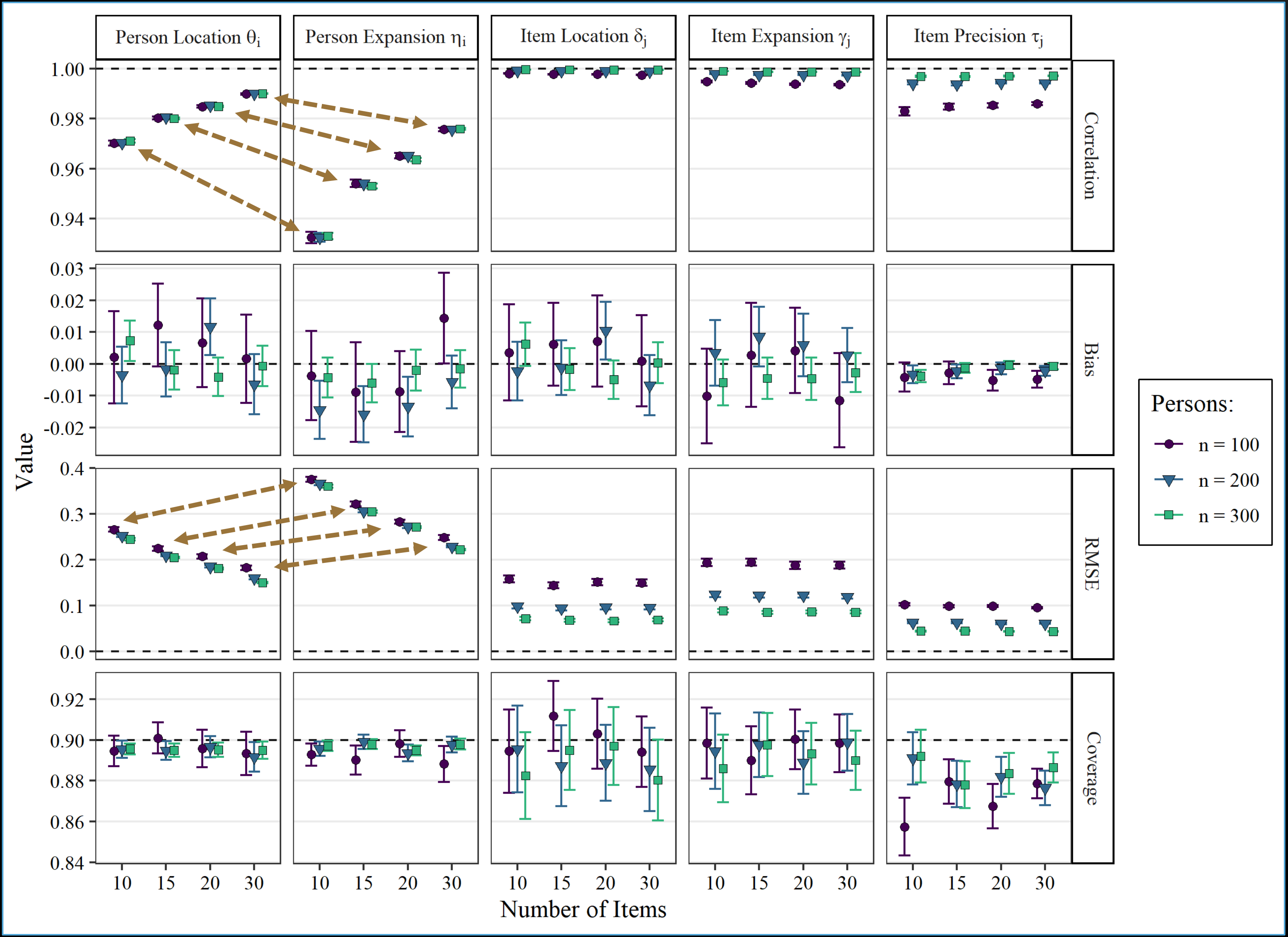
- Correlation: true vs. estimated
- Mean Signed Difference (Bias)
- Root Mean Squared Error (RMSE)
- Coverage: 90%CIs

RESULTS



Person Location θi
Person Expansion ηi
Item Location δj
Item Expansion γj
Item Precision τj
Correlation
Bias
RMSE
Coverage
Value
Number of Items
1.00
0.98
0.96
0.94
0.03
0.02
0.01
0.00
-0.01
-0.02
0.4
0.3
0.2
0.1
0.0
0.92
0.90
0.88
0.86
0.84
10 15 20 30
Persons:
n = 100
n = 200
n = 300

Results



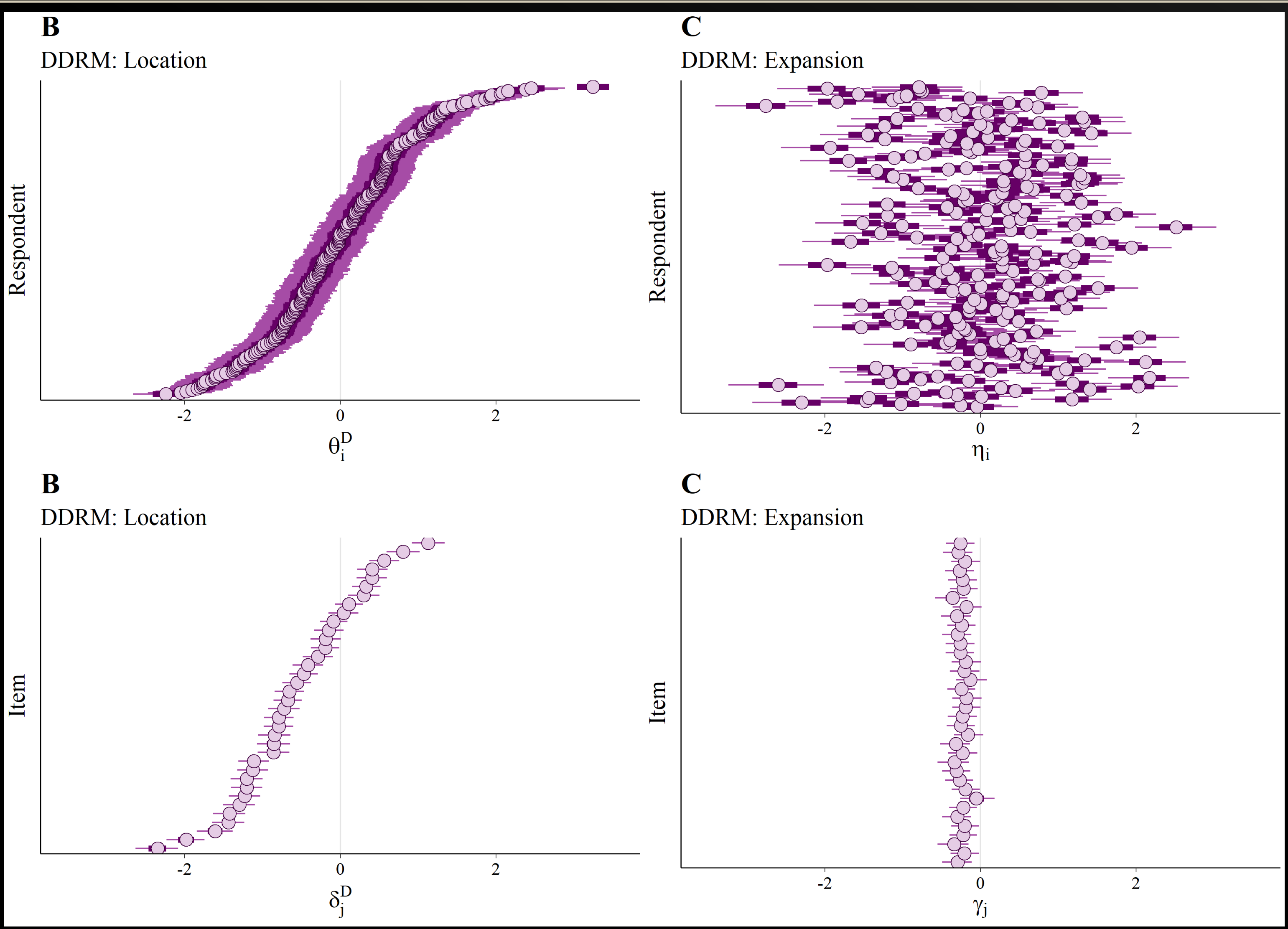
Person Location θ_i
Person Expansion η_i
Item Location δ_j
Item Expansion γ_j
Item Precision τ_j
Correlation
Bias
RMSE
Coverage
Value
Number of Items
Persons:
n = 100
n = 200
n = 300

Recommendations

- Use more than 200 persons
- Use more than 15 items

Parameter Estimates

Top: Person Bottom: Item



6 – Future Research

Future Research

- Application to rating- and forecasting data
  - Cultural consensus models
  - Are certain respondents more accurate?
- Test-retest reliability
- Discriminant validity:
  - useful information vs. response biases