A Multivariate Logit-Function for Modeling Continuous Bounded Interval Responses

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International Meeting of the Psychometric Society 2024

Motivation

When **ONE** Response Value is **NOT** Enough

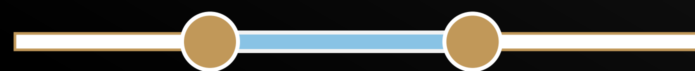
"What percentage of your daily work time did you spend on preparing for IMPS 2024 in the last week?"

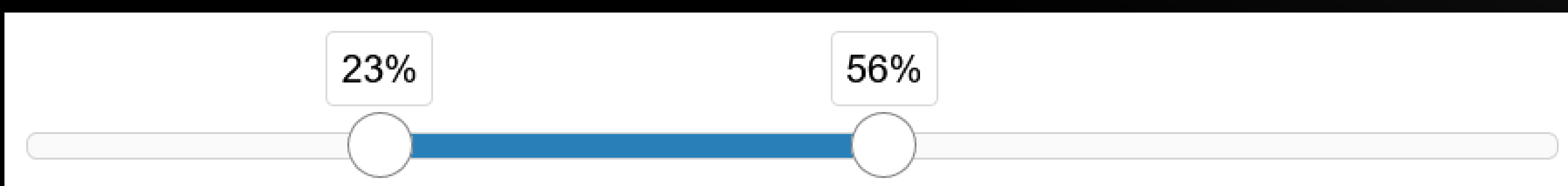
Interval Responses

Dual-range slider (**DRS**)

Interval Responses

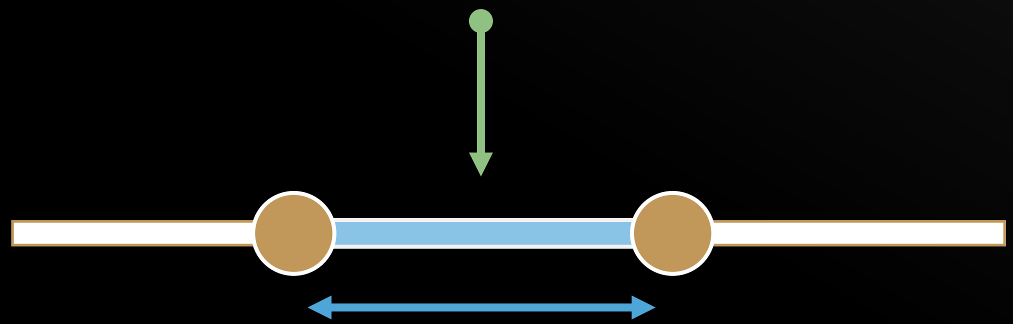
Dual-range slider (**DRS**)





INTERVAL RESPONSES

DRS **Location**: $\dfrac{y^{(L)}+y^{(U)}}{2}$



DRS **Width**: $y^{(U)} - y^{(L)}$

Interval Responses

Variability / plausible range:

- Self-ratings, stimuli

Uncertainty / expertise:

- Estimation (e.g, forecasting)

Ambiguity:

- Item content unclear
- No clear-cut true answer (e.g., verbal quantifiers like “seldom” or “likely”)

Topics of the Talk

What is an appropriate link function for interval responses?

- Smithson & Broomell (2024)

Application: consensus model

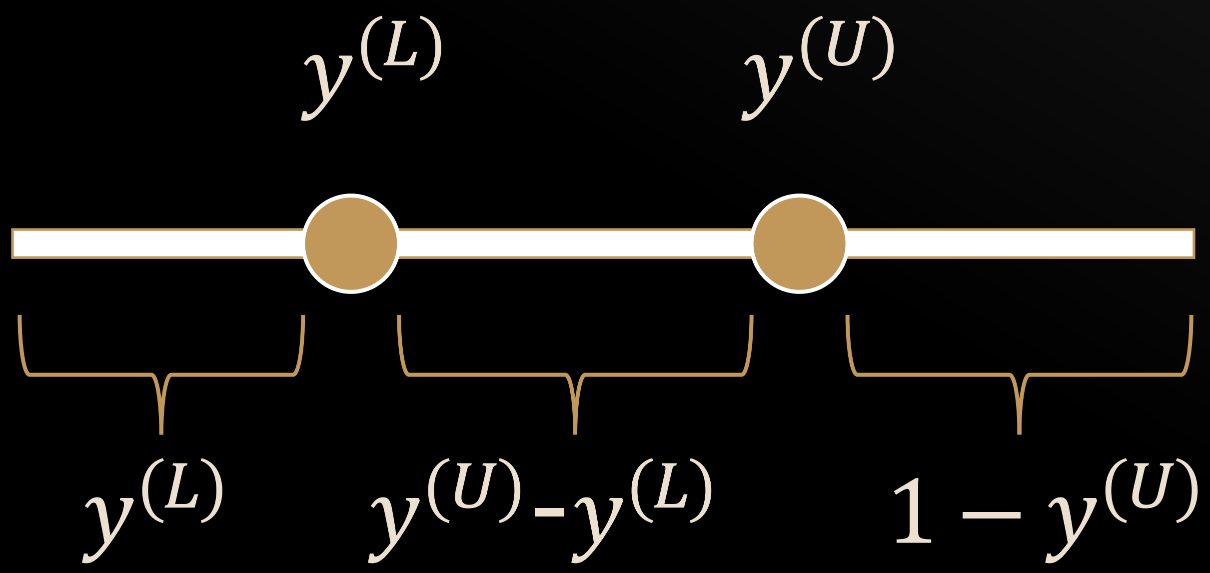
- Kloft et al. (2024, in preparation)

A Link Function for Interval Responses

Smithson & Broomell (2024)

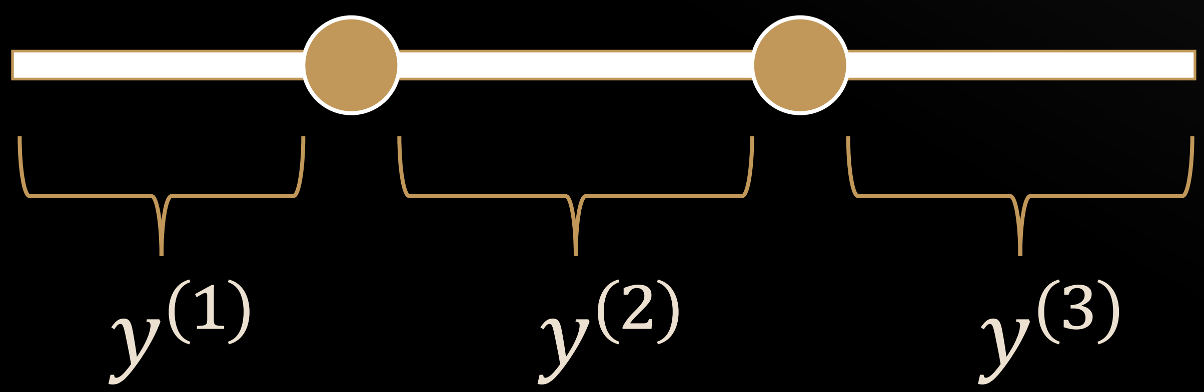
COMPOSITIONAL DATA

- Components must sum to one: simplex



COMPOSITIONAL DATA

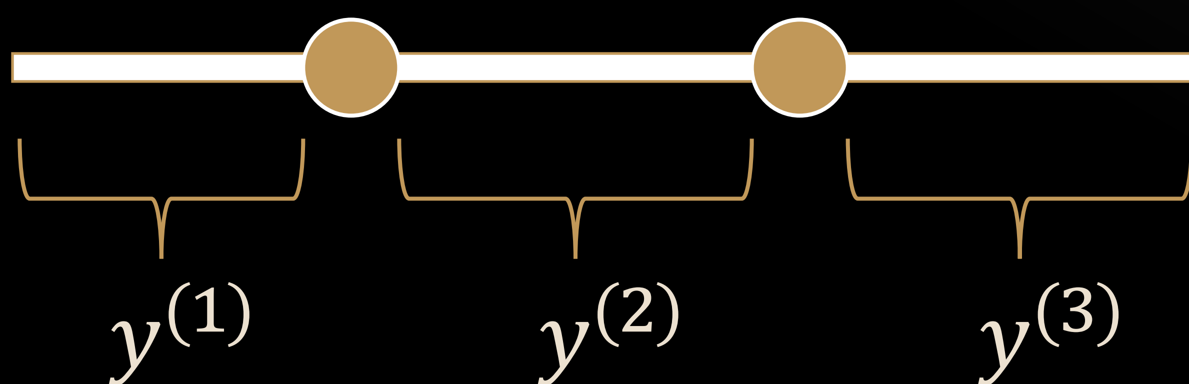
- Components must sum to one: simplex



Log-Ratios

Unbounded **Location**: $\log\left(\frac{y^{(1)}}{y^{(3)}}\right)$

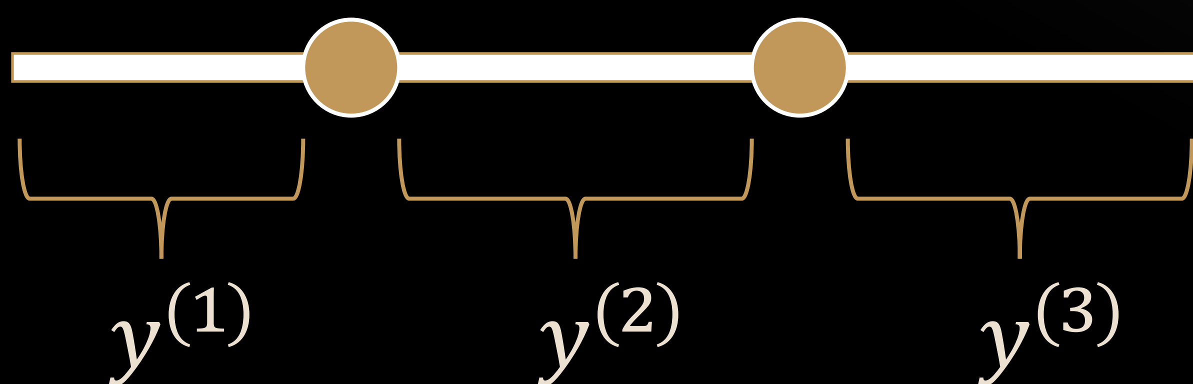
- Compares outer components



LOG-RATIOS

Unbounded Width: $\log\left(\frac{y^{(2)}}{\sqrt{y^{(1)} \times y^{(3)}}}\right)$

- Compares interval width to geometric mean of outer components



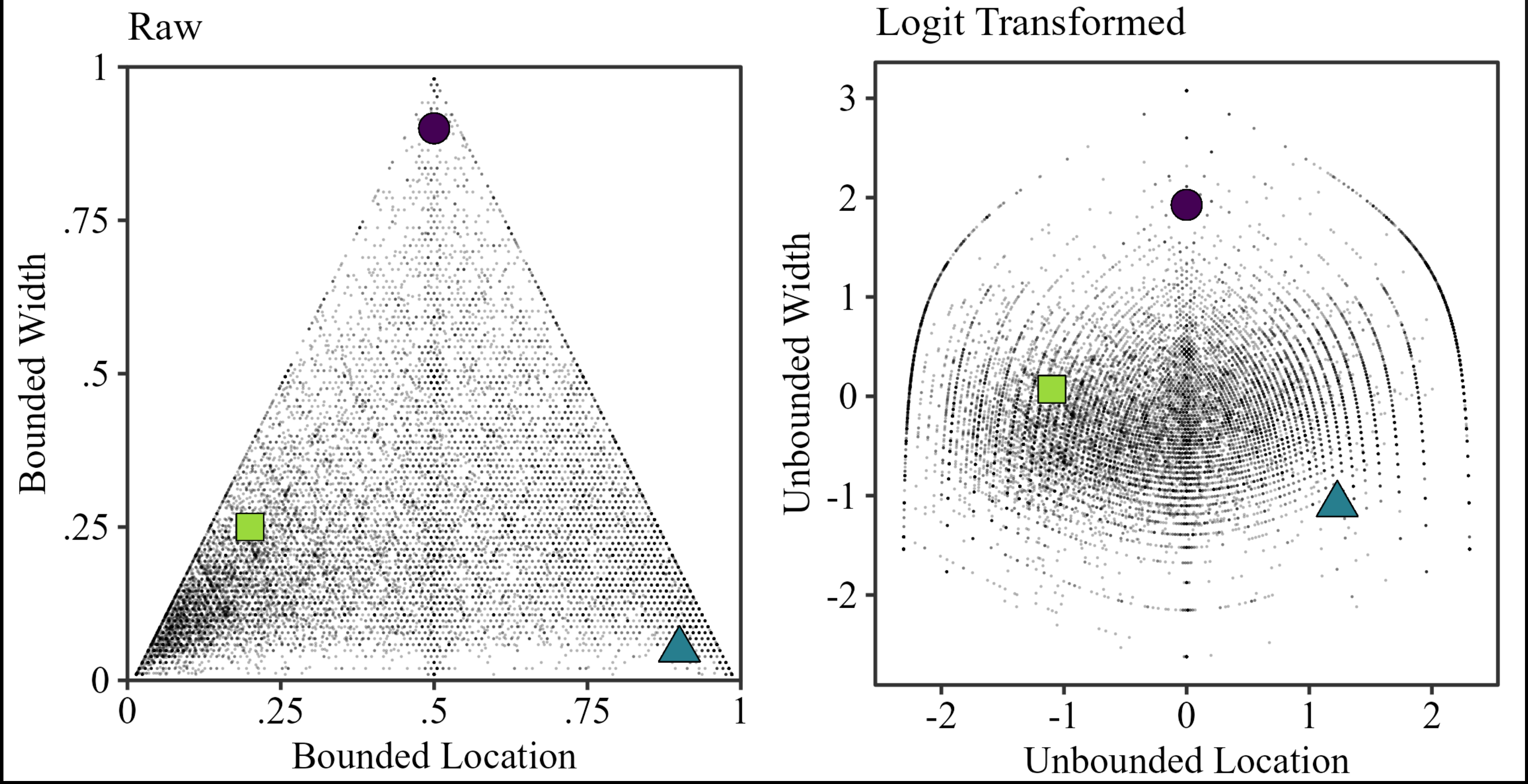
Isometric Log-Ratio Transformation

- Smithson & Broomel (2024)

$$\mathbf{z} = \begin{pmatrix} z^{loc} \\ z^{wid} \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{1}{2}} \log\left(\frac{y^{(1)}}{y^{(3)}}\right) \\ \sqrt{\frac{2}{3}} \log\left(\frac{y^{(2)}}{\sqrt{y^{(1)} \times y^{(3)}}}\right) \end{pmatrix}$$

Data Example

- More suitable for models using a normal distribution



Application: Consensus Model

Kloft et al. (2024, in preparation)

Applications of the Isometric Log-Ratio Transformation

Could just use transformed interval locations **or** widths

- Descriptive statistics
- Factor Analyses (Kloft & Heck, 2024)
  - **Better** model **fit** compared to untransformed intervals

Consensus Model

- **Joint** model for location and width

Interval Truth Model: Overview

- Extension of a univarite logit-normal model (Anders et al., 2014)

➢ Bivariate logit-normal model
  - Link function: isometric log-ratio

- Empirical applications:
  - Weighted aggregate of interval judgments
  - True value is an interval
  - Example: verbal quantifiers („seldom", „often", „likely")

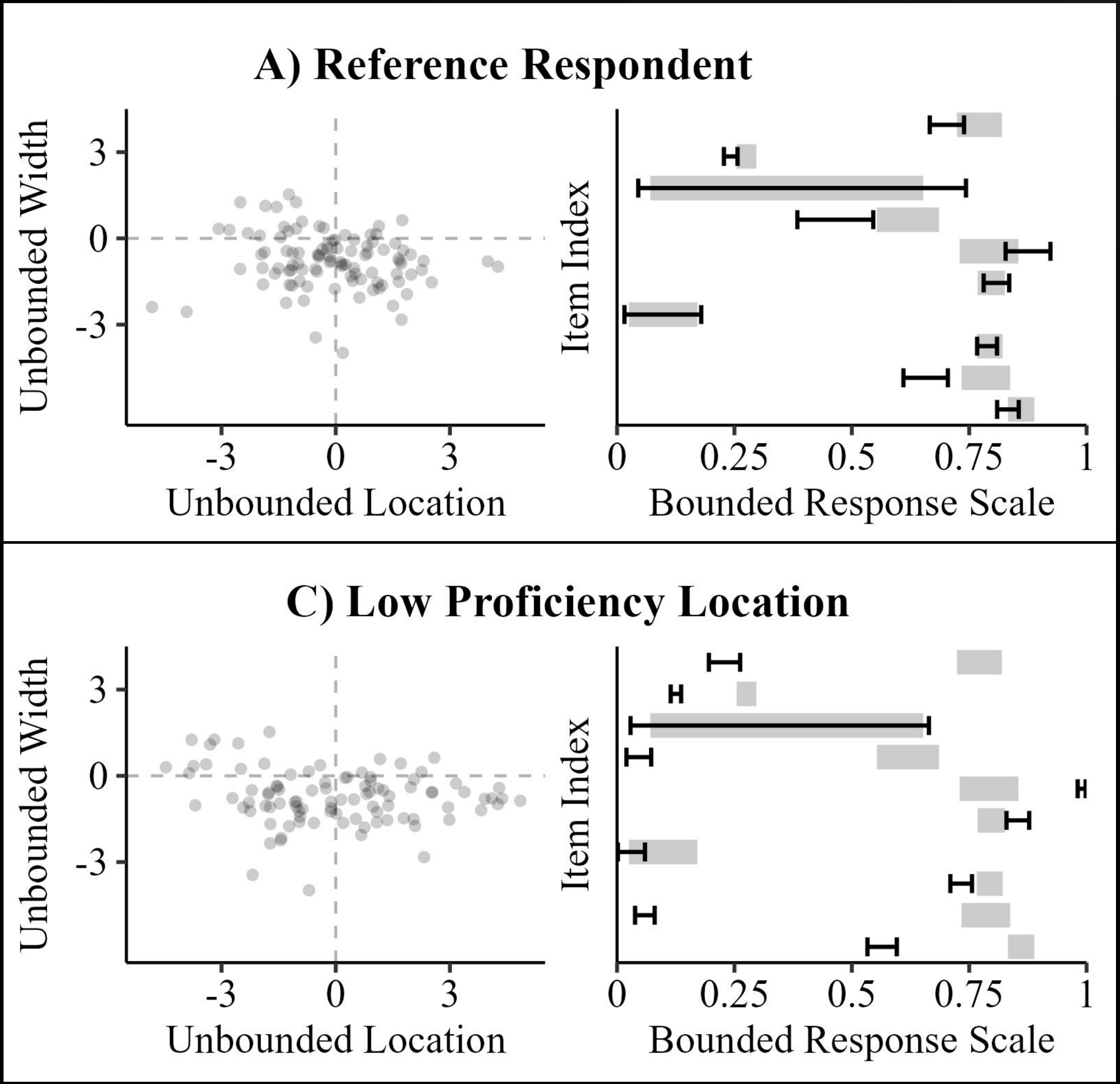
Assumptions & Prerequisites

- Respondent makes a **latent appraisal** of the **true consensus** interval for a group
- **Variance** in the **expertise** level
- Multiple judgments per respondent
- **Expertise** can be used to **weight** judgments in the aggregation of responses
  - Joint estimation of expertise and latent consensus

Model Mechanics: Latent Appraisal

- **Latent appraisal** of the latent consensus plus some error
- Precision based on:
  - **Proficiency** of the respondent
  - **Discernibility** of the item
- Bivariate normal distribution for appraisals
  - 2D: location and width
  - Expected value: latent consensus

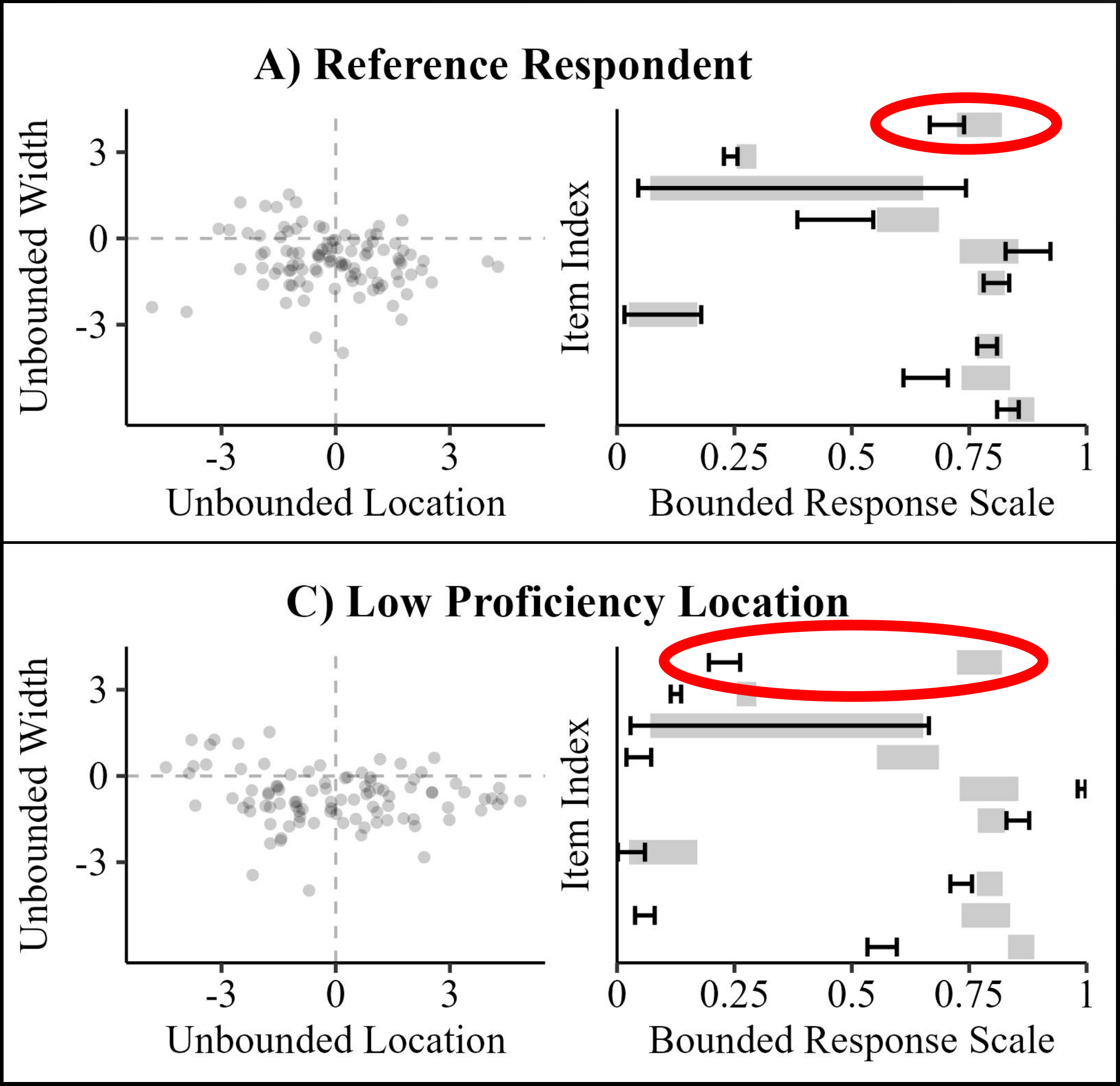
Model Mechanics: Latent Appraisal



High proficiency, **location**

Low proficiency, **location**

Model Mechanics: Latent Appraisal



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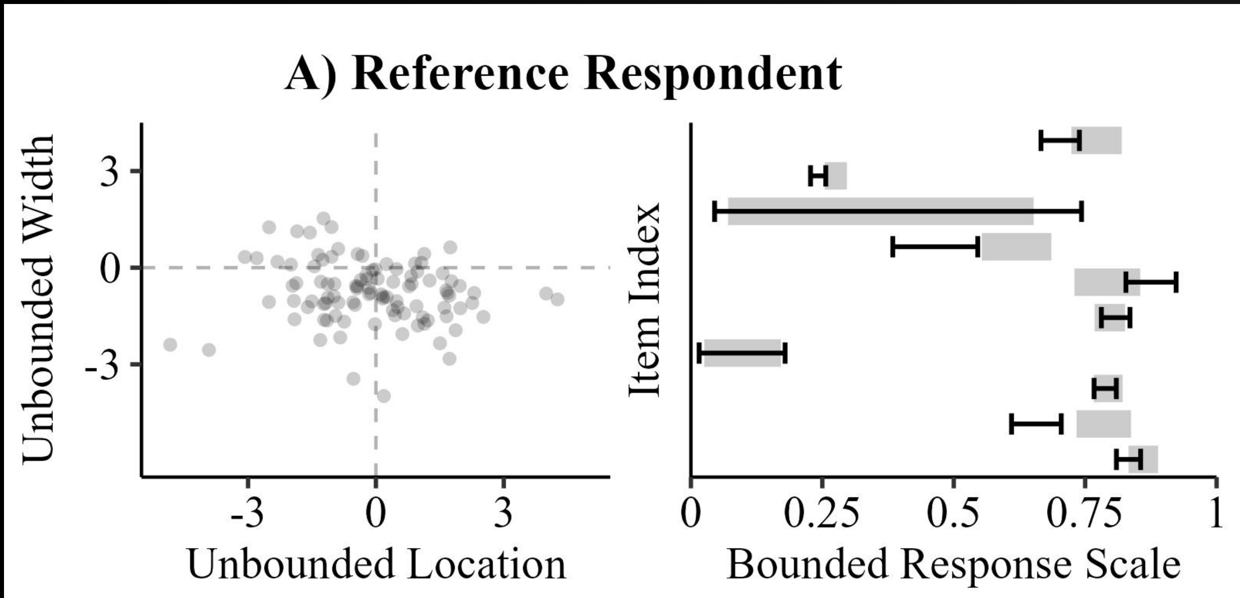
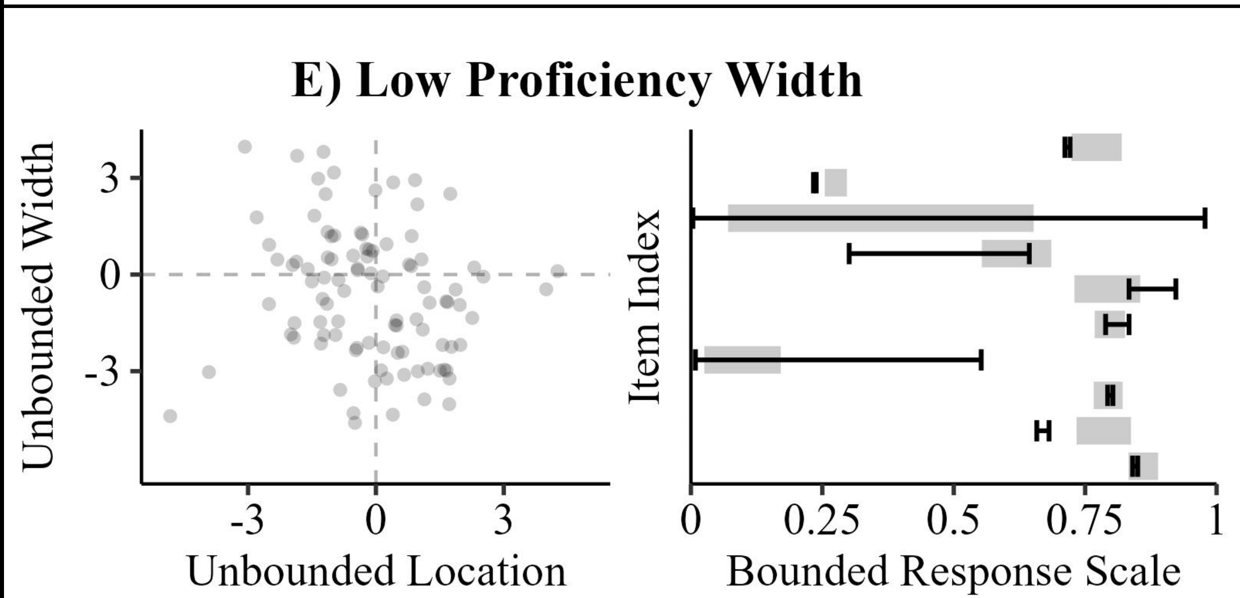
Model Mechanics: Latent Appraisal

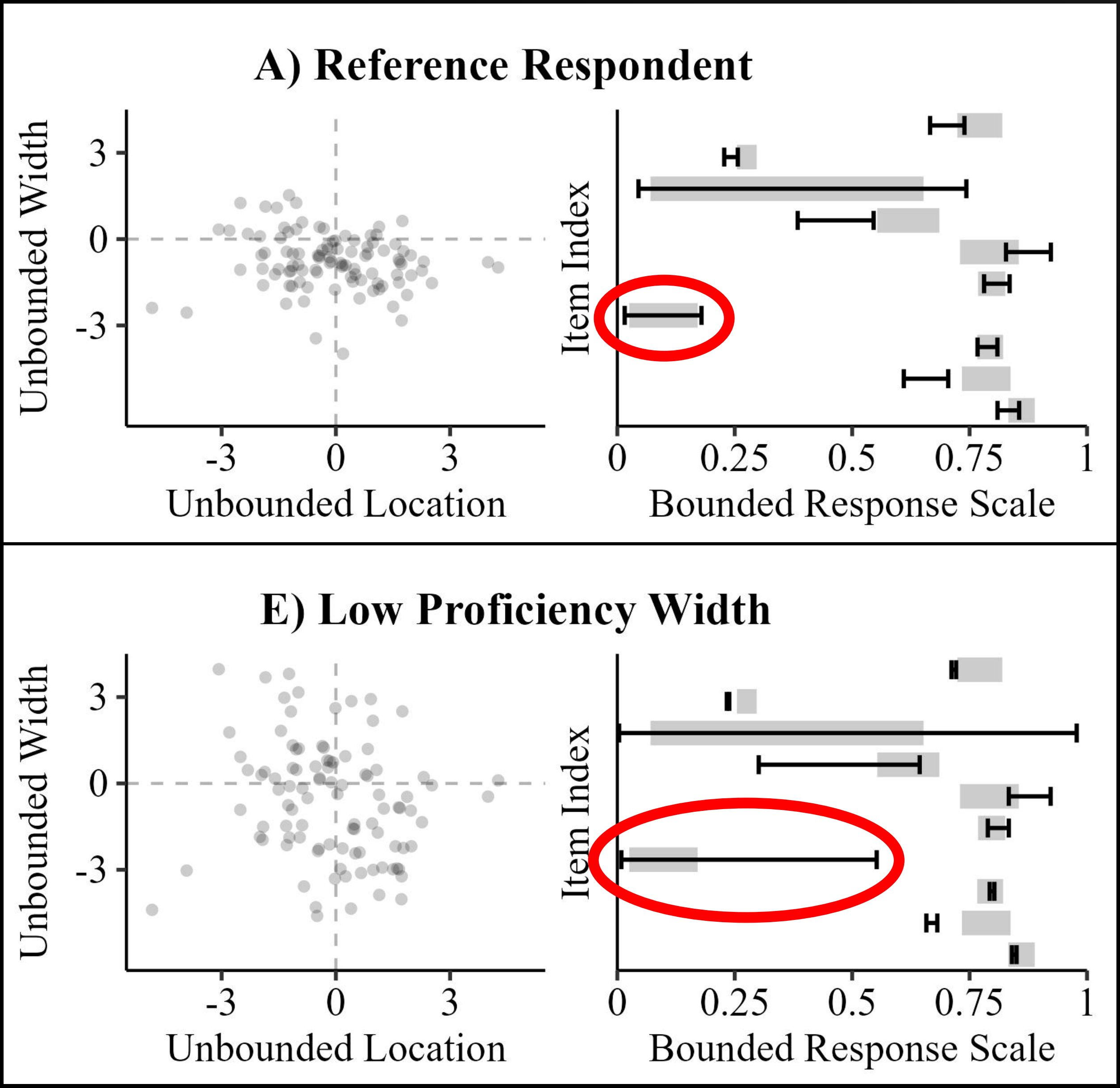




High proficiency, **width**

Low proficiency, **width**

Model Mechanics: Latent Appraisal



High proficiency, **width**

Low proficiency, **width**

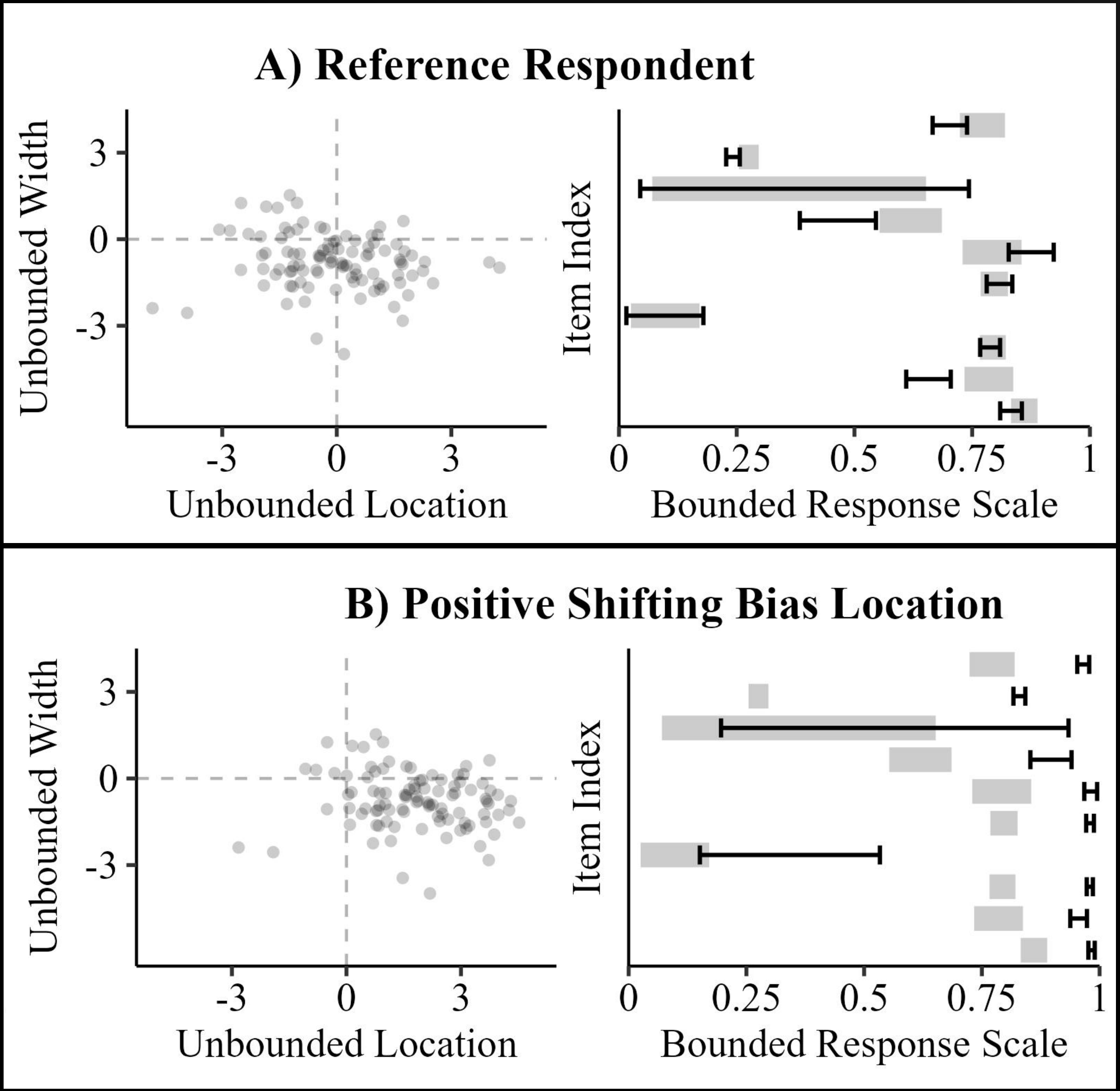
Model Mechanics: Biases

- Latent appraisal is scaled and shifted by respondent`s biases
  - Affect all responses of a particular respondent

- Shifting Biases
  - Interval **locations** shift to the **left/right** on the response scale
  - Interval **widths** get **wider/narrower**

- Scaling Bias (extremity bias):
  - Pushes/pulls the **location outwards/inwards** with respect to the response scale`s center

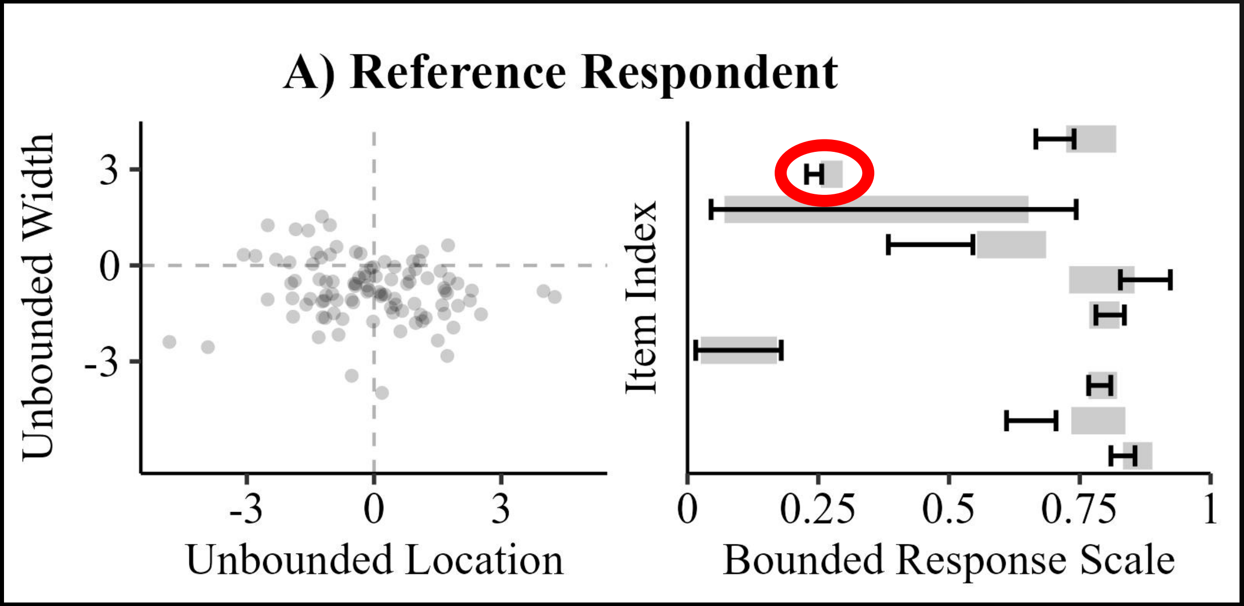
Model Mechanics: Biases



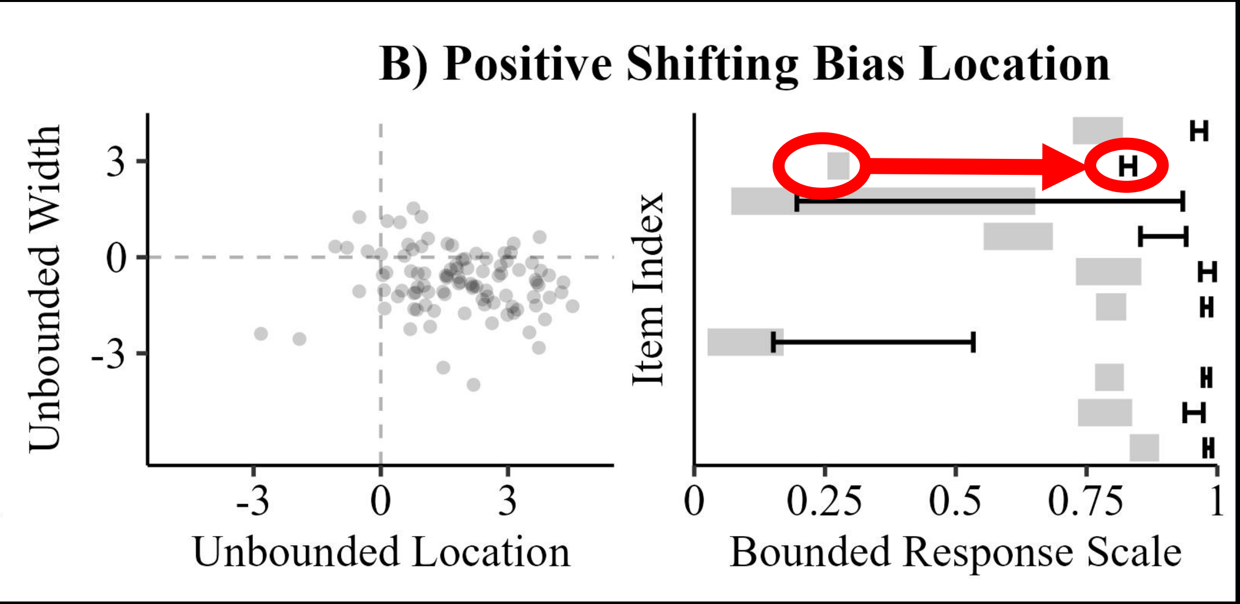
No biases

Positive **location** shifting bias

Model Mechanics: Biases

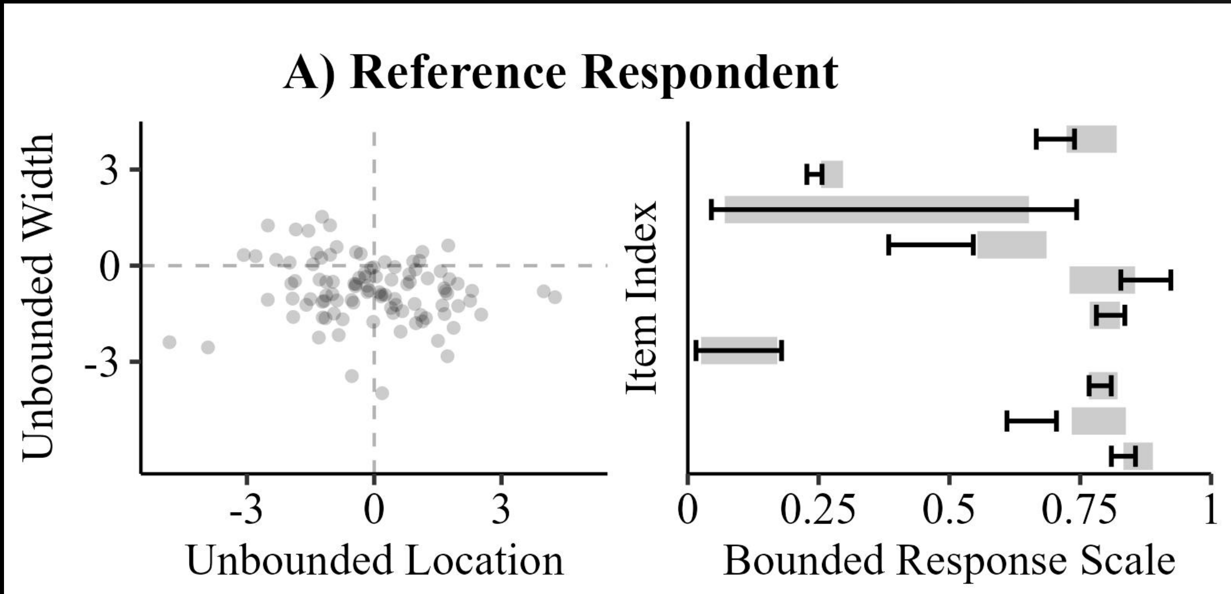


No biases

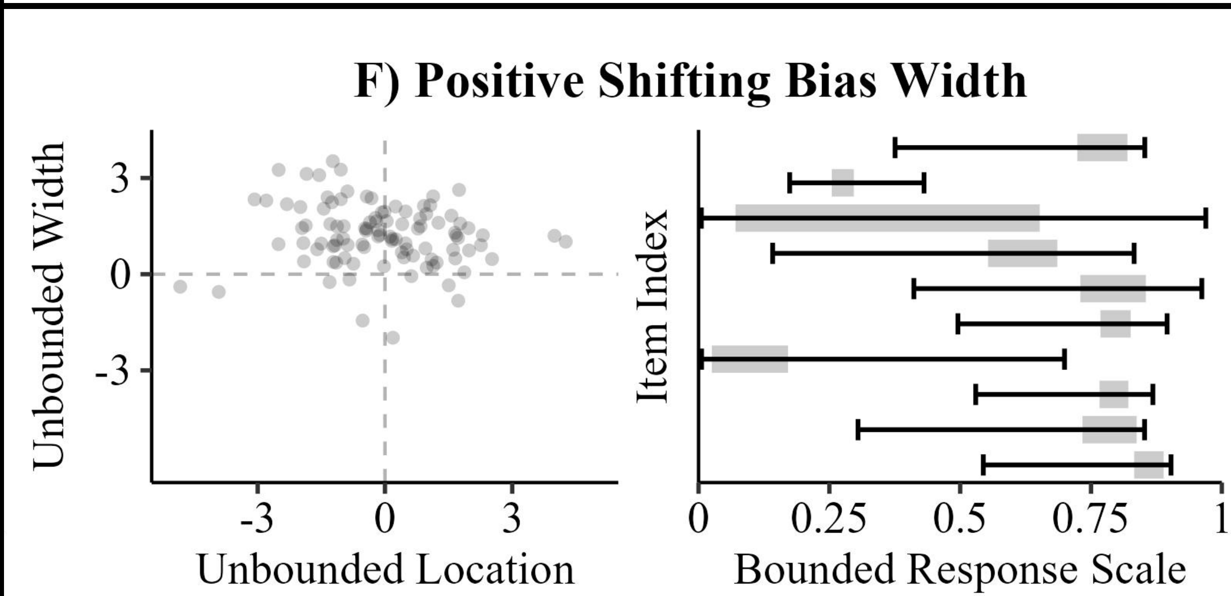


Positive **location** shifting bias

Model Mechanics: Biases

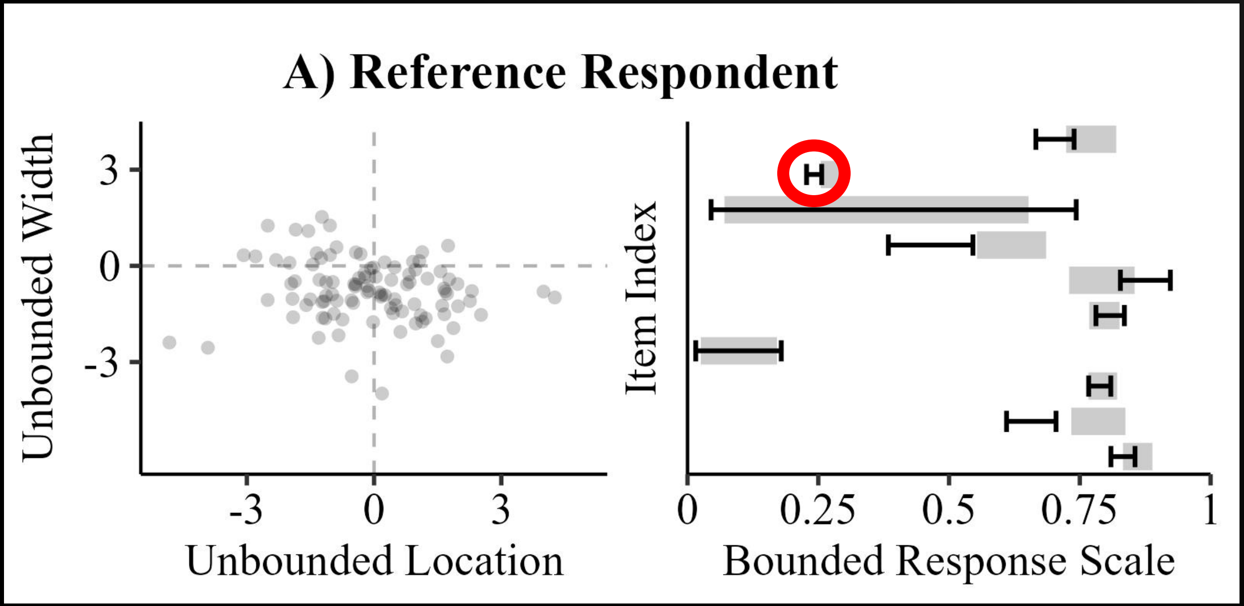


No biases

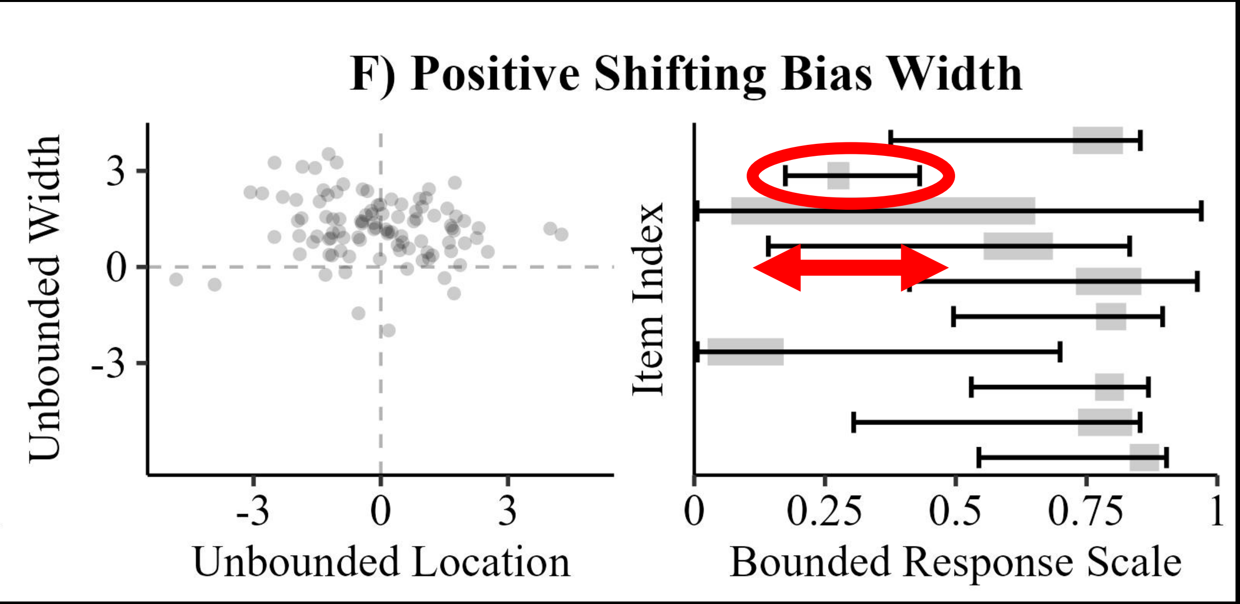


Positive **width** shifting bias

MODEL MECHANICS: BIASES

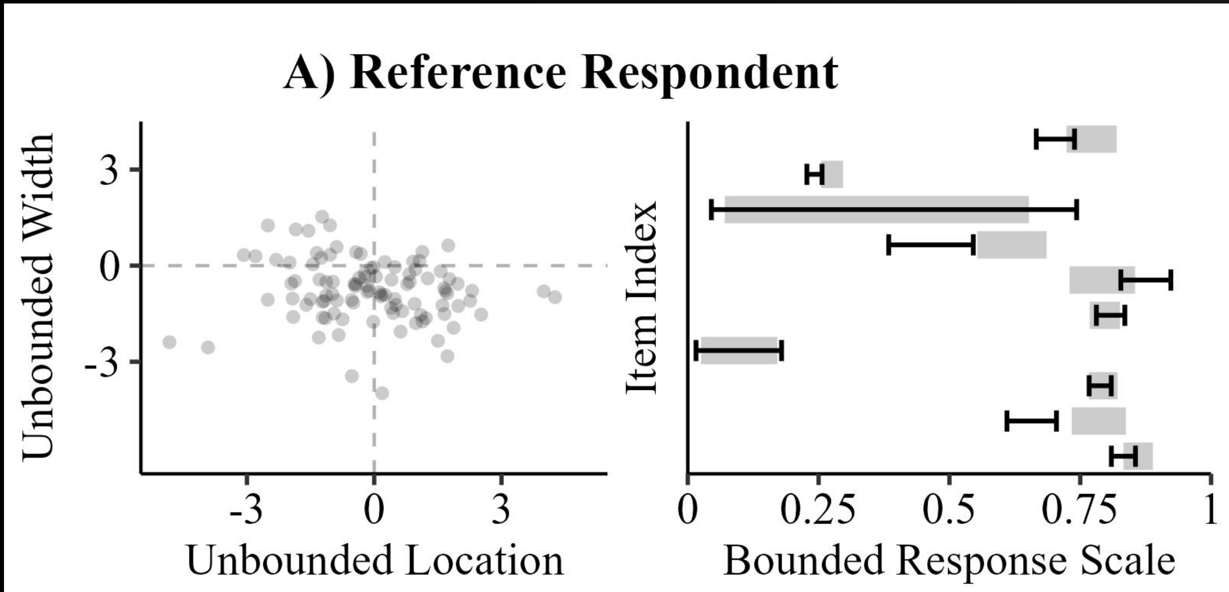


No biases

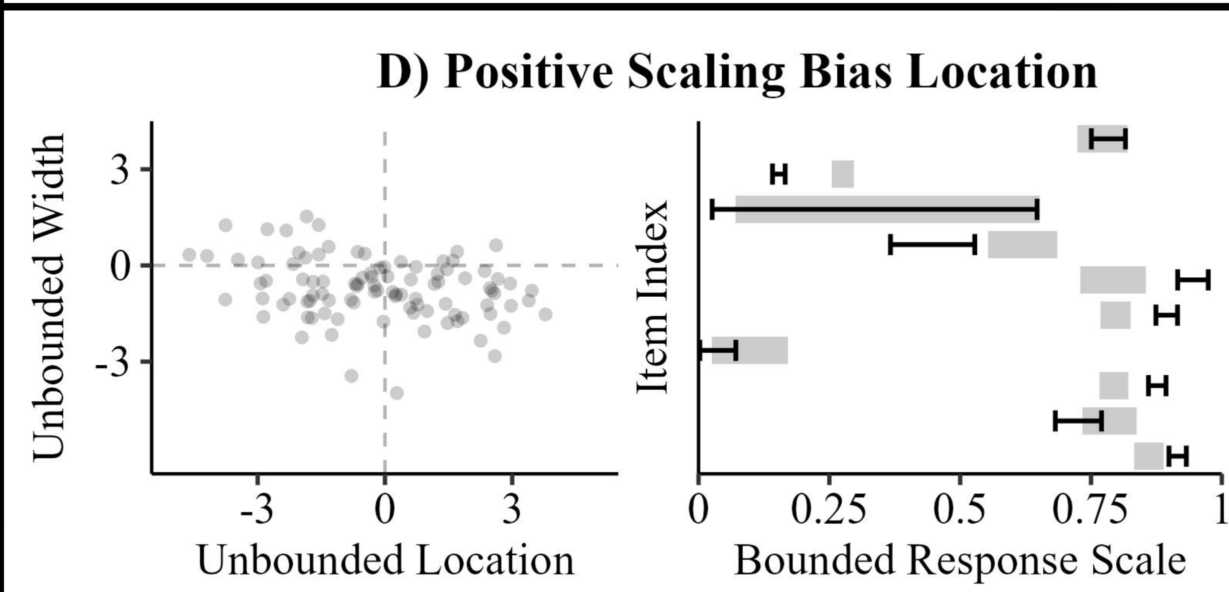


Positive **width** shifting bias

Model Mechanics: Biases

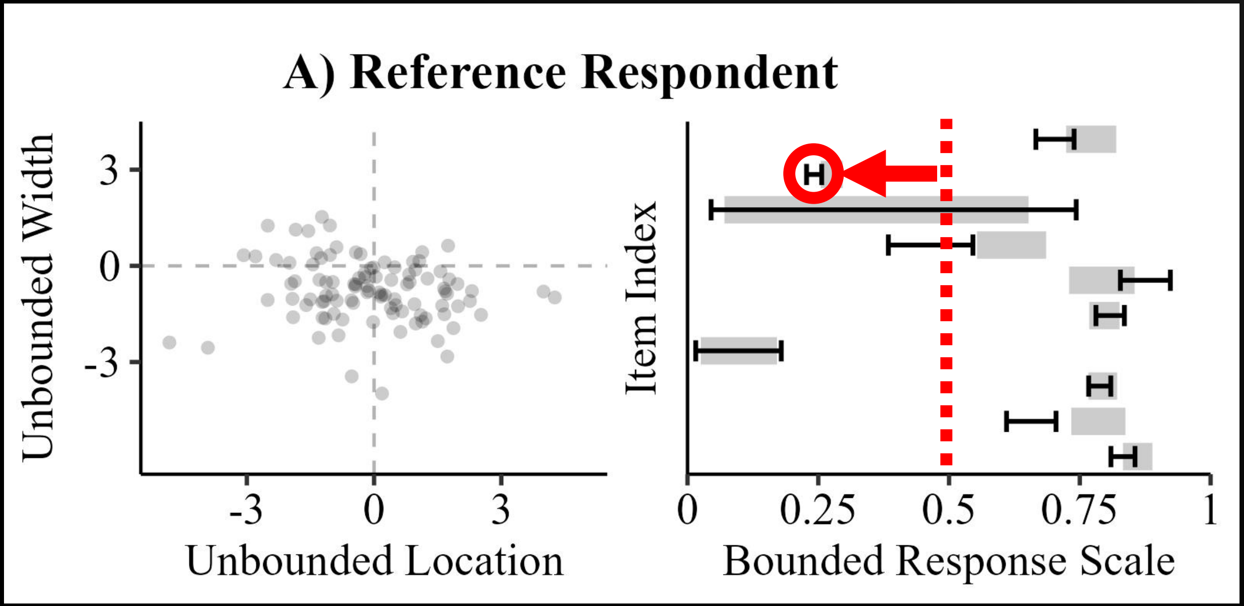


No biases

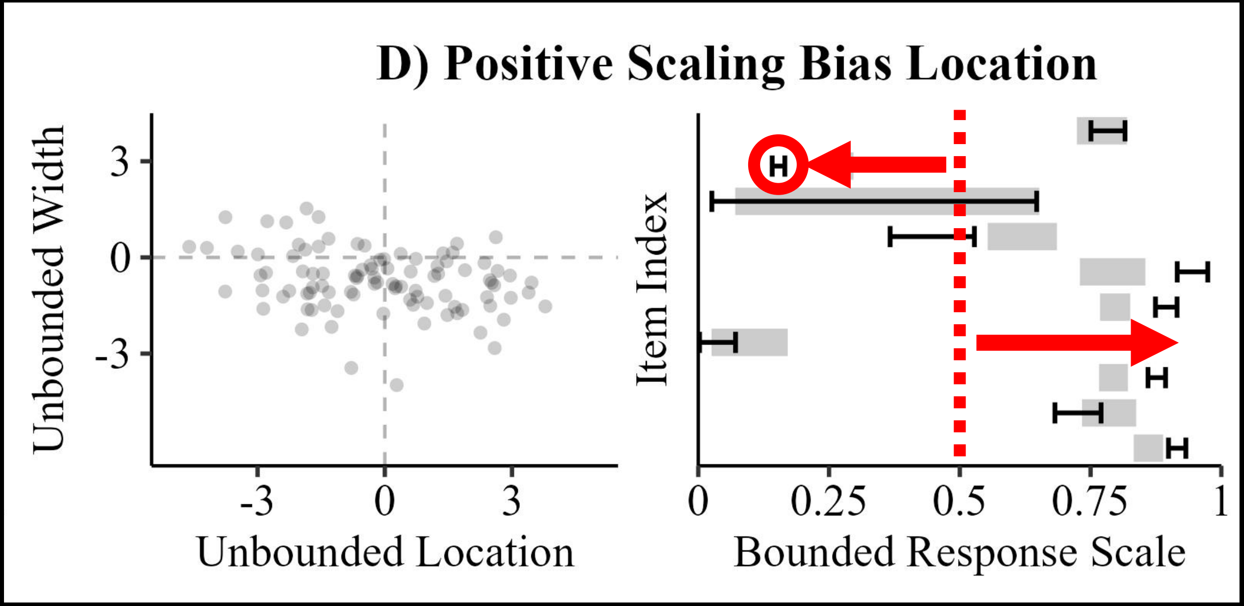


Positive **location** scaling bias

Model Mechanics: Biases



No biases

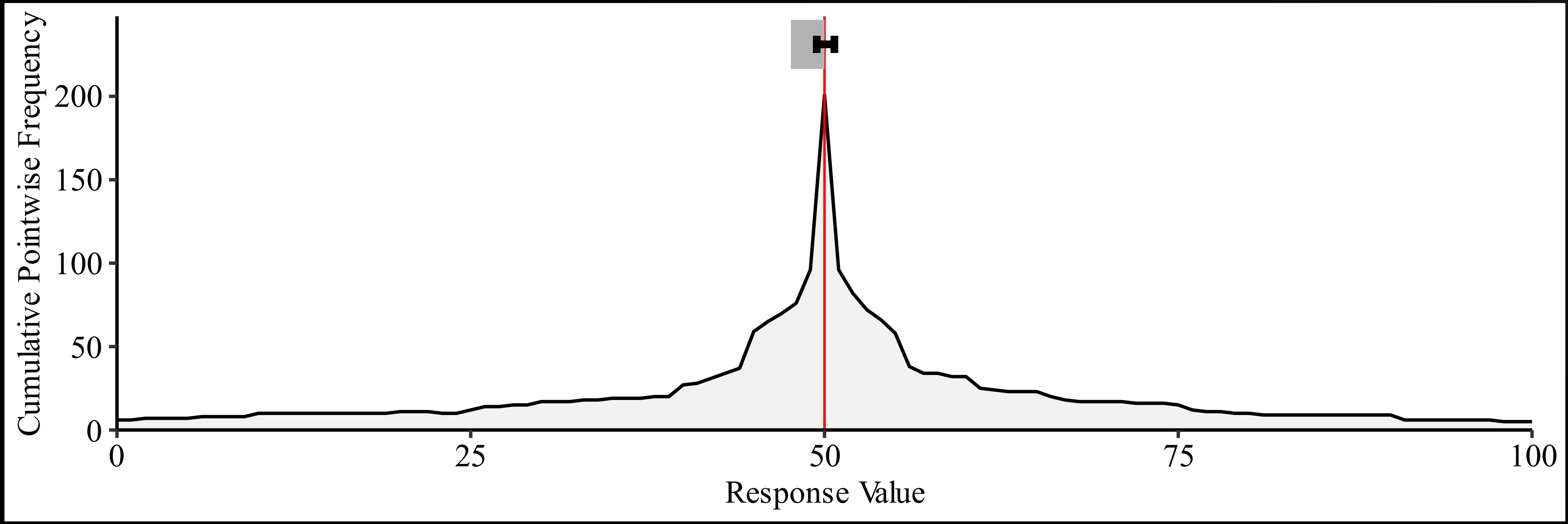


Positive **location** scaling bias

Emprical Example: Verbal Quantifiers

„50-50 Chance"

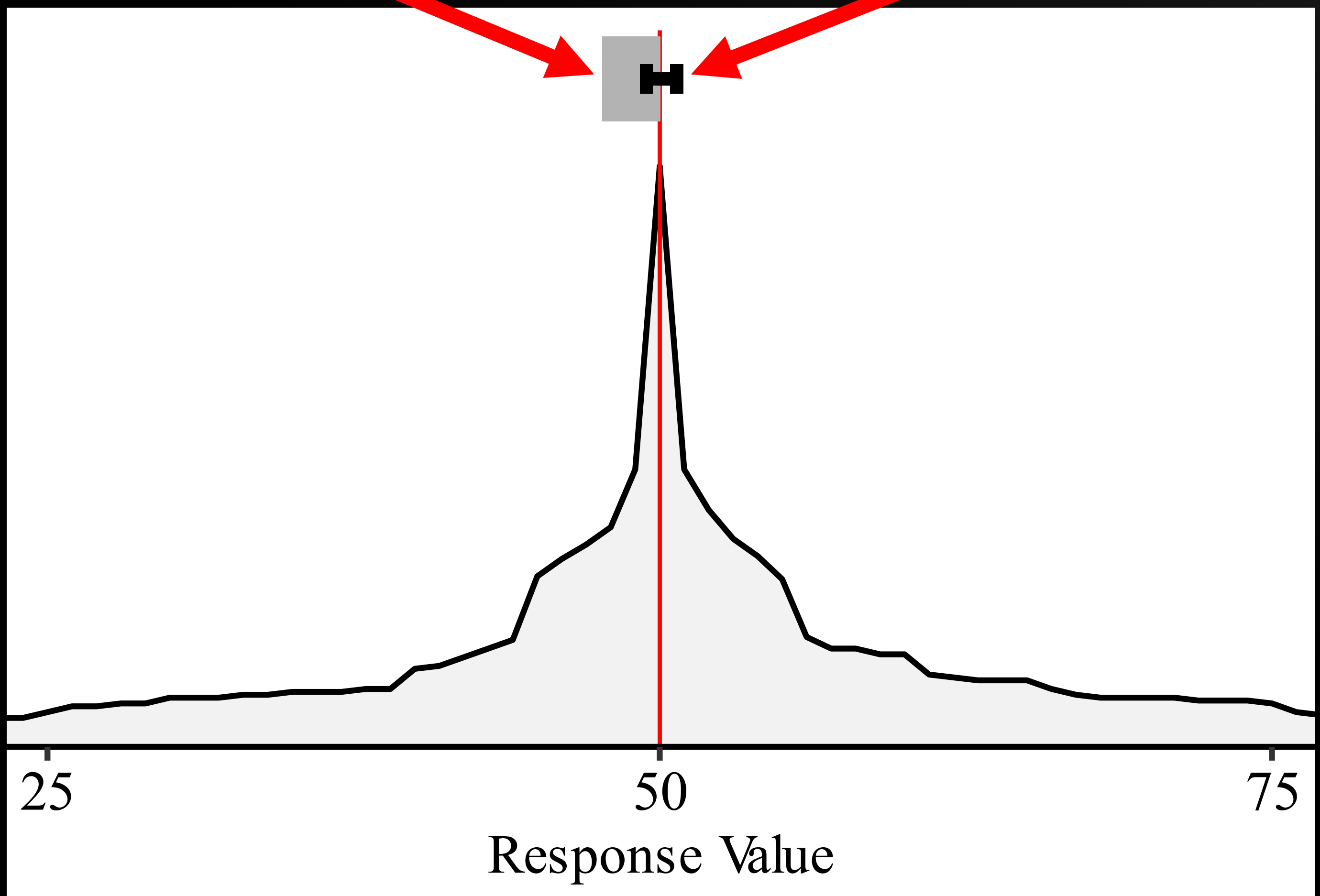
- Control Item in the study
- Should be centered on x = 50 and narrow



Emprical Example: Verba Quantifiers

Mean of logit-transformed location and width

Estimated consensus interval



Take-Home Points

- Isometric log-ratio transformation makes interval responses **more suitable** for modeling frameworks using **normal** distributions
- We can **adapt existing models** to interval reponses by using the isometric log-ratio transformation as a link function

THANKS TO:

- Prof. Dr. Daniel W. Heck

- Björn Siepe

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Slides:

https://github.com/matthiaskloft/

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Additional Slides

Model Equations

Interval Truth Model

Bivariate logit-normal model:

$$\mathrm{ILR}(\mathbf{y}_{\mathrm{ij}}) \sim BVN\left(\boldsymbol{\mu}_{ij}, \boldsymbol{\Sigma}_{ij}\right)$$

Interval Truth Model

Latent Appraisal

$$\begin{pmatrix} \mathrm{A}_{\mathrm{ij}}^{\mathrm{loc}} \\ \mathrm{A}_{\mathrm{ij}}^{\mathrm{wid}} \end{pmatrix} = \begin{pmatrix} \mathrm{T}_{\mathrm{j}}^{\mathrm{loc}} \\ \mathrm{T}_{\mathrm{j}}^{\mathrm{wid}} \end{pmatrix} + \begin{pmatrix} \epsilon_{ij}^{loc} \\ \epsilon_{\mathrm{ij}}^{\mathrm{wid}} \end{pmatrix}$$

$$\begin{pmatrix} \epsilon_{ij}^{loc} \\ \epsilon_{\mathrm{ij}}^{\mathrm{wid}} \end{pmatrix} \sim BVN\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \boldsymbol{\Sigma}_{ij} \right)$$

Interval Truth Model

Latent Appraisal

$$\begin{pmatrix} \mathrm{A}_{\mathrm{ij}}^{\mathrm{loc}} \\ \mathrm{A}_{\mathrm{ij}}^{\mathrm{wid}} \end{pmatrix} = \begin{pmatrix} \mathrm{T}_{\mathrm{j}}^{\mathrm{loc}} \\ \mathrm{T}_{\mathrm{j}}^{\mathrm{wid}} \end{pmatrix} + \begin{pmatrix} \epsilon_{ij}^{loc} \\ \epsilon_{\mathrm{ij}}^{\mathrm{wid}} \end{pmatrix}$$

$$\begin{pmatrix} \epsilon_{ij}^{loc} \\ \epsilon_{\mathrm{ij}}^{\mathrm{wid}} \end{pmatrix} \sim BVN\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \boldsymbol{\Sigma}_{ij} \right)$$

Interval Truth Model

Latent True Consensus

$$\begin{pmatrix} A_{\mathrm{ij}}^{\mathrm{loc}} \\ A_{\mathrm{ij}}^{\mathrm{wid}} \end{pmatrix} = \begin{pmatrix} T_{\mathrm{j}}^{\mathrm{loc}} \\ T_{\mathrm{j}}^{\mathrm{wid}} \end{pmatrix} + \begin{pmatrix} \epsilon_{ij}^{loc} \\ \epsilon_{\mathrm{ij}}^{\mathrm{wid}} \end{pmatrix}$$

$$\begin{pmatrix} \epsilon_{ij}^{loc} \\ \epsilon_{\mathrm{ij}}^{\mathrm{wid}} \end{pmatrix} \sim BVN\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \boldsymbol{\Sigma}_{ij} \right)$$

Interval Truth Model

Error / Precision of appraisal

$$\begin{pmatrix} \mathrm{A}_{\mathrm{ij}}^{\mathrm{loc}} \\ \mathrm{A}_{\mathrm{ij}}^{\mathrm{wid}} \end{pmatrix} = \begin{pmatrix} \mathrm{T}_{\mathrm{j}}^{\mathrm{loc}} \\ \mathrm{T}_{\mathrm{j}}^{\mathrm{wid}} \end{pmatrix} + \begin{pmatrix} \epsilon_{ij}^{loc} \\ \epsilon_{\mathrm{ij}}^{\mathrm{wid}} \end{pmatrix}$$

$$\begin{pmatrix} \epsilon_{ij}^{loc} \\ \epsilon_{\mathrm{ij}}^{\mathrm{wid}} \end{pmatrix} \sim BVN\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \boldsymbol{\Sigma}_{ij} \right)$$

Interval Truth Model

Error / Precision of appraisal

$$\boldsymbol{\Sigma}_{ij} = \begin{pmatrix} \left(\frac{1}{E_i^{loc}} \frac{1}{\lambda_j^{loc}}\right)^2 & \rho_{ij} \\ \rho_{ij} & \left(\frac{1}{E_i^{wid}} \frac{1}{\lambda_j^{wid}}\right)^2 \end{pmatrix}$$

$$\rho_{ij} = \sqrt{\omega_j \left(\frac{1}{E_i^{loc}} \frac{1}{\lambda_j^{loc}}\right)^2 \left(\frac{1}{E_i^{wid}} \frac{1}{\lambda_j^{wid}}\right)^2}$$

Interval Truth Model

Person proficiency to detect consensus

$$\boldsymbol{\Sigma}_{ij} = \begin{pmatrix} \left(\frac{1}{E_i^{loc}}\frac{1}{\lambda_j^{loc}}\right)^2 & \rho_{ij} \\ \rho_{ij} & \left(\frac{1}{E_i^{wid}}\frac{1}{\lambda_j^{wid}}\right)^2 \end{pmatrix}$$

$$\rho_{ij} = \sqrt{\omega_j \left(\frac{1}{E_i^{loc}}\frac{1}{\lambda_j^{loc}}\right)^2 \left(\frac{1}{E_i^{wid}}\frac{1}{\lambda_j^{wid}}\right)^2}$$

Interval Truth Model

Item discernibility/difficulty

$$\boldsymbol{\Sigma}_{ij} = \begin{pmatrix} \left(\frac{1}{E_i^{loc}}\frac{1}{\lambda_j^{loc}}\right)^2 & \rho_{ij} \\ \rho_{ij} & \left(\frac{1}{E_i^{wid}}\frac{1}{\lambda_j^{wid}}\right)^2 \end{pmatrix}$$

$$\rho_{ij} = \sqrt{\omega_j \left(\frac{1}{E_i^{loc}}\frac{1}{\lambda_j^{loc}}\right)^2 \left(\frac{1}{E_i^{wid}}\frac{1}{\lambda_j^{wid}}\right)^2}$$

Interval Truth Model

Residual correlation between location and width

$$\boldsymbol{\Sigma}_{ij} = \begin{pmatrix} \left(\frac{1}{E_i^{loc}}\frac{1}{\lambda_j^{loc}}\right)^2 & \rho_{ij} \\ \rho_{ij} & \left(\frac{1}{E_i^{wid}}\frac{1}{\lambda_j^{wid}}\right)^2 \end{pmatrix}$$

$$\rho_{ij} = \sqrt{\omega_j \left(\frac{1}{E_i^{loc}}\frac{1}{\lambda_j^{loc}}\right)^2 \left(\frac{1}{E_i^{wid}}\frac{1}{\lambda_j^{wid}}\right)^2}$$

Interval Truth Model

A respondent`s biases shift and scale the latent appraisal

$$\mathrm{ILR}(\mathbf{y}_{\mathrm{ij}}) = \left(\mathrm{A}_{\mathrm{ij}}^{\mathrm{loc}} a_i^{loc} + b_i^{loc},\ \ \mathrm{A}_{\mathrm{ij}}^{\mathrm{wid}} + b_i^{wid}\right)^{\top}$$

INTERVAL TRUTH MODEL

Shifting biases

$$\mathrm{ILR}(\mathbf{y}_{\mathrm{ij}}) = \left(\mathrm{A}_{\mathrm{ij}}^{\mathrm{loc}} a_i^{loc} + b_i^{loc},\ \ \mathrm{A}_{\mathrm{ij}}^{\mathrm{wid}} + b_i^{wid}\right)^{\top}$$

Interval Truth Model

Scaling bias

$$\mathrm{ILR}(\mathbf{y}_{\mathrm{ij}}) = \left(\mathrm{A}_{\mathrm{ij}}^{\mathrm{loc}} a_i^{loc} + b_i^{loc},\ \mathrm{A}_{\mathrm{ij}}^{\mathrm{wid}} + b_i^{wid}\right)^{\top}$$