Energy-Based Modeling and Tracking Control of Rigid Body Systems with Practical Multicopter Applications

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Abstract

Many machines, vehicles and robots may be modeled as rigid body systems, i.e. a number of interconnected, undeformable bodies subject to inertia, gravity, and other forces. Energy-based methods for derivation of their equations of motion, like the Lagrange formalism, are standard in engineering education and well established in the dedicated literature. These algorithms commonly rely on the use of a minimal set of generalized coordinates. This is appropriate for many applications, e.g. machines containing only one-dimensional joints. For systems whose configuration space is nonlinear, e.g. mobile robots whose configiguration space contains the rigid body attitude, the use of minimal coordinates necessarily leads to singularities. From the point of view of differential geometry, this is a well known fact.

This work resolves this problem by the use of (possibly) redundant configuration coordinates and (possibly) nonholonomic velocity coordinates. The second chapter reviews several established formalisms of analytical mechanics and states them in terms of these more general coordinates. The third chapter applies these results to rigid body systems. Though inertia is the crucial part of the dynamics, this work also investigates dissipation and stiffness. Finally, it presents an algorithm for the derivation of global equations of motion of general rigid body systems.

The literature states computed-torque is a standard approach for tracking control of fully actuated mechanical systems. However, this recipe relies on minimal coordinates and consequently suffers from the problems mentioned above. There is no established "standard" approach for the control of underactuated systems.

This work presents three sightly different algorithms for tracking control of general rigid body systems by means of static state feedback. These essentially minimize the distance between the actual realizable acceleration of the model and a desired acceleration computed from a stable prototype system. The prototype system shares the geometry and kinematics of the actual model, but may have different constitutive properties (inertia, damping, stiffness). The resulting control law can be computed globally and explicitly for any rigid body system. The resulting closed loop system (which may differ from the prototype in the underactuated case), is invariant to the chosen coordinates, i.e. its formulation is covariant. However, so far, there is no general proof of stability. The performance of the proposed approaches are discussed on several examples and simulation results.

The last chapter of this work discusses the experimental realization of the control approach to two small UAVs. The performance is demonstrated on tracking control for several aerobatic maneuvers.

Contents

Chapter 1

Introduction

1.1 context and state of the art

Modeling. As the title suggests this work deals with modeling and control of rigid body systems. The procedure of physical modeling is illustrated in Figure 1.1. It starts by approximating the system under consideration by a mechanical model. The mathematical part requires the choice of coordinates $\boldsymbol{z}$ to capture the state (positions and velocities) of the model. Combining this with principles of mechanics, we may derive a set of ordinary differential equations that capture its motion. This work mainly deals with this second part, i.e. the derivation of *equations of motion*.

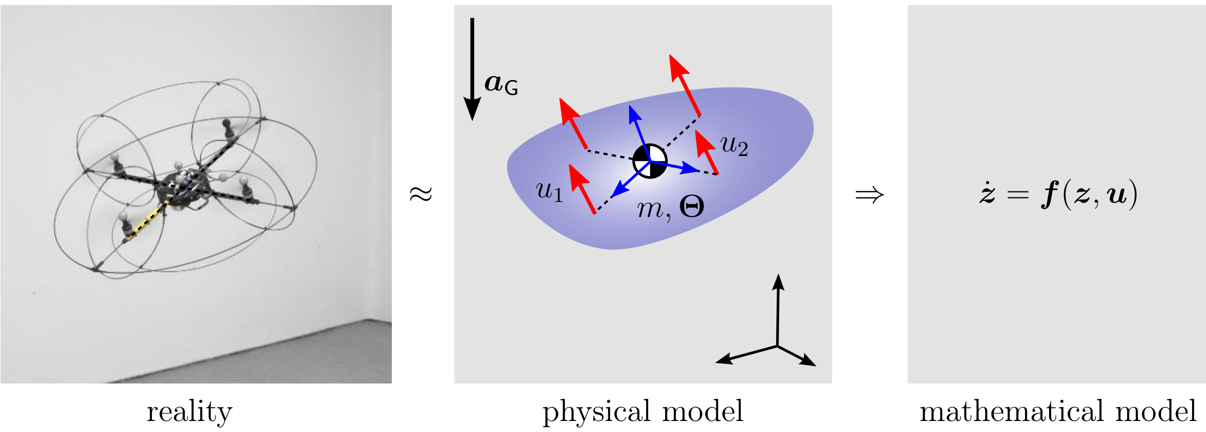

Figure 1.1: Modeling illustration

A very common approach for deriving equations of motion of finite-demensional, holonomic mechanical systems is the so called *Lagrange formalism*: First, the system is parameterized by so called *generalized coordinates* $\boldsymbol{q}$. Then the kinetic energy $\mathcal{T}$, the potential energy $\mathcal{V}$ and virtual work $\delta\mathcal{W}$ of external forces are formulated in terms of these coordinates and their derivatives $\dot{\boldsymbol{q}} = \mathrm{d}\boldsymbol{q}/\mathrm{d}t$:

$$\mathcal{L}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \mathcal{T}(\boldsymbol{q}, \dot{\boldsymbol{q}}) - \mathcal{V}(\boldsymbol{q}), \qquad \delta\mathcal{W}^{\mathrm{E}} = (\delta\boldsymbol{q})^{\top}\boldsymbol{f}^{\mathrm{E}}. \tag{1.1}$$

The equations of motion are derived from the Lagrangian $\mathcal{L}$ as

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}} - \frac{\partial \mathcal{L}}{\partial \boldsymbol{q}} = \boldsymbol{f}^{\mathrm{E}}. \tag{1.2}$$

The kinetic energy for a time-invariant, mechanical system is always strictly quadratic $\mathcal{T} = \frac{1}{2}\dot{\boldsymbol{q}}^\top \boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}}$. Furthermore, we assume that the external forces $\boldsymbol{f}^{\mathrm{E}}$ is an affine function of the control inputs $\boldsymbol{u}$. Then the equations of motion take the structure

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{b}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \boldsymbol{B}(\boldsymbol{q})\boldsymbol{u} \quad \Leftrightarrow \quad \tfrac{\mathrm{d}}{\mathrm{d}t}\underbrace{\begin{bmatrix}\boldsymbol{q}\\ \dot{\boldsymbol{q}}\end{bmatrix}}_{\boldsymbol{z}} = \underbrace{\begin{bmatrix}\dot{\boldsymbol{q}}\\ \boldsymbol{M}^{-1}(\boldsymbol{q})(\boldsymbol{B}(\boldsymbol{q})\boldsymbol{u} - \boldsymbol{b}(\boldsymbol{q}, \dot{\boldsymbol{q}}))\end{bmatrix}}_{\boldsymbol{f}(\boldsymbol{z},\boldsymbol{u})} \tag{1.3}$$

The right hand side of (1.3) is a standard form for simulation, i.e. numerical solution, or general control design. However, the structure of the left hand side of (1.3) can be exploited for a particular control design.

Feedback Control. From a mathematical point of view, a feedback controller is some map $\boldsymbol{u} = \boldsymbol{g}(\boldsymbol{z}, \boldsymbol{r})$ that computes the control input $\boldsymbol{u}$ based on the measured system state $\boldsymbol{z}$ and some reference input $\boldsymbol{r}$. The goal is that the resulting controlled system $\dot{\boldsymbol{z}} = \boldsymbol{f}(\boldsymbol{z}, \boldsymbol{g}(\boldsymbol{z}, \boldsymbol{r})) = \bar{\boldsymbol{f}}(\boldsymbol{z}, \boldsymbol{r})$ has desirable properties.

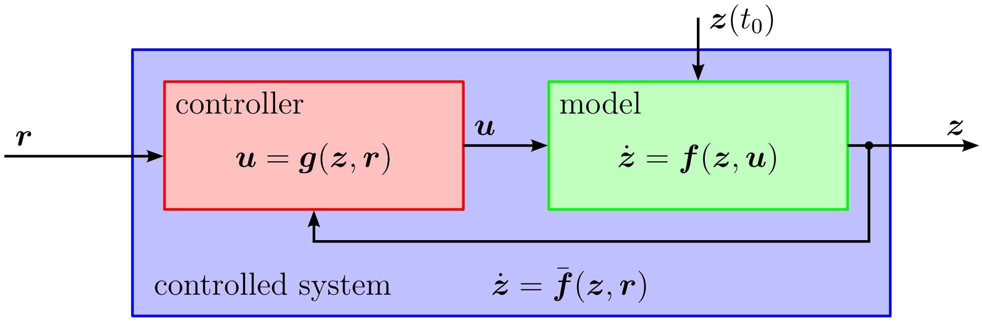

Figure 1.2: Model, controller and controlled system

For a system of the structure (1.3), a typical control objective is, that a given reference trajectory $t \mapsto \boldsymbol{q}_{\mathsf{R}}(t)$ is a stable trajectory of the controlled system. If there are as many control inputs as degrees of freedom $\dim \boldsymbol{u} = \dim \boldsymbol{q} = n$ and the input matrix $\boldsymbol{B}$ is invertible, there is a popular control approach, commonly called *computed torque*: Defining the error dynamics as

$$\ddot{\boldsymbol{e}} + \boldsymbol{\Lambda}_1\dot{\boldsymbol{e}} + \boldsymbol{\Lambda}_0\boldsymbol{e} = \boldsymbol{0}, \quad \boldsymbol{e} = \boldsymbol{q} - \boldsymbol{q}_{\mathsf{R}} \tag{1.4}$$

with the symmetric, positive definite matrices $\boldsymbol{\Lambda}_0, \boldsymbol{\Lambda}_1$ as tuning parameters. Combining this with the model (1.3) yields the control law

$$\boldsymbol{u} = \boldsymbol{B}^{-1}\big(\boldsymbol{M}(\boldsymbol{q})(\ddot{\boldsymbol{q}}_{\mathsf{R}} - \boldsymbol{\Lambda}_1\dot{\boldsymbol{e}} - \boldsymbol{\Lambda}_0\boldsymbol{e}) + \boldsymbol{b}(\boldsymbol{q}, \dot{\boldsymbol{q}})\big) = \boldsymbol{g}\big(\underbrace{\boldsymbol{q}, \dot{\boldsymbol{q}}}_{\boldsymbol{z}}, \underbrace{\boldsymbol{q}_{\mathsf{R}}, \dot{\boldsymbol{q}}_{\mathsf{R}}, \ddot{\boldsymbol{q}}_{\mathsf{R}}}_{\boldsymbol{r}}\big) \tag{1.5}$$

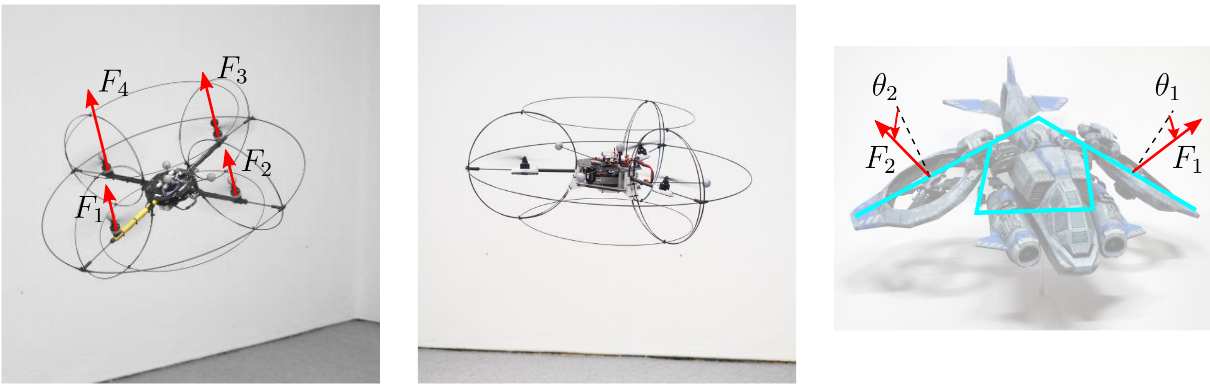

Figure 1.3: lsr-quadcopter (left), lsr-tricopter (middle) and a concept of a bicopter

Multicopters. In contrast to conventional helicopters, multicopters are aerial vehicles that use several rigid (fixed pitch) propellers to generate lift and maneuver. They are usually small and unmanned and are used to carry cameras or other sensors. In particular, the four propeller quadcopter configuration has became quite popular over the last two decades.

The *Chair of Systems Theory and Control Engineering* at Saarland University has developed realizations of a quadcopter and a tricopter with three tiltable propellers (see Figure 1.3). A bicopter with two titlable and inclined propellers has been studied through simulations.

From a mechanical point of view, these multicopters may be modelled as a free rigid body moving within Earth's gravity. The difference between them is only the placement of the actuators: the tricopter is fully-actuated, so it poses the easiest control task. The quadcopter only has four actuators for its six degrees of freedom, i.e. it is underactuated. However, its model is well known to be a configuration flat system and corresponding standard control design approaches may be applied. The bicopter is (probably) not a flat system and consequently, poses the toughest control task.

1.2 Motivation example

Consider a free rigid body as illustrated in the middle of Figure 1.1, but fixed at its center of mass, i.e. it may only rotate about this point. For simplicity, we also assume that the chosen frame coincides with its principle axis and there are three independent control torques about these axis. Then the coefficients of inertia are $\boldsymbol{\Theta} = \mathrm{diag}(\Theta_\mathsf{x}, \Theta_\mathsf{y}, \Theta_\mathsf{z})$.

Lagrange's equation. For application of the Lagrange formalism we need to parameterize the system by minimal generalized coordinates. A popular choice for the rigid body orientation are Euler angles in the *roll-pitch-yaw* convention:

$$\boldsymbol{R}(\boldsymbol{q}) = \begin{bmatrix} \mathrm{c}_\varphi \mathrm{c}_\beta & -\mathrm{s}_\varphi \mathrm{c}_\alpha + \mathrm{c}_\varphi \mathrm{s}_\beta \mathrm{s}_\alpha & \mathrm{s}_\varphi \mathrm{s}_\alpha + \mathrm{c}_\varphi \mathrm{s}_\beta \mathrm{c}_\alpha \\ \mathrm{s}_\varphi \mathrm{c}_\beta & \mathrm{c}_\varphi \mathrm{c}_\alpha + \mathrm{s}_\varphi \mathrm{s}_\beta \mathrm{s}_\alpha & -\mathrm{c}_\varphi \mathrm{s}_\alpha + \mathrm{s}_\varphi \mathrm{s}_\beta \mathrm{c}_\alpha \\ -\mathrm{s}_\beta & \mathrm{c}_\beta \mathrm{s}_\alpha & \mathrm{c}_\beta \mathrm{c}_\alpha \end{bmatrix}, \tag{1.6a}$$

$$\boldsymbol{\omega}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \underbrace{\begin{bmatrix} 1 & 0 & -\mathrm{s}_\beta \\ 0 & \mathrm{c}_\alpha & \mathrm{c}_\beta \mathrm{s}_\alpha \\ 0 & -\mathrm{s}_\alpha & \mathrm{c}_\beta \mathrm{c}_\alpha \end{bmatrix}}_{\boldsymbol{Y}(\boldsymbol{q})} \underbrace{\begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\varphi} \end{bmatrix}}_{\dot{\boldsymbol{q}}}. \tag{1.6b}$$

The shortcut notation $\mathrm{s}_\varphi := \sin(\varphi)$ and $\mathrm{c}_\varphi := \cos(\varphi)$ used here, will be used throughout this text. With this, we may formulate the kinetic energy $\mathcal{T}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \frac{1}{2}(\boldsymbol{\omega}(\boldsymbol{q}, \dot{\boldsymbol{q}}))^\top \boldsymbol{\Theta} \boldsymbol{\omega}(\boldsymbol{q}, \dot{\boldsymbol{q}})$ which coincides with the Lagrangian since there is no potential energy. Evaluation of Lagrange's equation (1.2) yields the equations of motion

$$\underbrace{\begin{bmatrix} \Theta_\mathsf{x} & 0 & -\Theta_\mathsf{x} \mathrm{s}_\beta \\ 0 & \Theta_\mathsf{y} \mathrm{c}_\alpha^2 + \Theta_\mathsf{z} \mathrm{s}_\alpha^2 & (\Theta_\mathsf{y} - \Theta_\mathsf{z}) \mathrm{c}_\alpha \mathrm{s}_\alpha \mathrm{c}_\beta \\ -\Theta_\mathsf{x} \mathrm{s}_\beta & (\Theta_\mathsf{y} - \Theta_\mathsf{z}) \mathrm{c}_\alpha \mathrm{s}_\alpha \mathrm{c}_\beta & \Theta_\mathsf{x} \mathrm{s}_\beta^2 + (\Theta_\mathsf{y} \mathrm{s}_\alpha^2 + \Theta_\mathsf{z} \mathrm{c}_\alpha^2) \mathrm{c}_\beta^2 \end{bmatrix}}_{\boldsymbol{M}(\boldsymbol{q})} \underbrace{\begin{bmatrix} \ddot{\alpha} \\ \ddot{\beta} \\ \ddot{\varphi} \end{bmatrix}}_{\ddot{\boldsymbol{q}}}$$
$$+ \underbrace{\begin{bmatrix} (\Theta_\mathsf{y} - \Theta_\mathsf{z}) \mathrm{c}_\alpha \mathrm{s}_\alpha \dot{\beta}^2 + \cdots \\ 2(\Theta_\mathsf{z} - \Theta_\mathsf{y}) \mathrm{s}_\alpha \mathrm{c}_\alpha \dot{\beta} \dot{\alpha} + \cdots \\ (2(\Theta_\mathsf{y} - \Theta_\mathsf{z}) \mathrm{c}_\alpha^2 - \Theta_\mathsf{x} - \Theta_\mathsf{y} + \Theta_\mathsf{z}) \mathrm{c}_\beta \dot{\beta} \dot{\alpha} + \cdots \end{bmatrix}}_{\boldsymbol{b}(\boldsymbol{q}, \dot{\boldsymbol{q}})} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \mathrm{c}_\alpha & -\mathrm{s}_\alpha \\ -\mathrm{s}_\beta & \mathrm{c}_\beta \mathrm{s}_\alpha & \mathrm{c}_\beta \mathrm{c}_\alpha \end{bmatrix}}_{\boldsymbol{Y}^\top(\boldsymbol{q})} \underbrace{\begin{bmatrix} \tau_\mathsf{x} \\ \tau_\mathsf{y} \\ \tau_\mathsf{z} \end{bmatrix}}_{\boldsymbol{u}}. \tag{1.7}$$

The entries in $\boldsymbol{b}$ are not displayed here since they would fill several lines and are actually not of relevance here. What is crucial here is that the model has singularities at $\beta = \pm\frac{\pi}{2}$: $\det \boldsymbol{M} = \Theta_\mathsf{x} \Theta_\mathsf{y} \Theta_\mathsf{z} \mathrm{c}_\beta^2$ and $\det \boldsymbol{Y} = \mathrm{c}_\beta$. Consequently this model is neither suited for a global simulation nor for application of computed torque control.

Euler's rotation equations. For this particular example one finds another formulation of the equations of motion directly in most textbooks on mechanics, e.g. [Arnold, 1989, p. 143] or [Roberson and Schwertassek, 1988, p. 145]:

$$\frac{\mathrm{d}}{\mathrm{d}t} \underbrace{\begin{bmatrix} R_\mathsf{x}^\mathsf{x} & R_\mathsf{y}^\mathsf{x} & R_\mathsf{z}^\mathsf{x} \\ R_\mathsf{x}^\mathsf{y} & R_\mathsf{y}^\mathsf{y} & R_\mathsf{z}^\mathsf{y} \\ R_\mathsf{x}^\mathsf{z} & R_\mathsf{y}^\mathsf{z} & R_\mathsf{z}^\mathsf{z} \end{bmatrix}}_{\boldsymbol{R}} = \underbrace{\begin{bmatrix} R_\mathsf{x}^\mathsf{x} & R_\mathsf{y}^\mathsf{x} & R_\mathsf{z}^\mathsf{x} \\ R_\mathsf{x}^\mathsf{y} & R_\mathsf{y}^\mathsf{y} & R_\mathsf{z}^\mathsf{y} \\ R_\mathsf{x}^\mathsf{z} & R_\mathsf{y}^\mathsf{z} & R_\mathsf{z}^\mathsf{z} \end{bmatrix}}_{\boldsymbol{R}} \underbrace{\begin{bmatrix} 0 & -\omega_\mathsf{z} & \omega_\mathsf{y} \\ \omega_\mathsf{z} & 0 & -\omega_\mathsf{x} \\ -\omega_\mathsf{y} & \omega_\mathsf{x} & 0 \end{bmatrix}}_{\mathrm{wed}\,\boldsymbol{\omega}}, \tag{1.8a}$$

$$\underbrace{\begin{bmatrix} \Theta_\mathsf{x} & 0 & 0 \\ 0 & \Theta_\mathsf{y} & 0 \\ 0 & 0 & \Theta_\mathsf{z} \end{bmatrix}}_{\boldsymbol{\Theta}} \underbrace{\begin{bmatrix} \dot{\omega}_\mathsf{x} \\ \dot{\omega}_\mathsf{y} \\ \dot{\omega}_\mathsf{z} \end{bmatrix}}_{\dot{\boldsymbol{\omega}}} + \underbrace{\begin{bmatrix} 0 & -\omega_\mathsf{z} & \omega_\mathsf{y} \\ \omega_\mathsf{z} & 0 & -\omega_\mathsf{x} \\ -\omega_\mathsf{y} & \omega_\mathsf{x} & 0 \end{bmatrix}}_{\mathrm{wed}\,\boldsymbol{\omega}} \underbrace{\begin{bmatrix} \Theta_\mathsf{x} & 0 & 0 \\ 0 & \Theta_\mathsf{y} & 0 \\ 0 & 0 & \Theta_\mathsf{z} \end{bmatrix}}_{\boldsymbol{\Theta}} \underbrace{\begin{bmatrix} \omega_\mathsf{x} \\ \omega_\mathsf{y} \\ \omega_\mathsf{z} \end{bmatrix}}_{\boldsymbol{\omega}} = \underbrace{\begin{bmatrix} \tau_\mathsf{x} \\ \tau_\mathsf{y} \\ \tau_\mathsf{z} \end{bmatrix}}_{\boldsymbol{u}}. \tag{1.8b}$$

The 9 coefficients of the matrix $\boldsymbol{R}$ have to obey the constraint $\boldsymbol{R}^\top \boldsymbol{R} = \mathbf{I}_3$. However, it can be shown that if this is fulfilled for the initial condition $\boldsymbol{R}(t_0)$, then the kinematic equation (1.8a) ensures that the condition remains fulfilled. This formulation has no singularities and is well suited for global simulation. Moreover, loosely speaking, its mathematical structure reflects the physical symmetries of the model. The obvious draw-back is that due to the lack of generalized coordinates, its unclear how a method like computed torque could be applied.

Discussion. Euler's equations (1.8) and (1.7) describe the same system. In fact one may plug (1.6b) into (1.8b) and multiply it by $\boldsymbol{Y}^\top$ to obtain (1.7).

Each of these formulations has its advantages and draw-backs: Euler's equations are more compact and have a symmetric structure in contrast to (1.7). The downside is that they require 9 coordinates, the coefficients of the rotation matrix $\boldsymbol{R}$, to parameterize the attitude, whereas the Euler-angles $\boldsymbol{q}$ only require 3. The crucial advantage of Lagrange's equation (1.2) is, that it holds for *any* finite dimensional and holonomic mechanical system, whereas Euler's equations only hold for this particular example. However, for this example, the equations (1.7) are quite cumbersome and lack an obvious structure. Probably the worst fact is that the inertia matrix $\boldsymbol{M}(\boldsymbol{q})$ is singular at the point $\beta = \pm\frac{\pi}{2}$ and consequently $\ddot{\boldsymbol{q}}$ is undefined at these points.

It should be stressed that there is no physical reason for the singularity in the Lagrangian version (1.7), it is rather a consequence of an unsuitable parameterization of the system. This is pointed out in [Roberson and Schwertassek, 1988, sec. 1.1.1] as: *[The scientists of the Eighteenth Century] recognized that there was something about rotation [...] which somehow made the analysis of rotation a problem of higher order difficulty. We now know that the problem is in the mathematics, not the physics, but the problem is still with us.*

The set of 3-dimensional rotation matrices $\boldsymbol{R} \in \mathbb{SO}(3)$ captures the actual configuration space of the rigid body orientation well, but it is difficult to work with since the coefficients of the rotation matrix are not independent. Since $\boldsymbol{R} \in \mathbb{SO}(3)$ is compact while $\mathbb{R}^3$ is not, there is no bijection between them, see also [Frankel, 1997, sec. 1.1d]. The chosen set of Euler angles may be regarded as a surjective but not injective function from $\mathbb{R}^3$ to $\boldsymbol{R} \in \mathbb{SO}(3)$ in a similar manner as latitude and longitude serve as coordinates for Earth's surface. The Euler angles fail are unsuited at the so called gimbal lock ($\beta = \pm 90°$) just as longitude fails at Earths poles.

1.3 Goal and outline of this work

The first chapter reviews established methods of analytical mechanics with the addition of allowing redundant coordinates (like the coefficients of a rotation matrix $\boldsymbol{R}$) and non-holonomic velocity coordinates (like the coefficients of the angular velocity $\boldsymbol{\omega}$). It will present a formulation that can derive both of the presented equations of motion for the motivation example, but holds for general finite-dimensional mechanical systems.

The second chapter specializes to rigid body systems, i.e. systems that consist of several interconnected rigid bodies. It presents an algorithm that derives the equations of motion based on chosen coordinates and given constitutive parameters. Furthermore, natural formulations of stiffness, dissipation and inertia for a rigid body are established.

The third chapter proposes tracking control algorithms for rigid body systems. Based on the findings of the second chapter it will present three control algorithms motivated by defining desired stiffness, damping and inertia of the resulting controlled system. Furthermore, these algorithms are extended to tackle underactuated systems. These algorithms are discussed through several examples.

The fourth chapter presents the developed quadcopter and tricopter and their performance for real control of aerobatic maneuvers.

Chapter 2

Some math

This chapter reviews some established mathematical concepts in particular for the context of redundant coordinates.

2.1 Coordinates

The example of the rigid body orientation showed that, though its degree of freedom is $n = 3$, it cannot be *globally* parameterized by 3 coordinates without having singularities. In other words, the configuration space of the rigid body orientation is not isomorphic to $\mathbb{R}^3$ and is called a nonlinear manifold.

If interested in a global parameterization of a n dimensional nonlinear manifold, there are two common approaches:

1. Choose a finite number of overlapping local charts with with *minimal* coordinates $\boldsymbol{q} \in \mathbb{R}^n$, e.g. four distinct sets of Euler angles for the rigid body attitude [Grafarend and Kühnel, 2011]. As this is the common way of defining a smooth manifold, this is always possible.

2. Choose one parameterization with *redundant* coordinates $\boldsymbol{x} \in \mathbb{R}^\nu$, i.e. coordinates that are constrained by smooth equations of the form $\boldsymbol{\phi}(\boldsymbol{x}) = \mathbf{0}$. E.g. the coefficients of the rotation matrix as done in (1.8). *Whitney embedding theorem* states that this is always possible with at least $\nu = 2n$ coordinates.

Both approaches have benefits and drawbacks depending on the application, but the first approach and the use of minimal coordinates is far more dominant in the literature. This work utilizes the second approach.

2.1.1 Redundant configuration coordinates

In the notation of this work, we use $\nu > 0$ coordinates $\boldsymbol{x}(t) = [x^1(t), \dots, x^\nu(t)]^\top \in \mathbb{R}^\nu$ that might be constrained by $c \geq 0$ smooth functions of the form $\boldsymbol{\phi}(\boldsymbol{x}) = [\phi^1(\boldsymbol{x}), \dots, \phi^c(\boldsymbol{x})]^\top =$

$\mathbf{0}$. For $c > 0$ these coordinates are not independent and are commonly called *redundant*. The set of mutually admissible coordinates is called the configuration space $\mathbb{X}$:

$$\mathbb{X} = \{\boldsymbol{x} \in \mathbb{R}^\nu \,|\, \boldsymbol{\phi}(\boldsymbol{x}) = \mathbf{0}\}. \tag{2.1}$$

Assuming that the rank of $\frac{\partial \boldsymbol{\phi}}{\partial \boldsymbol{x}}$ is constant, the dimension of the configuration space is

$$n = \dim \mathbb{X} = \nu - \operatorname{rank} \frac{\partial \boldsymbol{\phi}}{\partial \boldsymbol{x}}. \tag{2.2}$$

For holonomic systems, n is also called its degree of freedom.

Whitney embedding theorem (see e.g. [Lee, 2003, Theo. 6.14]) states that: *Every smooth manifold of dimension n can be smoothly embedded in the Euclidean space $\mathbb{R}^{2n}$.* The number $2n$ is a worst case bound, i.e. for a particular example a lower dimension for the embedding space $\mathbb{R}^\nu$ might work and a higher dimension is permitted anyway. For this work, it essentially guarantees the existence of a global parameterization by the set $\mathbb{X}$ for any smooth manifold.

2.1.2 Minimal velocity coordinates

For the following it is crucial to note that a geometric constraint is equivalent to its derivative supplemented with a suitable initial condition

$$\phi^\kappa(\boldsymbol{x}) = 0 \tag{2.3a}$$

$$\Leftrightarrow \qquad \frac{\partial \phi^\kappa}{\partial x^\alpha}(\boldsymbol{x})\dot{x}^\alpha = 0, \qquad \phi^\kappa(\boldsymbol{x}_0) = 0 \tag{2.3b}$$

$$\Leftrightarrow \qquad \frac{\partial \phi^\kappa}{\partial x^\alpha}(\boldsymbol{x})\ddot{x}^\alpha + \frac{\partial^2 \phi^\kappa}{\partial x^\alpha \partial x^\beta}(\boldsymbol{x})\dot{x}^\beta \dot{x}^\alpha = 0, \qquad \phi^\kappa(\boldsymbol{x}_0) = 0,\ \frac{\partial \phi^\kappa}{\partial x^\alpha}(\boldsymbol{x}_0)\dot{x}_0^\alpha = 0 \tag{2.3c}$$

$$\dots$$

where $\boldsymbol{x}_0 = \boldsymbol{x}(t_0)$. Even though (2.3a) might be nonlinear, its derivative (2.3b) is always *linear* in the velocities $\dot{\boldsymbol{x}}$. So here it is reasonable to choose *minimal velocity coordinates*: Let $\boldsymbol{A}(\boldsymbol{x}) \in \mathbb{R}^{\nu \times n}$ be a matrix with the properties $\frac{\partial \boldsymbol{\phi}}{\partial \boldsymbol{x}}\boldsymbol{A} = \mathbf{0}$ and $\operatorname{rank} \boldsymbol{A} = n$. The first property of $\boldsymbol{A}(\boldsymbol{x})$ is that these columns of $\boldsymbol{A}(\boldsymbol{x})$ are orthogonal to the rows of $\frac{\partial \boldsymbol{\phi}}{\partial \boldsymbol{x}}(\boldsymbol{x})$. The second property implies that the columns of $\boldsymbol{A}(\boldsymbol{x})$ are linearly independent. So the columns of $\boldsymbol{A}(\boldsymbol{x})$ can be interpreted as a *basis vectors* for the tangent space $\mathsf{T}_{\boldsymbol{x}}\mathbb{X}$. We can capture all allowed velocities $\dot{\boldsymbol{x}}(t)$ by the minimal velocity coordinates $\boldsymbol{\xi}(t) \in \mathbb{R}^n$ through

$$\dot{\boldsymbol{x}} = \boldsymbol{A}(\boldsymbol{x})\boldsymbol{\xi} \tag{2.4}$$

This kinematic relation (2.4) ensures that the time derivative (2.3b) of the geometric constraint is fulfilled, and consequently the geometric constraint only has to be imposed on the initial condition $\boldsymbol{\phi}(\boldsymbol{x}(t_0)) = \mathbf{0}$.

Example 1. Consider a single particle constrained to a circle of radius ρ as illustrated in Figure 2.1.

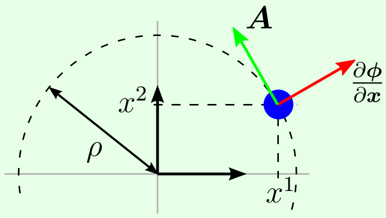

Figure 2.1: Particle on a circle

We use the its Cartesian position $[x^1, x^2]^\top \in \mathbb{R}^2$ constrained by $\phi = (x^1)^2 + (x^2)^2 - \rho^2 = 0$ as configuration coordinates. A reasonable choice for the kinematics matrix $\boldsymbol{A}$ is

$$\underbrace{\begin{bmatrix} 2x^1 & 2x^2 \end{bmatrix}}_{\frac{\partial \phi}{\partial \boldsymbol{x}}} \underbrace{\begin{bmatrix} -x^2 \\ x^1 \end{bmatrix}}_{\boldsymbol{A}} = 0 \tag{2.5}$$

Example 2. Consider the motivation example of the rigid body orientation from section 1.2. Instead of parameterizing the rotation matrix $\boldsymbol{R}$ by minimal coordinates, we take its 9 coefficients $\boldsymbol{x} = [R^{\mathsf{x}}_{\mathsf{x}}, R^{\mathsf{y}}_{\mathsf{x}}, R^{\mathsf{z}}_{\mathsf{x}}, R^{\mathsf{x}}_{\mathsf{y}}, R^{\mathsf{y}}_{\mathsf{y}}, R^{\mathsf{z}}_{\mathsf{y}}, R^{\mathsf{x}}_{\mathsf{z}}, R^{\mathsf{y}}_{\mathsf{z}}, R^{\mathsf{z}}_{\mathsf{z}}]^\top \in \mathbb{R}^9$ as configuration coordinates. The constraints $\boldsymbol{R}^\top \boldsymbol{R} = \mathbf{I}_3$ and $\det \boldsymbol{R} = 1$ read

$$\boldsymbol{\phi}(\boldsymbol{x}) = \begin{bmatrix} (R^{\mathsf{x}}_{\mathsf{x}})^2 + (R^{\mathsf{y}}_{\mathsf{x}})^2 + (R^{\mathsf{z}}_{\mathsf{x}})^2 - 1 \\ (R^{\mathsf{x}}_{\mathsf{y}})^2 + (R^{\mathsf{y}}_{\mathsf{y}})^2 + (R^{\mathsf{z}}_{\mathsf{y}})^2 - 1 \\ (R^{\mathsf{x}}_{\mathsf{z}})^2 + (R^{\mathsf{y}}_{\mathsf{z}})^2 + (R^{\mathsf{z}}_{\mathsf{z}})^2 - 1 \\ R^{\mathsf{x}}_{\mathsf{y}} R^{\mathsf{x}}_{\mathsf{z}} + R^{\mathsf{y}}_{\mathsf{y}} R^{\mathsf{y}}_{\mathsf{z}} + R^{\mathsf{z}}_{\mathsf{y}} R^{\mathsf{z}}_{\mathsf{z}} \\ R^{\mathsf{x}}_{\mathsf{x}} R^{\mathsf{x}}_{\mathsf{z}} + R^{\mathsf{y}}_{\mathsf{x}} R^{\mathsf{y}}_{\mathsf{z}} + R^{\mathsf{z}}_{\mathsf{x}} R^{\mathsf{z}}_{\mathsf{z}} \\ R^{\mathsf{x}}_{\mathsf{x}} R^{\mathsf{x}}_{\mathsf{y}} + R^{\mathsf{y}}_{\mathsf{x}} R^{\mathsf{y}}_{\mathsf{y}} + R^{\mathsf{z}}_{\mathsf{x}} R^{\mathsf{z}}_{\mathsf{y}} \\ R^{\mathsf{x}}_{\mathsf{x}} R^{\mathsf{y}}_{\mathsf{y}} R^{\mathsf{z}}_{\mathsf{z}} + R^{\mathsf{x}}_{\mathsf{y}} R^{\mathsf{y}}_{\mathsf{z}} R^{\mathsf{z}}_{\mathsf{x}} + R^{\mathsf{x}}_{\mathsf{z}} R^{\mathsf{y}}_{\mathsf{x}} R^{\mathsf{z}}_{\mathsf{y}} - R^{\mathsf{x}}_{\mathsf{x}} R^{\mathsf{y}}_{\mathsf{z}} R^{\mathsf{z}}_{\mathsf{y}} - R^{\mathsf{x}}_{\mathsf{y}} R^{\mathsf{y}}_{\mathsf{x}} R^{\mathsf{z}}_{\mathsf{z}} - R^{\mathsf{x}}_{\mathsf{z}} R^{\mathsf{y}}_{\mathsf{y}} R^{\mathsf{z}}_{\mathsf{x}} - 1 \end{bmatrix} = \mathbf{0}. \tag{2.6}$$

The 9 conditions $\boldsymbol{R}^\top \boldsymbol{R} = \mathbf{I}_3$ yields due to symmetry only 6 constraints and already imply $\det \boldsymbol{R} = \pm 1$. Since the determinant is a smooth function, the corresponding manifold must consist of two disjoint components, one with $\det \boldsymbol{R} = +1$ (proper rotations) and one with $\det \boldsymbol{R} = -1$ (rotations with reflection). So the additional constraint $\det \boldsymbol{R} = +1$ does not change the dimension of the configuration space. Formally this means $\operatorname{rank} \frac{\partial \phi}{\partial \boldsymbol{x}} = 6$ and consequently $\dim \mathbb{X} = 9 - 6 = 3$. A kinematics matrix with $\frac{\partial \phi}{\partial \boldsymbol{x}} \boldsymbol{A} = \mathbf{0}$ and $\operatorname{rank} \boldsymbol{A} = 3$

is given by

$$\boldsymbol{A}(\boldsymbol{x}) = \begin{bmatrix} 0 & -R^{\mathsf{x}}_{\mathsf{z}} & R^{\mathsf{x}}_{\mathsf{y}} \\ 0 & -R^{\mathsf{y}}_{\mathsf{z}} & R^{\mathsf{y}}_{\mathsf{y}} \\ 0 & -R^{\mathsf{z}}_{\mathsf{z}} & R^{\mathsf{z}}_{\mathsf{y}} \\ R^{\mathsf{x}}_{\mathsf{z}} & 0 & -R^{\mathsf{x}}_{\mathsf{x}} \\ R^{\mathsf{y}}_{\mathsf{z}} & 0 & -R^{\mathsf{y}}_{\mathsf{x}} \\ R^{\mathsf{z}}_{\mathsf{z}} & 0 & -R^{\mathsf{z}}_{\mathsf{x}} \\ -R^{\mathsf{x}}_{\mathsf{y}} & R^{\mathsf{x}}_{\mathsf{x}} & 0 \\ -R^{\mathsf{y}}_{\mathsf{y}} & R^{\mathsf{y}}_{\mathsf{x}} & 0 \\ -R^{\mathsf{z}}_{\mathsf{y}} & R^{\mathsf{z}}_{\mathsf{x}} & 0 \end{bmatrix}. \tag{2.7}$$

The resulting kinematic equation $\dot{\boldsymbol{x}} = \boldsymbol{A}(\boldsymbol{x})\boldsymbol{\xi}$ can be reordered to the matrix equation $\dot{\boldsymbol{R}} = \boldsymbol{R}\,\mathrm{wed}(\boldsymbol{\xi})$ by introducing the *wedge operator* defined as

$$\mathrm{wed}\begin{bmatrix} \xi^1 \\ \xi^2 \\ \xi^3 \end{bmatrix} = \begin{bmatrix} 0 & -\xi^3 & \xi^2 \\ \xi^3 & 0 & -\xi^1 \\ -\xi^2 & \xi^1 & 0 \end{bmatrix}. \tag{2.8}$$

Pseudoinverse For any matrix $\boldsymbol{S} \in \mathbb{R}^{m\times n}$ there exists a unique *(Moore-Penrose) pseudoinverse* $\boldsymbol{S}^+ \in \mathbb{R}^{n\times m}$ determined by the following conditions [Penrose, 1955, Theo. 1]:

$$\boldsymbol{S}\boldsymbol{S}^+\boldsymbol{S} = \boldsymbol{S}, \tag{2.9a}$$

$$\boldsymbol{S}^+\boldsymbol{S}\boldsymbol{S}^+ = \boldsymbol{S}^+, \tag{2.9b}$$

$$(\boldsymbol{S}\boldsymbol{S}^+)^\top = \boldsymbol{S}\boldsymbol{S}^+, \tag{2.9c}$$

$$(\boldsymbol{S}^+\boldsymbol{S})^\top = \boldsymbol{S}^+\boldsymbol{S}. \tag{2.9d}$$

If the matrix $\boldsymbol{S}$ has linearly independent columns, its pseudoinverse is $\boldsymbol{S}^+ = (\boldsymbol{S}^\top\boldsymbol{S})^{-1}\boldsymbol{S}^\top$. Similarly, if $\boldsymbol{S}$ has linearly independent rows, its pseudoinverse is $\boldsymbol{S}^+ = \boldsymbol{S}^\top(\boldsymbol{S}\boldsymbol{S}^\top)^{-1}$. Consequently, if $\boldsymbol{S}$ is invertible (independent rows and columns) the pseudoinverse coincides with the inverse $\boldsymbol{S}^+ = \boldsymbol{S}^{-1}$.

Some identities involving the pseudo-inverse. Define $\boldsymbol{Y}(\boldsymbol{x}) \in \mathbb{R}^{n\times\nu}$ as $\boldsymbol{Y} = \boldsymbol{A}^+ = (\boldsymbol{A}^\top\boldsymbol{A})^{-1}\boldsymbol{A}^\top$, i.e. the pseudoinverse of the kinematics matrix $\boldsymbol{A}$. Note that this implies $\boldsymbol{Y}\boldsymbol{A} = \mathbf{I}_n$, but $\boldsymbol{A}\boldsymbol{Y} \neq \mathbf{I}_\nu$. We also introduce the matrices $\boldsymbol{\Phi} = \frac{\partial\phi}{\partial\boldsymbol{x}}$ and $\boldsymbol{\Psi} = \boldsymbol{\Phi}^+$. With $\boldsymbol{\Phi}\boldsymbol{A} = \mathbf{0}$ and the Penrose conditions (2.9), we can show[1] that $\boldsymbol{\Psi}^\top\boldsymbol{A} = \mathbf{0}$ and $\boldsymbol{Y}^\top\boldsymbol{\Phi} = \mathbf{0}$. Furthermore, since $\operatorname{rank}\boldsymbol{\Psi} = \operatorname{rank}\boldsymbol{\Phi} = \nu - n$ the columns of $\boldsymbol{\Psi}(\boldsymbol{x})$ span the complementary space $(\mathsf{T}_{\boldsymbol{x}}\mathbb{X})^\perp$ though they might not be a basis since the columns might not be linearly independent.

The matrix $\boldsymbol{P} = \boldsymbol{A}\boldsymbol{Y}$ is an *orthogonal projector*, i.e. $\boldsymbol{P}^2 = \boldsymbol{P}$ and $\boldsymbol{P}^\top = \boldsymbol{P}$ which result directly from the Penrose conditions (2.9). Since $\boldsymbol{\Psi}$ spans the complementary

[1] $\boldsymbol{\Psi}^\top\boldsymbol{A} = (\boldsymbol{\Psi}\boldsymbol{\Phi}\boldsymbol{\Psi})^\top\boldsymbol{A} = \boldsymbol{\Psi}^\top(\boldsymbol{\Psi}\boldsymbol{\Phi})^\top\boldsymbol{A} = \boldsymbol{\Psi}^\top\boldsymbol{\Psi}\boldsymbol{\Phi}\boldsymbol{A} = \mathbf{0}$

space $(\mathsf{T}_{\boldsymbol{x}}\mathbb{X})^{\perp}$, the unique orthogonal projector from $\mathbb{R}^{\nu}$ to $(\mathsf{T}_{\boldsymbol{x}}\mathbb{X})^{\perp}$ can be expressed as $\boldsymbol{P}^{\perp} = \boldsymbol{\Psi\Phi}$. The identity $\boldsymbol{P} + \boldsymbol{P}^{\perp} = \mathbf{I}_{\nu}$ implies

$$\boldsymbol{AY} + \boldsymbol{\Psi\Phi} = \mathbf{I}_{\nu}. \tag{2.10}$$

2.2 Calculus

This section reviews some of the established tools of calculus for the context of redundant coordinates and nonholonomic velocity coordinates as introduced in the previous section.

2.2.1 Directional derivative and Hessian

Consider a function $\mathcal{V} : \mathbb{R}^{\nu} \to \mathbb{R}$ and a curve $\boldsymbol{x} : \mathbb{R} \to \mathbb{X}$. Since $\mathbb{X} \subset \mathbb{R}^{\nu}$, their composition $\mathcal{V} \circ \boldsymbol{x} = f : \mathbb{R} \to \mathbb{R}$ is a scalar function and has the the Taylor expansion

$$\begin{aligned}\underbrace{\mathcal{V}(\boldsymbol{x}(t))}_{f(t)} = \underbrace{\mathcal{V}(\boldsymbol{x}(0))}_{f(0)} &+ t\underbrace{\frac{\partial \mathcal{V}}{\partial x^{\alpha}}(\boldsymbol{x}(0))\dot{x}^{\alpha}(0)}_{\dot{f}(0)} \\ &+ \tfrac{1}{2}t^2 \underbrace{\Big(\frac{\partial^2 \mathcal{V}}{\partial x^{\beta}\partial x^{\alpha}}(\boldsymbol{x}(0))\dot{x}^{\alpha}(0)\dot{x}^{\beta}(0) + \frac{\partial \mathcal{V}}{\partial x^{\alpha}}(\boldsymbol{x}(0))\ddot{x}^{\alpha}(0)\Big)}_{\ddot{f}(0)} + \mathcal{O}(t^3).\end{aligned} \tag{2.11}$$

Now let the curve be parameterized by $\dot{\boldsymbol{x}}(t) = \boldsymbol{A}(\boldsymbol{x}(t))\boldsymbol{\xi}(t)$ and we use the shorthand notations $\bar{\boldsymbol{x}} = \boldsymbol{x}(0)$, $\bar{\boldsymbol{\xi}} = \boldsymbol{\xi}(0)$ and $\bar{\boldsymbol{A}} = \boldsymbol{A}(\boldsymbol{x}(0))$ to write

$$\begin{aligned}\mathcal{V}(\boldsymbol{x}(t)) &= \mathcal{V}(\bar{\boldsymbol{x}}) + t\frac{\partial \mathcal{V}}{\partial x^{\alpha}}(\bar{\boldsymbol{x}})\bar{A}_i^{\alpha}\bar{\xi}^i \\ &+ \tfrac{1}{2}t^2\Big(\frac{\partial^2 \mathcal{V}}{\partial x^{\beta}\partial x^{\alpha}}(\bar{\boldsymbol{x}})\bar{A}_i^{\alpha}\bar{A}_j^{\beta}\bar{\xi}^i\bar{\xi}^j + \frac{\partial \mathcal{V}}{\partial x^{\alpha}}(\bar{\boldsymbol{x}})\Big(\frac{\partial A_i^{\alpha}}{\partial x^{\beta}}(\bar{\boldsymbol{x}})\bar{A}_j^{\beta}\bar{\xi}^i\bar{\xi}^j + \bar{A}_i^{\alpha}\dot{\bar{\xi}}^i\Big)\Big) + \mathcal{O}(t^3)\end{aligned} \tag{2.12}$$

Introducing the notation

$$\partial_i = A_i^{\alpha}\frac{\partial}{\partial x^{\alpha}},\ \ i = 1, \dots, n \tag{2.13}$$

for the derivative in the direction of the i-th basis vector, we can state the Taylor expansion as

$$\mathcal{V}(\boldsymbol{x}(t)) = \mathcal{V}(\bar{\boldsymbol{x}}) + t\,\partial_i\mathcal{V}(\bar{\boldsymbol{x}})\bar{\xi}^i + \tfrac{1}{2}t^2\big(\partial_i\partial_j\mathcal{V}(\bar{\boldsymbol{x}})\bar{\xi}^i\bar{\xi}^j + \partial_i\mathcal{V}(\bar{\boldsymbol{x}})\dot{\bar{\xi}}^i\big) + \mathcal{O}(t^3). \tag{2.14}$$

There are two more things we can derive from this equation:

- If $\partial_i\mathcal{V}(\bar{\boldsymbol{x}}) = 0, i = 1, \dots, n$ then $\bar{\boldsymbol{x}}$ is called a *critical point* of $\mathcal{V}$. At a critical point the expansion (2.14) reduces to

$$\mathcal{V}(\boldsymbol{x}(t)) = \mathcal{V}(\bar{\boldsymbol{x}}) + \tfrac{1}{2}t^2\underbrace{(\partial_i\partial_j\mathcal{V})(\bar{\boldsymbol{x}})}_{\bar{H}_{ij}}\bar{\xi}^i\bar{\xi}^j + \mathcal{O}(t^3). \tag{2.15}$$

  This relation holds for any sufficiently smooth curve $t \mapsto \boldsymbol{x}(t)$ through $\bar{\boldsymbol{x}}$ and consequently for any velocity vector $\bar{\boldsymbol{\xi}}$ at the critical point. So if the matrix $\bar{\boldsymbol{H}}$ is positive (negative) definite, then $\bar{\boldsymbol{x}}$ is a local minimum (maximum) of $\mathcal{V}$.

- Assume the curve $t \mapsto \boldsymbol{x}(t)$ is a *geodesic*, i.e. $\dot{\xi}^i = -\Gamma^i_{jk}\xi^j\xi^k$ with the connection coefficients Γ^i_{jk} that will be discussed later. Plugging this into (2.14) we find a coordinate form of the *Hessian tensor* $\boldsymbol{\nabla}^2\mathcal{V}$ of the potential:

$$\mathcal{V}(\boldsymbol{x}(t)) = \mathcal{V}(\bar{\boldsymbol{x}}) + t(\partial_i\mathcal{V})(\bar{\boldsymbol{x}})\bar{\xi}^i + \tfrac{1}{2}t^2 \underbrace{\left(\partial_i\partial_j\mathcal{V} - \Gamma^k_{ij}\partial_k\mathcal{V}\right)}_{(\boldsymbol{\nabla}^2\mathcal{V})_{ij}}(\bar{\boldsymbol{x}})\bar{\xi}^i\bar{\xi}^j + \mathcal{O}(t^3). \tag{2.16}$$

At a critical point $\bar{\boldsymbol{x}}$, the Hessian of the potential is independent of the connection coefficients Γ^i_{jk} and consequently of the underlying metric. There it coincides with the matrix $\bar{\boldsymbol{H}}$ defined in (2.15).

2.2.2 Commutation coefficients

For a function $\mathcal{V} : \mathbb{R}^\nu \to \mathbb{R}$ we are used to the fact that partial derivatives commute, i.e. $\partial^2\mathcal{V}/\partial x^\alpha \partial x^\beta = \partial^2\mathcal{V}/\partial x^\beta \partial x^\alpha$. Unfortunately this is (in general) not the case for a directional derivatives like ∂_i defined in (2.13). Consequently we investigate the following commutation relation

$$\begin{aligned}
\partial_i\partial_j\mathcal{V} - \partial_j\partial_i\mathcal{V} &= A^\alpha_i \frac{\partial}{\partial x^\alpha}\left(A^\beta_j \frac{\partial\mathcal{V}}{\partial x^\beta}\right) - A^\beta_j \frac{\partial}{\partial x^\beta}\left(A^\alpha_i \frac{\partial\mathcal{V}}{\partial x^\alpha}\right) \\
&= A^\alpha_i \frac{\partial A^\beta_j}{\partial x^\alpha}\frac{\partial\mathcal{V}}{\partial x^\beta} - A^\beta_j \frac{\partial A^\alpha_i}{\partial x^\beta}\frac{\partial\mathcal{V}}{\partial x^\alpha} + A^\alpha_i A^\beta_j \underbrace{\left(\frac{\partial^2\mathcal{V}}{\partial x^\alpha\partial x^\beta} - \frac{\partial^2\mathcal{V}}{\partial x^\beta\partial x^\alpha}\right)}_{=0} \\
&= \left(A^\beta_i \frac{\partial A^\alpha_j}{\partial x^\beta} - A^\beta_j \frac{\partial A^\alpha_i}{\partial x^\beta}\right)\frac{\partial\mathcal{V}}{\partial x^\alpha}.
\end{aligned} \tag{2.17}$$

Now using the identity (2.10) with $\Phi^\kappa_\alpha = \frac{\partial\phi^\kappa}{\partial x^\alpha}$ and $\Phi^\kappa_\alpha A^\alpha_i = 0 \Rightarrow \Phi^\kappa_\alpha \frac{\partial A^\alpha_i}{\partial x^\beta} = -\frac{\partial\Phi^\kappa_\alpha}{\partial x^\beta}A^\alpha_i$ to shape this expressions a bit further

$$\begin{aligned}
\partial_i\partial_j\mathcal{V} - \partial_j\partial_i\mathcal{V} &= \left(A^\beta_i \frac{\partial A^\alpha_j}{\partial x^\beta} - A^\beta_j \frac{\partial A^\alpha_i}{\partial x^\beta}\right)\overbrace{\left(A^\sigma_k Y^k_\alpha + \Psi^\sigma_\kappa \Phi^\kappa_\alpha\right)}^{\delta^\sigma_\alpha}\frac{\partial\mathcal{V}}{\partial x^\sigma} \\
&= \left(A^\beta_i \frac{\partial A^\alpha_j}{\partial x^\beta} - A^\beta_j \frac{\partial A^\alpha_i}{\partial x^\beta}\right)Y^k_\alpha A^\sigma_k \frac{\partial\mathcal{V}}{\partial x^\sigma} - \left(A^\beta_i A^\alpha_j \frac{\partial\Phi^\kappa_\alpha}{\partial x^\beta} - A^\beta_j A^\alpha_i \frac{\partial\Phi^\kappa_\alpha}{\partial x^\beta}\right)\Psi^\sigma_\kappa \frac{\partial\mathcal{V}}{\partial x^\sigma} \\
&= \underbrace{\left(A^\beta_i \frac{\partial A^\alpha_j}{\partial x^\beta} - A^\beta_j \frac{\partial A^\alpha_i}{\partial x^\beta}\right)Y^k_\alpha}_{\gamma^k_{ij}} \underbrace{A^\sigma_k \frac{\partial\mathcal{V}}{\partial x^\sigma}}_{\partial_k\mathcal{V}} - A^\beta_j A^\alpha_i \underbrace{\left(\frac{\partial^2\phi^\kappa}{\partial x^\alpha\partial x^\beta} - \frac{\partial^2\phi^\kappa}{\partial x^\beta\partial x^\alpha}\right)}_{=0}\Psi^\sigma_\kappa \frac{\partial\mathcal{V}}{\partial x^\sigma}.
\end{aligned} \tag{2.18}$$

Since this relation holds for any function $\mathcal{V}$ we can state it in operator form and introduce the *commutation coefficients* γ^k_{ij} as

$$\partial_i\partial_j - \partial_j\partial_i = \gamma^k_{ij}\partial_k, \qquad \gamma^k_{ij} = \left(\partial_i A^\alpha_j - \partial_j A^\alpha_i\right)(A^+)^k_\alpha. \tag{2.19}$$

Note the skew symmetry $\gamma^k_{ij} = -\gamma^k_{ji}$.

Example 3. The commutation coefficients γ_{ij}^k associated with the kinematics matrix $\boldsymbol{A}$ from (2.7) are

$$\begin{aligned}
\gamma_{23}^1 = \gamma_{31}^2 = \gamma_{12}^3 &= +1,\\
\gamma_{32}^1 = \gamma_{13}^2 = \gamma_{21}^3 &= -1,\\
\gamma_{11}^1 = \gamma_{12}^1 = \gamma_{13}^1 = \gamma_{21}^1 = \gamma_{22}^1 = \gamma_{31}^1 = \gamma_{33}^1 &= 0,\\
\gamma_{11}^2 = \gamma_{12}^2 = \gamma_{21}^2 = \gamma_{22}^2 = \gamma_{23}^2 = \gamma_{32}^2 = \gamma_{33}^2 &= 0,\\
\gamma_{11}^3 = \gamma_{13}^3 = \gamma_{22}^3 = \gamma_{23}^3 = \gamma_{31}^3 = \gamma_{32}^3 = \gamma_{33}^3 &= 0.
\end{aligned}$$

This coincides with the three dimensional Levi-Civita symbol commonly defined as

$$\gamma_{ij}^k = \begin{cases} +1, & (i,j,k) \text{ even permutation of } (1,2,3) \\ -1, & (i,j,k) \text{ odd permutation of } (1,2,3) \\ 0, & \text{else} \end{cases} . \tag{2.20}$$

It is related to the 3 dimensional *cross product* by $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3 : [\gamma_{ij}^k a^i b^j]_{k=1..3} = \boldsymbol{a} \times \boldsymbol{b}$ and to the previously defined wed operator by $\boldsymbol{a} \in \mathbb{R}^3 : [\gamma_{ij}^k a^i]_{j,k=1..3} = \operatorname{wed}(\boldsymbol{a})$.

The right hand side of (2.19) appears in the context of Lagrange's equation in [Boltzmann, 1902, p. 687] and [Hamel, 1904a, p. 10] for the case of minimal configuration coordinates and consequently with a square matrix $\boldsymbol{A}$. In the contemporary literature on this context, the quantities γ_{ij}^k are sometimes called the *Boltzmann three-index symbols* [Lurie, 2002, sec. 1.8] or *Hamel coefficients* [Bremer, 2008, p. 75]. The left hand side of (2.19) appears in the context of tensor algebra in [Misner et al., 1973, Box 8.4] where γ_{ij}^k are called the *commutation coefficients*. From the way γ_{ij}^k is defined here, this naming seems most fitting and will be used throughout this work.

The case of redundant configuration coordinates and consequently a non-square matrix $\boldsymbol{A}$ as derived above, is not established in the literature to the best of the authors knowledge.

It is worth noting that the commutation coefficients are *invariant* to the choice of configuration coordinates $\boldsymbol{x}$, even though the coordinates appear explicitly in the definition: For a change of configuration coordinates $\boldsymbol{x} = f(\hat{\boldsymbol{x}})$ the commutation symbols transform like $\hat{\gamma}_{ij}^k(\hat{\boldsymbol{x}}) = \gamma_{ij}^k(f(\hat{\boldsymbol{x}}))$. This might be obvious from a geometric point of view, but the explicit calculation of the coordinate transformation is shown in see [sec:TrafoRules]. It will even turn out that for most of our examples the coefficients will be constants.

The commutation coefficients γ_{jk}^i vanish if the corresponding velocity coordinates ξ^i are *integrable*, i.e.

$$\begin{aligned}
\exists\, \pi^i : \dot{\pi}^i = \xi^i = Y_\alpha^i \dot{x}^\alpha \quad &\Rightarrow \quad Y_\alpha^i = \frac{\partial \pi^i}{\partial x^\beta} \\
&\Rightarrow \quad \frac{\partial Y_\alpha^i}{\partial x^\alpha} = \frac{\partial^2 \pi^i}{\partial x^\beta \partial x^\alpha} = \frac{\partial Y_\beta^i}{\partial x^\alpha}, \quad \Rightarrow \quad \gamma_{jk}^i = 0.
\end{aligned} \tag{2.21}$$

This is not the case in general. Nevertheless the quantities π are introduced as *nonholonomic coordinates* in [Boltzmann, 1902] or as *quasi coordinates* in [Lurie, 2002, sec. 1.5].

Then we could write $\partial_i(\partial_j f) - \partial_j(\partial_i f) = \partial^2 f/\partial\pi^i\partial\pi^j - \partial^2 f/\partial\pi^j\partial\pi^i \neq 0$ what might lead to the conception that partial derivatives do not commute. The commutativity clearly holds, the issue is rather π are no proper coordinates. To avoid confusion of this kind we do not pick up this notation here. See also [Hamel, 1904b] for an extensive discussion on this topic.

2.2.3 Linearization about a trajectory

Let $\bar{\boldsymbol{x}} : [t_1, t_2] \to \mathbb{X}$ be a smooth curve with the velocity coordinates $\bar{\boldsymbol{\xi}} : [t_1, t_2] \to \mathbb{R}^n : t \mapsto \boldsymbol{A}^+(\bar{\boldsymbol{x}}(t))\dot{\bar{\boldsymbol{x}}}(t)$. For a small deviation $\boldsymbol{x} \approx \bar{\boldsymbol{x}}$ with $\boldsymbol{x} \in \mathbb{X}$ we may approximate the geometric constraint as

$$\boldsymbol{\phi}(\boldsymbol{x}) \approx \underbrace{\boldsymbol{\phi}(\bar{\boldsymbol{x}})}_{=\boldsymbol{0}} + \frac{\partial\boldsymbol{\phi}}{\partial\boldsymbol{x}}(\bar{\boldsymbol{x}})(\boldsymbol{x} - \bar{\boldsymbol{x}}) = \boldsymbol{0}. \tag{2.22}$$

Since this constraint is affine w.r.t. $\boldsymbol{x}$ it is reasonable to use a the basis $\boldsymbol{\varepsilon}(t) \in \mathbb{R}^n$ for the deviated configuration coordinates:

$$\boldsymbol{x} = \bar{\boldsymbol{x}} + \boldsymbol{A}(\bar{\boldsymbol{x}})\boldsymbol{\varepsilon}, \qquad \boldsymbol{\varepsilon} = \boldsymbol{A}^+(\bar{\boldsymbol{x}})(\boldsymbol{x} - \bar{\boldsymbol{x}}), \tag{2.23}$$

For the velocity coordinates $\boldsymbol{\xi}$ of the deviated curve $\boldsymbol{x}$ we use again the first order approximation and $\boldsymbol{Y} = \boldsymbol{A}^+$:

$$\begin{aligned}
\xi^i &= Y^i_\alpha(\boldsymbol{x})\dot{x}^\alpha \\
&\approx Y^i_\alpha(\bar{\boldsymbol{x}} + \boldsymbol{A}(\bar{\boldsymbol{x}})\boldsymbol{\varepsilon})\tfrac{\mathrm{d}}{\mathrm{d}t}\big(\bar{x}^\alpha + A^\alpha_j(\bar{\boldsymbol{x}})\,\varepsilon^j\big) \\
&\approx Y^i_\alpha(\bar{\boldsymbol{x}})\big(\dot{x}^\alpha_{\mathsf{R}} + \tfrac{\partial A^\alpha_j}{\partial x^\beta}(\bar{\boldsymbol{x}})\,\dot{x}^\beta_{\mathsf{R}}\varepsilon^j + A^\alpha_j(\bar{\boldsymbol{x}})\,\dot{\varepsilon}^j\big) + \tfrac{\partial Y^i_\alpha}{\partial x^\beta}(\bar{\boldsymbol{x}})A^\beta_j(\bar{\boldsymbol{x}})\,\varepsilon^j\dot{x}^\alpha_{\mathsf{R}} \\
&= \bar{\xi}^i + \dot{\varepsilon}^i + \gamma^i_{kj}(\bar{\boldsymbol{x}})\bar{\xi}^k\varepsilon^j
\end{aligned} \tag{2.24}$$

Using these results we may formulate an approximation of a general smooth function f along the trajectory $t \mapsto \bar{\boldsymbol{x}}(t)$ as

$$\begin{aligned}
f(\boldsymbol{x}, \boldsymbol{\xi}, \dot{\boldsymbol{\xi}}) &\approx f(\bar{\boldsymbol{x}}, \bar{\boldsymbol{\xi}}, \dot{\bar{\boldsymbol{\xi}}}) + \frac{\partial f}{\partial x^\alpha}(\bar{\boldsymbol{x}}, \bar{\boldsymbol{\xi}}, \dot{\bar{\boldsymbol{\xi}}})(x^\alpha - \bar{x}^\alpha) + \frac{\partial f}{\partial \xi^i}(\bar{\boldsymbol{x}}, \bar{\boldsymbol{\xi}}, \dot{\bar{\boldsymbol{\xi}}})(\xi^i - \bar{\xi}^i) \\
&\quad + \frac{\partial f}{\partial \dot{\xi}^i}(\bar{\boldsymbol{x}}, \bar{\boldsymbol{\xi}}, \dot{\bar{\boldsymbol{\xi}}})(\dot{\xi}^i - \dot{\bar{\xi}}^i) \\
&\approx f(\bar{\boldsymbol{x}}, \bar{\boldsymbol{\xi}}, \dot{\bar{\boldsymbol{\xi}}}) + (\partial_i f)(\bar{\boldsymbol{x}}, \bar{\boldsymbol{\xi}}, \dot{\bar{\boldsymbol{\xi}}})\varepsilon^i + \frac{\partial f}{\partial \xi^i}(\bar{\boldsymbol{x}}, \bar{\boldsymbol{\xi}}, \dot{\bar{\boldsymbol{\xi}}})(\dot{\varepsilon}^i + \gamma^i_{kj}(\bar{\boldsymbol{x}})\bar{\xi}^k\varepsilon^j) \\
&\quad + \frac{\partial f}{\partial \dot{\xi}^i}(\bar{\boldsymbol{x}}, \bar{\boldsymbol{\xi}}, \dot{\bar{\boldsymbol{\xi}}})(\ddot{\varepsilon}^i + \gamma^i_{kj}(\bar{\boldsymbol{x}})\bar{\xi}^k\dot{\varepsilon}^j + \gamma^i_{kj}(\bar{\boldsymbol{x}})\dot{\bar{\xi}}^k\varepsilon^j + \partial_l\gamma^i_{kj}(\bar{\boldsymbol{x}})\bar{\xi}^l\bar{\xi}^k\varepsilon^j) \\
&= \bar{f} + \bar{F}^0_i\varepsilon^i + \bar{F}^1_i\dot{\varepsilon}^i + \bar{F}^2_i\ddot{\varepsilon}^i
\end{aligned} \tag{2.25}$$

where

$$\begin{aligned}
\bar{f} &= f(\bar{\boldsymbol{x}}, \bar{\boldsymbol{\xi}}, \dot{\bar{\boldsymbol{\xi}}}),\\
\bar{F}_i^0 &= (\partial_i f)(\bar{\boldsymbol{x}}, \bar{\boldsymbol{\xi}}, \dot{\bar{\boldsymbol{\xi}}}) + \frac{\partial f}{\partial \xi^j}(\bar{\boldsymbol{x}}, \bar{\boldsymbol{\xi}}, \dot{\bar{\boldsymbol{\xi}}})\gamma^j_{ki}(\bar{\boldsymbol{x}})\bar{\xi}^k + \frac{\partial f}{\partial \dot{\xi}^j}(\bar{\boldsymbol{x}}, \bar{\boldsymbol{\xi}}, \dot{\bar{\boldsymbol{\xi}}})\big(\gamma^j_{ki}(\bar{\boldsymbol{x}})\dot{\bar{\xi}}^k + \partial_l \gamma^j_{ki}(\bar{\boldsymbol{x}})\bar{\xi}^l\bar{\xi}^k\big),\\
\bar{F}_i^1 &= \frac{\partial f}{\partial \xi^i}(\bar{\boldsymbol{x}}, \bar{\boldsymbol{\xi}}, \dot{\bar{\boldsymbol{\xi}}}) + \frac{\partial f}{\partial \dot{\xi}^j}(\bar{\boldsymbol{x}}, \bar{\boldsymbol{\xi}}, \dot{\bar{\boldsymbol{\xi}}})\gamma^j_{ki}(\bar{\boldsymbol{x}})\bar{\xi}^k,\\
\bar{F}_i^2 &= \frac{\partial f}{\partial \dot{\xi}^i}(\bar{\boldsymbol{x}}, \bar{\boldsymbol{\xi}}, \dot{\bar{\boldsymbol{\xi}}}).
\end{aligned}$$

Evidently, the expressions simplify significantly is the velocity coordinates are holonomic, i.e. $\gamma = 0$, or if the approximation is about a static point $\bar{\boldsymbol{x}} = const. \Rightarrow \boldsymbol{\xi} = \boldsymbol{0}$.

2.2.4 Calculus of variations

The calculus of variations is concerned with the extremals of functionals, i.e. functions of functions. For the particular context of classical mechanics we are interested in the curves $t \mapsto \boldsymbol{x}(t)$ for which the functional

$$\mathcal{J}[\boldsymbol{x}] = \int_{t_1}^{t_2} \mathcal{L}(\boldsymbol{x}(t), \boldsymbol{\xi}(t), t)\,\mathrm{d}t \tag{2.26}$$

for given boundary conditions $\boldsymbol{x}(t_1)$ and $\boldsymbol{x}(t_2)$ is *stationary.* The *Lagrangian* $\mathcal{L}$ is here a function of the configuration coordinates $\boldsymbol{x}$, its derivatives $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{\xi}$ parameterized in the velocity coordinates $\boldsymbol{\xi}$ and may depend explicitly on the time t as well.

For the standard case, $\boldsymbol{x} = \boldsymbol{q}$ and $\boldsymbol{\xi} = \dot{\boldsymbol{q}}$, a derivation of may be found in e.g. [Courant and Hilbert, 1924, chap. 4, §3], [Lanczos, 1986, ch. II] or [Arnold, 1989, sec. 12]. For the present case we modify the well known derivation slightly: Suppose that $\boldsymbol{x} : [t_1, t_2] \mapsto \mathbb{X}$ is the solution to the variational problem. With the function $\boldsymbol{\chi}(t) \in \mathbb{R}^\nu$ and the parameter $\varepsilon \in \mathbb{R}$ we define a perturbation to it by

$$\bar{\boldsymbol{x}} = \boldsymbol{x} + \varepsilon\boldsymbol{\chi}. \tag{2.27}$$

We need $\bar{\boldsymbol{x}}(t) \in \mathbb{X}$ and consequently $\boldsymbol{\phi}(\bar{\boldsymbol{x}}) = \boldsymbol{0}$. Assuming ε to be sufficiently small, we may use the first order approximation analog to subsection 2.2.3: With the *variation coordinates* $\boldsymbol{h} : [t_1, t_2] \to \mathbb{R}^n$ we parameterize $\boldsymbol{\chi} = \boldsymbol{A}(\boldsymbol{x})\boldsymbol{h}$. Using the inverse kinematic relation $\boldsymbol{\xi} = \boldsymbol{Y}(\boldsymbol{x})\dot{\boldsymbol{x}}$ we can write the functional for the varied path as

$$\mathcal{J}[\bar{\boldsymbol{x}}] = \int_{t_1}^{t_2} \mathcal{L}\big(\boldsymbol{x} + \varepsilon\boldsymbol{A}(\boldsymbol{x})\boldsymbol{h}, \boldsymbol{Y}(\boldsymbol{x} + \varepsilon\boldsymbol{A}(\boldsymbol{x})\boldsymbol{h})\tfrac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{x} + \varepsilon\boldsymbol{A}(\boldsymbol{x})\boldsymbol{h}), t\big)\,\mathrm{d}t =: \mathcal{P}(\varepsilon) \tag{2.28}$$

Now if $\boldsymbol{x}(t)$ is indeed the solution to the variational problem, then $\mathcal{P}(\varepsilon)$ must have a minimum at $\mathcal{P}(0)$ and consequently $\partial\mathcal{P}/\partial\varepsilon(0) = 0$. Evaluation of this "ordinary" differentiation yields

$$\begin{aligned}
0 = \frac{\partial \mathcal{P}}{\partial \varepsilon}\Big|_{\varepsilon=0} &= \int_{t_1}^{t_2} \left(\frac{\partial \mathcal{L}}{\partial x^\alpha}A^\alpha_i h^i + \frac{\partial \mathcal{L}}{\partial \xi^i}\left(\frac{\partial Y^i_\alpha}{\partial x^\beta}A^\beta_j h^j \dot{x}^\alpha + Y^i_\alpha \frac{\partial A^\alpha_j}{\partial x^\beta} h^j \dot{x}^\beta + \dot{h}^i\right)\right)\mathrm{d}t\\
&= \int_{t_1}^{t_2} \left(\partial_i \mathcal{L}\, h^i + \frac{\partial \mathcal{L}}{\partial \xi^i}\big(\gamma^i_{kj} h^j \xi^k + \dot{h}^i\big)\right)\mathrm{d}t
\end{aligned} \tag{2.29}$$

where we have found again the commutation coefficients γ^i_{kj} previously derived in (2.19). Integrating by parts with the boundary conditions $\boldsymbol{h}(t_1) = \boldsymbol{h}(t_2) = \boldsymbol{0}$ gives

$$\int_{t_1}^{t_2} h^i \left(A^\alpha_i \frac{\partial \mathcal{L}}{\partial x^\alpha} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \mathcal{L}}{\partial \xi^i} - \gamma^k_{ij} \xi^j \frac{\partial \mathcal{L}}{\partial \xi^k} \right) \mathrm{d}t = 0. \tag{2.30}$$

Since the variation coordinates $h^i, i = 1, \dots, n$ are independent by definition, the *fundamental lemma of the calculus of variations* (see e.g. [Courant and Hilbert, 1924, p. 166] or [Arnold, 1989, p. 57]) states that, for the integral to vanish, the terms in the brackets have to vanish. Together with the kinematic equation, we have the following necessary conditions for the functional (2.26) to be stationary:

$$\dot{x}^\alpha = A^\alpha_i \xi^i, \qquad \alpha = 1 \dots \nu, \tag{2.31a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \mathcal{L}}{\partial \xi^i} + \gamma^k_{ij} \xi^j \frac{\partial \mathcal{L}}{\partial \xi^k} - \partial_i \mathcal{L} = 0, \qquad i = 1, \dots, n. \tag{2.31b}$$

For the special case $\boldsymbol{x}(t) = \boldsymbol{q}(t) \in \mathbb{R}^n$ and $\boldsymbol{\xi}(t) = \dot{\boldsymbol{q}}(t)$ we have $\boldsymbol{A} = \mathbf{I}_n$ and $\gamma = 0$. Then (2.31) coincides with the *Euler-Lagrange equation* (1.2).

Example 4. Consider again the configuration coordinates $\boldsymbol{x} = [\boldsymbol{R}_\mathsf{x}^\top, \boldsymbol{R}_\mathsf{y}^\top, \boldsymbol{R}_\mathsf{z}^\top]^\top$ and the velocity coordinates $\boldsymbol{\xi} = \boldsymbol{\omega}$ related by $\dot{\boldsymbol{R}} = \boldsymbol{R}\,\mathrm{wed}(\boldsymbol{\omega})$ as discussed in the previous example (2.7). The commutation coefficients were derived in (2.20).

For the Lagrangian

$$\mathcal{L} = \tfrac{1}{2} \boldsymbol{\omega}^\top \boldsymbol{\Theta} \boldsymbol{\omega} \tag{2.32}$$

we obain

$$\left[\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \mathcal{L}}{\partial \xi^i} + \gamma^k_{ij} \xi^j \frac{\partial \mathcal{L}}{\partial \xi^k} - A^\alpha_i \frac{\partial \mathcal{L}}{\partial x^\alpha} \right]_{i=1,2,3} = \boldsymbol{\Theta} \dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \boldsymbol{\Theta} \boldsymbol{\omega}. \tag{2.33}$$

2.2.5 Hamilton's equations

Legendre transformation. Define the *generalized momentum* $\boldsymbol{p}$ as

$$p_i = \frac{\partial \mathcal{L}}{\partial \xi^i}, \quad i = 1, \dots, n. \tag{2.34}$$

and assume that these relations can be inverted to express the velocity $\boldsymbol{\xi} = \boldsymbol{\zeta}(\boldsymbol{x}, \boldsymbol{p}, t)$ in terms of the momentum. Then define the *Hamiltonian* $\mathcal{H}$ as

$$\mathcal{H}(\boldsymbol{x}, \boldsymbol{p}, t) = \Big[p_i \xi^i - \mathcal{L}(\boldsymbol{x}, \boldsymbol{\xi}, t) \Big]_{\boldsymbol{\xi} = \boldsymbol{\zeta}(\boldsymbol{x}, \boldsymbol{p}, t)} = p_i \zeta^i(\boldsymbol{x}, \boldsymbol{p}, t) - \mathcal{L}(\boldsymbol{x}, \boldsymbol{\zeta}(\boldsymbol{x}, \boldsymbol{p}, t), t). \tag{2.35}$$

The definitions (2.34) and (2.35) describe the *Legendre transformation* $(\mathcal{L}, \boldsymbol{\xi}) \to (\mathcal{H}, \boldsymbol{p})$, see [Lanczos, 1986, ch. VI.1] or [Arnold, 1989, sec. 14] for some geometric background. Note that the configuration coordinates $\boldsymbol{x}$ and the time t do not participate in the transformation.

Hamilton's canonical equations. Evaluation of the differentials of (2.35), we get the relations

$$\partial_j \mathcal{H} = p_i \partial_j \zeta^i - \partial_j \mathcal{L} - \frac{\partial \mathcal{L}}{\partial \xi^i} \partial_j \zeta^i = -\partial_j \mathcal{L} \tag{2.36a}$$

$$\frac{\partial \mathcal{H}}{\partial p_j} = \zeta^j + p_i \frac{\partial \zeta^i}{\partial p_j} - \frac{\partial \mathcal{L}}{\partial \xi^i} \frac{\partial \zeta^i}{\partial p_j} = \xi^j \tag{2.36b}$$

$$\frac{\partial \mathcal{H}}{\partial t} = p_i \frac{\partial \zeta^i}{\partial t} - \frac{\partial \mathcal{L}}{\partial \xi^i} \frac{\partial \zeta^i}{\partial t} - \frac{\partial \mathcal{L}}{\partial t} = -\frac{\partial \mathcal{L}}{\partial t}, \tag{2.36c}$$

With this we can express the Euler-Lagrange equation (2.31) in terms of the generalized momentum $\boldsymbol{p}$ and the Hamiltonian $\mathcal{H}$ as

$$\dot{x}^\alpha = A_i^\alpha \frac{\partial \mathcal{H}}{\partial p_i}, \qquad \alpha = 1, \ldots, \nu, \tag{2.37a}$$

$$\dot{p}_i + \gamma_{ij}^k \frac{\partial \mathcal{H}}{\partial p_j} p_k + \partial_i \mathcal{H} = 0, \qquad i = 1, \ldots, n. \tag{2.37b}$$

For the special case of minimal configuration coordinates $\boldsymbol{q}$ and velocity coordinates $\boldsymbol{\xi} = \dot{\boldsymbol{q}}$ we have $\boldsymbol{A} = \mathbf{I}_n$ and (2.37) is called *Hamilton's canonical equations.*

Conservation law. The time derivative of the Hamiltonian along the the solutions of (2.37) is

$$\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} = \frac{\partial \mathcal{H}}{\partial x^\alpha} \dot{x}^\alpha + \frac{\partial \mathcal{H}}{\partial p_i} \dot{p}_i + \frac{\partial \mathcal{H}}{\partial t} = \underbrace{\frac{\partial \mathcal{H}}{\partial x^\alpha} A_i^\alpha \frac{\partial \mathcal{H}}{\partial p_i} - \frac{\partial \mathcal{H}}{\partial p_i} A_i^\alpha \frac{\partial \mathcal{H}}{\partial x^\alpha}}_{0} - p_k \underbrace{\gamma_{ij}^k \frac{\partial \mathcal{H}}{\partial p_i} \frac{\partial \mathcal{H}}{\partial p_j}}_{0} + \frac{\partial \mathcal{H}}{\partial t} \tag{2.38}$$

and consequently

$$\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} = \frac{\partial \mathcal{H}}{\partial t} = -\frac{\partial \mathcal{L}}{\partial t}. \tag{2.39}$$

This is the well known conservation law for the Hamiltonian, see e.g. [Lanczos, 1986, ch. VI.6]. The remarkable aspect of the conservation law (and for the Legendre transformation) is that there is no particular assumption on the structure of the Lagrangian $\mathcal{L}$.

Example 5. Consider a Lagrangian as

$$\mathcal{L}(\boldsymbol{x}, \boldsymbol{\xi}, t) = \tfrac{1}{2} M_{ij}(\boldsymbol{x}, t) \xi^j \xi^i + b_i(\boldsymbol{x}, t) \xi^i + c(\boldsymbol{x}, t). \tag{2.40}$$

The corresponding generalized momentum and Hamiltonian are

$$p_i = \frac{\partial \mathcal{L}}{\partial \xi^i} = M_{ij} \xi^i + b_i \qquad \Leftrightarrow \qquad \xi^i = M^{ij}(p_j - b_j) \tag{2.41}$$

$$\mathcal{H} = \tfrac{1}{2} M^{ij}(p_i - b_i)(p_j - b_j) - c. \tag{2.42}$$

Evaluation of (2.37) yields

$$\dot{x}^\alpha = A^\alpha_i M^{ij}(p_j - b_j), \tag{2.43a}$$
$$\dot{p}_i + \big(\gamma^k_{il} M^{lj} p_k + \tfrac{1}{2}\partial_i M^{kj}(p_k - b_k) - M^{kj}\partial_i b_k\big)(p_j - b_j) + \partial_i c = 0, \tag{2.43b}$$

The Euler-Lagrange equation evaluates to

$$\dot{x}^\alpha = A^\alpha_i \xi^i, \tag{2.44a}$$
$$M_{ij}\dot{\xi}^j + \big(\partial_k M_{ij} + \gamma^l_{ij} M_{lk} - \tfrac{1}{2}\partial_i M_{kj}\big)\xi^j\xi^k + \big(\frac{\partial M_{ij}}{\partial t} + \gamma^k_{ij} b_k\big)\xi^j + \frac{\partial b_i}{\partial t} - \partial_i c \tag{2.44b}$$

Note that the Hamiltonian in terms of Lagrangian coordinates reads

$$\mathcal{H} = \tfrac{1}{2} M_{ij}\xi^i\xi^j - c. \tag{2.45}$$

2.3 Linear algebra

2.3.1 Matrix sets

The following sets of real matrices that are frequently used in the work:

$$\text{(symmetric)} \quad \mathbb{SYM}(n) = \{\boldsymbol{A} \in \mathbb{R}^{n\times n} \,|\, \boldsymbol{A} = \boldsymbol{A}^\top\}, \tag{2.46a}$$
$$\text{(symmetric, pos. def.)} \quad \mathbb{SYM}^+(n) = \{\boldsymbol{A} \in \mathbb{SYM}(n) \,|\, \boldsymbol{x}^\top\boldsymbol{A}\boldsymbol{x} > 0\,\forall\,\boldsymbol{x} \in \mathbb{R}^n\backslash\{\boldsymbol{0}\}\}, \tag{2.46b}$$
$$\text{(sym., pos. semi-def.)} \quad \mathbb{SYM}^+_0(n) = \{\boldsymbol{A} \in \mathbb{SYM}(n) \,|\, \boldsymbol{x}^\top\boldsymbol{A}\boldsymbol{x} \geq 0\,\forall\,\boldsymbol{x} \in \mathbb{R}^n\backslash\{\boldsymbol{0}\}\}, \tag{2.46c}$$

$$\text{(unit sphere)} \quad \mathbb{S}^n = \{\boldsymbol{a} \in \mathbb{R}^n \,|\, \boldsymbol{a}^\top\boldsymbol{a} = 1\}, \tag{2.46d}$$
$$\text{(orthogonal)} \quad \mathbb{O}(n) = \{\boldsymbol{A} \in \mathbb{R}^{n\times n} \,|\, \boldsymbol{A}^\top\boldsymbol{A} = \mathbf{I}_n\}, \tag{2.46e}$$
$$\text{(special orthogonal)} \quad \mathbb{SO}(n) = \{\boldsymbol{R} \in \mathbb{O}(n) \,|\, \det\boldsymbol{R} = +1\}, \tag{2.46f}$$

2.3.2 Inner product, norm and metric

Inner product. For matrices $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}^{n\times m}$ and a symmetric, positive definite matrix $\boldsymbol{K} \in \mathbb{SYM}^+(n)$, define an *inner product* as

$$\langle\boldsymbol{A}, \boldsymbol{B}\rangle_{\boldsymbol{K}} = \mathrm{tr}(\boldsymbol{A}^\top\boldsymbol{K}\boldsymbol{B}). \tag{2.47}$$

Setting $\boldsymbol{K} = \mathbf{I}_n$ in the definition (2.47) is called the *Frobenius inner product* in [Horn and Johnson, 1985, sec. 5.2] or *Hilbert-Schmidt inner product* in [Hall, 2015, sec. A.6]. Furthermore, for $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}^{n\times 1}$ it coincides with the common *dot product*.

Norm. The inner product (2.47) induces the norm

$$\|\boldsymbol{A}\|_{\boldsymbol{K}} = \sqrt{\langle \boldsymbol{A}, \boldsymbol{A} \rangle_{\boldsymbol{K}}}. \tag{2.48}$$

Again for $\boldsymbol{K} = \mathbf{I}_n$ this coincides with the established *Frobenius norm*, e.g. [Golub and Loan, 1996, p 55]. Furthermore, for $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}^{n\times 1}$ it coincides with the common *Euclidian norm* or 2-norm.

Metric. The norm (2.48) induces the metric

$$d_{\boldsymbol{K}}(\boldsymbol{A}, \boldsymbol{B}) = \|\boldsymbol{A} - \boldsymbol{B}\|_{\boldsymbol{K}}. \tag{2.49}$$

Again for $\boldsymbol{K} = \mathbf{I}_n$ this coincides with the established *Euclidean metric.*

The introduced inner product, norm and metric may be regarded as weighted versions of their established forms. For $\boldsymbol{K} = \mathbf{I}_n$, this work uses the shorthand notation

$$\langle \cdot, \cdot \rangle \equiv \langle \cdot, \cdot \rangle_{\mathbf{I}_n}, \qquad \| \cdot \| \equiv \| \cdot \|_{\mathbf{I}_n}, \qquad d(\cdot, \cdot) \equiv d_{\mathbf{I}_n}(\cdot, \cdot). \tag{2.50}$$

2.3.3 Vee and wedge

Established definitions. Define the wedge operator as

$$\operatorname{wed} : \mathbb{R}^3 \to \mathbb{R}^{3\times 3} \; : \; \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \mapsto \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \tag{2.51a}$$

Its inverse is denoted $\operatorname{vee}(\cdot)$, i.e. $\operatorname{vee}(\operatorname{wed}(\boldsymbol{\omega}) = \boldsymbol{\omega}$. The wed and vee operators are well established in the literature, see e.g. [Murray et al., 1994, sec. 2.3.2], and have already been used in previous sections.

New definitions. The following operators are not established in the literature, but will prove quite useful for this work. Define the vee2 operator through: $\boldsymbol{M} \in \mathbb{R}^{3\times 3}$:

$$\operatorname{tr}\left(\boldsymbol{M}(\operatorname{wed}\boldsymbol{\omega})^\top\right) = \boldsymbol{\omega}^\top \operatorname{vee2}(\boldsymbol{M}), \tag{2.52}$$

this is

$$\operatorname{vee2} : \mathbb{R}^{3\times 3} \to \mathbb{R}^3 \; : \; \boldsymbol{M} \mapsto \operatorname{vee}(\boldsymbol{M} - \boldsymbol{M}^\top) \tag{2.53}$$

Note that we have $\operatorname{vee2}(\operatorname{wed}\boldsymbol{\omega}) = 2\operatorname{vee}(\operatorname{wed}\boldsymbol{\omega}) = 2\boldsymbol{\omega}$, thus giving the motivation for the name.

Define the Vee operator through

$$\operatorname{tr}\left(\operatorname{wed}\boldsymbol{\omega}\,\boldsymbol{M}(\operatorname{wed}\boldsymbol{\eta})^\top\right) = \boldsymbol{\eta}^\top(\operatorname{Vee}\boldsymbol{M})\boldsymbol{\omega}, \tag{2.54}$$

this is

$$\operatorname{Vee} : \mathbb{R}^{3\times 3} \to \mathbb{R}^{3\times 3} \; : \boldsymbol{M} \mapsto \operatorname{tr}(\boldsymbol{M})\mathbf{I}_3 - \boldsymbol{M} \tag{2.55}$$

Noting that $\operatorname{tr}(\operatorname{Vee}(\boldsymbol{M})) = 2\operatorname{tr}(\boldsymbol{M})$, we may write the inverse $\operatorname{Wed}(\operatorname{Vee}(\boldsymbol{M})) = \boldsymbol{M}$ as

$$\operatorname{Wed} : \mathbb{R}^{3\times 3} \to \mathbb{R}^{3\times 3} \; : \boldsymbol{M} \mapsto \tfrac{1}{2}\operatorname{tr}(\boldsymbol{M})\mathbf{I}_3 - \boldsymbol{M}. \tag{2.56}$$

Identities. For $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \mathbb{R}^3$ and $\boldsymbol{M} \in \mathbb{R}^{3\times 3}$, the following identities may be checked by direct computation:

$$\operatorname{wed}(\boldsymbol{a})^\top = -\operatorname{wed}(\boldsymbol{a}), \tag{2.57a}$$
$$\operatorname{wed}(\boldsymbol{a})\boldsymbol{b} = \operatorname{vee2}(\boldsymbol{b}\boldsymbol{a}^\top), \tag{2.57b}$$
$$\operatorname{wed}(\boldsymbol{a})\operatorname{wed}(\boldsymbol{b}) = \boldsymbol{b}\boldsymbol{a}^\top - (\boldsymbol{b}^\top\boldsymbol{a})\mathbf{I}_3 = -\operatorname{Vee}(\boldsymbol{b}\boldsymbol{a}^\top), \tag{2.57c}$$
$$\operatorname{wed}(\boldsymbol{a})\operatorname{wed}(\boldsymbol{b})\boldsymbol{c} + \operatorname{wed}(\boldsymbol{b})\operatorname{wed}(\boldsymbol{c})\boldsymbol{a} + \operatorname{wed}(\boldsymbol{c})\operatorname{wed}(\boldsymbol{a})\boldsymbol{b} = \mathbf{0}, \tag{2.57d}$$
$$\operatorname{vee2}(\operatorname{wed}(\boldsymbol{b})\boldsymbol{M}) = \operatorname{Vee}(\boldsymbol{M})\boldsymbol{b}. \tag{2.57e}$$

2.3.4 Singular value decomposition

Definition [Golub and Loan, 1996, Theo. 2.5.2]: For any matrix $\boldsymbol{M} \in \mathbb{R}^{n\times m}$ there exist orthonormal matrices $\boldsymbol{X} \in \mathbb{O}(n)$, $\boldsymbol{Y} \in \mathbb{O}(m)$ and a matrix $\boldsymbol{\Sigma} \in \mathbb{R}^{n\times m}$ with $i \neq j : \Sigma_{ij} = 0$, $\Sigma_{ii} = \sigma_i$ with $\sigma_1 \geq \ldots, \geq \sigma_p \geq 0$, $p = \max(n, m)$ such that $\boldsymbol{M} = \boldsymbol{X}\boldsymbol{\Sigma}\boldsymbol{Y}^\top$.

- The columns of $\boldsymbol{X}$ are eigenvectors of $\boldsymbol{M}\boldsymbol{M}^\top = \boldsymbol{X}(\boldsymbol{\Sigma}\boldsymbol{\Sigma}^\top)\boldsymbol{X}^\top$
- The columns of $\boldsymbol{Y}$ are eigenvectors of $\boldsymbol{M}^\top\boldsymbol{M} = \boldsymbol{Y}(\boldsymbol{\Sigma}^\top\boldsymbol{\Sigma})\boldsymbol{Y}^\top$
- The squared singular values σ_i^2 coincide with the eigenvalues of $\boldsymbol{M}^\top\boldsymbol{M}$ and $\boldsymbol{M}^\top\boldsymbol{M}$

Due to the descending order of the singular values, the matrix $\boldsymbol{\Sigma}$ is unique. The matrices $\boldsymbol{X}$ and $\boldsymbol{Y}$ are unique up to orthogonal transformations of the subspaces of each singular value and the kernel and co-kernel of $\boldsymbol{M}$.

2.4 An important function on the special orthogonal group

Motivation. In the context of satellite navigation, the following problem [Wahba, 1965] arose, now commonly called *Wahba's problem*: given $\boldsymbol{u}_k, \boldsymbol{v}_k \in \mathbb{R}^3$, find $\boldsymbol{R} \in \mathbb{SO}(3)$ that minimizes

$$\begin{aligned}\mathcal{V}_1(\boldsymbol{R}) = \sum_k \|\boldsymbol{u}_k - \boldsymbol{R}\boldsymbol{v}_k\|^2 &= \sum_k \left(\|\boldsymbol{u}_k\|^2 + \|\boldsymbol{v}_k\|^2 - \langle \boldsymbol{u}_k, \boldsymbol{R}\boldsymbol{v}_k\rangle\right) \\ &= \underbrace{\sum_k \left(\|\boldsymbol{u}_k\|^2 + \|\boldsymbol{v}_k\|^2\right)}_{const.} - \operatorname{tr}\big(\boldsymbol{R}\underbrace{\sum_k \boldsymbol{v}_k\boldsymbol{u}_k^\top}_{\boldsymbol{P}_1}\big).\end{aligned} \tag{2.58}$$

In [Koditschek, 1989] the following function with parameters $\boldsymbol{K} \in \mathbb{SYM}^+(3)$ and $\boldsymbol{R}_\mathsf{R} \in \mathbb{SO}(3)$ is called a *navigation function on* $\mathbb{SO}(3)$:

$$\mathcal{V}_2(\boldsymbol{R}) = \operatorname{tr}\big(\boldsymbol{K}(\mathbf{I}_3 - \boldsymbol{R}_\mathsf{R}^\top\boldsymbol{R})\big) = \underbrace{\operatorname{tr}\boldsymbol{K}}_{const.} - \operatorname{tr}(\underbrace{\boldsymbol{K}\boldsymbol{R}_\mathsf{R}^\top}_{\boldsymbol{P}_2}\boldsymbol{R}). \tag{2.59}$$

Using the metric from (2.49) with a weight $\boldsymbol{K} \in \mathbb{SYM}^{+}(3)$ we may ask for the rotation matrix $\boldsymbol{R} \in \mathbb{SO}(3)$ which is closest to a given matrix $\boldsymbol{Q} \in \mathbb{R}^{3\times 3}$, i.e. which minimizes

$$\begin{aligned}\mathcal{V}_3(\boldsymbol{R}) = \tfrac{1}{2} d_{\boldsymbol{K}}^2(\boldsymbol{Q}, \boldsymbol{R}) = \tfrac{1}{2}\operatorname{tr}\left((\boldsymbol{Q}-\boldsymbol{R})^\top \boldsymbol{K}(\boldsymbol{Q}-\boldsymbol{R})\right)& \\ = \underbrace{\tfrac{1}{2}\operatorname{tr}\left(\boldsymbol{Q}^\top\boldsymbol{K}\boldsymbol{Q}+\boldsymbol{K}\right)}_{const.} - \operatorname{tr}\big(\underbrace{\boldsymbol{Q}^\top\boldsymbol{K}}_{\boldsymbol{P}_3}\boldsymbol{R}\big)& \end{aligned} \tag{2.60}$$

Each of these problems essentially asks for an $\boldsymbol{R} \in \mathbb{SO}(3)$ wich minimizes $\mathcal{V} = -\operatorname{tr}(\boldsymbol{P}\boldsymbol{R})$ for some given $\boldsymbol{P} \in \mathbb{R}^{3\times 3}$.

Solutions to Wahba's problem are given in [Davenport, 1968] using attitude quaternions and in [Kabsch, 1976] using the singular value decomposition. A proof of a unique minimum in [Bullo and Murray, 1999] relies on $\boldsymbol{P}$ having distinct singular values.

In the following we extend theses results in the sense that, we do not impose assumptions on $\boldsymbol{P}$ and are also interested other extrema of $\mathcal{V}$.

Problem definition. Similar problems to the ones above will also appear in this work. So we are interested in the extrema, and their nature, of the of the function

$$\mathcal{V} : \mathbb{SO}(3) \to \mathbb{R} \; : \; \boldsymbol{R} \mapsto -\operatorname{tr}(\boldsymbol{P}\boldsymbol{R}) \tag{2.61}$$

with the parameter $\boldsymbol{P} \in \mathbb{R}^{3\times 3}$.

Coordinate transformation. Consider a singular value decomposition $\boldsymbol{P} = \boldsymbol{X}\boldsymbol{\Sigma}\boldsymbol{Y}^\top$ with $\boldsymbol{X}, \boldsymbol{Y} \in \mathbb{O}(3)$ and $\boldsymbol{\Sigma} = \operatorname{diag}(\sigma_1, \sigma_2, \sigma_3)$, $\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq 0$. [Kabsch, 1976]: Define

$$\bar{\boldsymbol{X}} = \boldsymbol{X} \operatorname{diag}(1, 1, \det \boldsymbol{X}) \in \mathbb{SO}(3), \tag{2.62a}$$
$$\bar{\boldsymbol{Y}} = \boldsymbol{Y} \operatorname{diag}(1, 1, \det \boldsymbol{Y}) \in \mathbb{SO}(3), \tag{2.62b}$$
$$\bar{\boldsymbol{\Sigma}} = \operatorname{diag}(\sigma_1, \sigma_2, \bar{\sigma}_3), \;\; \bar{\sigma}_3 = \det \boldsymbol{X} \det \boldsymbol{Y} \sigma_3 \tag{2.62c}$$

which yields a decomposition $\boldsymbol{P} = \bar{\boldsymbol{X}}\bar{\boldsymbol{\Sigma}}\bar{\boldsymbol{Y}}^\top$ with proper rotations. Using the cyclic permutation property of the trace we get

$$\mathcal{V}(\boldsymbol{R}) = -\operatorname{tr}(\bar{\boldsymbol{X}}\bar{\boldsymbol{\Sigma}}\bar{\boldsymbol{Y}}^\top\boldsymbol{R}) = -\operatorname{tr}(\bar{\boldsymbol{\Sigma}}\underbrace{\bar{\boldsymbol{Y}}^\top\boldsymbol{R}\bar{\boldsymbol{X}}}_{\bar{\boldsymbol{R}}}) =: \bar{\mathcal{V}}(\bar{\boldsymbol{R}}) \tag{2.63}$$

Since the SVD is not unique in general, the transformed function $\bar{\mathcal{V}}$ is neither. However, since the coordinate transformation $\boldsymbol{R} = \bar{\boldsymbol{Y}}\bar{\boldsymbol{R}}\bar{\boldsymbol{X}}^\top$ is bijective, no information is lost here.

Critical points. Using the operators defined above, we may formulate the differential and Hessian of the transformed function as

$$\boldsymbol{\nabla}\bar{\mathcal{V}}(\bar{\boldsymbol{R}}) = \operatorname{vee2}(\bar{\boldsymbol{\Sigma}}\bar{\boldsymbol{R}}), \tag{2.64}$$
$$\boldsymbol{\nabla}^2\bar{\mathcal{V}}(\bar{\boldsymbol{R}}) = \operatorname{Vee}(\bar{\boldsymbol{\Sigma}}\bar{\boldsymbol{R}})^\top. \tag{2.65}$$

For a critical point $\bar{\boldsymbol{R}}_0 : \boldsymbol{\nabla}\bar{\mathcal{V}}(\bar{\boldsymbol{R}}_0) = \mathbf{0}$ we need the matrix $\bar{\boldsymbol{\Sigma}}\bar{\boldsymbol{R}}_0$ to be symmetric. For the following it will be useful to substitute the entries/eigenvalues of $\mathrm{Vee}(\bar{\boldsymbol{\Sigma}}) = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3) = \boldsymbol{\Lambda}$ as

$$\left.\begin{aligned} \lambda_1 &= \sigma_2 + \bar{\sigma}_3, \\ \lambda_2 &= \bar{\sigma}_3 + \sigma_1, \\ \lambda_3 &= \sigma_1 + \sigma_2 \end{aligned}\right\} \quad \Leftrightarrow \quad \left\{\begin{aligned} \sigma_1 &= \tfrac{1}{2}(\lambda_2 + \lambda_3 - \lambda_1), \\ \sigma_2 &= \tfrac{1}{2}(\lambda_3 + \lambda_1 - \lambda_2), \\ \bar{\sigma}_3 &= \tfrac{1}{2}(\lambda_1 + \lambda_2 - \lambda_3) \end{aligned}\right. \tag{2.66}$$

Note that $\sigma_1 \geq \sigma_2 \geq |\bar{\sigma}_3| \geq 0$ implies $\lambda_3 \geq \lambda_2 \geq \lambda_1 \geq 0$. Depending on the constellation of the eigenvalues we have the following critical points:

- Distinct eigenvalues: $\lambda_3 > \lambda_2 > \lambda_1 > 0$: We have the critical points

$$\begin{aligned} &\bar{\boldsymbol{R}}_0 = \mathbf{I}_3 \ : \\ &\quad \mathcal{V}(\bar{\boldsymbol{R}}_0) = -\tfrac{\lambda_1}{2} - \tfrac{\lambda_2}{2} - \tfrac{\lambda_3}{2}, \quad \mathrm{eig}(\boldsymbol{\nabla}^2\bar{\mathcal{V}}(\bar{\boldsymbol{R}}_0)) = \{\lambda_3, \lambda_2, \lambda_1\} \end{aligned} \tag{2.67a}$$

$$\begin{aligned} &\bar{\boldsymbol{R}}_1 = \mathrm{diag}(1, -1, -1) \ : \\ &\quad \mathcal{V}(\bar{\boldsymbol{R}}_1) = \tfrac{3\lambda_1 - \lambda_2 - \lambda_3}{2}, \quad \mathrm{eig}(\boldsymbol{\nabla}^2\bar{\mathcal{V}}(\bar{\boldsymbol{R}}_1)) = \{\lambda_3 - \lambda_1, \lambda_2 - \lambda_1, -\lambda_1\} \end{aligned} \tag{2.67b}$$

$$\begin{aligned} &\bar{\boldsymbol{R}}_2 = \mathrm{diag}(-1, 1, -1) \ : \\ &\quad \mathcal{V}(\bar{\boldsymbol{R}}_2) = \tfrac{3\lambda_2 - \lambda_1 - \lambda_3}{2}, \quad \mathrm{eig}(\boldsymbol{\nabla}^2\bar{\mathcal{V}}(\bar{\boldsymbol{R}}_2)) = \{\lambda_3 - \lambda_2, \lambda_1 - \lambda_2, -\lambda_2\} \end{aligned} \tag{2.67c}$$

$$\begin{aligned} &\bar{\boldsymbol{R}}_3 = \mathrm{diag}(-1, -1, 1) \ : \\ &\quad \mathcal{V}(\bar{\boldsymbol{R}}_3) = \tfrac{3\lambda_3 - \lambda_1 - \lambda_2}{2}, \quad \mathrm{eig}(\boldsymbol{\nabla}^2\bar{\mathcal{V}}(\bar{\boldsymbol{R}}_3)) = \{\lambda_2 - \lambda_3, \lambda_1 - \lambda_3, -\lambda_3\} \end{aligned} \tag{2.67d}$$

  so $\bar{\boldsymbol{R}}_0$ is a minimum, $\bar{\boldsymbol{R}}_1$ and $\bar{\boldsymbol{R}}_2$ are saddle points, and $\bar{\boldsymbol{R}}_3$ is a maximum.

- Double eigenvalue: $\lambda_3 > \lambda_2 = \lambda_1 > 0$: We have a minimum at $\bar{\boldsymbol{R}}_0$, a maximum at $\bar{\boldsymbol{R}}_3$ and a saddle on the circular manifold

$$\bar{\boldsymbol{R}}_4 = \begin{bmatrix} -c & s & 0 \\ s & c & 0 \\ 0 & 0 & -1 \end{bmatrix}, \ c^2 + s^2 = 1 \ :$$

$$\mathcal{V}(\bar{\boldsymbol{R}}_4) = \lambda_1 - \tfrac{\lambda_3}{2}, \quad \mathrm{eig}(\boldsymbol{\nabla}^2\bar{\mathcal{V}}(\bar{\boldsymbol{R}}_4)) = \{\lambda_3 - \lambda_1, 0, -\lambda_1\} \tag{2.67e}$$

  which includes the points $\bar{\boldsymbol{R}}_1$ and $\bar{\boldsymbol{R}}_2$.

- Double eigenvalue: $\lambda_3 = \lambda_2 > \lambda_1 > 0$: Analog to above we have a minimum at $\bar{\boldsymbol{R}}_0$, a saddle at $\bar{\boldsymbol{R}}_1$ and a maximum on the circular manifold

$$\bar{\boldsymbol{R}}_5 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & c & s \\ 0 & s & -c \end{bmatrix}, \ c^2 + s^2 = 1 \ :$$

$$\mathcal{V}(\bar{\boldsymbol{R}}_5) = \lambda_2 - \tfrac{\lambda_1}{2}, \quad \mathrm{eig}(\boldsymbol{\nabla}^2\bar{\mathcal{V}}(\bar{\boldsymbol{R}}_5)) = \{0, \lambda_1 - \lambda_2, -\lambda_2\} \tag{2.67f}$$

  which includes the points $\bar{\boldsymbol{R}}_2$ and $\bar{\boldsymbol{R}}_3$.

- Triple eigenvalue: $\lambda_3 = \lambda_2 = \lambda_1 > 0$: Minimum at $\bar{\boldsymbol{R}}_0$ and a maximum on the spherical manifold

$$\bar{\boldsymbol{R}}_6 = \begin{bmatrix} q_{\mathsf{x}}^2 - q_{\mathsf{y}}^2 - q_{\mathsf{z}}^2 & 2q_{\mathsf{x}}q_{\mathsf{y}} & 2q_{\mathsf{x}}q_{\mathsf{z}} \\ 2q_{\mathsf{x}}q_{\mathsf{y}} & q_{\mathsf{y}}^2 - q_{\mathsf{x}}^2 + q_{\mathsf{z}}^2 & 2q_{\mathsf{y}}q_{\mathsf{z}} \\ 2q_{\mathsf{x}}q_{\mathsf{z}} & 2q_{\mathsf{y}}q_{\mathsf{z}} & q_{\mathsf{z}}^2 - q_{\mathsf{x}}^2 - q_{\mathsf{y}}^2 \end{bmatrix}, \ q_{\mathsf{x}}^2 + q_{\mathsf{y}}^2 + q_{\mathsf{z}}^2 = 1 :$$

$$\mathcal{V}(\bar{\boldsymbol{R}}_6) = \tfrac{\lambda_1}{2}, \quad \mathrm{eig}(\boldsymbol{\nabla}^2\bar{\mathcal{V}}(\bar{\boldsymbol{R}}_6)) = \{0, 0, -\lambda_1\} \tag{2.67g}$$

which includes the points $\bar{\boldsymbol{R}}_1$, $\bar{\boldsymbol{R}}_2$ and $\bar{\boldsymbol{R}}_3$ and the circles $\bar{\boldsymbol{R}}_4$ and $\bar{\boldsymbol{R}}_5$. It corresponds to a 180° rotation about an arbitrary axis $[q_\mathsf{x}, q_\mathsf{y}, q_\mathsf{z}]^\top \in \mathbb{S}^2$.

- One zero eigenvalue: $\lambda_3 > \lambda_2 > \lambda_1 = 0$: We have a minimum on the circular manifold

$$\bar{\boldsymbol{R}}_7 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & -s \\ 0 & s & c \end{bmatrix}, \; c^2 + s^2 = 1 \; :$$
$$\mathcal{V}(\bar{\boldsymbol{R}}_7) = -\tfrac{\lambda_2+\lambda_3}{2}, \qquad \operatorname{eig}(\boldsymbol{\nabla}^2\bar{\mathcal{V}}(\bar{\boldsymbol{R}}_7)) = \{\lambda_3, \lambda_2, 0\} \tag{2.67h}$$

  which includes $\bar{\boldsymbol{R}}_0$ and $\bar{\boldsymbol{R}}_1$. Furthermore we have a saddle point at $\bar{\boldsymbol{R}}_2$ and a maximum at $\bar{\boldsymbol{R}}_3$.

- Double eigenvalue and zero eigenvalue: $\lambda_3 = \lambda_2 > \lambda_1 = 0$: We have a minimum on $\bar{\boldsymbol{R}}_7$ and a maximum on $\bar{\boldsymbol{R}}_5$.

- Two zero eigenvalues: $\lambda_3 > \lambda_2 = \lambda_1 = 0$: We have a minimum on the spherical manifold

$$\bar{\boldsymbol{R}}_8 = \begin{bmatrix} q_\mathsf{w}^2 + q_\mathsf{x}^2 - q_\mathsf{y}^2 & 2q_\mathsf{x}q_\mathsf{y} & 2q_\mathsf{w}q_\mathsf{y} \\ 2q_\mathsf{x}q_\mathsf{y} & q_\mathsf{w}^2 - q_\mathsf{x}^2 + q_\mathsf{y}^2 & -2q_\mathsf{w}q_\mathsf{x} \\ -2q_\mathsf{w}q_\mathsf{x} & 2q_\mathsf{w}q_\mathsf{x} & q_\mathsf{w}^2 - q_\mathsf{x}^2 - q_\mathsf{y}^2 \end{bmatrix}, \; q_\mathsf{w}^2 + q_\mathsf{x}^2 + q_\mathsf{y}^2 = 1 \; :$$
$$\mathcal{V}(\bar{\boldsymbol{R}}_8) = -\tfrac{\lambda_3}{2}, \qquad \operatorname{eig}(\boldsymbol{\nabla}^2\bar{\mathcal{V}}(\bar{\boldsymbol{R}}_8)) = \{\lambda_3, 0, 0\} \tag{2.67i}$$

  which includes $\bar{\boldsymbol{R}}_0$, $\bar{\boldsymbol{R}}_1$ and $\bar{\boldsymbol{R}}_2$ and corresponds to an arbitrary rotation about an axis $[q_\mathsf{x}, q_\mathsf{y}, 0]^\top$. Furthermore we have a maximum at $\bar{\boldsymbol{R}}_3$.

- All zero Eigenvalues: $\lambda_3 = \lambda_2 = \lambda_1 = 0$: for this we have $\bar{\boldsymbol{\Sigma}} = \boldsymbol{P} = \boldsymbol{0}$ and the function is $\mathcal{V} = 0$.

We may conclude that the function $\bar{\mathcal{V}}$ has a minimum at $\bar{\boldsymbol{R}}_0 = \mathbf{I}_3$ and a maximum at $\bar{\boldsymbol{R}}_3 = \operatorname{diag}(-1, -1, 1)$, though they may not be strict. The minimum is strict if and only if $\lambda_1 > 0$. The maximum is strict if and only if $\lambda_3 > \lambda_2$.

It should also be noted that the results of this paragraph would be much more "symmetric" if we would not have required the descending order of the singular values σ_i. This did however reduce the number of cases to distinguish.

Original coordinates. The original function $\mathcal{V}$ has a minimum at $\boldsymbol{R}_0 = \bar{\boldsymbol{Y}}\bar{\boldsymbol{X}}^\top$. The minimum $\boldsymbol{R}_0$ is strict, if, and only if, $\lambda_i > 0, i = 1, 2, 3$ or equivalently if $\boldsymbol{K} \in \mathbb{SYM}_0^+(3)$ is positive definite:

$$\boldsymbol{K} = \boldsymbol{\nabla}^2\mathcal{V}(\boldsymbol{R}_0) = \operatorname{Vee}(\boldsymbol{P}\boldsymbol{R}_0) = \operatorname{Vee}(\bar{\boldsymbol{X}}\bar{\boldsymbol{\Sigma}}\bar{\boldsymbol{Y}}^\top\bar{\boldsymbol{Y}}\bar{\boldsymbol{X}}^\top) = \bar{\boldsymbol{X}}\operatorname{Vee}(\bar{\boldsymbol{\Sigma}})\bar{\boldsymbol{X}}^\top = \bar{\boldsymbol{X}}\boldsymbol{\Lambda}\bar{\boldsymbol{X}}^\top. \tag{2.68}$$

Special polar decomposition. From the results of this section we may also conclude the following: For any matrix $\boldsymbol{M} \in \mathbb{R}^{3\times 3}$ there is a matrix $\boldsymbol{R} \in \mathbb{SO}(3)$ and a unique matrix $\boldsymbol{K} \in \mathbb{SYM}_0^+(3)$, such that $\boldsymbol{M} = \boldsymbol{R}\operatorname{Wed}(\boldsymbol{K})$. The matrix $\boldsymbol{R}$ is unique if, and only if, the matrix $\boldsymbol{K}$ is positive definite. Within this work, this will be called the *special polar decomposition*.

Chapter 3

Analytical mechanics of particle systems

Goal. Established approaches of analytical mechanics commonly rely on the parameterization of the system in terms of *minimal* generalized coordinates $\boldsymbol{q}$ and their derivatives $\dot{\boldsymbol{q}}$. In this section we like to generalize this to handle redundant configuration coordinates $\boldsymbol{x}(t) \in \mathbb{X}$ and nonholonomic velocity coordinates $\boldsymbol{\xi}(t) \in \mathbb{R}^n$ as introduced in the previous section. The resulting formulations might become more cumbersome, but with some examples we like to show that it is worth it.

3.1 A single free particle

[Landau and Lifshitz, 1960, §1]: *One of the fundamental concepts of mechanics is that of a particle, also called material point.* This abstracts a body whose dimensions may be neglected and all its mass $\mathfrak{m}$ is located at a point with the Cartesian coordinates $\mathfrak{r}(t) \in \mathbb{R}^3$ at time t. Its motion obeys Newton's second law [Newton, 1687, p. 13, lex II], english translation [Newton, 1846, p. 83]: *The alternation of motion is ever proportional to the motive force impressed; and is made in the direction of the right line in which that force is impressed.* The contemporary version reads (e.g. [Lurie, 2002, eq. 6.1.1] or [Goldstein, 1951, eq. 1.3])

$$\mathfrak{m}\ddot{\mathfrak{r}} = \mathfrak{F}^{\mathsf{A}}. \tag{3.1}$$

where $\ddot{\mathfrak{r}} \equiv \mathrm{d}^2\mathfrak{r}/\mathrm{d}t^2$ is Newton's notation of differentiation and the applied force $\mathfrak{F}^{\mathsf{A}}$ collects all other (non inertial) influences on the particle. In this work we will investigate three sources of applied forces: gravity, linear springs and viscous friction.

Gravity. For far most engineering applications we are dealing with systems that move close to the surface of the Earth and where Galilei's gravitation principle [Galilei, 1638, Day 3] holds. In a contemporary formulation it states that a particle with mass $\mathfrak{m}_p$ is subject to the gravitational force

$$\mathfrak{F}^{\mathsf{G}} = \mathfrak{m}\boldsymbol{a}_{\mathsf{G}} \tag{3.2}$$

where $\boldsymbol{a}_\mathsf{G}$ are the coefficients of the gravitational acceleration of the earth w.r.t. the chosen reference frame. Commonly the reference frame is chosen such that the $\boldsymbol{e}_\mathsf{z}$ axis is opposing gravity and we have $\boldsymbol{a}_\mathsf{G} = [0, 0, -g]^\top$ with the *gravity of earth* $g = 9.8\,\frac{\mathrm{m}}{\mathrm{s}^2}$.

Linear spring. Let the particle be connected with a spring to a point $\mathfrak{r}_0$. The simplest model of a spring is that of Hooke's law [Hooke, 1678]: The force $\mathfrak{F}^\mathsf{K}$ on the particle is opposite and proportional by a factor $\mathfrak{k} \in \mathbb{R}^+$ to the spring displacement $\mathfrak{r} - \mathfrak{r}_0$, i.e.

$$\mathfrak{F}^\mathsf{K} = -\mathfrak{k}(\mathfrak{r} - \mathfrak{r}_0). \tag{3.3}$$

Viscous friction. [Rayleigh, 1877, §81]: *There is another group of forces whose existence is often advantageous to recognize specially, namely those arising from friction or viscosity. [..] we suppose that each particle is retarded by forces proportional to its component velocities.* We may think of the particle to be immersed in a viscous fluid which, at the particle position, has the velocity $\mathfrak{v}_0$. The force on the particle is

$$\mathfrak{F}^\mathsf{D} = -\mathfrak{d}(\dot{\mathfrak{r}} - \mathfrak{v}_0) \tag{3.4}$$

with the damping parameter $\mathfrak{d} \in \mathbb{R}^+$.

Equation of motion. A single free particle that is subject to all the aforementioned forces and a general, not further specified external force $\mathfrak{F}^\mathsf{E}$, i.e. $\mathfrak{F}^\mathsf{A} = \mathfrak{F}^\mathsf{G} + \mathfrak{F}^\mathsf{K} + \mathfrak{F}^\mathsf{D} + \mathfrak{F}^\mathsf{E}$, has the equation of motion

$$\mathfrak{m}(\ddot{\mathfrak{r}} - \boldsymbol{a}_\mathsf{G}) + \mathfrak{d}(\dot{\mathfrak{r}} - \mathfrak{v}_0) + \mathfrak{k}(\mathfrak{r} - \mathfrak{r}_0) = \mathfrak{F}^\mathsf{E}. \tag{3.5}$$

Control engineering. From a control engineering perspective the structure of this system is already nice enough to consider it as a desired closed loop dynamics: If we want the particle to track a sufficiently smooth reference trajectory $t \mapsto \mathfrak{r}_\mathsf{R}(t)$ a reasonable desired closed loop dynamics is

$$\mathfrak{r}_\mathrm{E} = \mathfrak{r} - \mathfrak{r}_0, \qquad \mathfrak{m}\ddot{\mathfrak{r}}_\mathrm{E} + \bar{\mathfrak{d}}\dot{\mathfrak{r}}_\mathrm{E} + \bar{\mathfrak{k}}\mathfrak{r}_\mathrm{E} = \mathbf{0}. \tag{3.6}$$

This is essentially the same as above (3.5), but we replaced the spring origin $\mathfrak{r}_0$ with the reference position $\mathfrak{r}_\mathsf{R}$, the fluid velocity $\mathfrak{v}_0$ with the reference velocity $\dot{\mathfrak{r}}_\mathsf{R}$ and the free-fall acceleration $\boldsymbol{a}_\mathsf{G}$ with the reference acceleration $\ddot{\mathfrak{r}}_\mathsf{R}$. Furthermore, we replaced the spring stiffness $\mathfrak{k}$ and viscosity $\mathfrak{d}$ by analog tuning parameters $\bar{\mathfrak{k}}, \bar{\mathfrak{d}} \in \mathbb{R}^+$. Plugging the desired dynamics (3.6) into the plant dynamics (3.5) yields the required control law

$$\mathfrak{F}^\mathsf{E} = \mathfrak{m}(\ddot{\mathfrak{r}}_\mathsf{R} - \boldsymbol{a}_\mathsf{G}) + \mathfrak{d}(\dot{\mathfrak{r}} - \mathfrak{v}_0) + \mathfrak{k}(\mathfrak{r} - \mathfrak{r}_0) - \bar{\mathfrak{d}}(\dot{\mathfrak{r}} - \dot{\mathfrak{r}}_\mathsf{R}) - \bar{\mathfrak{k}}(\mathfrak{r} - \mathfrak{r}_\mathsf{R}) - \mathfrak{m}\boldsymbol{a}_\mathsf{G}. \tag{3.7}$$

As this control approach is so closely related to basic mechanics, it could be more intuitive for an engineer than other generic mathematical approaches.

3.2 Systems of constrained particles

System under consideration. For this section we consider a system of $\mathfrak{N}$ particles under geometric constraints: The *position* of a particle with respect to a given inertial frame at a given time t is $\mathfrak{r}_p(t) \in \mathbb{R}^3, p = 1, \ldots, \mathfrak{N}$ and the collection of all particle positions is $\mathfrak{x} = [\mathfrak{r}_1^\top, \ldots, \mathfrak{r}_\mathfrak{N}^\top]^\top \in \mathbb{R}^{3\mathfrak{N}}$. *Geometric constraints* on the particles are captured in $\mathfrak{H} \geq 0$ smooth functions of the form $\mathfrak{c}(\mathfrak{x}) = [\mathfrak{c}^1(\mathfrak{x}), \ldots, \mathfrak{c}^\mathfrak{H}(\mathfrak{x})]^\top = \mathbf{0}$. The set of all mutually admissible particle positions

$$\mathfrak{X} = \{\mathfrak{x} \in \mathbb{R}^{3\mathfrak{N}} \mid \mathfrak{c}(\mathfrak{x}) = \mathbf{0}\} \tag{3.8}$$

is called the *configuration space*. We require $\frac{\partial \mathfrak{c}}{\partial \mathfrak{x}}(\mathfrak{x})$ to have a constant, though not necessarily full rank.

3.2.1 First principles

Principle of constraint release. The principle of constraint release (see e.g. [Hamel, 1949, sec. 32] or [Lurie, 2002, sec. 6.1]) states that the motion of system of geometrically constrained particles is governed by

$$\mathfrak{c}(\mathfrak{x}) = \mathbf{0}, \quad \mathfrak{m}_p \ddot{\mathfrak{r}}_p = \mathfrak{F}_p^\mathsf{A} + \lambda_\kappa \frac{\partial \mathfrak{c}^\kappa}{\partial \mathfrak{r}_p}, \quad p = 1, \ldots, \mathfrak{N}. \tag{3.9}$$

Lagrange-d'Alembert principle. For a system of geometrically constrained particles states (e.g. [Goldstein, 1951, sec. 1.4] or [Lurie, 2002, sec. 6.3]):

$$\sum\nolimits_{p=1}^{\mathfrak{N}} \langle \delta \mathfrak{r}_p, \mathfrak{F}_p^\mathsf{A} - \mathfrak{m}_p \ddot{\mathfrak{r}}_p \rangle = 0 \qquad \forall \quad \frac{\partial \mathfrak{c}}{\partial \mathfrak{x}} \delta \mathfrak{x} = \mathbf{0}. \tag{3.10}$$

The *virtual displacements* $\delta \mathfrak{r}_p$ are tangents to possible motions: For particle positions constrained by $\mathfrak{c}(\mathfrak{x}) = \mathbf{0}$ the displacements have to fulfill $\frac{\partial \mathfrak{c}}{\partial \mathfrak{x}} \delta \mathfrak{x} = \mathbf{0}$.

Gauß principle. Gauss's principle of least constraint was originally described in [Gauß, 1829] in words rather than equations. Maybe because of this, one finds somewhat different mathematical formulations in more contemporary sources, e.g. [Päsler, 1968, sec. 7], [Lanczos, 1986, sec. IV.8], [Bremer, 2008, sec. 2.2].

For the system given above, Gauss' principle states that the particle accelerations $\ddot{\mathfrak{r}}_p, p = 1, \ldots, \mathfrak{N}$ minimize the so-called Gaussian constrain $\mathcal{G}$:

$$\begin{aligned} \min_{\ddot{\mathfrak{x}} \in \mathbb{R}^{3\mathfrak{N}}} \quad & \mathcal{G} = \tfrac{1}{2} \sum\nolimits_{p=1}^{\mathfrak{N}} \mathfrak{m}_p \|\ddot{\mathfrak{r}}_p - \ddot{\mathfrak{r}}_p^\mathsf{f}\|^2 \\ \text{s. t.} \quad & \ddot{\mathfrak{c}}(\mathfrak{x}, \dot{\mathfrak{x}}, \ddot{\mathfrak{x}}) = \mathbf{0} \end{aligned} \tag{3.11}$$

where $\ddot{\mathfrak{r}}_p^\mathsf{f}$ are the *unconstrained* particle accelerations, i.e. Newtons's second law $\ddot{\mathfrak{r}}_p^\mathsf{f} = \frac{\mathfrak{F}_p^\mathsf{A}}{\mathfrak{m}_p}$. Its crucial to note that the constraint equations $\ddot{\mathfrak{c}}(\mathfrak{x}, \dot{\mathfrak{x}}, \ddot{\mathfrak{x}}) = \mathbf{0}$ are *linear* in the accelerations $\ddot{\mathfrak{x}}$. Consequently, as stressed in [Gauß, 1829], the principle (3.11) is a (static) quadratic optimization problem with linear constraints.

Hamilton's principle. [Lanczos, 1986, p. 113]: *Hamilton's principle states that the motion of a mechanical system occurs in such a way that the action* $\mathcal{A} = \int_{t_0}^{t_1} \mathcal{L}\,\mathrm{d}t$ *becomes stationary for arbitrary possible variations of the configuration of the system, provided the initial and final conditions are prescribed. The Lagrangian* $\mathcal{L} = \mathcal{T} - \mathcal{V}$ *is the excess of kinetic energy* $\mathcal{T}$ *over potential energy* $\mathcal{V}$. For the system considered here, the kinetic energy is $\mathcal{T} = \frac{1}{2}\sum_{p=1}^{\mathfrak{N}} \mathfrak{m}_p \|\dot{\mathfrak{r}}_p\|^2$. The potential energy may have various origins, some of them will be discussed later.

The remarkable quality of the principle of stationary action is that it is used interdisciplinary: With appropriate formulation of the Lagrangian $\mathcal{L}$ it has applications in all branches of physics e.g. general relativity [Einstein, 1916], electromagnetic field theory [Landau and Lifshitz, 1971, ch. 4] or optics, the original context of Hamilton [Klein, 1926, p. 192]. Apart form physics, there is optimal control that builds up on essentially the same idea e.g. [Bryson, 1975].

For the context of this work, the statement above is actually not that useful since it does not allow for generic impressed forces $\mathfrak{F}_p^{\mathsf{A}}$. To mend this one finds a similar statement in e.g. [Lurie, 2002, eq. 12.2.14] or [Szabó, 1956, sec. I.3]:

$$\delta'\mathcal{A} = \int_{t_1}^{t_2} \big(\delta\mathcal{T} - \delta'\mathcal{W}\big)\mathrm{d}t = 0, \qquad \mathcal{T} = \tfrac{1}{2}\sum\nolimits_p \mathfrak{m}_p \|\dot{\mathfrak{r}}_p\|^2,\ \delta'\mathcal{W} = \langle \delta\mathfrak{r}_p, \mathfrak{F}_p^{\mathsf{A}}\rangle. \tag{3.12}$$

Here $\delta'\mathcal{W}$ is an variational quantity, but in general, there is no $\mathcal{W}$ that it is the variation of, unless the impressed forces $\mathfrak{F}_p^{\mathsf{A}}$ may be derived from a potential. Consequently, there is generally no quantity $\mathcal{A}$ which becomes stationary, thus the naming is not suitable here.

3.2.2 Coordinates

Generalized coordinates. In most cases we are not really interested in the motion of the individual particles but rather in the system as a whole. Using the constraint equations it is possible to capture the configuration of the system by $\dim \mathfrak{X} = 3\mathfrak{N} - \operatorname{rank} \frac{\partial \mathfrak{c}}{\partial \mathfrak{r}} = n$ coordinates, commonly called *generalized coordinates* and commonly denoted by $\boldsymbol{q}$. Its components, in contrast to the Cartesian particle coordinates, may be lengths, angles or some completely generic quantity stressed by the term "generalized". What is more crucial but usually implicit, is that they have to be independent from another to parameterize the n dimensional configuration space with its n components. Whenever used in this work, these coordinates are referred to as *minimal* generalized coordinates $\boldsymbol{q} \in \mathbb{R}^n$.

Redundant configuration coordinates and velocity coordinates. As motivated in section 2.1, in some cases it can be beneficial to use use a slightly larger number of *redundant* generalized coordinates $\boldsymbol{x}(t) \in \mathbb{X} = \{\boldsymbol{x} \in \mathbb{R}^\nu \mid \boldsymbol{\phi}(\boldsymbol{x}) = \boldsymbol{0}\}$ and *minimal* velocity coordinates $\boldsymbol{\xi}(t) \in \mathbb{R}^n$ related by $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{\xi}$. Of course this includes common minimal parameterization $\boldsymbol{x} = \boldsymbol{q} \in \mathbb{R}^n$ and $\boldsymbol{\xi} = \dot{\boldsymbol{q}} \in \mathbb{R}^n$ as the special case $\mathbb{X} = \mathbb{R}^n$ and $\boldsymbol{A} = \mathbf{I}_n$.

Particle parameterization. Let the admissible particle positions $\mathfrak{r} \in \mathfrak{X}$ be parameterized $\mathfrak{r}_p = \mathfrak{r}_p(\boldsymbol{x}, t)$ by possibly redundant coordinates $\boldsymbol{x} \in \mathbb{X}$. This means $\boldsymbol{\phi}(\boldsymbol{x}) =$

$\mathbf{0} \Rightarrow \mathfrak{c}(\mathfrak{x}(\boldsymbol{x},t)) = \mathbf{0}$ and consequently $\boldsymbol{x} \in \mathbb{X} \Rightarrow \mathfrak{x}(\boldsymbol{x}) \in \mathfrak{X}$. The particle velocities and accelerations in terms of the coordinates we have

$$\dot{\mathfrak{r}}_p = \partial_i \mathfrak{r}_p \xi^i + \frac{\partial \mathfrak{r}_p}{\partial t} \tag{3.13a}$$

$$\ddot{\mathfrak{r}}_p = \partial_i \mathfrak{r}_p \dot{\xi}^i + \partial_j \partial_i \mathfrak{r}_p \xi^i \xi^j + \underbrace{\Big(\partial_i \frac{\partial \mathfrak{r}_p}{\partial t} + \frac{\partial}{\partial t} \partial_i \mathfrak{r}_p\Big)\xi^i + \frac{\partial^2 \mathfrak{r}_p}{\partial t^2}}_{\mathfrak{a}_p^{\mathsf{E}}}. \tag{3.13b}$$

The following relations will be useful for the next steps of this section:

- From (3.13) it is evident that

$$\partial_i \mathfrak{r}_p = \frac{\partial \dot{\mathfrak{r}}_p}{\partial \xi^i} = \frac{\partial \ddot{\mathfrak{r}}_p}{\partial \dot{\xi}^i}. \qquad i = 1, \dots, n. \tag{3.14}$$

- From $\frac{\mathrm{d}}{\mathrm{d}t}\partial_i \mathfrak{r}_p = \partial_j \partial_i \mathfrak{r}_p \xi^j + \frac{\partial}{\partial t}\partial_i \mathfrak{r}_p = (\partial_i \partial_j - \gamma_{ij}^k \partial_k)\mathfrak{r}_p \xi^j + \partial_i \frac{\partial \mathfrak{r}_p}{\partial t} = \partial_i \dot{\mathfrak{r}}_p - \gamma_{ij}^k \partial_k \mathfrak{r}_p \xi^j$ with the commutation coefficients γ_{ij}^k established in subsection 2.2.2, we obtain the commutation relation

$$\partial_i \dot{\mathfrak{r}}_p - \frac{\mathrm{d}}{\mathrm{d}t}\partial_i \mathfrak{r}_p = \gamma_{ij}^k \partial_k \mathfrak{r}_p \xi^j, \qquad i = 1, \dots, n. \tag{3.15}$$

- As $\frac{\mathrm{d}}{\mathrm{d}t}\mathfrak{c}^\kappa = \sum_p \frac{\partial \mathfrak{c}^\kappa}{\partial \mathfrak{r}_p}\dot{\mathfrak{r}}_p = \sum_p \frac{\partial \mathfrak{c}^\kappa}{\partial \mathfrak{r}_p}\big(\partial_i \mathfrak{r}_p \xi^i + \frac{\partial \mathfrak{r}_p}{\partial t}\big) = 0$ has to hold for any $\boldsymbol{\xi}$ we have

$$\sum\nolimits_p \frac{\partial \mathfrak{c}^\kappa}{\partial \mathfrak{r}_p}\partial_i \mathfrak{r}_p = 0, \qquad \kappa = 1, \dots, \mathfrak{H},\ i = 1, \dots, n \tag{3.16a}$$

$$\sum\nolimits_p \frac{\partial \mathfrak{c}^\kappa}{\partial \mathfrak{r}_p}\frac{\partial \mathfrak{r}_p}{\partial t} = 0, \qquad \kappa = 1, \dots, \mathfrak{H}. \tag{3.16b}$$

Application to the principle of constraint release. Summing up the projections of (3.9) on $\partial_i \mathfrak{r}_p$ eliminates the constraint forces $\boldsymbol{\lambda}$ due to (3.16a):

$$\underbrace{\sum\nolimits_p \langle \partial_i \mathfrak{r}_p, \mathfrak{m}_p \ddot{\mathfrak{r}}_p \rangle}_{f_i^{\mathrm{M}}} = \underbrace{\sum\nolimits_p \langle \partial_i \mathfrak{r}_p, \mathfrak{F}_p^{\mathsf{A}} \rangle}_{f_i^{\mathrm{A}}} + \lambda_\kappa \underbrace{\sum\nolimits_p \langle \partial_i \mathfrak{r}_p, \frac{\partial \mathfrak{c}^\kappa}{\partial \mathfrak{r}_p} \rangle}_{0}, \quad i = 1, \dots, n. \tag{3.17}$$

We call $\boldsymbol{f}^{\mathrm{M}}$ the generalized inertia force and $\boldsymbol{f}^{\mathrm{A}}$ the generalized applied force.

Parameterizing the particle accelerations $\ddot{\mathfrak{r}}_p$ in terms of the chosen coordinates (3.13b) yields

$$\begin{aligned} & \sum\nolimits_p \mathfrak{m}_p \langle \partial_i \mathfrak{r}_p, \overbrace{\partial_j \mathfrak{r}_p \dot{\xi}^j + \partial_k \partial_j \mathfrak{r}_p \xi^k \xi^j + \mathfrak{a}_p^{\mathsf{E}}}^{\ddot{\mathfrak{r}}_p} \rangle = \sum\nolimits_p \langle \partial_i \mathfrak{r}_p, \mathfrak{F}_p^{\mathsf{A}} \rangle, \quad i = 1, \dots, n \\ \Leftrightarrow \quad & \underbrace{\sum\nolimits_p \mathfrak{m}_p \langle \partial_i \mathfrak{r}_p, \partial_j \mathfrak{r}_p \rangle}_{M_{ij}} \dot{\xi}^j = \underbrace{\sum\nolimits_p \langle \partial_i \mathfrak{r}_p, \mathfrak{F}_p^{\mathsf{A}} - \mathfrak{m}_p\big(\partial_k \partial_j \mathfrak{r}_p \xi^k \xi^j + \mathfrak{a}_p^{\mathsf{E}}\big) \rangle}_{b_i}, \quad i = 1, \dots, n. \end{aligned} \tag{3.18}$$

Mathematically, the matrix $\boldsymbol{M}$ is symmetric and (assuming $\mathfrak{m}_p \geq 0$) is positive semidefinite. Physically, we may assume that $\boldsymbol{M}$ is actually positive definite, otherwise there is a degree of freedom with no inertia attached which would rather be an error in modeling. For this work we will assume that $\boldsymbol{M}$ is symmetric positive definite and consequently invertible. Then the acceleration coordinates can be expressed as

$$\dot{\boldsymbol{\xi}} = \boldsymbol{M}^{-1}\boldsymbol{b}. \tag{3.19}$$

Application to the Lagrange-d'Alembert principle. Analog to the velocity, we may parameterize virtual displacements as $\delta\mathfrak{r}_p = \partial_i \mathfrak{r}_p h^i, p = 1, \dots, \mathfrak{N}$ in terms of *minimal* displacements coordinates $\boldsymbol{h} \in \mathbb{R}^n$. Plugging this into (3.10) we get

$$h^i \sum\nolimits_p \langle \partial_i \mathfrak{r}_p, \mathfrak{F}_p^{\mathsf{A}} - \mathfrak{m}_p \ddot{\mathfrak{r}}_p \rangle = 0 \qquad \forall \quad \boldsymbol{h} \in \mathbb{R}^n. \tag{3.20}$$

Since this has to hold for any $\boldsymbol{h}$, we have the identical result as above (3.17).

Application to the Gauß principle. Parameterizing the particle accelerations $\ddot{\mathfrak{r}}_p = \ddot{\mathfrak{r}}_p(\boldsymbol{x}, \boldsymbol{\xi}, \dot{\boldsymbol{\xi}})$ in terms of the chosen coordinates (3.13b) in Gauß' principle (3.11) eliminates the constraints. So it essentially transforms it to an *unconstrained* minimization problem. The Gaussian constraint now reads

$$\begin{aligned}
\mathcal{G} &= \tfrac{1}{2} \sum\nolimits_p \mathfrak{m}_p \| \overbrace{\partial_j \mathfrak{r}_p \dot{\xi}^j + \partial_k \partial_j \mathfrak{r}_p \xi^k \xi^j + \mathfrak{a}_p^{\mathsf{E}}}^{\ddot{\mathfrak{r}}_p} - \tfrac{\mathfrak{F}_p^{\mathsf{A}}}{\mathfrak{m}_p} \|^2 \\
&= \tfrac{1}{2} \underbrace{\sum\nolimits_p \mathfrak{m}_p \langle \partial_i \mathfrak{r}_p, \partial_j \mathfrak{r}_p \rangle}_{M_{ij}} \dot{\xi}^i \dot{\xi}^j - \underbrace{\sum\nolimits_p \langle \partial_i \mathfrak{r}_p, \mathfrak{F}_p^{\mathsf{A}} - \mathfrak{m}_p(\partial_k \partial_j \mathfrak{r}_p \xi^k \xi^j + \mathfrak{a}_p^{\mathsf{E}}) \rangle}_{b_i} \dot{\xi}^i \\
&\qquad + \underbrace{\tfrac{1}{2} \sum\nolimits_p \tfrac{1}{\mathfrak{m}_p} \| \mathfrak{F}_p^{\mathsf{A}} - \mathfrak{m}_p(\partial_k \partial_j \mathfrak{r}_p \xi^k \xi^j + \mathfrak{a}_p^{\mathsf{E}}) \|^2}_{\mathcal{G}_0} \\
&= \tfrac{1}{2} \dot{\boldsymbol{\xi}}^\top \boldsymbol{M} \dot{\boldsymbol{\xi}} - \dot{\boldsymbol{\xi}}^\top \boldsymbol{b} + \mathcal{G}_0.
\end{aligned} \tag{3.21}$$

The necessary condition for a critical point is

$$\frac{\partial \mathcal{G}}{\partial \dot{\boldsymbol{\xi}}} = \boldsymbol{0} \qquad \Leftrightarrow \qquad \boldsymbol{M}\dot{\boldsymbol{\xi}} = \boldsymbol{b}. \tag{3.22}$$

Which is, again, the same result as above. Since $\partial^2 \mathcal{G} / \partial \dot{\boldsymbol{\xi}} \partial \dot{\boldsymbol{\xi}} = \boldsymbol{M}$ is positive definite, the solution $\dot{\boldsymbol{\xi}} = \boldsymbol{M}^{-1}\boldsymbol{b}$ is a minimum of the Gaussian constraint $\mathcal{G}$.

Application to Hamilton's principle. As discussed above, Hamilton's principle of stationary action is not applicable for generic forces as assumed here. However, it might still be instructive to apply it to our system when setting $\boldsymbol{f}^{\mathrm{A}} = \boldsymbol{0}$ and $\mathcal{L} = \mathcal{T}$.

The kinetic energy $\mathcal{T}$ in terms of the chosen coordinates is

$$\mathcal{T}(\boldsymbol{x}, \boldsymbol{\xi}, t) = \tfrac{1}{2} \sum\nolimits_p \mathfrak{m}_p \| \dot{\mathfrak{r}}_p(\boldsymbol{x}, \boldsymbol{\xi}, t) \|^2, \qquad \dot{\mathfrak{r}}_p(\boldsymbol{x}, \boldsymbol{\xi}, t) = \partial_i \mathfrak{r}_p(\boldsymbol{x}) \xi^i + \frac{\partial \mathfrak{r}_p}{\partial t}(\boldsymbol{x}, t) \tag{3.23}$$

As this is the structure assumed in (2.26) we may use the result (2.31) from the calculus of variations to obtain

$$f_i^{\mathrm{M}} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{T}}{\partial \xi^i} + \gamma_{ij}^k \xi^j \frac{\partial \mathcal{T}}{\partial \xi^k} - \partial_i \mathcal{T} = 0 \quad i = 1, \ldots, n. \tag{3.24}$$

Evaluation and some rearrangement using the identities (3.14) and (3.15) yields

$$\begin{aligned} f_i^{\mathrm{M}} &= \sum\nolimits_p \mathfrak{m}_p \Big(\frac{\mathrm{d}}{\mathrm{d}t} \langle \frac{\partial \dot{\mathfrak{r}}_p}{\partial \xi^i}, \dot{\mathfrak{r}}_p \rangle + \gamma_{ij}^k \xi^j \langle \frac{\partial \dot{\mathfrak{r}}_p}{\partial \xi^k}, \dot{\mathfrak{r}}_p \rangle - \langle \partial_i \dot{\mathfrak{r}}_p, \dot{\mathfrak{r}}_p \rangle \Big) \\ &= \sum\nolimits_p \mathfrak{m}_p \Big(\langle \partial_i \mathfrak{r}_p, \ddot{\mathfrak{r}}_p \rangle + \langle \underbrace{\frac{\mathrm{d}}{\mathrm{d}t} \partial_i \mathfrak{r}_p + \gamma_{ij}^k \xi^j \partial_k \mathfrak{r}_p - \partial_i \dot{\mathfrak{r}}_p}_{0}, \dot{\mathfrak{r}}_p \rangle \Big) \end{aligned} \tag{3.25}$$

Which matches the generalized inertial force derived above (3.17).

3.2.3 Inertia

In (3.17) we have introduced the generalized inertia force $\boldsymbol{f}^{\mathrm{M}} = \sum_p \langle \partial_i \mathfrak{r}_p, \mathfrak{F}_p^{\mathsf{M}} \rangle$ as the projection of the particle inertia forces $\mathfrak{F}_p^{\mathsf{M}} = \mathfrak{m}_p \ddot{\mathfrak{r}}_p$. With this we will review some established formalisms and extend them for the use of redundant configuration coordinates.

Gibbs-Appell formulation. Using the identity of the differentials (3.14) we may formulate

$$f_i^{\mathrm{M}} = \sum\nolimits_p \langle \frac{\partial \ddot{\mathfrak{r}}_p}{\partial \dot{\xi}^i}, -\mathfrak{m}_p \ddot{\mathfrak{r}}_p \rangle = -\frac{\partial}{\partial \dot{\xi}^i} \underbrace{\Big(\tfrac{1}{2} \sum\nolimits_p \mathfrak{m}_p \|\ddot{\mathfrak{r}}_p\|^2 \Big)}_{\mathcal{S}}, \qquad i = 1, \ldots, n. \tag{3.26}$$

So the generalized inertia force $\boldsymbol{f}^{\mathrm{M}}$ may be derived from the function $\mathcal{S} = \mathcal{S}(\boldsymbol{x}, \boldsymbol{\xi}, \dot{\boldsymbol{\xi}}, t)$ which is commonly called the *acceleration energy* though its dimension is *not* that of energy.

This formulation was first proposed by [Gibbs, 1879] for Cartesian, minimal coordinates and by [Appell, 1900] also using nonholonomic velocity coordinates. Some historic overview is given in [Lewis, 1996, sec. 1].

Euler-Lagrange formulation. Using the identity of the differentials (3.14), the commutation relation (3.15) and the product rule of differentiation we may formulate

$$\begin{aligned} f_i^{\mathrm{M}} &= \sum\nolimits_p \langle \frac{\partial \dot{\mathfrak{r}}_p}{\partial \xi^i}, \frac{\mathrm{d}}{\mathrm{d}t} (\mathfrak{m}_p \dot{\mathfrak{r}}_p) \rangle \\ &= \sum\nolimits_p \mathfrak{m}_p \Big(\frac{\mathrm{d}}{\mathrm{d}t} \langle \frac{\partial \dot{\mathfrak{r}}_p}{\partial \xi^i}, \dot{\mathfrak{r}}_p \rangle - \langle \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \dot{\mathfrak{r}}_p}{\partial \xi^i}, \dot{\mathfrak{r}}_p \rangle \Big) \\ &= \sum\nolimits_p \mathfrak{m}_p \Big(\frac{\mathrm{d}}{\mathrm{d}t} \langle \frac{\partial \dot{\mathfrak{r}}_p}{\partial \xi^i}, \dot{\mathfrak{r}}_p \rangle - \langle \partial_i \dot{\mathfrak{r}}_p - \gamma_{ij}^k \frac{\partial \dot{\mathfrak{r}}_p}{\partial \xi^k} \xi^j, \dot{\mathfrak{r}}_p \rangle \Big) \\ &= \Big(\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial}{\partial \xi^i} + \gamma_{ij}^k \xi^j \frac{\partial}{\partial \xi^k} - \partial_i \Big) \underbrace{\Big(\tfrac{1}{2} \sum\nolimits_p \mathfrak{m}_p \|\dot{\mathfrak{r}}_p\|^2 \Big)}_{\mathcal{T}}, \qquad i = 1, \ldots, n. \end{aligned} \tag{3.27}$$

So the generalized inertia force $\boldsymbol{f}^{\mathrm{M}}$ may be derived from the *kinetic energy* $\mathcal{T} = \mathcal{T}(\boldsymbol{x}, \boldsymbol{\xi}, t)$ formulated in terms of the chosen coordinates. This has already been shown in (3.24) and the computation is essentially (3.25) backwards.

For the special case of minimal configuration coordinates $\boldsymbol{x} = \boldsymbol{q}$ and the holonomic velocity coordinates $\boldsymbol{\xi} = \dot{\boldsymbol{q}}$, which implies $\gamma = 0$, this formulation and its derivation may be found in any graduate textbook on mechanics and is commonly called the *Euler-Lagrange equation*. A nearly identical form as (3.27) can be found in [Hamel, 1904a, p. 17]. The difference is that, in contrast to this work, the directional derivative ∂_i and the commutation coefficients γ_{ij}^k are therein restricted to minimal configuration coordinates.

It is worth noting that the quatitiy $2\mathcal{T}$ which appeared in this context, was called *vis viva* in older publications and translated to *living force* or *lebendige Kraft* [Hamel, 1904a]. The contemporary term *kinetic energy* seems to have established in the early 20th century.

Levi-Civita formulation. Formulation of the particle accelerations $\ddot{\mathfrak{r}}_p$ explicitly in terms of the chosen coordinates (3.13b) yields

$$
\begin{aligned}
f_i^{\mathrm{M}} &= \sum\nolimits_p \langle \partial_i \mathfrak{r}_p, \mathfrak{m}_p \overbrace{\left(\partial_j \mathfrak{r}_p \dot{\xi}^j + \partial_k \partial_j \mathfrak{r}_p \xi^k \xi^j + \mathfrak{a}_p^{\mathsf{E}}\right)}^{\ddot{\mathfrak{r}}_p} \rangle \\
&= \underbrace{\sum\nolimits_p \mathfrak{m}_p \langle \partial_i \mathfrak{r}_p, \partial_j \mathfrak{r}_p \rangle}_{M_{ij}} \dot{\xi}^j + \underbrace{\sum\nolimits_p \mathfrak{m}_p \langle \partial_i \mathfrak{r}_p, \partial_k \partial_j \mathfrak{r}_p \rangle}_{\Gamma_{ijk}} \xi^k \xi^j + \sum\nolimits_p \mathfrak{m}_p \langle \partial_i \mathfrak{r}_p, \mathfrak{a}_p^{\mathsf{E}} \rangle, \ i = 1, \dots, n.
\end{aligned}
\tag{3.28}
$$

The inertia matrix $\boldsymbol{M}$ was already discussed above. Here we are interested in the terms denoted by Γ_{ijk}. Based on their definition in (3.28) one may validate the following identities

$$
\partial_k M_{ij} = \Gamma_{ijk} + \Gamma_{jik}, \tag{3.29a}
$$

$$
\gamma_{ij}^s M_{sk} = \Gamma_{kji} - \Gamma_{kij}. \tag{3.29b}
$$

Plugging these together while permuting the indices, we find

$$
\begin{aligned}
\Gamma_{ijk} &= \partial_k M_{ij} - \Gamma_{jik} \\
&= \partial_k M_{ij} + \gamma_{ik}^s M_{sj} - \Gamma_{jki} \\
&= \partial_k M_{ij} + \gamma_{ik}^s M_{sj} - \partial_i M_{kj} + \Gamma_{kji} \\
&= \partial_k M_{ij} + \gamma_{ik}^s M_{sj} - \partial_i M_{kj} + \gamma_{ij}^s M_{sk} + \Gamma_{kij} \\
&= \partial_k M_{ij} + \gamma_{ik}^s M_{sj} - \partial_i M_{kj} + \gamma_{ij}^s M_{sk} + \partial_j M_{ik} - \Gamma_{ikj} \\
&= \partial_k M_{ij} + \gamma_{ik}^s M_{sj} - \partial_i M_{kj} + \gamma_{ij}^s M_{sk} + \partial_j M_{ik} - \gamma_{kj}^s M_{si} - \Gamma_{ijk}
\end{aligned}
\tag{3.30}
$$

$$
\Leftrightarrow \quad \Gamma_{ijk} = \tfrac{1}{2}\left(\partial_k M_{ij} + \partial_j M_{ik} - \partial_i M_{jk} + \gamma_{ij}^s M_{sk} + \gamma_{ik}^s M_{sj} - \gamma_{jk}^s M_{si}\right). \tag{3.31}
$$

This means the coefficients Γ_{ijk} are completely determined by the inertia matrix $\boldsymbol{M}$ and the geometric matrix $\boldsymbol{A}$ which determines the directional derivative ∂_i and the commutation coefficients $\boldsymbol{\gamma}$.

A similar, coordinate free derivation can be found in [Abraham and Marsden, 1978, proof of Theorem 2.7.6] for the proof of the fundamental theorem of Riemannian geometry, i.e. the existence and uniqueness of the *Levi-Civita connection*. However, all coordinate versions therein are restricted to minimal holonomic coordinates. For this *special case*, i.e. $\boldsymbol{x} = \boldsymbol{q}$, $\boldsymbol{\xi} = \dot{\boldsymbol{q}}$, $\boldsymbol{A} = \mathbf{I}_n$ and consequently $\gamma = 0$, (3.31) simplifies to the familiar definition of the *Christoffel symbols* $\mathit{\Gamma}_{ijk} = \frac{1}{2}\big(\frac{\partial M_{ij}}{\partial q^k} + \frac{\partial M_{ik}}{\partial q^j} - \frac{\partial M_{jk}}{\partial q^i}\big)$, see e.g. [Abraham and Marsden, 1978, p. 145] or [Spivak, 1999, Vol. 2, p. 221]. In [Frankel, 1997, sec. 9.2] it is pointed out that the name Christoffel symbols is exclusive for the holonomic case, whereas in general $\mathit{\Gamma}_{ijk}$ are referred to as the *(Levi-Civita) connection coefficients*. To the best of the authors knowledge, the only popular source that states the coordinate version (3.31) explicitly is [Misner et al., 1973, eq. 8.24], though restricted to minimal coordinates and in a rather different context of relativistic point masses. Since the directional derivative ∂_i and the commutation coefficients γ^s_{ij} are defined in a setting supporting redundant coordinates, so does (3.31) as definition of the connection coefficients.

3.2.4 Gravitation

Earth's gravity acts on a system of particles just as on a single particle (3.2), i.e. with a force $\mathfrak{F}^{\mathsf{G}}_p = \mathfrak{m}_p \boldsymbol{a}_{\mathsf{G}}$ on each particle. The resulting generalized force on the system is

$$f^{\mathrm{G}}_i = \sum\nolimits_p \langle \partial_i \mathfrak{r}_p, \mathfrak{F}^{\mathsf{G}}_p \rangle = \sum\nolimits_p \mathfrak{m}_p \langle \partial_i \mathfrak{r}_p, \boldsymbol{a}_{\mathsf{G}} \rangle = \partial_i \underbrace{\sum\nolimits_p \mathfrak{m}_p \langle \mathfrak{r}_p, -\boldsymbol{a}_{\mathsf{G}} \rangle}_{\mathcal{V}^{\mathrm{G}}} . \tag{3.32}$$

The force may also be derived from the *gravitational potential* $\mathcal{V}^{\mathrm{G}}$ of the system which is simply the sum of the potentials of the individual particles.

The *gravitational mass* $\mathfrak{m}_p$ in (3.32) takes the same value as the *inertial mass* from the previous section. This is sometimes referred to as the *(Galilean) equivalence principle* and is crucial topic for general relativity, see e.g., [Misner et al., 1973, chap. 16]. In this context, there is no *absolute* acceleration $\ddot{\mathfrak{r}}_p$, instead, we are interested in the deviation from the free fall acceleration $\boldsymbol{a}_{\mathsf{G}}$. This motivates the following formulation for the sum of generalized inertial and gravitational force

$$\begin{aligned} f^{\mathrm{M}}_i + f^{\mathrm{G}}_i = \sum\nolimits_p \mathfrak{m}_p \langle \partial_i \mathfrak{r}_p, \ddot{\mathfrak{r}}_p - \boldsymbol{a}_{\mathsf{G}} \rangle &= \sum\nolimits_p \mathfrak{m}_p \langle \frac{\partial \ddot{\mathfrak{r}}_p}{\partial \dot{\xi}^i}, \ddot{\mathfrak{r}}_p - \boldsymbol{a}_{\mathsf{G}} \rangle \\ &= \frac{\partial}{\partial \dot{\xi}^i} \underbrace{\Big(\tfrac{1}{2} \sum\nolimits_p \mathfrak{m}_p \| \ddot{\mathfrak{r}}_p - \boldsymbol{a}_{\mathsf{G}} \|^2 \Big)}_{\mathcal{S}^{\mathrm{G}}}, \qquad i = 1, \dots, n. \end{aligned} \tag{3.33}$$

The quantity $\mathcal{S}^{\mathrm{G}}$ may be regarded as a metric for the deviation of the system to its natural acceleration, the free fall.

Taking this one step further, we may consider the free fall velocities $\mathfrak{v}_{p\mathsf{G}}(t) = \boldsymbol{a}_{\mathsf{G}} t + \mathfrak{v}_{p0}$

with arbitrary initial velocities $\mathfrak{v}_{p0} \in \mathbb{R}^3, p = 1, \dots, \mathfrak{N}$ and compute analog to (3.27):

$$\begin{aligned} f_i^{\mathrm{M}} + f_i^{\mathrm{G}} &= \sum\nolimits_p \mathfrak{m}_p \langle \frac{\partial \dot{\mathfrak{r}}_p}{\partial \xi^i}, \frac{\mathrm{d}}{\mathrm{d}t}\big(\dot{\mathfrak{r}}_p - \mathfrak{v}_{p\mathsf{G}}\big)\rangle \\ &= \sum\nolimits_p \mathfrak{m}_p \Big(\frac{\mathrm{d}}{\mathrm{d}t}\langle \frac{\partial \dot{\mathfrak{r}}_p}{\partial \xi^i}, \dot{\mathfrak{r}}_p - \mathfrak{v}_{p\mathsf{G}}\rangle - \langle \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \dot{\mathfrak{r}}_p}{\partial \xi^i}, \dot{\mathfrak{r}}_p - \mathfrak{v}_{p\mathsf{G}}\rangle\Big) \\ &= \sum\nolimits_p \mathfrak{m}_p \Big(\frac{\mathrm{d}}{\mathrm{d}t}\langle \frac{\partial \dot{\mathfrak{r}}_p}{\partial \xi^i}, \dot{\mathfrak{r}}_p - \mathfrak{v}_{p\mathsf{G}}\rangle - \langle \partial_i \dot{\mathfrak{r}}_p - \gamma_{ij}^k \frac{\partial \dot{\mathfrak{r}}_p}{\partial \xi^k}\xi^j, \dot{\mathfrak{r}}_p - \mathfrak{v}_{p\mathsf{G}}\rangle\Big) \\ &= \Big(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \xi^i} + \gamma_{ij}^k \xi^j \frac{\partial}{\partial \xi^k} - \partial_i\Big) \underbrace{\Big(\tfrac{1}{2}\sum\nolimits_p \mathfrak{m}_p \|\dot{\mathfrak{r}}_p - \mathfrak{v}_{p\mathsf{G}}\|^2\Big)}_{\mathcal{T}^{\mathrm{G}}}, \qquad i = 1, \dots, n. \end{aligned} \tag{3.34}$$

3.2.5 Stiffness

Consider that *one* particle $\mathfrak{r}_p$ of the system is connected by a linear spring with stiffness $\mathfrak{k}_p$ to a position $\mathfrak{p}_p$. The force on this one particle, as proposed in (3.3), is $\mathfrak{F}_p^{\mathsf{K}} = \mathfrak{k}_p(\mathfrak{p}_p - \mathfrak{r}_p)$. The generalized force on the system is

$$f_i^{\mathrm{K}} = \langle \partial_i \mathfrak{r}_p, \mathfrak{F}_p^{\mathsf{K}}\rangle = \mathfrak{k}_p \langle \partial_i \mathfrak{r}_p, \mathfrak{p}_p - \mathfrak{r}_p\rangle = -\partial_i \underbrace{\big(\tfrac{1}{2}\mathfrak{k}_p \|\mathfrak{r}_p - \mathfrak{p}_p\|^2\big)}_{\mathcal{V}_p^{\mathrm{K}}}. \tag{3.35}$$

We may also consider a spring with stiffness $\mathfrak{k}_{pq}$ between two particles of the system: Then we have the force $\mathfrak{F}_p^{\mathsf{K}} = \mathfrak{k}_{pq}(\mathfrak{r}_q - \mathfrak{r}_p)$ on particle $\mathfrak{r}_p$ and the opposite force $\mathfrak{F}_q^{\mathsf{K}} = \mathfrak{k}_{pq}(\mathfrak{r}_p - \mathfrak{r}_q)$ on particle $\mathfrak{r}_q$. The generalized force on the system is

$$f_i^{\mathrm{K}} = \langle \partial_i \mathfrak{r}_p, \mathfrak{F}_p^{\mathsf{K}}\rangle + \langle \partial_i \mathfrak{r}_q, \mathfrak{F}_q^{\mathsf{K}}\rangle = -\mathfrak{k}_{pq}\langle \partial_i(\mathfrak{r}_p - \mathfrak{r}_q), \mathfrak{r}_p - \mathfrak{r}_q\rangle = -\partial_i \underbrace{\big(\tfrac{1}{2}\mathfrak{k}_{pq}\|\mathfrak{r}_p - \mathfrak{r}_q\|^2\big)}_{\mathcal{V}_{pq}^{\mathrm{K}}}. \tag{3.36}$$

In both cases the generalized force can be derived from a potential $\mathcal{V}^{\mathrm{K}}$.

For a system with an arbitrary number of linear springs, one may simply sum up the individual potentials to obtain the stiffness potential $\mathcal{V}^{\mathrm{K}}$ and derive the corresponding generalized force $\boldsymbol{f}^{\mathrm{K}} = \boldsymbol{\nabla}\mathcal{V}^{\mathrm{K}}$. Note that non-negativity of the spring constants $\mathfrak{k} \geq 0$ implies non-negativity of the potential $\mathcal{V}^{\mathrm{K}} \geq 0$.

3.2.6 Dissipation

Let the system of particles move within a viscous fluid with velocities $\mathfrak{v}_p^{\mathsf{D}}$ at the positions $\mathfrak{r}_p$ of the particles. Then, as proposed in (3.4), each particle is subject to a friction force $\mathfrak{F}_p^{\mathsf{D}} = -\mathfrak{d}_p(\dot{\mathfrak{r}}_p - \mathfrak{v}_p^{\mathsf{D}})$. The generalized force on the the system is

$$f_i^{\mathsf{D}} = \sum\nolimits_p \langle \partial_i \mathfrak{r}_p, \mathfrak{F}_p^{\mathsf{D}}\rangle = -\sum\nolimits_p \mathfrak{d}_p \langle \frac{\partial \dot{\mathfrak{r}}_p}{\partial \xi^i}, \dot{\mathfrak{r}}_p - \mathfrak{v}_p^{\mathsf{D}}\rangle = -\frac{\partial}{\partial \xi^i} \underbrace{\Big(\tfrac{1}{2}\sum\nolimits_p \mathfrak{d}_p \|\dot{\mathfrak{r}}_p - \mathfrak{v}_p^{\mathsf{D}}\|^2\Big)}_{\mathcal{R}}. \tag{3.37}$$

This dissipative force may be derived from $\mathcal{R}$, which is commonly called *Rayleigh dissipation function*, e.g. [Goldstein, 1951, p. 24]. Its dimension is that of power, i.e. watts. Note that non-negativity of the damping parameters $\mathfrak{k} \geq 0$ implies non-negativity of the dissipation function $\mathcal{V}^{\mathrm{K}} \geq 0$.

3.2.7 Energy

Total energy. The time derivative of the kinetic energy may be formulated as

$$\dot{\mathcal{T}} = \sum\nolimits_p \mathfrak{m}_p \langle \dot{\mathfrak{r}}_p, \ddot{\mathfrak{r}}_p \rangle = \xi^i \underbrace{\sum\nolimits_p \mathfrak{m}_p \langle \partial_i \mathfrak{r}_p, \ddot{\mathfrak{r}}_p \rangle}_{f_i^{\mathrm{M}}} + \sum\nolimits_p \mathfrak{m}_p \langle \frac{\partial \mathfrak{r}_p}{\partial t}, \ddot{\mathfrak{r}}_p \rangle. \tag{3.38}$$

In subsection 3.2.4 and subsection 3.2.4 we have seen potentials of the form $\mathcal{V}(\boldsymbol{x}, t)$ and their associated generalized force $f_i^{\mathrm{V}} = \partial_i \mathcal{V}$. The time derivative of this potential is

$$\dot{\mathcal{V}} = \xi^i \underbrace{\partial_i \mathcal{V}}_{f_i^{\mathrm{V}}} + \frac{\partial \mathcal{V}}{\partial t}. \tag{3.39}$$

The sum $\mathcal{W} = \mathcal{T} + \mathcal{V}$ is commonly called the *total energy*. Its time derivative is

$$\dot{\mathcal{W}} = \boldsymbol{\xi}^\top \big(\boldsymbol{f}^{\mathrm{M}} + \boldsymbol{f}^{\mathrm{V}}\big) + \sum\nolimits_p \mathfrak{m}_p \langle \frac{\partial \mathfrak{r}_p}{\partial t}, \ddot{\mathfrak{r}}_p \rangle + \frac{\partial \mathcal{V}}{\partial t}. \tag{3.40}$$

Note that the equation of motion implies $\boldsymbol{f}^{\mathrm{M}} + \boldsymbol{f}^{\mathrm{V}} = \boldsymbol{f}^{\mathrm{E}} - \boldsymbol{f}^{\mathrm{D}}$.

A mechanical system is called skleronomic (otherwise rheonomic) if it does not contain explicit time dependency. For this important case the change of total energy may be exressed by the external and dissipative forces alone

$$\dot{\mathcal{W}} = \boldsymbol{\xi}^\top \big(\boldsymbol{f}^{\mathrm{E}} - \boldsymbol{f}^{\mathrm{D}}\big). \tag{3.41}$$

Lagrangian and Hamiltonian. Define the *Lagrangian* as $\mathcal{L} = \mathcal{T} - \mathcal{V}$ which according to subsection 2.2.5 implies the generalized momentum and *Hamiltonian* as

$$p_i = \frac{\partial \mathcal{L}}{\partial \xi^i} = \sum\nolimits_p \mathfrak{m}_p \langle \partial_i \mathfrak{r}_p, \dot{\mathfrak{r}}_p \rangle, \quad i = 1, \ldots, n, \tag{3.42}$$

$$\mathcal{H} = p_i \xi^i - \mathcal{L} = \underbrace{\tfrac{1}{2} \sum\nolimits_p \mathfrak{m}_p \|\dot{\mathfrak{r}}_p\|^2 + \mathcal{V}}_{\mathcal{W}} - \sum\nolimits_p \mathfrak{m}_p \langle \frac{\partial \mathfrak{r}_p}{\partial t}, \dot{\mathfrak{r}}_p \rangle. \tag{3.43}$$

Extending the derivation from subsection 2.2.5 with the nonconservative forces $\boldsymbol{f}^{\mathrm{D}}$ and $\boldsymbol{f}^{\mathrm{E}}$ we find the change of the Hamiltonian along the solutions of the equations of motion as

$$\dot{\mathcal{H}} = \boldsymbol{\xi}^\top \big(\boldsymbol{f}^{\mathrm{E}} - \boldsymbol{f}^{\mathrm{D}}\big) - \frac{\partial \mathcal{L}}{\partial t} = \boldsymbol{\xi}^\top \big(\boldsymbol{f}^{\mathrm{E}} - \boldsymbol{f}^{\mathrm{D}}\big) - \sum\nolimits_p \mathfrak{m}_p \langle \frac{\partial \dot{\mathfrak{r}}_p}{\partial t}, \dot{\mathfrak{r}}_p \rangle + \frac{\partial \mathcal{V}}{\partial t} \tag{3.44}$$

Note that for a skleronomic system, the Hamiltonian $\mathcal{H}$ coincides with the total energy $\mathcal{W}$.

3.3 A single free rigid body

Established textbooks on physics (e.g. [Goldstein, 1951, chap. 4], [Landau and Lifshitz, 1960, §31] or [Boltzmann, 1897, §44]) define a *rigid body* as a a system of a *finite* number $\mathfrak{N}$ particles such that the distances $d_{pq} = \|\mathfrak{r}_p - \mathfrak{r}_q\|$ between their positions $\mathfrak{r}_p$ are constant. Textbooks that are more focused on engineering like [Hamel, 1949, sec. 8, § 1], [Bremer, 2008, sec. 4.1] or [Roberson and Schwertassek, 1988, sec. 6.1.1] rather define a rigid body as a rigid volume over which mass is continuously distributed. Both modeling assumptions eventually lead to the same equations of motion when using the same generalized coordinates. They differ in the computation of the inertial parameters of total mass, center of mass and moment of inertia: The physics perspective uses a finite sum over the particles, whereas the engineering point of view requires an integral over the body volume. This work will consider a finite number of particles.

In contrast to the sources mentioned above, this section will investigate apart from inertia and gravitation, also stiffness and damping for a rigid body. In particular for the latter two parts, the model of concentrated particles is more intuitive in the authors humble opinion.

3.3.1 Coordinates

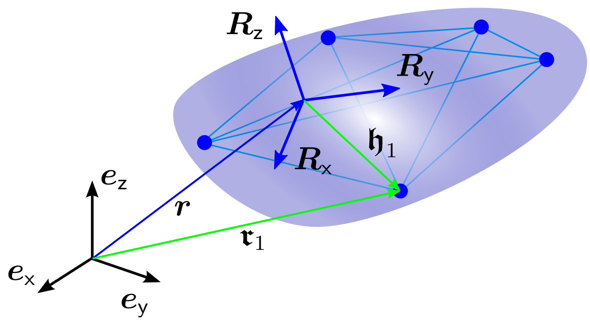

Figure 3.1: body fixed frame and particle positions

Body fixed frame. A common approach for modeling the rigid body (used in all sources above) is the use of a *body fixed frame*. This is the choice of a position $\boldsymbol{r} \in \mathbb{R}^3$ and a triple of orthonormal, right handed vectors $[\boldsymbol{R}_\mathrm{x}, \boldsymbol{R}_\mathrm{y}, \boldsymbol{R}_\mathrm{z}] = \boldsymbol{R} \in \mathbb{SO}(3)$ which are rigidly attached to the body, i.e. move with it, see Figure 3.1. With this, the position of any particle of the body may then be written as

$$\mathfrak{r}_p = \boldsymbol{r} + \boldsymbol{R}\mathfrak{h}_p, \quad p = 1, \dots, \mathfrak{N} \tag{3.45}$$

where the relative particle positions $\mathfrak{h}_p \in \mathbb{R}^3$ to the body fixed frame are *constant*. Consequently, the motion of the rigid body is completely captured by the position $\boldsymbol{r}(t) \in \mathbb{R}^3$ and orientation $\boldsymbol{R}(t) \in \mathbb{SO}(3)$ of the body fixed frame.

Usefulness of these particular coordinates. This is an example par excellence for the use of redundant parameters as discussed in the previous chapter: The system of particles has $\mathfrak{N}$ has $\nu = 3\mathfrak{N}$ coordinates, the coefficients of the particle positions $\mathfrak{r}_p, p = 1, \ldots, \mathfrak{N}$, and $c = \frac{1}{2}\mathfrak{N}(\mathfrak{N}-1)$ fixing their distances to each other. Actually, just fixing the distances between the particles still allows a mirroring of the rigid body, which from a physical perspective is not permitted. To resolve this there should be additional constraints on the "handedness" of the particles. Even without the "handedness" constraints, the number of distance constraints c surpasses the number of coordinates ν for larger number of particles. Consequently, the distance constraints cannot be independent. All these issues discourage us from working with the particle positions as configuration parameters.

On the other hand, since the configuration space of the rigid body $\mathbb{X} \cong \mathbb{R}^3 \times \mathbb{SO}(3)$ contains $\mathbb{SO}(3)$, any set of $n = \dim \mathbb{X} = 6$ minimal generalized coordinates will lead to singularities as discussed in section 1.2.

This particular choice of coordinates $(\boldsymbol{r}, \boldsymbol{R})$ may be regarded as a trade-off between these two extremes: It uses a fixed number of $\nu = 12$ coordinates and constraints and respects the topology of the configuration space. Furthermore, the interpretation of $\boldsymbol{r}$ and the columns of $\boldsymbol{R}$ as a body fixed frame are quite intuitive for practical applications.

Velocity. For parameterization of the velocity of the rigid body, we will use the body fixed velocity $\boldsymbol{v}(t) \in \mathbb{R}^3$ and the angular velocity $\boldsymbol{\omega}(t) \in \mathbb{R}^3$ which are related to the configuration by

$$\dot{\boldsymbol{r}} = \boldsymbol{R}\boldsymbol{v}, \qquad \dot{\boldsymbol{R}} = \boldsymbol{R}\,\mathrm{wed}(\boldsymbol{\omega}). \tag{3.46}$$

With this we may express the velocity and accelerations of the body particles as

$$\dot{\mathfrak{r}}_p = \boldsymbol{R}\big(\boldsymbol{v} - \mathrm{wed}(\mathfrak{h}_p)\boldsymbol{\omega}\big), \tag{3.47a}$$

$$\ddot{\mathfrak{r}}_p = \boldsymbol{R}\big(\dot{\boldsymbol{v}} - \mathrm{wed}(\mathfrak{h}_p)\dot{\boldsymbol{\omega}} + \mathrm{wed}(\boldsymbol{\omega})(\boldsymbol{v} - \mathrm{wed}(\mathfrak{h}_p)\boldsymbol{\omega})\big), \quad p = 1, \ldots, \mathfrak{N} \tag{3.47b}$$

Compliance with the framework. In order to comply to the framework from the previous chapter we may group the configuration coordinates as $\boldsymbol{x} = [\boldsymbol{r}^\top, \boldsymbol{R}_\mathsf{x}^\top, \boldsymbol{R}_\mathsf{y}^\top, \boldsymbol{R}_\mathsf{z}^\top]^\top \in \mathbb{X} \subset \mathbb{R}^{12}$ and their geometric constraint as $\boldsymbol{\phi}(\boldsymbol{x}) = \boldsymbol{0}$ are the constraints of $\mathbb{SO}(3)$ as given in (2.6). The vector form of the kinematic relation (3.46) is

$$\underbrace{\begin{bmatrix} \dot{\boldsymbol{r}} \\ \dot{\boldsymbol{R}}_\mathsf{x} \\ \dot{\boldsymbol{R}}_\mathsf{y} \\ \dot{\boldsymbol{R}}_\mathsf{z} \end{bmatrix}}_{\dot{\boldsymbol{x}}} = \underbrace{\begin{bmatrix} \boldsymbol{R}_\mathsf{x} & \boldsymbol{R}_\mathsf{y} & \boldsymbol{R}_\mathsf{z} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{R}_\mathsf{z} & \boldsymbol{R}_\mathsf{y} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{R}_\mathsf{z} & \boldsymbol{0} & -\boldsymbol{R}_\mathsf{x} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{R}_\mathsf{y} & \boldsymbol{R}_\mathsf{x} & \boldsymbol{0} \end{bmatrix}}_{\boldsymbol{A}} \underbrace{\begin{bmatrix} v_\mathsf{x} \\ v_\mathsf{y} \\ v_\mathsf{z} \\ \omega_\mathsf{x} \\ \omega_\mathsf{y} \\ \omega_\mathsf{z} \end{bmatrix}}_{\boldsymbol{\xi}}. \tag{3.48}$$

Commutation coefficients. Plugging the kinematic matrix $\boldsymbol{A}$ from (3.48) into the definition (2.19) of the commutation symbols $\boldsymbol{\gamma}$ yields

$$\begin{aligned} &\gamma^1_{26} = \gamma^1_{53} = \gamma^2_{34} = \gamma^2_{61} = \gamma^3_{15} = \gamma^3_{42} = \gamma^4_{56} = \gamma^5_{64} = \gamma^6_{45} = 1\ , \\ &\gamma^1_{62} = \gamma^1_{35} = \gamma^2_{43} = \gamma^2_{16} = \gamma^3_{51} = \gamma^3_{24} = \gamma^4_{65} = \gamma^5_{46} = \gamma^6_{54} = -1 \end{aligned} \tag{3.49}$$

and the remaining coefficients vanish. With this we have

$$\left[\gamma_{ij}^{k}\xi^{j}\right]_{i=1\dots6}^{k=1\dots6} = \begin{bmatrix} \operatorname{wed}(\boldsymbol{\omega}) & 0 \\ \operatorname{wed}(\boldsymbol{v}) & \operatorname{wed}(\boldsymbol{\omega}) \end{bmatrix} = -\operatorname{ad}_{\boldsymbol{\xi}}^{\top} \tag{3.50}$$

whose naming will be discussed later.

3.3.2 Inertia

The previous section derived several formulations for the generalized inertial force $\boldsymbol{f}^{\mathsf{M}}$. These will now be applied to the the rigid body and the chosen coordinates.

Kinetic energy. With the particle velocities $\dot{\mathfrak{r}}_p$ in terms of the chosen coordinates (3.47a) we obtain the kinetic energy $\mathcal{T}$ of a free rigid body as

$$\begin{aligned}\mathcal{T} &= \tfrac{1}{2}\sum_p \mathfrak{m}_p \| \overbrace{\boldsymbol{R}\big(\boldsymbol{v} - \operatorname{wed}(\mathfrak{h}_p)\boldsymbol{\omega}\big)}^{\dot{\mathfrak{r}}_p} \|^2 \\ &= \tfrac{1}{2}\underbrace{\sum_p \mathfrak{m}_p}_{m} \|\boldsymbol{v}\|^2 - \boldsymbol{v}^{\top} \underbrace{\sum_p \mathfrak{m}_p \operatorname{wed}(\mathfrak{h}_p)}_{m\operatorname{wed}(\boldsymbol{s})} \boldsymbol{\omega} + \tfrac{1}{2}\boldsymbol{\omega}^{\top} \underbrace{\sum_p \mathfrak{m}_p \operatorname{wed}(\mathfrak{h}_p)^{\top} \operatorname{wed}(\mathfrak{h}_p)}_{\boldsymbol{\Theta}} \boldsymbol{\omega} \\ &= \tfrac{1}{2} \underbrace{\begin{bmatrix} \boldsymbol{v}^{\top} & \boldsymbol{\omega}^{\top} \end{bmatrix}}_{\boldsymbol{\xi}^{\top}} \underbrace{\begin{bmatrix} m\mathbf{I}_3 & m\operatorname{wed}(\boldsymbol{s})^{\top} \\ m\operatorname{wed}(\boldsymbol{s}) & \boldsymbol{\Theta} \end{bmatrix}}_{\mathsf{M}} \underbrace{\begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\omega} \end{bmatrix}}_{\boldsymbol{\xi}}. \end{aligned} \tag{3.51}$$

Here we have substituted some well established inertia parameters: the total mass m, the center of mass $\boldsymbol{s} = m^{-1}\sum_p \mathfrak{m}_p \mathfrak{h}_p$ and the moment of inertia $\boldsymbol{\Theta} = \boldsymbol{\Theta}^{\top}$. Assuming that the particle masses are positive $\mathfrak{m}_p > 0, p = 1, \dots, \mathfrak{N}$ implies that the total mass is positive $m > 0$. Furthermore, if the rigid body has at least three particles that do not lie on a line, the inertia matrix is positive definite $\boldsymbol{\Theta} > 0$. It is important to notice that the inertia matrix M for the chosen coordinates is *constant*[1].

Plugging the the kinetic energy (3.51) into the corresponding formulation (3.27) of the generalized inertia force and using the commutation symbols from (3.50) yields

$$\boldsymbol{f}^{\mathrm{M}} = \underbrace{\begin{bmatrix} m\mathbf{I}_3 & m\operatorname{wed}(\boldsymbol{s})^{\top} \\ m\operatorname{wed}(\boldsymbol{s}) & \boldsymbol{\Theta} \end{bmatrix}}_{\mathsf{M}} \underbrace{\begin{bmatrix} \dot{\boldsymbol{v}} \\ \dot{\boldsymbol{\omega}} \end{bmatrix}}_{\dot{\boldsymbol{\xi}}} + \underbrace{\begin{bmatrix} \operatorname{wed}(\boldsymbol{\omega}) & 0 \\ \operatorname{wed}(\boldsymbol{v}) & \operatorname{wed}(\boldsymbol{\omega}) \end{bmatrix}}_{-\operatorname{ad}_{\boldsymbol{\xi}}^{\top}} \underbrace{\begin{bmatrix} m\mathbf{I}_3 & m\operatorname{wed}(\boldsymbol{s})^{\top} \\ m\operatorname{wed}(\boldsymbol{s}) & \boldsymbol{\Theta} \end{bmatrix}}_{\mathsf{M}} \underbrace{\begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\omega} \end{bmatrix}}_{\boldsymbol{\xi}} \tag{3.52}$$

[1] One reason for the choice of $\boldsymbol{v}$ as velocity coordinates it the fact that the inertia matrix M is *constant*. If we choose instead $\dot{\boldsymbol{r}}$ as velocity coordinates we have

$$\mathcal{T} = \tfrac{1}{2}[\dot{\boldsymbol{r}}^{\top}, \boldsymbol{\omega}^{\top}] \begin{bmatrix} m\mathbf{I}_3 & m\boldsymbol{R}\operatorname{wed}(\boldsymbol{s})^{\top} \\ m\operatorname{wed}(\boldsymbol{s})\boldsymbol{R}^{\top} & \boldsymbol{\Theta} \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{r}} \\ \boldsymbol{\omega} \end{bmatrix}.$$

Obviously the body inertia matrix depends on the orientation $\boldsymbol{R}$ of the body and is not constant unless the reference position $\boldsymbol{r}$ coincides with the *center of mass*, i.e. $\boldsymbol{s} = 0$. Actually many textbooks, e.g. [Murray et al., 1994, p. 167] or [Shabana, 2005, p. 153], restrict to this case for their expressions of the kinetic energy. In the next section on rigid body systems we will see that it can be quite useful to use *geometrically* meaningful body fixed frames rather than restricting to the center of mass.

Acceleration energy. With the particle accelerations $\ddot{\mathfrak{r}}_p$ in terms of the coordinates (3.47b) and using the Jacobi identity (2.57d) we find the acceleration energy $\mathcal{S}$ for the free rigid body as

$$
\begin{aligned}
\mathcal{S} &= \tfrac{1}{2}\sum\nolimits_p \mathfrak{m}_p \| \overbrace{\boldsymbol{R}\big(\dot{\boldsymbol{v}} - \operatorname{wed}(\mathfrak{h}_p)\dot{\boldsymbol{\omega}} + \operatorname{wed}(\boldsymbol{\omega})(\boldsymbol{v} - \operatorname{wed}(\mathfrak{h}_p)\boldsymbol{\omega})\big)}^{\ddot{\mathfrak{r}}_p} \|^2 \\
&= \tfrac{1}{2}\underbrace{\sum\nolimits_p \mathfrak{m}_p}_{m} \|\dot{\boldsymbol{v}}\|^2 - \dot{\boldsymbol{v}}^\top \underbrace{\sum\nolimits_p \mathfrak{m}_p \operatorname{wed}(\mathfrak{h}_p)}_{m \operatorname{wed}(\boldsymbol{s})} \dot{\boldsymbol{\omega}} + \tfrac{1}{2}\dot{\boldsymbol{\omega}}^\top \underbrace{\sum\nolimits_p \mathfrak{m}_p \operatorname{wed}(\mathfrak{h}_p)^\top \operatorname{wed}(\mathfrak{h}_p)}_{\boldsymbol{\Theta}} \dot{\boldsymbol{\omega}} \\
&\quad + \dot{\boldsymbol{v}}^\top \operatorname{wed}(\boldsymbol{\omega})\Big(\underbrace{\sum\nolimits_p \mathfrak{m}_p}_{m} \boldsymbol{v} - \underbrace{\sum\nolimits_p \mathfrak{m}_p \operatorname{wed}(\mathfrak{h}_p)}_{m \operatorname{wed}(\boldsymbol{s})} \boldsymbol{\omega}\Big) \\
&\quad + \dot{\boldsymbol{\omega}}^\top \Big(\underbrace{\sum\nolimits_p \mathfrak{m}_p \operatorname{wed}(\mathfrak{h}_p)}_{m \operatorname{wed}(\boldsymbol{s})} \operatorname{wed}(\boldsymbol{\omega})\boldsymbol{v} + \operatorname{wed}(\boldsymbol{\omega}) \underbrace{\sum\nolimits_p \mathfrak{m}_p \operatorname{wed}(\mathfrak{h}_p)^\top \operatorname{wed}(\mathfrak{h}_p)}_{\boldsymbol{\Theta}} \boldsymbol{\omega}\Big) \\
&\quad + \tfrac{1}{2}\underbrace{\sum\nolimits_p \mathfrak{m}_p}_{m} \| \operatorname{wed}(\boldsymbol{\omega})\boldsymbol{v}\|^2 + \boldsymbol{v}^\top \operatorname{wed}(\boldsymbol{\omega})^2 \underbrace{\sum\nolimits_p \mathfrak{m}_p \operatorname{wed}(\mathfrak{h}_p)}_{m \operatorname{wed}(\boldsymbol{s})} \boldsymbol{\omega} \\
&\quad + \tfrac{1}{2}\operatorname{tr}\Big(\underbrace{\sum\nolimits_p \mathfrak{m}_p \mathfrak{h}_p \mathfrak{h}_p^\top}_{\boldsymbol{\Theta}' = \frac{1}{2}\operatorname{tr}(\boldsymbol{\Theta})\mathbf{I}_3 - \boldsymbol{\Theta}'} \operatorname{wed}(\boldsymbol{\omega})^4\Big)
\end{aligned}
\tag{3.53}
$$

Not at all surprisingly, we found the same inertia parameters m, $\boldsymbol{s}$ and $\boldsymbol{\Theta}$ as for the kinetic energy $\mathcal{T}$ in (3.51). Collecting these further in the inertia matrix M we have

$$
\begin{aligned}
\mathcal{S} = \tfrac{1}{2} &\underbrace{\begin{bmatrix}\dot{\boldsymbol{v}}^\top & \dot{\boldsymbol{\omega}}^\top\end{bmatrix}}_{\dot{\boldsymbol{\xi}}^\top} \underbrace{\begin{bmatrix} m\mathbf{I}_3 & m \operatorname{wed}(\boldsymbol{s})^\top \\ m \operatorname{wed}(\boldsymbol{s}) & \boldsymbol{\Theta}\end{bmatrix}}_{\mathsf{M}} \underbrace{\begin{bmatrix}\dot{\boldsymbol{v}} \\ \dot{\boldsymbol{\omega}}\end{bmatrix}}_{\dot{\boldsymbol{\xi}}} \\
&+ \underbrace{\begin{bmatrix}\dot{\boldsymbol{v}}^\top & \dot{\boldsymbol{\omega}}^\top\end{bmatrix}}_{\dot{\boldsymbol{\xi}}^\top} \underbrace{\begin{bmatrix}\operatorname{wed}(\boldsymbol{\omega}) & \mathbf{0} \\ \operatorname{wed}(\boldsymbol{v}) & \operatorname{wed}(\boldsymbol{\omega})\end{bmatrix}}_{-\operatorname{ad}_{\boldsymbol{\xi}}^\top} \underbrace{\begin{bmatrix} m\mathbf{I}_3 & m \operatorname{wed}(\boldsymbol{s})^\top \\ m \operatorname{wed}(\boldsymbol{s}) & \boldsymbol{\Theta}\end{bmatrix}}_{\mathsf{M}} \underbrace{\begin{bmatrix}\boldsymbol{v} \\ \boldsymbol{\omega}\end{bmatrix}}_{\boldsymbol{\xi}} \\
&+ \underbrace{\tfrac{1}{2}m\| \operatorname{wed}(\boldsymbol{\omega})\boldsymbol{v}\|^2 + \boldsymbol{v}^\top \operatorname{wed}(\boldsymbol{\omega})^2 m \operatorname{wed}(\boldsymbol{s})\boldsymbol{\omega} + \tfrac{1}{2}\operatorname{tr}\big(\boldsymbol{\Theta}' \operatorname{wed}(\boldsymbol{\omega})^4\big)}_{\mathcal{S}_0}.
\end{aligned}
\tag{3.54}
$$

Plugging this into the corresponding formulation (3.26) of the generalized inertia force, i.e. $\boldsymbol{f}^{\mathsf{M}} = \partial\mathcal{S}/\partial\dot{\boldsymbol{\xi}}$, we obviously find the same expression as above (3.52). Note that $\mathcal{S}_0$ is independent of the generalized acceleration $\dot{\boldsymbol{\xi}}$ and consequently does not contribute to the inertia force.

Inertia matrix and connection coefficients. We can compute the Jacobian of the particle positions from

$$
\boldsymbol{\nabla}\mathfrak{r}_p = \Big[\partial_i \mathfrak{r}_p\Big]_{i=1,\dots,6} = \left[\frac{\partial \dot{\mathfrak{r}}_p}{\partial \xi^i}\right]_{i=1,\dots,6} = \begin{bmatrix}\frac{\partial}{\partial \boldsymbol{v}} \\ \frac{\partial}{\partial \boldsymbol{\omega}}\end{bmatrix}^\top \overbrace{\boldsymbol{R}\big(\boldsymbol{v} - \operatorname{wed}(\mathfrak{h}_p)\boldsymbol{\omega}\big)}^{\dot{\mathfrak{r}}_p} = \boldsymbol{R}\begin{bmatrix}\mathbf{I}_3 & \operatorname{wed}(\mathfrak{h}_p)^\top\end{bmatrix}. \tag{3.55}
$$

With this, the rigid body inertia matrix may be written as

$$\begin{aligned}\boldsymbol{M} &= \Big[\sum\nolimits_p \mathfrak{m}_p \langle \partial_i \mathfrak{r}_p, \partial_j \mathfrak{r}_p \rangle\Big]_{i,j=1,\dots,6} = \sum\nolimits_p \mathfrak{m}_p (\boldsymbol{\nabla}\mathfrak{r}_p)^\top \boldsymbol{\nabla}\mathfrak{r}_p \\ &= \sum\nolimits_p \mathfrak{m}_p \begin{bmatrix} \mathbf{I}_3 & \operatorname{wed}(\mathfrak{h}_p)^\top \\ \operatorname{wed}(\mathfrak{h}_p) & \operatorname{wed}(\mathfrak{h}_p)\operatorname{wed}(\mathfrak{h}_p)^\top \end{bmatrix} = \begin{bmatrix} m\mathbf{I}_3 & m\operatorname{wed}(\boldsymbol{s})^\top \\ m\operatorname{wed}(\boldsymbol{s}) & \boldsymbol{\Theta} \end{bmatrix}\end{aligned} \tag{3.56}$$

which obviously coincides with what we found from the kinetic energy (3.51) and from the acceleration energy (3.54).

As already pointed out above, for the chosen velocity coordinates $\boldsymbol{\xi} = (\boldsymbol{v}, \boldsymbol{\omega})$, the coefficients of the rigid body inertia matrix M_{ij} are constant. Consequently, since $\partial_k M_{ij} \equiv 0$, the corresponding connection coefficients Γ_{ijk} only consist of the terms with the commutation coefficients γ:

$$\Gamma_{ijk} = \tfrac{1}{2}\big(\gamma_{ij}^{s} \mathsf{M}_{sk} + \gamma_{ik}^{s} \mathsf{M}_{sj} - \gamma_{jk}^{s} \mathsf{M}_{si}\big) = -\Gamma_{jik}. \tag{3.57}$$

Using the commutation coefficients γ given in (3.49) and taking into account the skew symmetry above, the non-zero connection coefficients are

$$\begin{aligned}
\Gamma_{324} = \Gamma_{135} = \Gamma_{216} &= m, && (3.58a)\\
\Gamma_{254} = \Gamma_{364} = \Gamma_{515} = \Gamma_{616} &= m s_{\mathsf{x}}, && (3.58b)\\
\Gamma_{424} = \Gamma_{145} = \Gamma_{365} = \Gamma_{626} &= m s_{\mathsf{y}}, && (3.58c)\\
\Gamma_{434} = \Gamma_{535} = \Gamma_{146} = \Gamma_{256} &= m s_{\mathsf{z}}, && (3.58d)\\
\Gamma_{654} &= \Theta'_{\mathsf{xx}} = \tfrac{1}{2}\big(\Theta_{\mathsf{yy}} + \Theta_{\mathsf{zz}} - \Theta_{\mathsf{xx}}\big), && (3.58e)\\
\Gamma_{465} &= \Theta'_{\mathsf{yy}} = \tfrac{1}{2}\big(\Theta_{\mathsf{xx}} + \Theta_{\mathsf{zz}} - \Theta_{\mathsf{yy}}\big), && (3.58f)\\
\Gamma_{546} &= \Theta'_{\mathsf{zz}} = \tfrac{1}{2}\big(\Theta_{\mathsf{xx}} + \Theta_{\mathsf{yy}} - \Theta_{\mathsf{zz}}\big), && (3.58g)\\
\Gamma_{464} = \Gamma_{655} &= \Theta'_{\mathsf{xy}} = -\Theta_{\mathsf{xy}}, && (3.58h)\\
\Gamma_{544} = \Gamma_{656} &= \Theta'_{\mathsf{xz}} = -\Theta_{\mathsf{xz}}, && (3.58i)\\
\Gamma_{545} = \Gamma_{466} &= \Theta'_{\mathsf{yz}} = -\Theta_{\mathsf{yz}}. && (3.58j)
\end{aligned}$$

Note that the quantity $\boldsymbol{\Theta}' = \frac{1}{2}\operatorname{tr}(\boldsymbol{\Theta})\mathbf{I}_3 - \boldsymbol{\Theta}$ also appeared above in the formulation of the acceleration energy (3.53).

Finally, assembling the terms $f_i^{\mathrm{M}} = M_{ij}\dot{\xi}^j + \mathit{\Gamma}_{ijk}\xi^j\xi^k$ we may check that this is indeed identical to (3.52).

3.3.3 Gravitation

The potential energy $\mathcal{V}^{\mathrm{G}}$ of a rigid body due to a gravitational acceleration $\boldsymbol{a}_{\mathsf{G}}$ according to (3.32) in terms of the chosen coordinates is

$$\mathcal{V}^{\mathrm{G}} = \sum\nolimits_p \langle \overbrace{\boldsymbol{r} + \boldsymbol{R}\mathfrak{h}_p}^{\mathfrak{r}_p}, -\mathfrak{m}_p \boldsymbol{a}_{\mathsf{G}} \rangle = -\langle \underbrace{\sum\nolimits_p \mathfrak{m}_p}_{m} \boldsymbol{r} + \boldsymbol{R} \underbrace{\sum\nolimits_p \mathfrak{m}_p \mathfrak{h}_p}_{m\boldsymbol{s}}, \boldsymbol{a}_{\mathsf{G}} \rangle = -m\langle \boldsymbol{r} + \boldsymbol{R}\boldsymbol{s}, \boldsymbol{a}_{\mathsf{G}} \rangle. \tag{3.59}$$

Note that the parameters of total mass m and center of mass $\boldsymbol{s}$ are the same as found above for the inertia matrix. The resulting generalized force is

$$\boldsymbol{f}^{\mathsf{G}} = \boldsymbol{\nabla}\mathcal{V}^{\mathrm{G}} = -m \begin{bmatrix} \boldsymbol{R}^\top \boldsymbol{a}_{\mathsf{G}} \\ \mathrm{wed}(\boldsymbol{s})\boldsymbol{R}^\top \boldsymbol{a}_{\mathsf{G}} \end{bmatrix}. \tag{3.60}$$

3.3.4 Stiffness

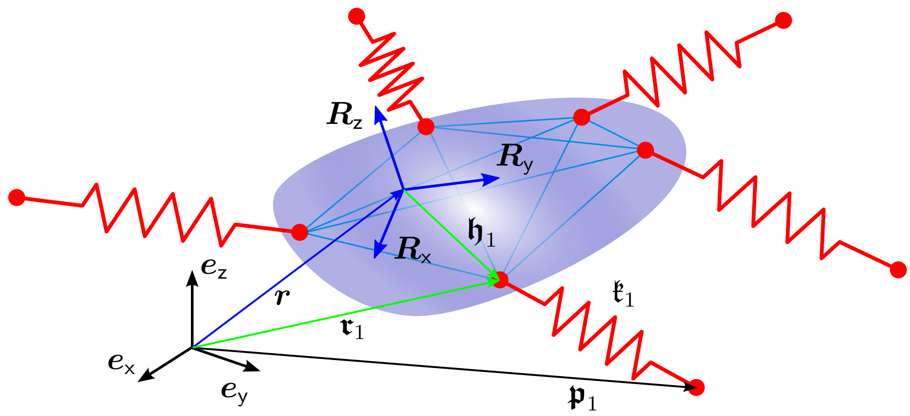

Figure 3.2: springs attached to a rigid body

Assume that every particle of the rigid body with position $\mathfrak{r}_p$ is connected to a position $\mathfrak{p}_p \in \mathbb{R}^3$ by a linear spring with stiffness $\mathfrak{k}_p \in \mathbb{R}^{(+)}$, see Figure 3.2. The resulting potential energy in terms of the rigid body coordinates $\boldsymbol{x} \cong (\boldsymbol{r}, \boldsymbol{R})$ is

$$\mathcal{V}^{\mathrm{K}}(\boldsymbol{x}) = \tfrac{1}{2}\sum\nolimits_p \mathfrak{k}_p \|\boldsymbol{r} + \boldsymbol{R}\mathfrak{h}_p - \mathfrak{p}_p\|^2. \tag{3.61}$$

Stiffness parameters. Using the identities above we may rearrange (3.61) to

$$\begin{aligned}
\mathcal{V}^{\mathrm{K}}(\boldsymbol{x}) &= \tfrac{1}{2}\sum\nolimits_p \mathfrak{k}_p \|\boldsymbol{r} + \boldsymbol{R}\mathfrak{h}_p - \mathfrak{p}_p\|^2 \\
&= \tfrac{1}{2}\sum\nolimits_p \mathfrak{k}_p \big(\|\boldsymbol{r}\|^2 + \underbrace{\|\boldsymbol{R}\mathfrak{h}_p\|^2}_{const.} + \underbrace{\|\mathfrak{p}_p\|^2}_{const.} + 2\langle \boldsymbol{r}, \boldsymbol{R}\mathfrak{h}_p\rangle - 2\langle \boldsymbol{r}, \mathfrak{p}_p\rangle - 2\langle \boldsymbol{R}\mathfrak{h}_p, \mathfrak{p}_p\rangle\big) \\
&= \tfrac{1}{2}k\|\boldsymbol{r}\|^2 + k\langle \boldsymbol{r}, \boldsymbol{R}\boldsymbol{h}\rangle - k\langle \boldsymbol{r}, \boldsymbol{p}\rangle - \mathrm{tr}(\boldsymbol{P}\boldsymbol{R}) + \underbrace{\tfrac{1}{2}\sum\nolimits_p \mathfrak{k}_p\big(\|\mathfrak{h}_p\|^2 + \|\mathfrak{p}_p\|^2\big)}_{\mathcal{V}_c^{\mathrm{K}} = const.}
\end{aligned} \tag{3.62}$$

with substitution of the constant parameters

$$k = \sum\nolimits_p \mathfrak{k}_p, \qquad \boldsymbol{h} = k^{-1}\sum\nolimits_p \mathfrak{k}_p \mathfrak{h}_p, \qquad \boldsymbol{p} = k^{-1}\sum\nolimits_p \mathfrak{k}_p \mathfrak{p}_p, \qquad \boldsymbol{P} = \sum\nolimits_p \mathfrak{k}_p \mathfrak{h}_p \mathfrak{p}_p^\top. \tag{3.63}$$

Note that there was no specific assumption for the particle and spring distribution, i.e. on the values of $\mathfrak{h}_p$, $\mathfrak{p}_p$ and $\mathfrak{k}_p$. Consequently, any constellation may be captured by the $1+3+3+9+1 = 17$ parameters within $(k, \boldsymbol{h}, \boldsymbol{p}, \boldsymbol{P}, \mathcal{V}_c^{\mathrm{K}})$.

Critical points. The time derivatives of the potential may be written as

$$\begin{aligned}\tfrac{\mathrm{d}}{\mathrm{d}t}\mathcal{V}^{\mathrm{K}} &= k\langle \boldsymbol{r}, \boldsymbol{R}\boldsymbol{v}\rangle + k\langle \boldsymbol{R}\boldsymbol{v}, \boldsymbol{R}\boldsymbol{h}\rangle + k\langle \boldsymbol{r}, \boldsymbol{R}\,\mathrm{wed}(\boldsymbol{\omega})\boldsymbol{h}\rangle - k\langle \boldsymbol{R}\boldsymbol{v}, \boldsymbol{p}\rangle - \mathrm{tr}(\boldsymbol{P}\boldsymbol{R}\,\mathrm{wed}(\boldsymbol{\omega}))\\ &= \boldsymbol{\xi}^{\top}\underbrace{\begin{bmatrix} k(\boldsymbol{R}^{\top}(\boldsymbol{r}-\boldsymbol{p})+\boldsymbol{h}) \\ k\,\mathrm{wed}(\boldsymbol{h})\boldsymbol{R}^{\top}\boldsymbol{r} + \mathrm{vee2}(\boldsymbol{P}\boldsymbol{R})\end{bmatrix}}_{\boldsymbol{\nabla}\mathcal{V}^{\mathrm{K}}}\end{aligned} \tag{3.64}$$

$$\tfrac{\mathrm{d}^2}{\mathrm{d}t^2}\mathcal{V}^{\mathrm{K}} = \dot{\boldsymbol{\xi}}^{\top}\boldsymbol{\nabla}\mathcal{V}^{\mathrm{K}} + \boldsymbol{\xi}^{\top}\underbrace{\begin{bmatrix} k\mathbf{I}_3 & k\,\mathrm{wed}\left(\boldsymbol{R}^{\top}(\boldsymbol{r}-\boldsymbol{p})\right) \\ k\,\mathrm{wed}(\boldsymbol{h}) & k\,\mathrm{wed}(\boldsymbol{h})\,\mathrm{wed}(\boldsymbol{R}^{\top}\boldsymbol{r}) + \mathrm{tr}(\boldsymbol{P}\boldsymbol{R})\mathbf{I}_3 - (\boldsymbol{P}\boldsymbol{R})^{\top}\end{bmatrix}}_{\boldsymbol{\nabla}^2\mathcal{V}^{\mathrm{K}}}\boldsymbol{\xi} \tag{3.65}$$

We are interested in configurations $\boldsymbol{x}_{\mathsf{R}} \cong (\boldsymbol{r}_{\mathsf{R}}, \boldsymbol{R}_{\mathsf{R}})$ at which the potential is stationary $\boldsymbol{\nabla}\mathcal{V}^{\mathrm{K}}(\boldsymbol{x}_{\mathsf{R}}) = \mathbf{0}$: From the upper part of (3.64) we get the condition

$$\boldsymbol{r}_{\mathsf{R}} = \boldsymbol{p} - \boldsymbol{R}_{\mathsf{R}}\boldsymbol{h}. \tag{3.66}$$

Plugging this into the lower part of (3.64) we obtain

$$k\,\mathrm{wed}(\boldsymbol{h})\boldsymbol{R}_{\mathsf{R}}^{\top}(\boldsymbol{p} - \boldsymbol{R}_{\mathsf{R}}\boldsymbol{h}) + \mathrm{vee2}(\boldsymbol{P}\boldsymbol{R}_{\mathsf{R}}) = \mathrm{vee2}(\underbrace{(\boldsymbol{P} - k\boldsymbol{h}\boldsymbol{p}^{\top})}_{\boldsymbol{P}_s}\boldsymbol{R}_{\mathsf{R}}) = \mathbf{0}. \tag{3.67}$$

The solution to this subproblem $\mathrm{vee2}(\boldsymbol{P}_s\boldsymbol{R}_{\mathsf{R}}) = \mathbf{0}, \boldsymbol{R}_{\mathsf{R}} \in \mathbb{SO}(3)$ is discussed in great detail in section 2.4: Let $\boldsymbol{P}_s^{\top} = \boldsymbol{X}\,\mathrm{Wed}(\boldsymbol{\Pi}_{\mathsf{s}})$ with $\boldsymbol{X} \in \mathbb{SO}(3)$, $\boldsymbol{\Pi}_{\mathsf{s}} \in \mathbb{SYM}_0^+(3)$ be a *special polar decomposition*. Then $\boldsymbol{R}_{\mathsf{R}} = \boldsymbol{X}$ is clearly a critical point. Plugging $\boldsymbol{P} = \mathrm{Wed}(\boldsymbol{\Pi}_{\mathsf{s}})\boldsymbol{R}_{\mathsf{R}}^{\top} + k\boldsymbol{h}\boldsymbol{p}^{\top}$ into the Hessian matrix, we have

$$\begin{aligned}\boldsymbol{\nabla}^2\mathcal{V}^{\mathrm{K}}(\boldsymbol{x}_{\mathsf{R}}) &= \begin{bmatrix} k\mathbf{I}_3 & k\,\mathrm{wed}(\boldsymbol{h})^{\top} \\ k\,\mathrm{wed}(\boldsymbol{h}) & \boldsymbol{\Pi}_{\mathsf{s}} - k\,\mathrm{wed}(\boldsymbol{h})^2\end{bmatrix}\\ &= \begin{bmatrix} \mathbf{I}_3 & \mathbf{0} \\ \mathrm{wed}(\boldsymbol{h}) & \mathbf{I}_3\end{bmatrix}\begin{bmatrix} k\mathbf{I}_3 & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Pi}_{\mathsf{s}}\end{bmatrix}\begin{bmatrix} \mathbf{I}_3 & \mathrm{wed}(\boldsymbol{h})^{\top} \\ \mathbf{0} & \mathbf{I}_3\end{bmatrix}\end{aligned} \tag{3.68}$$

By Sylvester's law of inertia, the definiteness of $\boldsymbol{\nabla}^2\mathcal{V}^{\mathrm{K}}(\boldsymbol{x}_{\mathsf{R}})$ coincides with the definiteness of $k \geq 0$ and $\boldsymbol{\Pi}_{\mathsf{s}} \geq 0$. Using the results from section 2.4 we may conclude that $\boldsymbol{x}_{\mathsf{R}}$ is a minimum, and it is strict and global, if, and only if, $k > 0$ and $\boldsymbol{\Pi}_{\mathsf{s}} > 0$.

Stiffness parameters cont'd. We may express the parameters $\boldsymbol{p}$ and $\boldsymbol{P}$ in terms of the minimum configuration $(\boldsymbol{r}_{\mathsf{R}}, \boldsymbol{R}_{\mathsf{R}})$ and the matrix $\boldsymbol{\Pi}_{\mathsf{s}}$ as

$$\boldsymbol{p} = \boldsymbol{r}_{\mathsf{R}} + \boldsymbol{R}_{\mathsf{R}}\boldsymbol{h}, \qquad \boldsymbol{P} = \mathrm{Wed}(\boldsymbol{\Pi}_{\mathsf{s}})\boldsymbol{R}_{\mathsf{R}}^{\top} + k\boldsymbol{h}(\boldsymbol{r}_{\mathsf{R}} + \boldsymbol{R}_{\mathsf{R}}\boldsymbol{h})^{\top} \tag{3.69}$$

Plugging this into (3.62) we may reformulate the potential energy as

$$\begin{aligned}\mathcal{V}^{\mathrm{K}}(\boldsymbol{x}) &= \tfrac{1}{2}k\|\boldsymbol{r}\|^2 + k\langle \boldsymbol{r}, \boldsymbol{R}\boldsymbol{h}\rangle - k\langle \boldsymbol{r}, \boldsymbol{r}_{\mathsf{R}} + \boldsymbol{R}_{\mathsf{R}}\boldsymbol{h}\rangle - k\langle \boldsymbol{r}_{\mathsf{R}} + \boldsymbol{R}_{\mathsf{R}}\boldsymbol{h}, \boldsymbol{R}\boldsymbol{h}\rangle\\ &\qquad - \mathrm{tr}(\mathrm{Wed}(\boldsymbol{\Pi}_{\mathsf{s}})\boldsymbol{R}_{\mathsf{R}}^{\top}\boldsymbol{R}) + \tfrac{1}{2}\sum\nolimits_p \mathfrak{k}_p\left(\|\mathfrak{h}_p\|^2 + \|\mathfrak{p}_p\|^2\right)\\ &= \tfrac{1}{2}k\|\boldsymbol{r} + \boldsymbol{R}\boldsymbol{h} - (\boldsymbol{r}_{\mathsf{R}} + \boldsymbol{R}_{\mathsf{R}}\boldsymbol{h})\|^2 + \mathrm{tr}\left(\mathrm{Wed}(\boldsymbol{\Pi}_{\mathsf{s}})(\mathbf{I}_3 - \boldsymbol{R}_{\mathsf{R}}^{\top}\boldsymbol{R})\right)\\ &\qquad \underbrace{-\tfrac{1}{2}k\|\boldsymbol{r}_{\mathsf{R}} + \boldsymbol{R}_{\mathsf{R}}\boldsymbol{h}\|^2 - \tfrac{1}{2}k\|\boldsymbol{h}\|^2 - \mathrm{tr}\left(\mathrm{Wed}(\boldsymbol{\Pi}_{\mathsf{s}})\right) + \tfrac{1}{2}\sum\nolimits_p \mathfrak{k}_p\left(\|\mathfrak{h}_p\|^2 + \|\mathfrak{p}_p\|^2\right)}_{\mathcal{V}_0^{\mathrm{K}} = \mathcal{V}^{\mathrm{K}}(\boldsymbol{x}_{\mathsf{R}})}\\ &= \tfrac{1}{2}k\|\boldsymbol{r} - \boldsymbol{r}_{\mathsf{R}}\|^2 + k\langle \boldsymbol{r} - \boldsymbol{r}_{\mathsf{R}}, (\boldsymbol{R} - \boldsymbol{R}_{\mathsf{R}})\boldsymbol{h}\rangle + \mathrm{tr}\left(\mathrm{Wed}(\boldsymbol{\Pi})(\mathbf{I}_3 - \boldsymbol{R}_{\mathsf{R}}^{\top}\boldsymbol{R})\right) + \mathcal{V}_0^{\mathrm{K}}\end{aligned} \tag{3.70}$$

where $\boldsymbol{\Pi} = \boldsymbol{\Pi}_{\mathsf{s}} + k\,\mathrm{wed}(\boldsymbol{h})\,\mathrm{wed}(\boldsymbol{h})^\top$ and $\mathcal{V}_0^{\mathrm{K}}$ is the minimal potential, i.e. the potential of the residual displacement of the springs at the minimum $\boldsymbol{x}_{\mathsf{R}}$. The differential may be written as

$$\nabla\mathcal{V}^{\mathrm{K}}(\boldsymbol{x}) = \begin{bmatrix} k\boldsymbol{R}^\top(\boldsymbol{r}-\boldsymbol{r}_{\mathsf{R}}) + (\mathbf{I}_3 - \boldsymbol{R}^\top\boldsymbol{R}_{\mathsf{R}})k\boldsymbol{h} \\ k\,\mathrm{wed}(\boldsymbol{h})\boldsymbol{R}^\top(\boldsymbol{r}-\boldsymbol{r}_{\mathsf{R}}) + \mathrm{vee2}\left(\mathrm{Wed}(\boldsymbol{\Pi})\boldsymbol{R}_{\mathsf{R}}^\top\boldsymbol{R}\right) \end{bmatrix}. \tag{3.71}$$

The Hessian at the minimum is

$$\nabla^2\mathcal{V}^{\mathrm{K}}(\boldsymbol{x}_{\mathsf{R}}) = \begin{bmatrix} k\mathbf{I}_3 & k\,\mathrm{wed}(\boldsymbol{h})^\top \\ k\,\mathrm{wed}(\boldsymbol{h}) & \boldsymbol{\Pi} \end{bmatrix} \geq 0 \tag{3.72}$$

Conclusion. The conclusion of this subsection is that any constellation of linear springs attached to a rigid body may be captured by the potential $\mathcal{V}^{\mathrm{K}}$ from (3.70) and the resulting force $\boldsymbol{f}^{\mathrm{K}} = \nabla\mathcal{V}^{\mathrm{K}}$ from (3.71). It is parameterized by 6 parameters within $(\boldsymbol{r}_{\mathsf{R}}, \boldsymbol{R}_{\mathsf{R}}) \in \mathbb{R}^3\times\mathbb{SO}(3)$ which describe the configuration at the minimum, the $1+3+6=10$ parameters within $k, \in \mathbb{R}_0^+$, $\boldsymbol{h} \in \mathbb{R}^3$ and $\boldsymbol{\Pi} \in \mathbb{SYM}_0^+(3)$, and the minimum $\mathcal{V}_0^{\mathrm{K}}$: $\mathcal{V}^{\mathrm{K}}(\boldsymbol{x}) \geq \mathcal{V}_0^{\mathrm{K}} \geq 0$. The minimum is strict and global if, and only if, $k > 0$ and $\boldsymbol{\Pi}_{\mathsf{s}} = \boldsymbol{\Pi} - k\,\mathrm{wed}(\boldsymbol{h})\,\mathrm{wed}(\boldsymbol{h})^\top > 0$.

The rigid body stiffness matrix $\mathbf{K} = \nabla^2\mathcal{V}^{\mathrm{K}}(\boldsymbol{x}_{\mathsf{R}})$ has the same structure as the inertia matrix $\mathbf{M} = \partial^2\mathcal{T}/\partial\boldsymbol{\xi}\partial\boldsymbol{\xi}$ for the chosen coordinates. Due to these analogies to the established inertia parameters, we refer to the rigid body stiffness parameters in the following as: total stiffness k, center of stiffness $\boldsymbol{h}$, moment of stiffness $\boldsymbol{\Pi}$, and moment of stiffness at the center of stiffness $\boldsymbol{\Pi}_{\mathsf{s}}$.

3.3.5 Dissipation

As motivated in the previous section we may motivate damping as particles moving within a viscous fluid which produce a drag force proportional to the particles velocity $\dot{\mathfrak{r}}_p$. Different volumes of the particles may motivate different drag coefficients $\mathfrak{d}_p$, see Figure 3.3.

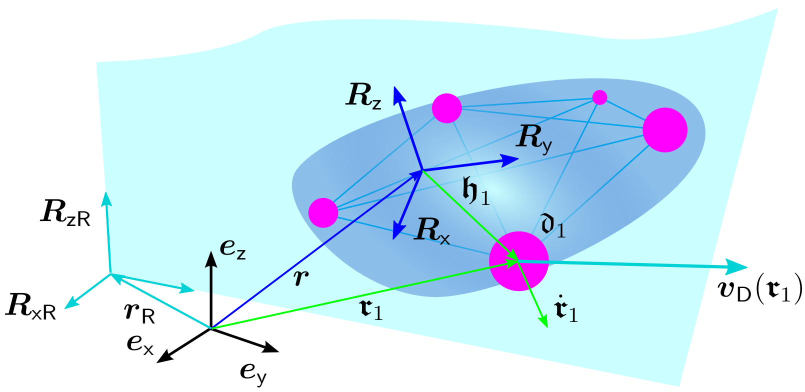

Figure 3.3: rigid body within viscous fluid

General fluid motion. Let the fluid at the position of p-th particle position have the velocity $\boldsymbol{v}_{\mathsf{D}p}(t) \in \mathbb{R}^3$ and let its drag coefficient be $\mathfrak{d}_p \in \mathbb{R}_0^+$. Overall, the dissipation function is

$$
\begin{aligned}
\mathcal{R} &= \tfrac{1}{2}\sum\nolimits_p \mathfrak{d}_p \| \overbrace{\boldsymbol{R}(\boldsymbol{v} - \mathrm{wed}(\mathfrak{h}_p)\boldsymbol{\omega})}^{\dot{\mathfrak{r}}_p} - \boldsymbol{v}_{\mathsf{D}p}) \|^2 \\
&= \tfrac{1}{2}\underbrace{\sum\nolimits_p \mathfrak{d}_p}_{d} \|\boldsymbol{v}\|^2 - \boldsymbol{v}^\top \underbrace{\sum\nolimits_p \mathfrak{d}_p \,\mathrm{wed}(\mathfrak{h}_p)}_{d\,\mathrm{wed}(\boldsymbol{l})} \boldsymbol{\omega} + \tfrac{1}{2}\boldsymbol{\omega}^\top \underbrace{\sum\nolimits_p \mathfrak{d}_p \,\mathrm{wed}(\mathfrak{h}_p)^\top \mathrm{wed}(\mathfrak{h}_p)}_{\boldsymbol{\Upsilon}} \boldsymbol{\omega} \\
&\quad - \boldsymbol{v}^\top \sum\nolimits_p \mathfrak{d}_p \boldsymbol{R}^\top \boldsymbol{v}_{\mathsf{D}p} + \boldsymbol{\omega}^\top \sum\nolimits_p \mathfrak{d}_p \,\mathrm{wed}(\mathfrak{h}_p)\boldsymbol{R}^\top \boldsymbol{v}_{\mathsf{D}p} + \tfrac{1}{2}\sum\nolimits_p \mathfrak{d}_p \|\boldsymbol{v}_{\mathsf{D}p}\|^2
\end{aligned}
\tag{3.73}
$$

The resulting generalized force is

$$
\boldsymbol{f}^{\mathsf{D}} = \frac{\partial \mathcal{R}}{\partial \boldsymbol{\xi}} = \underbrace{\begin{bmatrix} d\mathbf{I}_3 & d\,\mathrm{wed}(\boldsymbol{l})^\top \\ d\,\mathrm{wed}(\boldsymbol{l}) & \boldsymbol{\Upsilon} \end{bmatrix}}_{\mathbf{D}} \underbrace{\begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\omega} \end{bmatrix}}_{\boldsymbol{\xi}} + \sum\nolimits_p \mathfrak{d}_p \begin{bmatrix} -\mathbf{I}_3 \\ \mathrm{wed}(\mathfrak{h}_p) \end{bmatrix} \boldsymbol{R}^\top \boldsymbol{v}_{\mathsf{D}p}. \tag{3.74}
$$

Again we found parameters similar to the established inertia parameters. In analogy to them we call $d \in \mathbb{R}_0^+$ the total damping, $\boldsymbol{l} \in \mathbb{R}^3$ the center of damping, and $\boldsymbol{\Upsilon} \in \mathbb{SYM}_0^+(3)$ the moment of damping.

Rigid fluid motion. Let us consider a special case in which the fluid velocity also obeys a rigid body motion parameterized by $(\boldsymbol{r}_{\mathsf{R}}, \boldsymbol{R}_{\mathsf{R}})$ and the velocity $(\boldsymbol{v}_{\mathsf{R}}, \boldsymbol{\omega}_{\mathsf{R}})$, see Figure 3.3. The absolute fluid velocity at the particle position is then $\boldsymbol{v}_{\mathsf{D}p} = \boldsymbol{R}_{\mathsf{R}}(\boldsymbol{v}_{\mathsf{R}} + \mathrm{wed}(\boldsymbol{\omega}_{\mathsf{R}})\boldsymbol{R}_{\mathsf{R}}^\top(\mathfrak{r}_p - \boldsymbol{r}_{\mathsf{R}}))$. The we have the dissipation function

$$
\begin{aligned}
\mathcal{R} &= \tfrac{1}{2}\sum\nolimits_p \mathfrak{d}_p \| \overbrace{\boldsymbol{R}(\boldsymbol{v} - \mathrm{wed}(\mathfrak{h}_p)\boldsymbol{\omega})}^{\dot{\mathfrak{r}}_p} - \overbrace{\boldsymbol{R}_{\mathsf{R}}(\boldsymbol{v}_{\mathsf{R}} + \mathrm{wed}(\boldsymbol{\omega}_{\mathsf{R}})\boldsymbol{R}_{\mathsf{R}}^\top(\boldsymbol{r} + \boldsymbol{R}\mathfrak{h}_p - \boldsymbol{r}_{\mathsf{R}}))}^{\boldsymbol{v}_{\mathsf{D}p}} \|^2 \\
&= \tfrac{1}{2}\sum\nolimits_p \mathfrak{d}_p \| \underbrace{\boldsymbol{v} - \boldsymbol{R}^\top(\boldsymbol{R}_{\mathsf{R}}\boldsymbol{v}_{\mathsf{R}} - \mathrm{wed}(\boldsymbol{r} - \boldsymbol{r}_{\mathsf{R}})\boldsymbol{R}_{\mathsf{R}}\boldsymbol{\omega}_{\mathsf{R}})}_{\boldsymbol{v}_{\mathrm{E}}} - \mathrm{wed}(\mathfrak{h}_p)\underbrace{(\boldsymbol{\omega} - \boldsymbol{R}^\top \boldsymbol{R}_{\mathsf{R}}\boldsymbol{\omega}_{\mathsf{R}})}_{\boldsymbol{\omega}_{\mathrm{E}}} \|^2 \\
&= \tfrac{1}{2}\underbrace{[\boldsymbol{v}_{\mathrm{E}}^\top, \boldsymbol{\omega}_{\mathrm{E}}^\top]}_{\boldsymbol{\xi}_{\mathrm{E}}^\top} \underbrace{\begin{bmatrix} d\mathbf{I}_3 & d\,\mathrm{wed}(\boldsymbol{l})^\top \\ d\,\mathrm{wed}(\boldsymbol{l}) & \boldsymbol{\Upsilon} \end{bmatrix}}_{\mathbf{D}} \underbrace{\begin{bmatrix} \boldsymbol{v}_{\mathrm{E}} \\ \boldsymbol{\omega}_{\mathrm{E}} \end{bmatrix}}_{\boldsymbol{\xi}_{\mathrm{E}}}
\end{aligned}
\tag{3.75}
$$

The resulting generalized force is

$$
\boldsymbol{f}^{\mathsf{D}} = \frac{\partial \mathcal{R}}{\partial \boldsymbol{\xi}} = \frac{\partial \mathcal{R}}{\partial \boldsymbol{\xi}_{\mathrm{E}}} = \mathbf{D}\boldsymbol{\xi}_{\mathrm{E}} \tag{3.76}
$$

3.3.6 Summary and the special Euclidean group

Special Euclidean group. Instead of collecting the configuration coordinates in a tuple $\boldsymbol{x} = [\boldsymbol{r}^\top, \boldsymbol{R}_{\mathsf{x}}^\top, \boldsymbol{R}_{\mathsf{y}}^\top, \boldsymbol{R}_{\mathsf{z}}^\top]^\top \in \mathbb{X}$ as proposed in the previous chapter, it can also be useful to arrange them within a matrix:

$$
\mathbf{G} = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{r} \\ \mathbf{0} & 1 \end{bmatrix} \in \mathbb{SE}(3) = \left\{ \begin{bmatrix} \boldsymbol{R} & \boldsymbol{r} \\ \mathbf{0} & 1 \end{bmatrix} \,\middle|\, \boldsymbol{r} \in \mathbb{R}^3, \boldsymbol{R} \in \mathbb{SO}(3) \right\} \tag{3.77}
$$

which is commonly referred to as the *homogeneous representation*, e.g. [Murray et al., 1994, sec. 2.3.1]. The set $\mathbb{SE}(3)$ combined with matrix multiplication forms a Lie group which is called the *special Euclidean group*. Euclidean denotes to the fact that its transformations preserve the Euclidean distance, while special denotes to the fact that it does not permit reflections (analog to the special orthogonal group $\mathbb{SO}(3)$).

Operators. This section already used the wed on $\mathbb{R}^3$ quite extensively. On $\mathbb{R}^6$ we define it as

$$\text{wed} : \mathbb{R}^3 \to \mathfrak{so}(3) : \begin{bmatrix}\omega_1\\ \omega_2\\ \omega_3\end{bmatrix} \mapsto \begin{bmatrix}0 & -\omega_3 & \omega_2\\ \omega_3 & 0 & -\omega_1\\ -\omega_2 & \omega_1 & 0\end{bmatrix} \tag{3.78a}$$

$$\text{wed} : \mathbb{R}^6 \to \mathfrak{se}(3) : \begin{bmatrix}\boldsymbol{v}\\ \boldsymbol{\omega}\end{bmatrix} \mapsto \begin{bmatrix}\text{wed}\,\boldsymbol{\omega} & \boldsymbol{v}\\ \mathbf{0} & 0\end{bmatrix} \tag{3.78b}$$

Its inverse is denoted $\text{vee}(\cdot)$, i.e. $\text{vee}(\text{wed}(\boldsymbol{\xi})) = \boldsymbol{\xi}$. The wed and vee operators are well established in the literature, see e.g. [Murray et al., 1994, sec. 2.3.2].

Using the wed operator may rewrite the rigid body kinematics from (3.46) in matrix form as:

$$\underbrace{\begin{bmatrix}\dot{\boldsymbol{R}} & \dot{\boldsymbol{r}}\\ \mathbf{0} & 0\end{bmatrix}}_{\dot{\mathbf{G}}} = \underbrace{\begin{bmatrix}\boldsymbol{R} & \boldsymbol{r}\\ \mathbf{0} & 1\end{bmatrix}}_{\mathbf{G}} \underbrace{\begin{bmatrix}\text{wed}(\boldsymbol{\omega}) & \boldsymbol{v}\\ \mathbf{0} & 0\end{bmatrix}}_{\text{wed}(\boldsymbol{\xi})} \tag{3.79}$$

More operators. The following operators are not established in the literature, but will prove quite useful for this work. Define the vee2 operator through

$$\text{tr}\left(\boldsymbol{A}(\text{wed}\,\boldsymbol{\xi})^\top\right) = \boldsymbol{\xi}^\top\,\text{vee2}(\boldsymbol{A}), \tag{3.80}$$

this is

$$\text{vee2} : \mathbb{R}^{3\times 3} \to \mathbb{R}^3 : \boldsymbol{A} \mapsto \text{vee}(\boldsymbol{A} - \boldsymbol{A}^\top) \tag{3.81a}$$

$$\text{vee2} : \mathbb{R}^{4\times 4} \to \mathbb{R}^6 : \begin{bmatrix}\boldsymbol{A} & \boldsymbol{b}\\ * & *\end{bmatrix} \mapsto \begin{bmatrix}\boldsymbol{b}\\ \text{vee2}\,\boldsymbol{A}\end{bmatrix}. \tag{3.81b}$$

Note that for $\boldsymbol{\Omega} \in \mathfrak{so}(3) \subset \mathbb{R}^{3\times 3}$ we have $\text{vee2}(\boldsymbol{\Omega}) = 2\,\text{vee}(\boldsymbol{\Omega})$, thus giving the motivation for the name. Define the Vee operator through

$$\text{tr}\left(\text{wed}\,\boldsymbol{\xi}\,\boldsymbol{A}(\text{wed}\,\boldsymbol{\eta})^\top\right) = \boldsymbol{\eta}^\top(\text{Vee}\,\boldsymbol{A})\boldsymbol{\xi}, \tag{3.82}$$

this is

$$\text{Vee} : \mathbb{R}^{3\times 3} \to \mathbb{R}^{3\times 3} : \boldsymbol{A} \mapsto \text{tr}(\boldsymbol{A})\mathbf{I}_3 - \boldsymbol{A} \tag{3.83a}$$

$$\text{Vee} : \mathbb{R}^{4\times 4} \to \mathbb{R}^{6\times 6} : \begin{bmatrix}\boldsymbol{A} & \boldsymbol{b}\\ \boldsymbol{c}^\top & d\end{bmatrix} \mapsto \begin{bmatrix}d\mathbf{I}_3 & (\text{wed}\,\boldsymbol{b})^\top\\ \text{wed}\,\boldsymbol{c} & \text{Vee}\,\boldsymbol{A}\end{bmatrix}. \tag{3.83b}$$

Let $\text{Wed}(\cdot)$ denote its inverse. Combining the definitions (3.82) and (3.80) also yields

$$\text{vee2}\left(\text{wed}\,\boldsymbol{\xi}\,\boldsymbol{A}\right) = \text{Vee}(\boldsymbol{A})\boldsymbol{\xi} \tag{3.84}$$

Adjoint representation. Define

$$\mathbf{G} = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{r} \\ \mathbf{0} & 1 \end{bmatrix} : \qquad \mathrm{Ad}_{\mathbf{G}} = \begin{bmatrix} \boldsymbol{R} & \mathrm{wed}(\boldsymbol{r})\boldsymbol{R} \\ \mathbf{0} & \boldsymbol{R} \end{bmatrix} \tag{3.85a}$$

$$\boldsymbol{\xi} = \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\omega} \end{bmatrix} : \qquad \mathrm{ad}_{\boldsymbol{\xi}} = \begin{bmatrix} \mathrm{wed}(\boldsymbol{\omega}) & \mathrm{wed}(\boldsymbol{v}) \\ \mathbf{0} & \mathrm{wed}(\boldsymbol{\omega}) \end{bmatrix} \tag{3.85b}$$

The notation is due to the established notation for the *adjoint representation* for Lie groups and their associated Lie algebras, see e.g. [Hall, 2015, Def. 3.32 & 3.7]. Using this background we have the obvious relations:

$$\mathrm{Ad}_{\mathbf{G}_1\mathbf{G}_2} = \mathrm{Ad}_{\mathbf{G}_1}\mathrm{Ad}_{\mathbf{G}_1}, \qquad \mathrm{Ad}_{\mathbf{G}}^{-1} = \mathrm{Ad}_{\mathbf{G}^{-1}}, \qquad \mathrm{Ad}_{\mathbf{I}_4} = \mathbf{I}_6, \tag{3.86a}$$

$$\mathrm{ad}_{\boldsymbol{\xi}_1}\boldsymbol{\xi}_2 = -\mathrm{ad}_{\boldsymbol{\xi}_2}\boldsymbol{\xi}_1, \qquad \mathrm{ad}_{\boldsymbol{\xi}}\boldsymbol{\xi} = \mathbf{0}. \tag{3.86b}$$

Furthermore, for $\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{G} = \mathbf{G}\,\mathrm{wed}(\boldsymbol{\xi})$ we have

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\mathrm{Ad}_{\mathbf{G}} = \mathrm{Ad}_{\mathbf{G}}\mathrm{ad}_{\boldsymbol{\xi}}. \tag{3.87}$$

Though the Lie group theory can be extremely useful for rigid body mechanics, for this work it is sufficient to regard $\mathrm{Ad}_{(\cdot)}$ and $\mathrm{ad}_{(\cdot)}$ as simple algebraic operators with the identities (3.86).

Rigid body energies. Notice that for $\mathbf{x}, \mathfrak{h}_p \in \mathbb{R}^n$, $\mathbf{X} \in \mathbb{R}^{n\times n}$ we have

$$\begin{aligned}
\sum\nolimits_p \mathfrak{m}_p \|\mathbf{x} + \mathbf{X}\mathfrak{h}_p\|^2 &= \sum\nolimits_p \mathfrak{m}_p \,\mathrm{tr}\left((\mathbf{x} + \mathbf{X}\mathfrak{h}_p)(\mathbf{x} + \mathbf{X}\mathfrak{h}_p)^\top\right) \\
&= \sum\nolimits_p \mathfrak{m}_p \,\mathrm{tr}\left(\left(\begin{bmatrix} \mathbf{X} & \mathbf{x} \\ \mathbf{0} & 0 \end{bmatrix}\begin{bmatrix} \mathfrak{h}_p \\ 1 \end{bmatrix}\right)\left(\begin{bmatrix} \mathbf{X} & \mathbf{x} \\ \mathbf{0} & 0 \end{bmatrix}\begin{bmatrix} \mathfrak{h}_p \\ 1 \end{bmatrix}\right)^\top\right) \\
&= \mathrm{tr}\left(\underbrace{\begin{bmatrix} \mathbf{X} & \mathbf{x} \\ \mathbf{0} & 0 \end{bmatrix}}_{\boldsymbol{\Xi}}\underbrace{\left(\sum\nolimits_p \mathfrak{m}_p \begin{bmatrix} \mathfrak{h}_p\mathfrak{h}_p^\top & \mathfrak{h}_p^\top \\ \mathfrak{h}_p & 1 \end{bmatrix}\right)}_{\mathbf{M}'}\underbrace{\begin{bmatrix} \mathbf{X}^\top & \mathbf{0} \\ \mathbf{x}^\top & 0 \end{bmatrix}}_{\boldsymbol{\Xi}^\top}\right) \\
&= \|\boldsymbol{\Xi}^\top\|^2_{\mathbf{M}'}
\end{aligned} \tag{3.88}$$

with the weighted Frobenius norm motivated in (2.48). Furthermore, for the special case $\boldsymbol{\Xi} = \mathrm{wed}\,\boldsymbol{\xi}$ we have due to (3.82):

$$\|\,\mathrm{wed}(\boldsymbol{\xi})^\top\|^2_{\mathbf{M}'} = \|\boldsymbol{\xi}\|^2_{\mathrm{Vee}\,\mathbf{M}'}, \tag{3.89}$$

Using this, we may rewrite the kinetic energy $\mathcal{T}$ of a free rigid body (3.51), the acceleration energy $\mathcal{S}$ from (3.53), the dissipation function $\mathcal{R}$ in (3.75), the potential energy $\mathcal{V}$ due to linear springs (3.70) and the potential energy $\mathcal{V}^{\mathrm{G}}$ due to earth's gravitation from (3.59) as

$$\mathcal{T} = \tfrac{1}{2}\sum\nolimits_p \mathfrak{m}_p \| \overbrace{\dot{\boldsymbol{r}} + \dot{\boldsymbol{R}}\mathfrak{h}_p}^{\dot{\mathfrak{r}}_p} \|^2 = \tfrac{1}{2}\|\dot{\mathbf{G}}^\top\|^2_{\mathbf{M}'} = \tfrac{1}{2}\|\boldsymbol{\xi}\|^2_{\mathbf{M}} \tag{3.90a}$$

$$\mathcal{T}^{\mathrm{G}} = \tfrac{1}{2}\sum\nolimits_p \mathfrak{m}_p \| \overbrace{\dot{\boldsymbol{r}} + \dot{\boldsymbol{R}}\mathfrak{h}_p}^{\dot{\mathfrak{r}}_p} - \boldsymbol{a}_{\mathsf{G}} t\|^2 = \tfrac{1}{2}\|(\dot{\mathbf{G}} - \mathrm{wed}(\boldsymbol{\alpha}_{\mathsf{G}} t))^\top\|^2_{\mathbf{M}'} \qquad \boldsymbol{\alpha}_{\mathsf{G}} = \begin{bmatrix} \boldsymbol{a}_{\mathsf{G}} \\ \mathbf{0}_{3\times 1}\end{bmatrix} \tag{3.90b}$$

$$\mathcal{S} = \tfrac{1}{2}\sum\nolimits_p \mathfrak{m}_p \| \overbrace{\ddot{\boldsymbol{r}} + \ddot{\boldsymbol{R}}\mathfrak{h}_p}^{\ddot{\mathfrak{r}}_p} \|^2 = \tfrac{1}{2}\|\ddot{\mathbf{G}}^\top\|^2_{\mathbf{M}'} \tag{3.90c}$$

$$\mathcal{S}^{\mathrm{G}} = \tfrac{1}{2}\sum\nolimits_p \mathfrak{m}_p \| \overbrace{\ddot{\boldsymbol{r}} + \ddot{\boldsymbol{R}}\mathfrak{h}_p}^{\ddot{\mathfrak{r}}_p} - \boldsymbol{a}_{\mathsf{G}}\|^2 = \tfrac{1}{2}\|(\ddot{\mathbf{G}} - \mathrm{wed}(\boldsymbol{\alpha}_{\mathsf{G}}))^\top\|^2_{\mathbf{M}'} \tag{3.90d}$$

$$\mathcal{R} = \tfrac{1}{2}\sum\nolimits_p \mathfrak{d}_p \| \overbrace{\dot{\boldsymbol{r}} + \dot{\boldsymbol{R}}\mathfrak{h}_p}^{\dot{\mathfrak{r}}_p} \|^2 = \tfrac{1}{2}\|\dot{\mathbf{G}}^\top\|^2_{\mathbf{D}'} = \tfrac{1}{2}\|\boldsymbol{\xi}\|^2_{\mathbf{D}} \tag{3.90e}$$

$$\mathcal{V}^{\mathrm{K}} = \tfrac{1}{2}\sum\nolimits_p \mathfrak{k}_p \| \overbrace{\boldsymbol{r} + \boldsymbol{R}\mathfrak{h}_p}^{\mathfrak{r}_p} - (\overbrace{\boldsymbol{r}_{\mathsf{R}} + \boldsymbol{R}_{\mathsf{R}}\mathfrak{h}_p}^{\mathfrak{r}_{p\mathrm{R}}})\|^2 = \tfrac{1}{2}\|(\mathbf{G} - \mathbf{G}_{\mathsf{R}})^\top\|^2_{\mathbf{K}'} \tag{3.90f}$$

$$\mathcal{V}^{\mathrm{G}} = \sum\nolimits_p \langle \overbrace{\boldsymbol{r} + \boldsymbol{R}\mathfrak{h}_p}^{\mathfrak{r}_p}, -\mathfrak{m}_p \boldsymbol{a}_{\mathsf{G}}\rangle = \langle \mathbf{G}^\top, \mathrm{wed}(-\boldsymbol{\alpha}_{\mathsf{G}})^\top\rangle_{\mathbf{M}'}, \tag{3.90g}$$

where

$$\mathbf{M}' = \sum\nolimits_p \mathfrak{m}_p \begin{bmatrix} \mathfrak{h}_p\mathfrak{h}_p^\top & \mathfrak{h}_p^\top \\ \mathfrak{h}_p & 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Theta}' & m\boldsymbol{s} \\ m\boldsymbol{s}^\top & m \end{bmatrix} = \mathrm{Wed}(\mathbf{M}) \tag{3.91a}$$

$$\mathbf{D}' = \sum\nolimits_p \mathfrak{d}_p \begin{bmatrix} \mathfrak{h}_p\mathfrak{h}_p^\top & \mathfrak{h}_p^\top \\ \mathfrak{h}_p & 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Upsilon}' & d\boldsymbol{l} \\ d\boldsymbol{l}^\top & d \end{bmatrix} = \mathrm{Wed}(\mathbf{D}) \tag{3.91b}$$

$$\mathbf{K}' = \sum\nolimits_p \mathfrak{k}_p \begin{bmatrix} \mathfrak{h}_p\mathfrak{h}_p^\top & \mathfrak{h}_p^\top \\ \mathfrak{h}_p & 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Pi}' & k\boldsymbol{h} \\ k\boldsymbol{h}^\top & k \end{bmatrix} = \mathrm{Wed}(\mathbf{K}) \tag{3.91c}$$

Note that m, $m\boldsymbol{s}$ and $\boldsymbol{\Theta}'$ correspond to the zeroth, first and second *mathematical moments* of the distribution $\mathfrak{m}_p\mathfrak{h}_p, p = 1, \ldots, \mathfrak{N}$. Their collection in the matrix $\mathbf{M}' \in \mathbb{SYM}(4)$ is bijective to the previously encountered inertia matrix $\mathbf{M}' = \mathrm{Wed}(\mathbf{M}) \Leftrightarrow \mathbf{M} = \mathrm{Vee}(\mathbf{M}')$. Obviously, the same holds for the damping $\mathbf{K}'$ and stiffness matrix $\mathbf{D}'$.

Rigid body forces. The corresponding forces forces which were already derived in the previous subsections can be written in a more compact form:

$$\boldsymbol{f}^{\mathrm{M}} = \frac{\partial \mathcal{S}}{\partial \dot{\boldsymbol{\xi}}} = \mathrm{vee2}\left((\mathrm{wed}(\dot{\boldsymbol{\xi}}) + \mathrm{wed}(\boldsymbol{\xi})^2)\mathbf{M}'\right) = \mathbf{M}\dot{\boldsymbol{\xi}} - \mathrm{ad}^\top_{\boldsymbol{\xi}}\mathbf{M}\boldsymbol{\xi} \tag{3.92a}$$

$$\boldsymbol{f}^{\mathrm{D}} = \frac{\partial \mathcal{R}}{\partial \boldsymbol{\xi}} = \mathrm{vee2}\left(\mathrm{wed}(\boldsymbol{\xi})\mathbf{D}'\right) = \mathbf{D}\boldsymbol{\xi} \tag{3.92b}$$

$$\boldsymbol{f}^{\mathrm{K}} = \boldsymbol{\nabla}\mathcal{V}^{\mathrm{K}} = \mathrm{vee2}\left((\mathbf{I}_4 - \mathbf{G}^{-1}\mathbf{G}_{\mathsf{R}})\mathbf{K}'\right) \tag{3.92c}$$

$$\boldsymbol{f}^{\mathrm{G}} = \boldsymbol{\nabla}\mathcal{V}^{\mathrm{G}} = \mathrm{vee2}\left(\mathbf{G}^\top \mathrm{wed}(-\boldsymbol{\alpha}_{\mathsf{G}})\mathbf{M}'\right) = -\mathbf{M}\mathrm{Ad}^{-1}_{\mathbf{G}}\boldsymbol{\alpha}_{\mathsf{G}}. \tag{3.92d}$$

Equation of motion. Combining the results above, the equations of motion of a free rigid body subject to inertia, gravity, viscous friction, linear springs and a generalized force $\boldsymbol{f}^{\mathrm{E}}$ may be written as

$$\dot{\mathbf{G}} = \mathbf{G}\,\mathrm{wed}(\boldsymbol{\xi}), \tag{3.93a}$$
$$\dot{\boldsymbol{\xi}} = \mathrm{Ad}_{\mathbf{G}}^{-1}\boldsymbol{\alpha}_{\mathsf{G}} + \mathbf{M}^{-1}\big(\boldsymbol{f}^{\mathrm{E}} + (\mathrm{ad}_{\boldsymbol{\xi}}^{\top}\mathbf{M} - \mathbf{D})\boldsymbol{\xi} - \mathrm{vee2}\left((\mathbf{I}_4 - \mathbf{G}^{-1}\mathbf{G}_{\mathsf{R}})\mathbf{K}'\right)\big). \tag{3.93b}$$

Thee are three sets of ingredients:

- The chosen coordinates are collected within $\mathbf{G}(t) \in \mathbb{SE}(3)$ and $\boldsymbol{\xi}(t) \in \mathbb{R}^6$.
- The matrices $\mathbf{M}', \mathbf{D}', \mathbf{K}' \in \mathbb{SYM}(4)$ capture the distribution of mass, damping and stiffness.
- External influences are collected within the gravity wrench $\boldsymbol{\alpha}_{\mathsf{G}}^{\top} = [\boldsymbol{a}_{\mathsf{G}}^{\top}, \mathbf{0}_{1\times 3}]$, the equilibrium configuration of the springs $\mathbf{G}_{\mathsf{R}}(t) \in \mathbb{SE}(3)$ and the generalized external force $\boldsymbol{f}^{\mathrm{E}}(t) \in \mathbb{R}^6$.

3.4 Rigid body systems

A rigid body system is a system of $N \geq 1$ rigid bodies which may be constrained to each other and/or to the surrounding space. As before, this section restricts to geometric constraints.

There are many established textbooks on this subject e.g. [Roberson and Schwertassek, 1988], [Murray et al., 1994], [Kane and Levinson, 1985]. However, all these excellent texts restrict to *minimal* generalized coordinates or are even more restrictive by requiring Denavit-Hartenberg parameters [Denavit and Hartenberg, 1955]. While this is just fine when dealing only with one-dimensional joints, it may be too restrictive when dealing with multidimensional joints as e.g. mobile robots. Furthermore, the texts mentioned above mostly focus on inertia but not dissipation and stiffness.

This section deals with the derivation of equations of motion for rigid body systems subject to inertia, gravity, linear springs and viscous friction. It allows for a quite general parameterization as motivated in the previous sections.

3.4.1 Parameterization

Configuration coordinates. As motivated for the single rigid body, let there be a body fixed frame for each body of the system as illustrated in Figure 3.4. The components of the position of the b-th body w.r.t. the inertial frame are ${}^{0}_{b}\boldsymbol{r} \in \mathbb{R}^3$ and the components of its attitude are ${}^{0}_{b}\boldsymbol{R} = [{}^{0}_{b}\boldsymbol{R}_{\mathsf{x}}, {}^{0}_{b}\boldsymbol{R}_{\mathsf{y}}, {}^{0}_{b}\boldsymbol{R}_{\mathsf{z}}] \in \mathbb{SO}(3)$.

The configuration can also be expressed w.r.t. any other body: ${}^{a}_{b}\boldsymbol{r}$ is the position of the b-th frame w.r.t. the frame of the a-th body and analog of the attitude ${}^{a}_{b}\boldsymbol{R}$. The left side indices are used for readability but also to emphasize their different nature compared

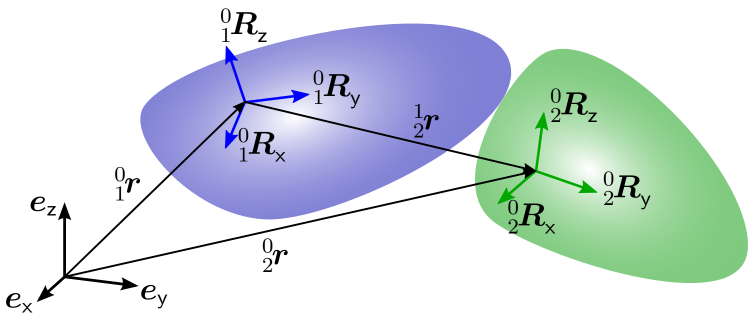

Figure 3.4: reference frame and body fixed frames

to the right side indices. The sum convention does not apply to these indices. For the positions and attitudes we have the following relations

$$
\begin{aligned}
{}^a_c\boldsymbol{r} &= {}^a_b\boldsymbol{r} + {}^a_b\boldsymbol{R}\,{}^b_c\boldsymbol{r}, & {}^a_c\boldsymbol{R} &= {}^a_b\boldsymbol{R}\,{}^b_c\boldsymbol{R}, & & \qquad (3.94a)\\
{}^b_a\boldsymbol{r} &= -{}^a_b\boldsymbol{R}^\top\,{}^a_b\boldsymbol{r}, & {}^b_a\boldsymbol{R} &= {}^a_b\boldsymbol{R}^\top, & & \qquad (3.94b)\\
{}^a_a\boldsymbol{r} &= \mathbf{0}, & {}^a_a\boldsymbol{R} &= \mathbf{I}_3, & a, b, c &= 0, \dots, N. \qquad (3.94c)
\end{aligned}
$$

As motivated in the previous section, it will be convenient to merge position ${}^0_b\boldsymbol{r} \in \mathbb{R}^3$ and rotation matrix ${}^0_b\boldsymbol{R} \in \mathbb{SO}(3)$ into the (rigid body) configuration matrix

$$
{}^a_b\mathsf{G} = \begin{bmatrix} {}^a_b\boldsymbol{R} & {}^a_b\boldsymbol{r} \\ 0 & 1 \end{bmatrix} \in \mathbb{SE}(3). \tag{3.95}
$$

Then (3.94) is equivalent to

$$
\begin{aligned}
{}^a_c\mathsf{G} &= {}^a_b\mathsf{G}\,{}^b_c\mathsf{G}, & & \qquad (3.96a)\\
{}^b_a\mathsf{G} &= {}^a_b\mathsf{G}^{-1}, & & \qquad (3.96b)\\
{}^a_a\mathsf{G} &= \mathbf{I}_4, & a, b, c &= 0, \dots, N. \qquad (3.96c)
\end{aligned}
$$

For a system of N body fixed frames and a reference frame there are $(N+1)^2$ transformations, but due to the rules (3.96), only N of them can be independent. So, a RBS can have at most $6N$ degrees of freedom, which is only the case if there are no constraints (like joints) between the bodies. Constraints of a joint between body a and b can be captured inside the corresponding transformation ${}^a_b\mathsf{G}$. We will discuss this in the following example.

Example 6. Tricopter with suspended load: configuration. Consider the Tricopter with a suspended load as shown in Figure 3.5. The top part of the figure shows the body fixed frames which are attached to geometrically meaningful points. The numbering of the bodies is rather arbitrary.

The Tricopter flies freely in space, i.e. there are no constraints between the reference frame and any body of the system. So we chose to describe the configuration of the central body w.r.t. the reference frame as

$$
{}^0_1\mathsf{G} = \begin{bmatrix}
{}^0_1R^\mathsf{x}_\mathsf{x} & {}^0_1R^\mathsf{x}_\mathsf{y} & {}^0_1R^\mathsf{x}_\mathsf{z} & {}^0_1r^\mathsf{x} \\
{}^0_1R^\mathsf{y}_\mathsf{x} & {}^0_1R^\mathsf{y}_\mathsf{y} & {}^0_1R^\mathsf{y}_\mathsf{z} & {}^0_1r^\mathsf{y} \\
{}^0_1R^\mathsf{z}_\mathsf{x} & {}^0_1R^\mathsf{z}_\mathsf{y} & {}^0_1R^\mathsf{z}_\mathsf{z} & {}^0_1r^\mathsf{z} \\
0 & 0 & 0 & 1
\end{bmatrix}. \tag{3.97a}
$$

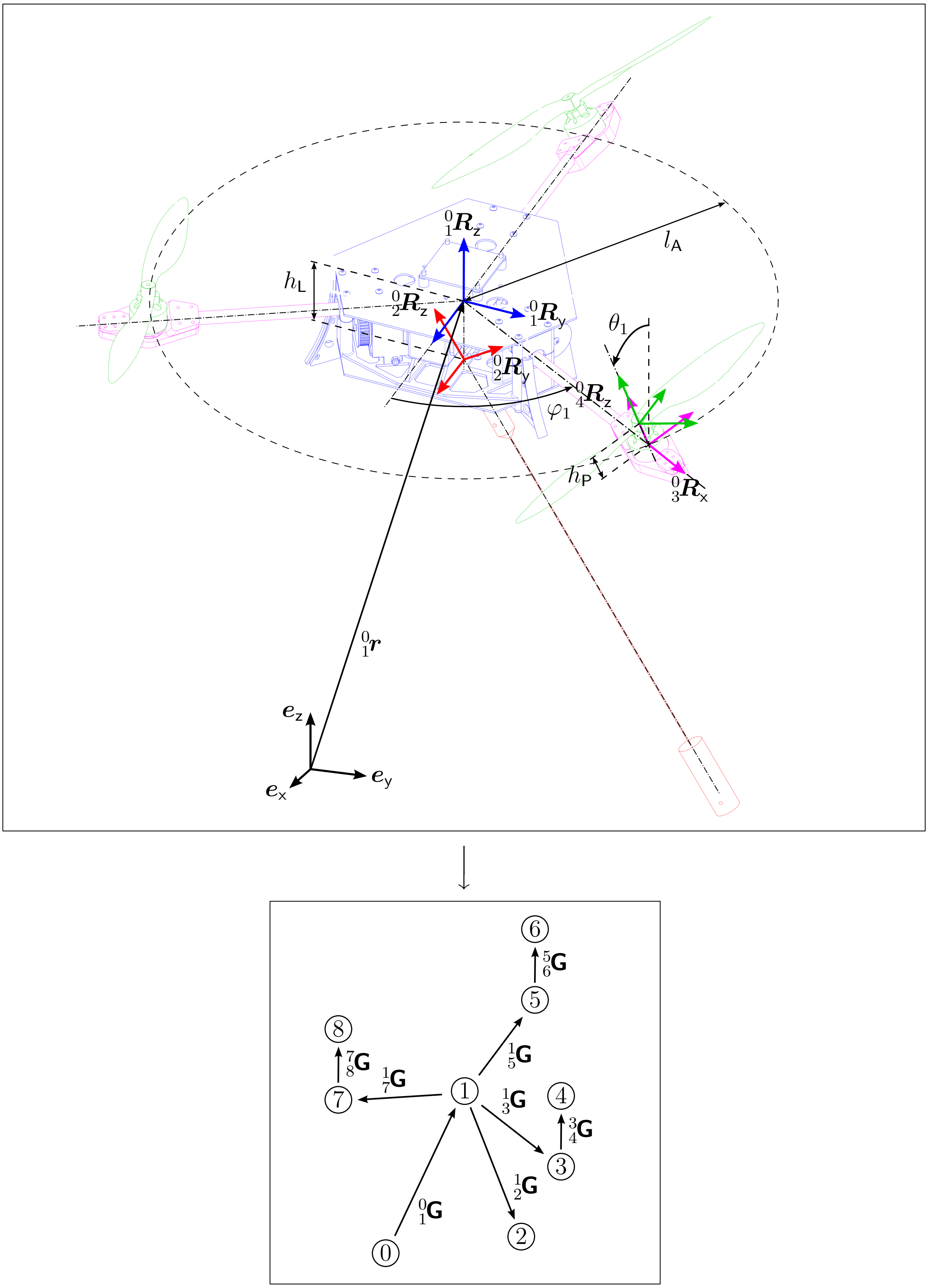

Figure 3.5: Frames attached to the Tricopter bodies (top) and the configuration graph (bottom)

The suspended load is a rigid body that is attached by a spherical joint to the central body. The body fixed frame of the load is placed in the center of this spherical joint. As a consequence, the position of the load (the position of its body fixed frame not the position of its center of mass) w.r.t. the central body is constant. This is reflected by the configuration

$$ {}^{1}_{2}\mathsf{G} = \begin{bmatrix} {}^{1}_{2}R^{\mathsf{x}}_{\mathsf{x}} & {}^{1}_{2}R^{\mathsf{x}}_{\mathsf{y}} & {}^{1}_{2}R^{\mathsf{x}}_{\mathsf{z}} & 0 \\ {}^{1}_{2}R^{\mathsf{y}}_{\mathsf{x}} & {}^{1}_{2}R^{\mathsf{y}}_{\mathsf{y}} & {}^{1}_{2}R^{\mathsf{y}}_{\mathsf{z}} & 0 \\ {}^{1}_{2}R^{\mathsf{z}}_{\mathsf{x}} & {}^{1}_{2}R^{\mathsf{z}}_{\mathsf{z}} & {}^{1}_{2}R^{\mathsf{z}}_{\mathsf{z}} & h_{\mathsf{L}} \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{3.97b} $$

The three arms are connected to the central body each by revolute joints with tilt angles θ_k, $k = 1, 2, 3$. The joint axis lie in the plane spanned by ${}^{0}_{1}\boldsymbol{R}_{\mathsf{x}}$ and ${}^{0}_{1}\boldsymbol{R}_{\mathsf{y}}$ and their angles to ${}^{0}_{1}\boldsymbol{R}_{\mathsf{x}}$ are $\varphi_1 = \frac{\pi}{3}, \varphi_2 = \pi, \varphi_3 = -\frac{\pi}{3}$. The body fixed axes are placed such that ${}^{\;\;\;0}_{2k+1}\boldsymbol{R}_{\mathsf{x}}$ coincide with the tilt axis and ${}^{\;\;\;0}_{2k+1}\boldsymbol{R}_{\mathsf{z}}$, $k = 1, 2, 3$ coincide with the propeller spinning axis. The configuration of the k-th arm w.r.t. the central body is

$$ {}^{\;\;\;1}_{2k+1}\mathsf{G} = \begin{bmatrix} \cos\varphi_k & -\sin\varphi_k\cos\theta_k & \sin\varphi_k\sin\theta_k & l_{\mathsf{A}}\cos\varphi_k \\ \sin\varphi_k & \cos\varphi_k\cos\theta_k & -\cos\varphi_k\sin\theta_k & l_{\mathsf{A}}\sin\varphi_k \\ 0 & \sin\theta_k & \cos\theta_k & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad k = 1, \ldots, 3. \tag{3.97c} $$

The propellers are connected by revolute joints to the arms. The body fixed frame is attached to the geometric center of the propeller (which will be an important point for its aerodynamic model). The configuration w.r.t. the corresponding arm is

$$ {}^{2k+1}_{2k+2}\mathsf{G} = \begin{bmatrix} \mathrm{c}_k & -\mathrm{s}_k & 0 & 0 \\ \mathrm{s}_k & \mathrm{c}_k & 0 & 0 \\ 0 & 0 & 1 & h_{\mathsf{P}} \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad k = 1, \ldots, 3. \tag{3.97d} $$

The set of configurations $\mathcal{G}_0 = \{{}^{0}_{1}\mathsf{G}, {}^{1}_{2}\mathsf{G}, {}^{1}_{3}\mathsf{G}, {}^{3}_{4}\mathsf{G}, {}^{1}_{5}\mathsf{G}, {}^{5}_{6}\mathsf{G}, {}^{1}_{7}\mathsf{G}, {}^{7}_{8}\mathsf{G}\}$ form a directed graph as shown at the bottom of Figure 3.5. With them and the rules from (3.96) we can compute the configuration ${}^{a}_{b}\mathsf{G}$ of any body w.r.t. any other body or the reference frame.

The configurations can be seen as functions ${}^{a}_{b}\mathsf{G}(\boldsymbol{x})$ of the system coordinates

$$ \boldsymbol{x} = \left[{}^{0}_{1}r^{\mathsf{x}}, {}^{0}_{1}r^{\mathsf{y}}, {}^{0}_{1}r^{\mathsf{z}}, {}^{0}_{1}R^{\mathsf{x}}_{\mathsf{x}}, \ldots, {}^{0}_{1}R^{\mathsf{z}}_{\mathsf{z}}, {}^{1}_{2}R^{\mathsf{x}}_{\mathsf{x}}, \ldots, {}^{1}_{2}R^{\mathsf{z}}_{\mathsf{z}}, \theta_1, \theta_2, \theta_3, \mathrm{c}_1, \mathrm{s}_1, \mathrm{c}_2, \mathrm{s}_2, \mathrm{c}_3, \mathrm{s}_3\right]^{\top} \in \mathbb{R}^{30} \tag{3.98} $$

and the constant parameters $h_{\mathsf{L}}, l_{\mathsf{A}}, \varphi_1, \varphi_2, \varphi_3, h_{\mathsf{P}}$. From the rules (3.96) emerge the geometric constraints

$$ \boldsymbol{\phi}(\boldsymbol{x}) = \mathbf{0} \qquad \cong \qquad \begin{cases} {}^{0}_{1}\boldsymbol{R}^{\top}\,{}^{0}_{1}\boldsymbol{R} = \mathbf{I}_3, \ \det{}^{0}_{1}\boldsymbol{R} = +1, \\ {}^{1}_{2}\boldsymbol{R}^{\top}\,{}^{1}_{2}\boldsymbol{R} = \mathbf{I}_3, \ \det{}^{1}_{2}\boldsymbol{R} = +1, \\ (\mathrm{c}_k)^2 + (\mathrm{s}_k)^2 = 1, \ k = 1, 2, 3 \end{cases} \tag{3.99} $$

The configuration space of the rigid body system is

$$ \mathbb{X} = \{\boldsymbol{x} \in \mathbb{R}^{30} \,|\, \boldsymbol{\phi}(\boldsymbol{x}) = 0\} \ \cong \ \mathbb{SE}(3) \times \mathbb{SO}(3) \times \mathbb{R}^3 \times (\mathbb{S}^1)^3. \tag{3.100} $$

This example was mainly chosen as the Tricopter will be discussed in the following chapters. However, it is also an example of a system that is complex enough that one probably does not want to derive the equations of motion without a formalism. It also covers the most common manifolds encountered in rigid body mechanics. Even though the revolute joints for the propeller tilt and the propeller spinning axes both imply a $\mathbb{S}^1$ manifold, the local parameterization by the angle $\theta_k, k = 1, 2, 3$ is chosen. This has a practical motivation: The tilt mechanism also twists the cables to the propeller motor and so $\theta_k = 0$ and $\theta_k = 2\pi$ are really different situations in practice. On the other hand it should also show that the following algorithm handles minimal coordinates just as fine.

A generalization of the example states: A rigid body system can be parameterized by a set of ν (possibly redundant) coordinates $\boldsymbol{x}$ which again parameterize a set of configurations ${}^{a}_{b}\mathbf{G}(\boldsymbol{x})$ which form a *connected* graph. The property connected is essential: it ensures that, with the rules (3.96), all remaining configurations of the graph can be computed i.e. the corresponding *complete* graph. Loops in the graph and the property ${}^{a}_{b}\mathbf{G} \in \mathbb{SE}(3)$ may imply geometric constraints.

The use of graph theory in the context of algorithms for rigid body systems is quite common, see e.g. [Roberson and Schwertassek, 1988, sec. 8.2] or [Wittenburg, 2008, sec. 5.3]. However, we will not go any deeper into this. All we need for the following is that any configuration ${}^{a}_{b}\mathbf{G}(\boldsymbol{x}), a, b = 0 \dots N$ can be expressed in terms of the configuration coordinates $\boldsymbol{x}$.

Body velocity. The previous section motivated particular velocity coordinates $\boldsymbol{\xi} = [\boldsymbol{v}^\top, \boldsymbol{\omega}^\top]^\top$ for the free rigid body, which did lead to a convenient mathematical expressions. In the context of rigid body systems we may associate a *body velocity* ${}^{a}_{b}\boldsymbol{\xi} = [{}^{a}_{b}\boldsymbol{v}^\top, {}^{a}_{b}\boldsymbol{\omega}^\top]^\top$ with any configuration ${}^{a}_{b}\mathbf{G}, a, b = 0, \dots, N$ defined by

$$
{}^{a}_{b}\boldsymbol{\xi} = \operatorname{vee}\left({}^{b}_{a}\mathbf{G}\,{}^{a}_{b}\dot{\mathbf{G}}\right), \quad a, b = 0, \dots, N. \tag{3.101}
$$

From the rules (3.96) for the configurations we can conclude similar rules for their velocities: For the composition ${}^{a}_{c}\mathbf{G} = {}^{a}_{b}\mathbf{G}\,{}^{b}_{c}\mathbf{G}$ we get

$$
{}^{a}_{c}\boldsymbol{\xi} = \operatorname{vee}\left({}^{c}_{b}\mathbf{G}\,{}^{b}_{a}\mathbf{G}({}^{a}_{b}\dot{\mathbf{G}}\,{}^{b}_{c}\mathbf{G} + {}^{a}_{b}\mathbf{G}\,{}^{b}_{c}\dot{\mathbf{G}})\right) = \operatorname{vee}\left({}^{c}_{b}\mathbf{G}\operatorname{wed}({}^{a}_{b}\boldsymbol{\xi})\,{}^{b}_{c}\mathbf{G}\right) + {}^{b}_{c}\boldsymbol{\xi} = \operatorname{Ad}_{{}^{c}_{b}\mathbf{G}}\,{}^{a}_{b}\boldsymbol{\xi} + {}^{b}_{c}\boldsymbol{\xi} \tag{3.102a}
$$

with the adjoint representation introduced in (3.85a). Differentiation of ${}^{a}_{b}\mathbf{G}\,{}^{b}_{a}\mathbf{G} = \mathbf{I}$ yields

$$
\tfrac{\mathrm{d}}{\mathrm{d}t}\left({}^{a}_{b}\mathbf{G}\,{}^{b}_{a}\mathbf{G}\right) = {}^{a}_{b}\dot{\mathbf{G}}\,{}^{b}_{a}\mathbf{G} + {}^{a}_{b}\mathbf{G}\,{}^{b}_{a}\dot{\mathbf{G}} = {}^{a}_{b}\mathbf{G}\operatorname{wed}({}^{a}_{b}\boldsymbol{\xi})\,{}^{b}_{a}\mathbf{G} + \operatorname{wed}({}^{b}_{a}\boldsymbol{\xi}) = \mathbf{0} \quad \Leftrightarrow \quad {}^{b}_{a}\boldsymbol{\xi} = -\operatorname{Ad}_{{}^{a}_{b}\mathbf{G}}\,{}^{a}_{b}\boldsymbol{\xi} \tag{3.102b}
$$

and obviously

$$
{}^{a}_{a}\boldsymbol{\xi} = \mathbf{0}. \tag{3.102c}
$$

System velocity and body Jacobians. Based on their definition (3.101), the body velocities ${}^{a}_{b}\boldsymbol{\xi}$ can be seen as a function of the system coordinates $\boldsymbol{x}$ and their derivatives

$\dot{\boldsymbol{x}} = \boldsymbol{A}(\boldsymbol{x})\boldsymbol{\xi}$. Crucially the velocity is linear in $\dot{\boldsymbol{x}}$ and consequently linear in the system velocity $\boldsymbol{\xi}$ and we can write

$$ {}^a_b\boldsymbol{\xi}(\boldsymbol{x},\boldsymbol{\xi}) = {}^a_b\boldsymbol{J}(\boldsymbol{x})\boldsymbol{\xi}, \qquad {}^a_b\boldsymbol{J}(\boldsymbol{x}) = \frac{\partial\, {}^a_b\boldsymbol{\xi}}{\partial \boldsymbol{\xi}}(\boldsymbol{x}) = \frac{\partial}{\partial \dot{\boldsymbol{x}}} \operatorname{vee}\left({}^b_a\mathbf{G}(\boldsymbol{x})\frac{\mathrm{d}}{\mathrm{d}t}\big({}^a_b\mathbf{G}(\boldsymbol{x})\big)\right)\boldsymbol{A}. \tag{3.103}$$

The matrix ${}^a_b\boldsymbol{J}(\boldsymbol{x}) \in \mathbb{R}^{6\times n}$ that maps the system velocity $\boldsymbol{\xi}$ to the body velocity ${}^a_b\boldsymbol{\xi}$ is commonly called the *body Jacobian*. An alternative formula for the body Jacobian, which might give additional geometric insight, is given in [eq:AppendixDefBodyJac]. The following rules emerge directly from (3.102):

$$ {}^a_c\boldsymbol{J} = \mathrm{Ad}_{{}^c_b\mathbf{G}}\, {}^a_b\boldsymbol{J} + {}^b_c\boldsymbol{J}, \qquad {}^b_a\boldsymbol{J} = -\mathrm{Ad}_{{}^a_b\mathbf{G}}\, {}^a_b\boldsymbol{J}, \qquad {}^a_a\boldsymbol{J} = \mathbf{0}. \tag{3.104}$$

Example 7. Tricopter with suspended load: kinematics. For the tricopter with load from Example 6 we chose the following velocity coordinates: The components of the body velocity ${}^0_1\boldsymbol{\xi}$ of the central body w.r.t. the inertial frame, the components of the angular velocity ${}^0_2\boldsymbol{\omega}$ of the load w.r.t. the inertial frame, the angular velocities $\dot{\theta}_k, k = 1,2,3$ of the arm tilt mechanism and the angular velocities $\varpi_k, k = 1,2,3$ of the propellers w.r.t. the arms. These velocity coordinates $\boldsymbol{\xi} = [{}^0_1\boldsymbol{\xi}^\top, {}^0_2\boldsymbol{\omega}^\top, \dot{\theta}_1, \dot{\theta}_2, \dot{\theta}_3, \varpi_1, \varpi_2, \varpi_3]^\top$ are related to the configuration coordinates $\boldsymbol{\xi}$ by the kinematic equation

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{\xi} \qquad \cong \qquad \begin{cases} {}^0_1\dot{\mathbf{G}} = {}^0_1\mathbf{G}\,\operatorname{wed}({}^0_1\boldsymbol{\xi}), \\ {}^1_2\dot{\boldsymbol{R}} = {}^1_2\boldsymbol{R}\,\operatorname{wed}({}^0_2\boldsymbol{\omega}) - \operatorname{wed}({}^0_1\boldsymbol{\omega})\,{}^1_2\boldsymbol{R}, \\ \dot{\theta}_k = \dot{\theta}_k,\ \ k = 1,2,3 \\ \dot{\mathrm{c}}_k = -\mathrm{s}_k\varpi_k,\ \ k = 1,2,3 \\ \dot{\mathrm{s}}_k = \mathrm{c}_k\varpi_k,\ \ k = 1,2,3 \end{cases}. \tag{3.105}$$

The relative velocity ${}^1_2\boldsymbol{\omega} = \operatorname{vee}({}^1_2\boldsymbol{R}^\top\, {}^1_2\dot{\boldsymbol{R}})$ of the load would be another possible and probably more obvious choice. The absolute velocity ${}^0_2\boldsymbol{\omega}$ is mainly chosen to demonstrate the flexibility of the presented approach but the use of absolute velocities also leads to less cumbersome terms in the system inertia matrix.

The body velocities associated with the configuration matrices from (3.97) are

$$ {}^0_1\boldsymbol{\xi} = \begin{bmatrix} {}^0_1v^{\mathsf{x}} \\ {}^0_1v^{\mathsf{y}} \\ {}^0_1v^{\mathsf{z}} \\ {}^0_1\omega^{\mathsf{x}} \\ {}^0_1\omega^{\mathsf{y}} \\ {}^0_1\omega^{\mathsf{z}} \end{bmatrix}, \qquad {}^1_2\boldsymbol{\xi} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ {}^0_2\omega^{\mathsf{x}} - {}^1_2R^{\mathsf{x}}_{\mathsf{x}}\,{}^0_2\omega^{\mathsf{x}} - {}^1_2R^{\mathsf{y}}_{\mathsf{x}}\,{}^0_2\omega^{\mathsf{y}} - {}^1_2R^{\mathsf{z}}_{\mathsf{x}}\,{}^0_2\omega^{\mathsf{z}} \\ {}^0_2\omega^{\mathsf{y}} - {}^1_2R^{\mathsf{x}}_{\mathsf{y}}\,{}^0_2\omega^{\mathsf{x}} - {}^1_2R^{\mathsf{y}}_{\mathsf{y}}\,{}^0_2\omega^{\mathsf{y}} - {}^1_2R^{\mathsf{z}}_{\mathsf{y}}\,{}^0_2\omega^{\mathsf{z}} \\ {}^0_2\omega^{\mathsf{z}} - {}^1_2R^{\mathsf{x}}_{\mathsf{z}}\,{}^0_2\omega^{\mathsf{x}} - {}^1_2R^{\mathsf{y}}_{\mathsf{z}}\,{}^0_2\omega^{\mathsf{y}} - {}^1_2R^{\mathsf{z}}_{\mathsf{z}}\,{}^0_2\omega^{\mathsf{z}} \end{bmatrix}, \tag{3.106a}$$

$$ {}^{\ \ \ \ 1}_{2k+1}\boldsymbol{\xi} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \dot{\theta}_k \\ 0 \\ 0 \end{bmatrix}, \qquad {}^{2k+1}_{2k+2}\boldsymbol{\xi} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \varpi_k \end{bmatrix}, \qquad k = 1,2,3. \tag{3.106b}$$

From this it should clear how the corresponding body Jacobians look like, e.g. ${}^0_1\boldsymbol{J} = [\mathbf{I}_6\ 0]$.

For the formulation of the kinetic energy in the following subsection we will need the body Jacobians ${}^0_b\boldsymbol{J}, b = 1, \ldots, N$. From the graph structure and the rules (3.104) we can compute them iteratively as

$$
\begin{aligned}
{}^0_2\boldsymbol{J} &= \mathrm{Ad}_{{}^2_1\mathsf{G}}\, {}^0_1\boldsymbol{J} + {}^1_2\boldsymbol{J}, && (3.107a)\\
{}_{2k+1}^{0}\boldsymbol{J} &= \mathrm{Ad}_{{}^{2k+1}_{1}\mathsf{G}}\, {}^0_1\boldsymbol{J} + {}_{2k+1}^{1}\boldsymbol{J}, && (3.107b)\\
{}_{2k+2}^{0}\boldsymbol{J} &= \mathrm{Ad}_{{}^{2k+2}_{2k+1}\mathsf{G}}\, {}_{2k+1}^{0}\boldsymbol{J} + {}^{2k+1}_{2k+2}\boldsymbol{J}. && (3.107c)
\end{aligned}
$$

These terms are significantly more cumbersome, so they are not displayed explicitly.

3.4.2 Inertia

Kinetic energy and inertia matrix. The kinetic energy $\mathcal{T}$ of a rigid body system is simply the sum of the kinetic energies of its bodies. Combining this with the kinetic energy (3.51) of a single free rigid body and the formulation of the absolute body velocities ${}^0_b\boldsymbol{\xi}$ in terms of the chosen coordinates using the body Jacobian ${}^0_b\boldsymbol{J}$ from (3.103), yields

$$
\mathcal{T} = \sum\nolimits_b \tfrac{1}{2}\, {}^0_b\boldsymbol{\xi}^\top\, {}^0_b\mathbf{M}\, {}^0_b\boldsymbol{\xi} = \tfrac{1}{2}\boldsymbol{\xi}^\top \underbrace{\sum\nolimits_b {}^0_b\boldsymbol{J}^\top\, {}^0_b\mathbf{M}\, {}^0_b\boldsymbol{J}}_{\boldsymbol{M}}\, \boldsymbol{\xi}. \tag{3.108}
$$

Recall from the previous section, that the constant *body* inertia matrix ${}^0_b\mathbf{M} \in \mathbb{R}^{6\times 6}$ collects the inertia parameters of the rigid body with index b and w.r.t. its body fixed frame. The matrix $\boldsymbol{M}(\boldsymbol{x}) \in \mathbb{R}^{n\times n}$ is called the *system* inertia matrix.

Connection coefficients. With a rather cumbersome computation (see [eq: ??]), it can be shown that the connection coefficients $\mathit{\Gamma}_{ijk}$ associated to the system inertia matrix $\boldsymbol{M}$ from (3.108) can be expressed in terms of the body Jacobians ${}^0_b\boldsymbol{J}$, the body inertia matrices ${}^0_b\mathbf{M}$ and the body connection coefficients ${}^0_b\Gamma_{pqr}$ from (3.58) or the body commutation coefficients Υ^h_{pq} from (3.49) as

$$
\mathit{\Gamma}_{ijk} = \sum\nolimits_b {}^0_bJ^p_i \big({}^0_b\mathsf{M}_{pq} \partial_k\, {}^0_bJ^q_j + \underbrace{\tfrac{1}{2}\big(\Upsilon^h_{pq}\, {}^0_b\mathsf{M}_{hr} + \Upsilon^h_{pr}\, {}^0_b\mathsf{M}_{hq} - \Upsilon^h_{qr}\, {}^0_b\mathsf{M}_{hp}\big)\, {}^0_bJ^q_j\, {}^0_bJ^r_k}_{{}^0_b\Gamma_{pqr}} \big) \tag{3.109}
$$

Acceleration energy. As with the kinetic energy, the acceleration energy of a rigid body system is simply the sum of the acceleration energies of its bodies. Summing up the body acceleration energies from (3.90c) and plugging in the body velocities ${}^0_b\boldsymbol{\xi} = {}^0_b\boldsymbol{J}\boldsymbol{\xi}$

yields

$$\begin{aligned}
\mathcal{S} &= \sum\nolimits_b \tfrac{1}{2}\|({}^0_b\ddot{\mathbf{G}})^\top\|^2_{{}^0_b\mathbf{M}'} \\
&= \sum\nolimits_b \tfrac{1}{2}\|(\mathrm{wed}({}^0_b\boldsymbol{J}\dot{\boldsymbol{\xi}} + {}^0_b\dot{\boldsymbol{J}}\boldsymbol{\xi}) + \mathrm{wed}({}^0_b\boldsymbol{J}\boldsymbol{\xi})^2)^\top\|^2_{{}^0_b\mathbf{M}'} \\
&= \sum\nolimits_b \tfrac{1}{2}\|(\mathrm{wed}({}^0_b\boldsymbol{J}\dot{\boldsymbol{\xi}}))^\top\|^2_{{}^0_b\mathbf{M}'} + \sum\nolimits_b \mathrm{tr}\left(\mathrm{wed}({}^0_b\boldsymbol{J}\dot{\boldsymbol{\xi}}){}^0_b\mathbf{M}'(\mathrm{wed}({}^0_b\dot{\boldsymbol{J}}\boldsymbol{\xi}) + \mathrm{wed}({}^0_b\boldsymbol{J}\boldsymbol{\xi})^2)^\top\right) \\
&\quad + \underbrace{\sum\nolimits_b \tfrac{1}{2}\|(\mathrm{wed}({}^0_b\dot{\boldsymbol{J}}\boldsymbol{\xi}) + \mathrm{wed}({}^0_b\boldsymbol{J}\boldsymbol{\xi})^2)^\top\|^2_{{}^0_b\mathbf{M}'}}_{\mathcal{S}_0} \\
&= \tfrac{1}{2}\dot{\boldsymbol{\xi}}^\top \underbrace{\sum\nolimits_b {}^0_b\boldsymbol{J}^\top{}^0_b\mathbf{M}\,{}^0_b\boldsymbol{J}}_{\boldsymbol{M}}\,\dot{\boldsymbol{\xi}} + \dot{\boldsymbol{\xi}}^\top \underbrace{\sum\nolimits_b {}^0_b\boldsymbol{J}^\top\big({}^0_b\mathbf{M}\,{}^0_b\dot{\boldsymbol{J}} - \mathrm{ad}^\top_{{}^0_b\boldsymbol{J}\boldsymbol{\xi}}\,{}^0_b\mathbf{M}\,{}^0_b\boldsymbol{J}\big)\boldsymbol{\xi}}_{\boldsymbol{c}} + \mathcal{S}_0.
\end{aligned} \tag{3.110}$$

Obviously, we found again the system inertia matrix $\boldsymbol{M}$ and one may check that indeed $c_i = \Gamma_{ijk}\xi^j\xi^k$ with the connection coefficients Γ_{ijk} from (3.109). Note that $\mathcal{S}_0$ is independent of $\dot{\boldsymbol{\xi}}$, so it does not contribute to the generalized inertia force.

Inertia force. As before, there are several equivalent ways for computing the inertia force $\boldsymbol{f}^{\mathrm{M}}$ of a rigid body system: We may use the Lagrange operator on the kinetic energy from (3.108), use the inertia matrix and connection coefficients from (3.109), or taking the differential of the acceleration energy from (3.110). Each of these approaches will yield

$$\boldsymbol{f}^{\mathrm{M}} = \underbrace{\sum\nolimits_b {}^0_b\boldsymbol{J}^\top{}^0_b\mathbf{M}\,{}^0_b\boldsymbol{J}}_{\boldsymbol{M}}\,\dot{\boldsymbol{\xi}} + \underbrace{\sum\nolimits_b {}^0_b\boldsymbol{J}^\top\big({}^0_b\mathbf{M}\,{}^0_b\dot{\boldsymbol{J}} - \mathrm{ad}^\top_{{}^0_b\boldsymbol{J}\boldsymbol{\xi}}\,{}^0_b\mathbf{M}\,{}^0_b\boldsymbol{J}\big)\boldsymbol{\xi}}_{\boldsymbol{c}}. \tag{3.111}$$

Similar results are called *the projection equation* in [Bremer, 2008, sec. 4.2.5] and *the Kane equations* [Kane and Levinson, 1985, chap. 6]. There is some controversy (starting in [Desloge, 1987]) about the naming, since the equations result rather directly (as shown above) from the Gibbs-Appell formulation. See [Lesser, 1992] or [Papastavridis, 2002, p. 714] for an overview.

In contrast to the sources above, the derivation here does allow for redundant configuration coordinates $\boldsymbol{x}$ and general velocity coordinates $\boldsymbol{\xi}$. This im mostly due to the more general formulation (3.103) of the body jacobian ${}^0_b\boldsymbol{J}$, wheras the formulation of the inertia matrix $\boldsymbol{M}$ and gyroscopic terms $\boldsymbol{c}$ should look familiar.

3.4.3 Gravitation

The potential energy of gravitation of a rigid body system is the sum of the potentials of the individual bodies (3.90g). This is

$$\mathcal{V}^{\mathrm{G}} = \sum\nolimits_b \langle({}^0_b\mathbf{G})^\top, \mathrm{wed}(\boldsymbol{\alpha}_{\mathsf{G}})^\top\rangle_{{}^0_b\mathbf{M}'}, \qquad \boldsymbol{\alpha}_{\mathsf{G}}^\top = [\boldsymbol{a}_{\mathsf{G}}^\top, \mathbf{0}_{1\times3}], \tag{3.112}$$

where ${}^0_b\mathbf{M}' = \mathrm{Vee}({}^0_b\mathbf{M})$ is the body inertia matrix and $\boldsymbol{\alpha}_{\mathsf{G}}$ is Earth's gravity wrench.

Finally, the generalized force of gravity on a rigid body system may be formulated as

$$\begin{aligned}\boldsymbol{f}^{\mathrm{G}} = \boldsymbol{\nabla}\mathcal{V}^{\mathrm{G}} = \frac{\partial \dot{\mathcal{V}}^{\mathrm{G}}}{\partial \boldsymbol{\xi}} &= \frac{\partial}{\partial \boldsymbol{\xi}} \sum\nolimits_b \operatorname{tr}\left({}^0_b\mathbf{G}\,\operatorname{wed}({}^0_b\boldsymbol{\xi})\,{}^0_b\mathbf{M}'\,\operatorname{wed}(\boldsymbol{\alpha}_{\mathsf{G}})^\top\right) \\ &= \sum\nolimits_b \left(\frac{\partial {}^0_b\boldsymbol{\xi}}{\partial \boldsymbol{\xi}}\right)^\top \operatorname{vee2}\left({}^0_b\mathbf{G}^\top \operatorname{wed}(\boldsymbol{\alpha}_{\mathsf{G}})\,{}^0_b\mathbf{M}'\right) = \sum\nolimits_b {}^0_b\boldsymbol{J}^\top\,{}^0_b\mathbf{M}\,\operatorname{Ad}^{-1}_{{}^0_b\mathbf{G}}\boldsymbol{\alpha}_{\mathsf{G}}. \end{aligned} \tag{3.113}$$

3.4.4 Stiffness

In subsection 3.3.4 we considered linear springs between arbitrary points of the body and the inertial frame. For a system of rigid bodies we may consider the same for each body, but additionally we may also consider springs connecting the bodies to each other, see Figure 3.6.

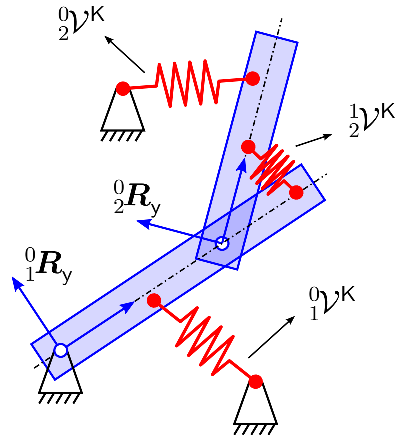

Figure 3.6: MultibodyStiffnessIllustration

The total potential energy $\mathcal{V}^{\mathsf{K}}$ is the sum of the potentials of the individual springs. Here it will make sense to group them differently: Let ${}^a_b\mathcal{V}^{\mathsf{K}}$ with $0 \leq a < b \leq N$ denote the combined potential of all springs connecting body a and b. The total energy is

$$\mathcal{V}^{\mathsf{K}} = \sum_{a=0}^{N-1} \sum_{b=a+1}^{N} {}^a_b\mathcal{V}^{\mathsf{K}} \tag{3.114}$$

Using the results from subsection 3.3.4 it may be shown that each potential can be formulated as

$${}^a_b\mathcal{V}^{\mathsf{K}} = \tfrac{1}{2}\|({}^a_b\mathbf{G} - {}^a_b\mathbf{G}_{\mathsf{R}})^\top\|^2_{{}^a_b\mathbf{K}'} = \tfrac{1}{2}\|({}^a_b\mathbf{G}_{\mathsf{R}}^{-1}\,{}^a_b\mathbf{G} - \mathbf{I}_4)^\top\|^2_{{}^a_b\mathbf{K}'} \tag{3.115}$$

with the constant parameters ${}^a_b\mathbf{K}' \in \mathbb{SYM}(4)$ and ${}^a_b\mathbf{G}_{\mathsf{R}} \in \mathbb{SE}(3)$ resulting from the particular spring distribution between body a and b.

Finally, the generalized force due to an arbitrary constellation of linear springs on a rigid body system may be formulated as

$$\boldsymbol{f}^{\mathrm{K}} = \boldsymbol{\nabla}\mathcal{V}^{\mathrm{K}} = \frac{\partial\dot{\mathcal{V}}^{\mathrm{K}}}{\partial\boldsymbol{\xi}} = \frac{\partial}{\partial\boldsymbol{\xi}}\sum\nolimits_{a,b}\operatorname{tr}\left(({}^a_b\mathbf{G} - {}^a_b\mathbf{G}_{\mathsf{R}}){}^a_b\mathbf{K}'({}^a_b\mathbf{G}\operatorname{wed}({}^a_b\boldsymbol{\xi}))^\top\right)$$
$$= \sum\nolimits_{a,b}\left(\frac{\partial{}^a_b\boldsymbol{\xi}}{\partial\boldsymbol{\xi}}\right)^\top \operatorname{vee2}\left({}^a_b\mathbf{G}^\top({}^a_b\mathbf{G} - {}^a_b\mathbf{G}_{\mathsf{R}}){}^a_b\mathbf{K}'\right)$$
$$= \sum\nolimits_{a,b}{}^a_b\boldsymbol{J}^\top \operatorname{vee2}\left((\mathbf{I}_4 - {}^a_b\mathbf{G}^{-1}{}^a_b\mathbf{G}_{\mathsf{R}}){}^a_b\mathbf{K}'\right) \quad (3.116)$$

3.4.5 Dissipation

Similar to the previous subsection we may consider viscous friction of the bodies to each other and to the inertial frame. Let the body with index b move through a viscous fluid that is attached to the body with index a. The corresponding dissipation function ${}^a_b\mathcal{R}$ was derived in (3.75):

$${}^a_b\mathcal{R} = \tfrac{1}{2}{}^a_b\boldsymbol{\xi}^\top{}^a_b\mathbf{D}\,{}^a_b\boldsymbol{\xi} \quad (3.117)$$

with the body dissipation matrix ${}^a_b\mathbf{D}$. Notice that in contrast to stiffness, the dissipation is, in general, not symmetric in the sense ${}^a_b\mathcal{R} \neq {}^b_a\mathcal{R}$. But due to ${}^a_a\boldsymbol{\xi} = \mathbf{0}$ we have ${}^a_a\mathcal{R} = 0$. For system of rigid bodies we have the dissipation function

$$\mathcal{R} = \sum_{a=0}^{N}\sum_{b=0,b\neq a}^{N}{}^a_b\mathcal{R} = \tfrac{1}{2}\boldsymbol{\xi}^\top\underbrace{\sum_{a=0}^{N}\sum_{b=0,b\neq a}^{N}{}^a_b\boldsymbol{J}^\top{}^a_b\mathbf{D}\,{}^a_b\boldsymbol{J}}_{\boldsymbol{D}}\,\boldsymbol{\xi} \quad (3.118)$$

where $\boldsymbol{D}$ is called the system dissipation matrix.

Finally, the generalized force due to viscous friction on a rigid body system may be formulated as

$$\boldsymbol{f}^{\mathrm{D}} = \frac{\partial\mathcal{R}}{\partial\boldsymbol{\xi}} = \boldsymbol{D}\boldsymbol{\xi} \quad (3.119)$$

3.4.6 Summary

There are three ingredients:

1. the chosen parameterization in the configuration coordinates $\boldsymbol{x}$, the velocity coordinates $\boldsymbol{\xi}$ and their relation captured by the kinematics matrix $\boldsymbol{A}$
2. the rigid body graph ${}^a_b\mathbf{G}$ that maps the coordinates to the rigid body configuration.
3. the constitutive parameters merged into the inertia matrices ${}^0_b\mathbf{M}'$, dissipation matrices ${}^a_b\mathbf{D}'$, stiffness matrices ${}^a_b\mathbf{K}'$, their corresponding minimum ${}^a_b\mathbf{G}_{\mathsf{R}}$ and the gravity vector $\boldsymbol{a}_{\mathsf{G}}$. It should be stressed that these are independent of the chosen coordinates.

Based on this input we may compute the equations of motion in three steps:

1. compute the body Jacobians for the given configurations:

$$ {}^{a}_{b}\boldsymbol{J}(\boldsymbol{x}) = \frac{\partial}{\partial \dot{\boldsymbol{x}}} \operatorname{vee}\Big({}^{b}_{a}\mathbf{G}(\boldsymbol{x})\frac{\mathrm{d}}{\mathrm{d}t}\big({}^{a}_{b}\mathbf{G}(\boldsymbol{x})\big)\Big)\boldsymbol{A}(\boldsymbol{x}) \tag{3.120} $$

2. use the group rules to compute the missing configurations and Jacobians

$$ {}^{a}_{c}\mathbf{G} = {}^{a}_{b}\mathbf{G}\,{}^{b}_{c}\mathbf{G}, \qquad {}^{b}_{a}\mathbf{G} = {}^{a}_{b}\mathbf{G}^{-1}, \tag{3.121a} $$

$$ {}^{a}_{c}\boldsymbol{J} = \mathrm{Ad}_{{}^{c}_{b}\mathbf{G}}\,{}^{a}_{b}\boldsymbol{J} + {}^{b}_{c}\boldsymbol{J}, \qquad {}^{b}_{a}\boldsymbol{J} = -\mathrm{Ad}_{{}^{a}_{b}\mathbf{G}}\,{}^{a}_{b}\boldsymbol{J}, \tag{3.121b} $$

$$ {}^{a}_{c}\dot{\boldsymbol{J}} = \mathrm{Ad}_{{}^{c}_{b}\mathbf{G}}\big({}^{a}_{b}\dot{\boldsymbol{J}} + \mathrm{ad}_{{}^{c}_{b}\boldsymbol{J}\boldsymbol{\xi}}\,{}^{a}_{b}\boldsymbol{J}\big) + {}^{b}_{c}\dot{\boldsymbol{J}}, \qquad {}^{b}_{a}\dot{\boldsymbol{J}} = -\mathrm{Ad}_{{}^{a}_{b}\mathbf{G}}\big({}^{a}_{b}\dot{\boldsymbol{J}} + \mathrm{ad}_{{}^{a}_{b}\boldsymbol{J}\boldsymbol{\xi}}\,{}^{a}_{b}\boldsymbol{J}\big), \tag{3.121c} $$

3. assemble the system matrices

$$ \boldsymbol{M} = \sum\nolimits_{b} {}^{0}_{b}\boldsymbol{J}^{\top}\operatorname{Vee}({}^{0}_{b}\mathbf{M}')\,{}^{0}_{b}\boldsymbol{J}, \tag{3.122a} $$

$$ \boldsymbol{c} = \sum\nolimits_{b} {}^{0}_{b}\boldsymbol{J}^{\top}\operatorname{vee2}\Big(\big(\operatorname{wed}({}^{0}_{b}\dot{\boldsymbol{J}}\boldsymbol{\xi}) + \operatorname{wed}({}^{0}_{b}\boldsymbol{J}\boldsymbol{\xi})^{2}\big)\,{}^{0}_{b}\mathbf{M}'\Big) \tag{3.122b} $$

$$ \boldsymbol{D} = \sum\nolimits_{a,b} {}^{a}_{b}\boldsymbol{J}^{\top}\operatorname{Vee}({}^{a}_{b}\mathbf{D}')\,{}^{a}_{b}\boldsymbol{J} \tag{3.122c} $$

$$ \boldsymbol{f}^{\mathrm{K}} = \sum\nolimits_{a,b} {}^{a}_{b}\boldsymbol{J}^{\top}\operatorname{vee2}\big((\mathbf{I}_{4} - {}^{b}_{a}\mathbf{G}\,{}^{a}_{b}\mathbf{G}_{\mathsf{R}})\,{}^{a}_{b}\mathbf{K}'\big) \tag{3.122d} $$

$$ \boldsymbol{f}^{\mathrm{G}} = \sum\nolimits_{b} {}^{0}_{b}\boldsymbol{J}^{\top}\operatorname{Vee}({}^{0}_{b}\mathbf{M}')\,\mathrm{Ad}_{{}^{b}_{0}\mathbf{G}}\boldsymbol{\alpha}_{\mathsf{G}}, \qquad \boldsymbol{\alpha}_{\mathsf{G}} = [\boldsymbol{a}_{\mathsf{G}}^{\top}, \mathbf{0}_{1\times 3}]^{\top} \tag{3.122e} $$

4. explicit equations of motion

$$ \dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{\xi}, \tag{3.123a} $$

$$ \dot{\boldsymbol{\xi}} = \boldsymbol{M}^{-1}(\boldsymbol{f}^{\mathrm{E}} - \boldsymbol{c} - \boldsymbol{D}\boldsymbol{\xi} - \boldsymbol{f}^{\mathrm{K}} - \boldsymbol{f}^{\mathrm{G}}). \tag{3.123b} $$

The first step (3.120) requires differentiation, so must be performed symbolically. The remaining steps only require basic linear algebra, so can be preformed numerically. For small systems it might be still reasonable to compute $\boldsymbol{M}(\boldsymbol{x})$ symbolically, but for larger systems the explicit expressions can be overwhelming even for contemporary computers.

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Chapter 4

Templates

4.1 Math fonts

Latin alphabet in math mode

default	$A\ B\ C\ D\ E\ F\ G\ H\ I\ J\ K\ L\ M\ N\ O\ P\ Q\ R\ S\ T\ U\ V\ W\ X\ Y\ Z$ $a\ b\ c\ d\ e\ f\ g\ h\ i\ j\ k\ l\ m\ n\ o\ p\ q\ r\ s\ t\ u\ v\ w\ x\ y\ z$
mathrm	$\mathrm{A\ B\ C\ D\ E\ F\ G\ H\ I\ J\ K\ L\ M\ N\ O\ P\ Q\ R\ S\ T\ U\ V\ W\ X\ Y\ Z}$ $\mathrm{a\ b\ c\ d\ e\ f\ g\ h\ i\ j\ k\ l\ m\ n\ o\ p\ q\ r\ s\ t\ u\ v\ w\ x\ y\ z}$
mathsf	$\mathsf{A\ B\ C\ D\ E\ F\ G\ H\ I\ J\ K\ L\ M\ N\ O\ P\ Q\ R\ S\ T\ U\ V\ W\ X\ Y\ Z}$ $\mathsf{a\ b\ c\ d\ e\ f\ g\ h\ i\ j\ k\ l\ m\ n\ o\ p\ q\ r\ s\ t\ u\ v\ w\ x\ y\ z}$
mathtt	$\mathtt{A\ B\ C\ D\ E\ F\ G\ H\ I\ J\ K\ L\ M\ N\ O\ P\ Q\ R\ S\ T\ U\ V\ W\ X\ Y\ Z}$ $\mathtt{a\ b\ c\ d\ e\ f\ g\ h\ i\ j\ k\ l\ m\ n\ o\ p\ q\ r\ s\ t\ u\ v\ w\ x\ y\ z}$
boldsymbol	$\boldsymbol{A\ B\ C\ D\ E\ F\ G\ H\ I\ J\ K\ L\ M\ N\ O\ P\ Q\ R\ S\ T\ U\ V\ W\ X\ Y\ Z}$ $\boldsymbol{a\ b\ c\ d\ e\ f\ g\ h\ i\ j\ k\ l\ m\ n\ o\ p\ q\ r\ s\ t\ u\ v\ w\ x\ y\ z}$
mathbf	$\mathbf{A\ B\ C\ D\ E\ F\ G\ H\ I\ J\ K\ L\ M\ N\ O\ P\ Q\ R\ S\ T\ U\ V\ W\ X\ Y\ Z}$ $\mathbf{a\ b\ c\ d\ e\ f\ g\ h\ i\ j\ k\ l\ m\ n\ o\ p\ q\ r\ s\ t\ u\ v\ w\ x\ y\ z}$
mathfrak	$\mathfrak{A\ B\ C\ D\ E\ F\ G\ H\ I\ J\ K\ L\ M\ N\ O\ P\ Q\ R\ S\ T\ U\ V\ W\ X\ Y\ Z}$ $\mathfrak{a\ b\ c\ d\ e\ f\ g\ h\ i\ j\ k\ l\ m\ n\ o\ p\ q\ r\ s\ t\ u\ v\ w\ x\ y\ z}$
mathcal	$\mathcal{A\ B\ C\ D\ E\ F\ G\ H\ I\ J\ K\ L\ M\ N\ O\ P\ Q\ R\ S\ T\ U\ V\ W\ X\ Y\ Z}$
mathbb	$\mathbb{A\ B\ C\ D\ E\ F\ G\ H\ I\ J\ K\ L\ M\ N\ O\ P\ Q\ R\ S\ T\ U\ V\ W\ X\ Y\ Z}$

Greek alphabet in math mode

default			Γ	Δ				Θ			Λ			Ξ		Π		Σ		Υ	Φ		Ψ	Ω
	α	β	γ	δ	ϵ	ζ	η	θ	ι	κ	λ	μ	ν	ξ	o	π	ρ	σ	τ	υ	ϕ	χ	ψ	ω
var	A	B	$\varGamma$	$\varDelta$	E	Z	H	$\varTheta$	I	K	$\varLambda$	M	N	$\varXi$	O	$\varPi$	P	$\varSigma$	T	$\varUpsilon$	$\varPhi$	X	$\varPsi$	$\varOmega$
					ε			ϑ								ϖ	ϱ	ς			φ			
boldsymbol			$\boldsymbol{\Gamma}$	$\boldsymbol{\Delta}$				$\boldsymbol{\Theta}$			$\boldsymbol{\Lambda}$			$\boldsymbol{\Xi}$		$\boldsymbol{\Pi}$		$\boldsymbol{\Sigma}$		$\boldsymbol{\Upsilon}$	$\boldsymbol{\Phi}$		$\boldsymbol{\Psi}$	$\boldsymbol{\Omega}$
	$\boldsymbol{\alpha}$	$\boldsymbol{\beta}$	$\boldsymbol{\gamma}$	$\boldsymbol{\delta}$	$\boldsymbol{\epsilon}$	$\boldsymbol{\zeta}$	$\boldsymbol{\eta}$	$\boldsymbol{\theta}$	$\boldsymbol{\iota}$	$\boldsymbol{\kappa}$	$\boldsymbol{\lambda}$	$\boldsymbol{\mu}$	$\boldsymbol{\nu}$	$\boldsymbol{\xi}$	$\boldsymbol{o}$	$\boldsymbol{\pi}$	$\boldsymbol{\rho}$	$\boldsymbol{\sigma}$	$\boldsymbol{\tau}$	$\boldsymbol{\upsilon}$	$\boldsymbol{\phi}$	$\boldsymbol{\chi}$	$\boldsymbol{\psi}$	$\boldsymbol{\omega}$
var	$\boldsymbol{A}$	$\boldsymbol{B}$	$\boldsymbol{\varGamma}$	$\boldsymbol{\varDelta}$	$\boldsymbol{E}$	$\boldsymbol{Z}$	$\boldsymbol{H}$	$\boldsymbol{\varTheta}$	$\boldsymbol{I}$	$\boldsymbol{K}$	$\boldsymbol{\varLambda}$	$\boldsymbol{M}$	$\boldsymbol{N}$	$\boldsymbol{\varXi}$	$\boldsymbol{O}$	$\boldsymbol{\varPi}$	$\boldsymbol{P}$	$\boldsymbol{\varSigma}$	$\boldsymbol{T}$	$\boldsymbol{\varUpsilon}$	$\boldsymbol{\varPhi}$	$\boldsymbol{X}$	$\boldsymbol{\varPsi}$	$\boldsymbol{\varOmega}$
					$\boldsymbol{\varepsilon}$			$\boldsymbol{\vartheta}$								$\boldsymbol{\varpi}$	$\boldsymbol{\varrho}$	$\boldsymbol{\varsigma}$			$\boldsymbol{\varphi}$			
mathsf	A	B	$\mathsf{\Gamma}$	$\mathsf{\Delta}$	E	Z	H	$\mathsf{\Theta}$	I	K	$\mathsf{\Lambda}$	M	N	$\mathsf{\Xi}$	O	$\mathsf{\Pi}$	P	$\mathsf{\Sigma}$	T	$\mathsf{\Upsilon}$	$\mathsf{\Phi}$	X	$\mathsf{\Psi}$	$\mathsf{\Omega}$
	$\mathsf{\alpha}$	$\mathsf{\beta}$	$\mathsf{\gamma}$	$\mathsf{\delta}$	$\mathsf{\varepsilon}$	$\mathsf{\zeta}$	$\mathsf{\eta}$	$\mathsf{\theta}$	$\mathsf{\iota}$	$\mathsf{\kappa}$	$\mathsf{\lambda}$	$\mathsf{\mu}$	$\mathsf{\nu}$	$\mathsf{\xi}$	o	$\mathsf{\pi}$	$\mathsf{\rho}$	$\mathsf{\sigma}$	$\mathsf{\tau}$	$\mathsf{\upsilon}$	$\mathsf{\varphi}$	$\mathsf{\chi}$	$\mathsf{\psi}$	$\mathsf{\omega}$
mathbf	$\mathbf{A}$	$\mathbf{B}$	$\mathbf{\Gamma}$	$\mathbf{\Delta}$	$\mathbf{E}$	$\mathbf{Z}$	$\mathbf{H}$	$\mathbf{\Theta}$	$\mathbf{I}$	$\mathbf{K}$	$\mathbf{\Lambda}$	$\mathbf{M}$	$\mathbf{N}$	$\mathbf{\Xi}$	$\mathbf{O}$	$\mathbf{\Pi}$	$\mathbf{P}$	$\mathbf{\Sigma}$	$\mathbf{T}$	$\mathbf{\Upsilon}$	$\mathbf{\Phi}$	$\mathbf{X}$	$\mathbf{\Psi}$	$\mathbf{\Omega}$

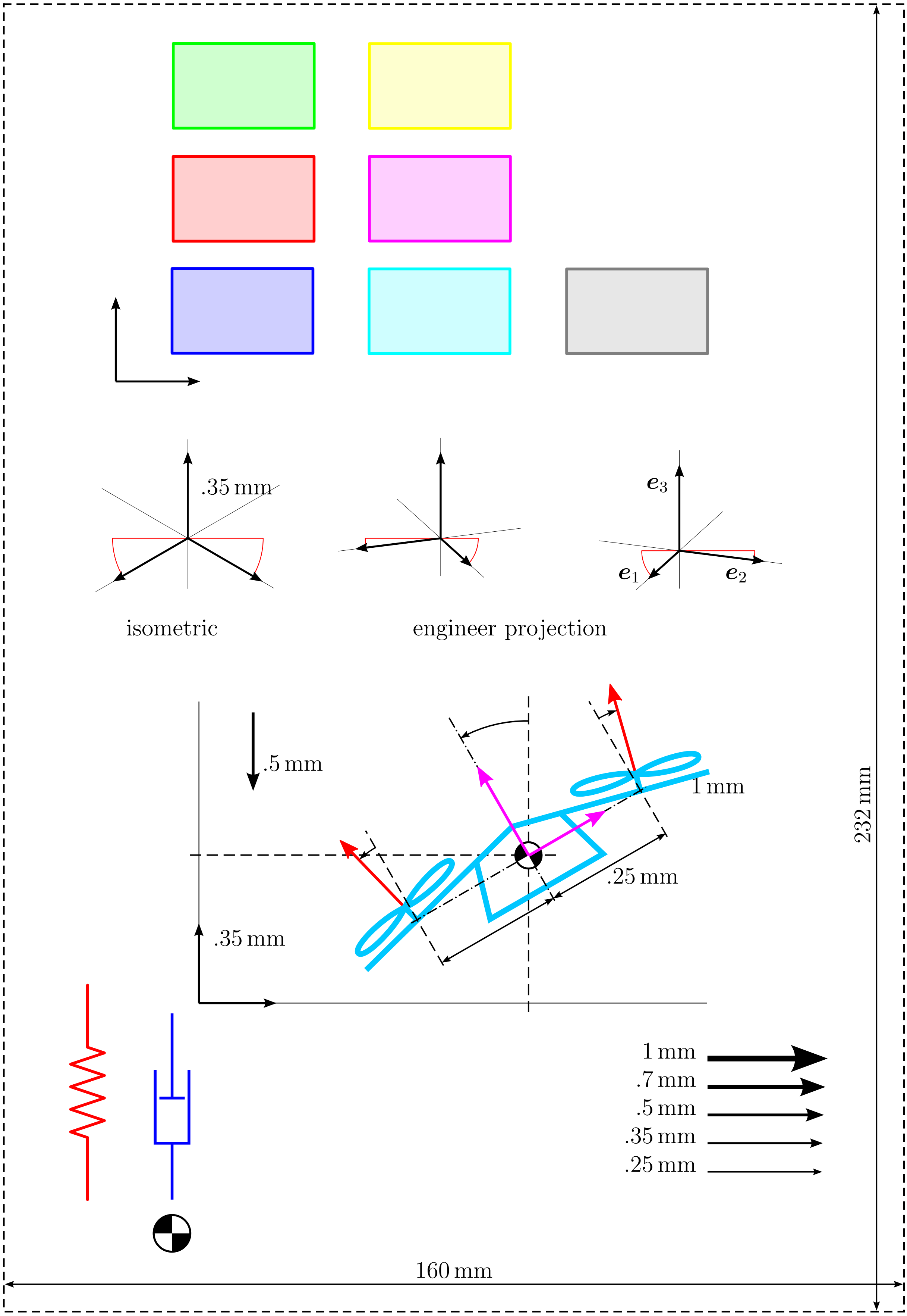

Figure 4.1: Inkscape figure template