Contents

1 $SL(2,\mathbb{Z})$

1.1 Eisenstein series and the spectral decomposition of $L^2(\Gamma\setminus\mathbb{H})$

Let $\Gamma=\mathrm{SL}(2,\mathbb{Z})$. This group has only a cusp at infinity. The stabilizer of the cusp ∞ in Γ is

$$\Gamma_\infty=\left\{\begin{pmatrix}1 & m\\ 0 & 1\end{pmatrix}\mid m\in\mathbb{Z}\right\}.$$

The Eisenstein series associated to this cusp, defined on $\mathbb{H}\times\mathbb{C}$, is defined by

$$E(z,s):=E_\infty(z,s)=\sum_{\gamma\in\Gamma_\infty\setminus\Gamma}\mathrm{Im}(\gamma z)^s. \tag{1}$$

Notice that $\mathrm{Im}(z)$ is Γ_∞-invariant, and the Eisenstein series $E(z,s)$ defines an automorphic function with respect to Γ, that is, it satisfies $E(\gamma z,s)=E(z,s)$ for all $\gamma\in\Gamma$. For $\mathrm{Re}(s)>1$, this series converges absolutely and uniformly on compact sets. Because $\Delta y^s=s(1-s)y^s$ and because Δ commutes with the Γ-action, the Eisenstein series is also an eigenfunction of the Laplacian

$$\Delta E(z,s)=s(1-s)E(z,s).$$

Automorphic functions which are eigenfunctions of the Laplace operator are called Maass forms. The Fourier expansion of $E(z,s)$ is given by

$$E(z,s)=y^s+\phi(s)y^{1-s}+2\sum_{m\neq 0}a_m y^{1/2}K_{s-1/2}(2\pi|m|y)e(mx),$$

where

$$\phi(s)=\pi^{1/2}\frac{\Gamma(s-1/2)}{\Gamma(s)}\frac{\zeta(2s-1)}{\zeta(2s)},$$
$$a_m=\frac{\pi^s}{\Gamma(s)\zeta(2s)|n|^{1/2}}\sum_{ab=|n|}\left(\frac{a}{b}\right)^{s-\frac12}.$$

The modified Bessel functions K_ν are exponentially decaying functions. In particular, $E(z,s)=y^s+\phi(s)y^{1-s}+O(e^{-cy})$, for some constant $c>0$. For a function to be in $L^2(\Gamma\setminus\mathbb{H})$ its growth need to be $O(y^{1/2})$. From the constant term of the Fourier expansion we see that the Eisenstein series is not in $L^2(\Gamma\setminus\mathbb{H})$. But for $\mathrm{Re}(s)=\frac12$ the Eisenstein series is “almost square integrable”, and this suggests to work on the line $\mathrm{Re}(s)=\frac12$. We now have to address two issues: We want to work with square integrable functions on $\Gamma\setminus\mathbb{H}$, and we need to meromorphically continue $E(z,s)$ to the line $\mathrm{Re}(s)=\frac12$. The meromorphic continuation of $E(z,s)$ follows from the Fourier expansion. In the half-plane $\mathrm{Re}(s) \geq \frac{1}{2}$ there is only a simple pole at $s = 1$ with residue $\frac{3}{\pi}$. The Eisenstein series enjoys the functional equation

$$E(z, 1-s) = \phi(1-s)E(z, s).$$

For a smooth, compactly supported function ψ on $\mathbb{R}^{>0}$, the incomplete Eisenstein series is

$$E(z|\psi) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \psi(\mathrm{Im}(\gamma z)).$$

The incomplete Eisenstein series $E(z|\psi)$ lies in $C_c^\infty(\Gamma \setminus \mathbb{H}) \subset L^2(\Gamma \setminus \mathbb{H})$, but it is not an eigenfunction of Δ. By Mellin inversion it can however be represented as an integral of Eisenstein series

$$E(z|\psi) = \frac{1}{2\pi i} \int_{(\sigma)} E(z, s)\widehat{\psi}(s)\mathrm{d}s, \tag{2}$$

where $\sigma > 1$ and

$$\widehat{\psi}(s) = \int_0^\infty \psi(y) y^{-s-1}\mathrm{d}y.$$

We denote by $\mathcal{E}(\Gamma \setminus \mathbb{H}) \subset L^2(\Gamma \setminus \mathbb{H})$ the space of incomplete Eisenstein series $E(z|\psi)$. The inner product of a function $f \in L^2(\Gamma \setminus \mathbb{H})$ with an incomplete Eisenstein series $E(z|\psi)$ is

$$\langle f, E(\cdot|\psi)\rangle = \int_0^\infty f_0(y)\overline{\psi}(y) y^{-2}\mathrm{d}y,$$

where f_0 is the constant term in the Fourier expansion of f. If f is orthogonal to $\mathcal{E}(\Gamma \setminus \mathbb{H})$, then the above integral is zero for all smooth functions ψ of compact support in $(0, \infty)$. Thus the orthogonal complement $\mathcal{C} = \mathcal{E}^\perp$ consists of functions whose constant term in the Fourier series is zero. The Laplace operator Δ has discrete spectrum in $\mathcal{C}$ and $\mathcal{C}$ is spanned by cusp forms. For an orthonormal basis of cusp forms $\{u_j\}$ every $f \in \mathcal{C}(\Gamma \setminus \mathbb{H})$ has the spectral expansion

$$f(z) = \sum_j \langle f, u_j\rangle u_j(z).$$

The spectrum of Δ in $\mathcal{E}(\Gamma \setminus \mathbb{H})$ turns out to consist of a continuous part spanned by the Eisenstein series $E(z, \frac{1}{2} + ir)$, and the zero eigenvalue corresponding to the constant function u_0.

For $\mathrm{Re}(s) \geq \frac{1}{2}$, the Eisenstein series has only a simple pole at $s = 1$. The Eisenstein series are holomorphic on the line $\mathrm{Re}(s) = \frac{1}{2}$ and are of polynomial growth in vertical strips $-\epsilon \leq \mathrm{Re}(s) \leq 1 + \epsilon$. One can then shift the integration in Equation 2 to the line $\mathrm{Re}(s) = \frac{1}{2}$, thereby picking up the residue of $E(z, s)$ at the pole $s = 1$. As a result we obtain

$$E(z|\psi) = \widehat{\psi}(1)\mathrm{Res}_{s=1}(E(z, s)) + \frac{1}{2\pi i} \int_{(1/2)} E(z, s)\widehat{\psi}(s)\mathrm{d}s.$$

The term $\widehat{\psi}(1)$ can be written as $\frac{\langle E(\cdot|\psi), u_0\rangle}{\langle u_0, u_0\rangle}$. We still need the projection of $E(z|\psi)$ onto $E(z, s)$. The functional equation along with

$$\langle E(\cdot|\psi), E(\cdot, 1/2 + ir)\rangle = \widehat{\psi}(1/2 + ir) + \phi(1/2 - ir)\widehat{\psi}(1/2 - ir)$$

yields the spectral decomposition of $E(z|\psi)$ onto the Eisenstein series

$$\frac{1}{2\pi i} \int_{(1/2)} E(z, s)\widehat{\psi}(s)\mathrm{d}s = \frac{1}{4\pi i} \int_{(1/2)} \langle E(\cdot|\psi), E(\cdot, s)\rangle E(z, s)\mathrm{d}s.$$

In conclusion, for an orthonormal basis of cusp forms $\{u_j\}$, every $f \in L^2(\Gamma \setminus \mathbb{H})$ has the spectral expansion

$$f(z) = \sum_j \langle f, u_j\rangle u_j(z) + \frac{1}{4\pi i} \int_{(1/2)} \langle E(\cdot|\psi), E(\cdot, s)\rangle E(z, s)\mathrm{d}s.$$

1.2 Hecke operators for $L^2(\mathrm{SL}(2,\mathbb{Z}) \setminus \mathbb{H})$

1.3 L-functions

Let

$$f(z) = \sum_{n\neq 0} a(n)\sqrt{y}K_{\nu_f}(2\pi|n|y)e^{2\pi inx} \tag{3}$$

be a cuspidal Maass form for $SL(2,\mathbb{Z})$.

Lemma 1. The coefficients $a(n)$ in the Fourier expansion of $f(z)$ satisfy

$$a(n) = O(\sqrt{|n|}).$$

Proof. Because f is a cusp form, it is bounded as $\mathrm{Im}(z) \to \infty$. Thus

$$\left|a(n)\sqrt{y}K_{\nu_f}(2\pi|n|y)\right| = \left|\int_0^1 f(x+iy)e^{-2\pi inx}\mathrm{d}x\right| \leq \int_0^1 |f(x+iy)|\mathrm{d}x \leq C,$$

for some constant C (that depends on f). If we choose $y = \frac{1}{|n|}$ the lemma is proved. □

For $\mathrm{Re}(s) \geq \frac{3}{2}$ we define the L-function $L_f(s)$ associated to $f(z)$ by the absolutely convergent series

$$L_f(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}. \tag{4}$$

Lemma 2. Let $f(z)$ be a cuspidal Maass form for $SL(2,\mathbb{Z})$. Then its L-function $L_f(s)$ can be meromorphically continued to all $\mathbb{C}$ and it satisfies the functional equation

$$\Lambda_f(s) = \pi^{-2}\Gamma\left(\frac{s+\epsilon+\nu_f}{2}\right)\Gamma\left(\frac{s+\epsilon-\nu_f}{2}\right)L_f(s) = (-1)^{\epsilon}\Lambda_f(1-s).$$

Proof. ... □

1.4 Rankin-Selberg convolution for $\mathrm{SL}(2,\mathbb{Z})$

Let

$$f(z) = \sum_{n\neq 0} a(n)\sqrt{y}K_{\nu_f}(2\pi|n|y)e^{2\pi inx}, \tag{5}$$

$$g(z) = \sum_{n\neq 0} b(n)\sqrt{y}K_{\nu_g}(2\pi|n|y)e^{2\pi inx}, \tag{6}$$

be cuspidal Maass forms for $SL(2,\mathbb{Z})$. Recall that $f(z)$ (resp. $g(z)$) is an eigenfunction for the Laplacian with eigenvalue $1/4-\nu_f$ (resp. $1/4-\nu_g$). For sufficiently large $\mathrm{Re}(s)$ we define the convolution function as the absolutely convergent series

$$L_{f\times g}(s) = \zeta(2s)\sum_{n=1}^{\infty}\frac{a(n)\overline{b(n)}}{n^s}.$$

We will prove that $L_{f\times g}$ can be expressed as an inner product of $f\overline{g}$ with an Eisenstein series. This construction gives the meromorphic continuation and functional equation for $L_{f\times g}$.

Theorem 3. *Let $f(z)$ and $g(z)$ be cuspidal Maass forms as in Equation 5 and 6. Then $L_{f\times g}$ can be meromorphically continued to all $s\in\mathbb{C}$ with at most a simple pole at $s=1$. Furthermore, we have the functional equation*

$$L^*_{f\times g}(s) = \pi^{-2s}G_{\nu_f,\nu_g}(s)L_{f\times g}(s) = L^*_{f\times g}(1-s),$$

where $G_{\nu_f,\nu_g}(s) = \prod\Gamma\left(\frac{s\pm\nu_f\pm\nu_g}{2}\right)$.

Proof. Let $E(z,s)$ be the non-holomorphic Eisenstein series as defined in Equation 1. For sufficiently large $\mathrm{Re}(s)$, we have

$$\begin{aligned}
\zeta(2s)\langle f\overline{g}, E(\cdot,\overline{s})\rangle &= \zeta(2s)\int_{SL(2,\mathbb{Z})\backslash\mathbb{H}} f(z)\overline{g(z)E(z,\overline{s})}\mathrm{d}\mu(z)\\
&= \zeta(2s)\sum_{\gamma\in\Gamma_\infty\backslash SL(2,\mathbb{Z}}\int_{SL(2,\mathbb{Z})\backslash\mathbb{H}} f(\gamma z)\overline{g(\gamma z)}\,\mathrm{Im}(\gamma z)^s\mathrm{d}\mu(\gamma z)\\
&= \zeta(2s)\int_{\Gamma_\infty\backslash\mathbb{H}} f(z)\overline{g(z)}\,\mathrm{Im}(z)^s\mathrm{d}\mu(z)\\
&= \zeta(2s)\int_0^\infty\int_0^1 f(z)\overline{g(z)}y^{s-2}\mathrm{d}x\mathrm{d}y\\
&= \zeta(2s)\sum_{n,m\neq 0} a(n)\overline{b(m)}\int_0^\infty K_{\nu_f}(2\pi|n|y)K_{\nu_g}(2\pi|m|y)y^{s-1}\int_0^1 e^{2\pi i(n-m)x}\mathrm{d}x\mathrm{d}y\\
&= \zeta(2s)\sum_{n\neq 0} a(n)\overline{b(n)}\int_0^\infty K_{\nu_f}(2\pi|n|y)K_{\nu_g}(2\pi|n|y)y^s\frac{\mathrm{d}y}{y}\\
&= \frac{\zeta(2s)}{(2\pi)^s}\sum_{n\neq 0}\frac{a(n)\overline{b(n)}}{n^s}\int_0^\infty K_{\nu_f}(y)K_{\nu_g}(y)y^s\frac{\mathrm{d}y}{y}\\
&= (2\pi)^{-s}\,L_{f\times g}(s)\int_0^\infty K_{\nu_f}(y)K_{\nu_g}(y)y^s\frac{\mathrm{d}y}{y}.
\end{aligned}$$

The Mellin transform of $K_\nu K_{\nu'}$ is given by

$$\int_0^\infty K_\nu(y)K_{\nu'}(y)y^s\frac{\mathrm{d}y}{y} = \frac{2^{s-3}}{\Gamma(s)}\prod\Gamma\left(\frac{s\pm\nu\pm\nu'}{2}\right).$$

From the calculation it follows that the convolution function $L_{f\times g}$ inherits the analytical properties of the Eisenstein series $E(z,s)$. This means that $L_{f\times g}$ can be meromorphically continued on $\mathbb{C}$. Because the Eisenstein series has a simple pole at $s=1$ and the Gamma function no zeros, it follows that $L_{f\times g}$ has a simple pole at $s=1$ if and only if $\langle f,g\rangle\neq 0$. The functional equation follows from the functional equation of the Eisenstein series

$$E^*(z,s) = \pi^{-s}\Gamma(s)\zeta(2s)E(z,s) = E^*(z,1-s).$$

□

Lemma 4. Let $\alpha_i,\beta_i\in\mathbb{C}$ for $i=1,2$. Then

$$\prod_{i=1}^2\prod_{j=1}^2(1-\alpha_i\beta_j x)^{-1} = \sum_{k=0}^\infty S_k(\alpha_1,\alpha_2)S_k(\beta_1,\beta_2)x^k(1-\alpha_1\alpha_2\beta_1\beta_2x^2)^{-1},$$

where $S_k(x_1,x_2)$ is the Schur polynomial

$$S_k(x_1,x_2) = \frac{\det\begin{pmatrix}x_1^{k+1} & x_2^{k+1}\\ 1 & 1\end{pmatrix}}{\det\begin{pmatrix}x_1 & x_2\\ 1 & 1\end{pmatrix}} = \frac{x_1^{k+1}-x_2^{k+1}}{x_1-x_2}.$$

Proof. □

Theorem 5. *Let $f(z)$ and $g(z)$ be cuspidal Maass forms as in Equation* 5 *and* 6*. Assume that L_f and L_g have Euler products*

$$L_f(s) = \prod_p\prod_{i=1}^2\left(1-\frac{\alpha_{i,p}}{p^s}\right)^{-1},\quad L_g(s) = \prod_p\prod_{j=1}^2\left(1-\frac{\beta_{j,p}}{p^s}\right)^{-1}.$$

Then $L_{f\times g}(s)$ admits the Euler product

$$L_{f\times g}(s) = \prod_p \prod_{i=1}^{2} \prod_{j=1}^{2} \left(1 - \frac{\alpha_{i,p}\,\overline{\beta}_{j,p}}{p^s}\right)^{-1}.$$

Proof. By assumption we have

$$L_{f\times g}(s) = \prod_p \frac{\sum_{k=1}^{\infty} a(p^k)\overline{b(p^k)}p^{-ks}}{(1-p^{-2s})}.$$

In view of Lemma 4, after choosing $x = p^{-s}$, it suffices to show that

$$a(p^k) = S_k(\alpha_{1,p}, \alpha_{2,p}), \quad b(p^k) = S_k(\beta_{1,p}, \beta_{2,p}).$$

The above equalities are obtained inductively from the relations

$$\begin{aligned} 1 &= \alpha_{1,p}\,\alpha_{2,p}, \\ a(p) &= \alpha_{1,p} + \alpha_{2,p}, \\ a(p^{k+1}) &= a(p)a(p^k) - a(p^{k-1}). \end{aligned}$$

□

2 $\Gamma_0(N)$

Let $\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) \,|\, c \equiv 0 \pmod N \right\}$.

2.1 Dirichlet characters

Let m be a positive integer. A Dirichlet character of modulus m is a function $\chi : \mathbb{Z} \to \mathbb{C}$ such that

- $\chi(ab) = \chi(a)\chi(b)$ for all $a, b \in \mathbb{Z}$,
- $\chi(a) = 0$ if and only if $(a, m) > 1$,
- If $a \equiv b \pmod m$, then $\chi(a) = \chi(b)$.

Let χ_1 and χ_2 be Dirichlet characters of modulus m_1 and m_2, respectively, with $m_1|m_2$. If $\chi_2(a) = \chi_1(a)$ for $a \in (\mathbb{Z}/m_2\mathbb{Z})^\times$, then χ_2 is said to be induced by χ_1. A Dirichlet character is called primitive if it is not induced by any Dirichlet character other than itself. A Dirichlet character induced by the identity function is called principal. If p is prime, then every nonprincipal Dirichlet character of modulus p is primitive.

2.2 Twisted Eisenstein series

In this section we assume that χ is an even ($\chi(-1) = 1$) primitive Dirichlet character mod q, and that $(q, N) = 1$. For $\mathrm{Re}(s) > 1$ we define the twisted Eisenstein series by the absolutely convergent series

$$E(z, s, \chi) = \frac{1}{2} \sum_\gamma \chi(d)\, \mathrm{Im}(\gamma z)^s,$$

where the sum goes over a set of coset representatives $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\infty \setminus \Gamma_0(Nq^2)$.

A function $f : \mathbb{H} \to \mathbb{C}$ is said to be automorphic for $\Gamma_0(q)$ with character χ if it satisfies the condition $f(\gamma z) = \chi(\gamma)f(z)$ for all $\gamma \in \Gamma_0(q), z \in \mathbb{H}$. One can easily see that $E(z, s, \chi)$ is an automorphic function for $\Gamma_0(Nq^2)$ with character χ. In particular, the function $E(z, s, \chi)$ is invariant under $z \mapsto z+1$. Hence it has a Fourier expansion. In order to determine its Fourier expansion we first establish a few identities.

We associate to a pair of coprime integers (c,d), subject to $Nq^2|c$, the set of all matrices in $\Gamma_0(Nq^2)$ whose bottom row is (c,d). Such a matrix represents a unique coset $\Gamma_\infty \setminus \Gamma_0(Nq^2)$. A pair $(m,n) \in \mathbb{Z} \setminus \{0,0\}$ can be uniquely written as (Mc, Md) for $(c,d)=1$ and for some $M>0$. As a consequence, we can write

$$\begin{aligned} E^*(z,s,\chi) &:= L(2s,\chi)E(z,s,\chi) \\ &= \frac{L(2s,\chi)}{2} \sum_{\gamma \in \Gamma_\infty \setminus \Gamma_0(Nq^2)} \chi(d)\, \mathrm{Im}(\gamma z)^s \\ &= \frac{L(2s,\chi)}{2} \sum_{\substack{(c,d)=1 \\ Nq^2|c}} \chi(d) \frac{y^s}{|cz+d|^{2s}} \\ &= \frac{1}{2} \sum_{\substack{m,n \in \mathbb{Z} \\ (m,n)\neq 0}} \chi(n) \frac{y^s}{|mNq^2 z+n|^{2s}}, \end{aligned} \tag{7}$$

where $L(s,\chi)$ is the Dirichlet series $L(s,\chi)=\sum_{n=1}^\infty \chi(n)n^{-s}$.

Lemma 6. If $\mathrm{Re}(s)>1/2$ and $r \in \mathbb{R}$, then

$$\int_{-\infty}^{\infty} \frac{e^{-2\pi i r x}}{(x^2+y^2)^s} \mathrm{d}x = \frac{\pi^s}{y^s \Gamma(s)} \begin{cases} \pi^{-s+\frac{1}{2}} \Gamma(s-\frac{1}{2}) y^{1-s} & \text{if } r=0 \\ 2|r|^{s-\frac{1}{2}} \sqrt{y} K_{s-\frac{1}{2}}(2\pi|r|y) & \text{if } r \neq 0 \end{cases}. \tag{8}$$

Proof. Recall the integral representation of the Gamma function

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} \mathrm{d}t.$$

Thus we have

$$\begin{aligned} \frac{y^s}{\pi^s}\Gamma(s) \int_{-\infty}^{\infty} \frac{e^{-2\pi i r x}}{(x^2+y^2)^s} \mathrm{d}x &= \int_0^\infty \int_{-\infty}^{\infty} e^{-t} \left(\frac{ty}{\pi(x^2+y^2)} \right)^s e^{-2\pi i r x} \mathrm{d}x \frac{\mathrm{d}t}{t} \\ &= \int_0^\infty \int_{-\infty}^{\infty} e^{-\pi\xi(x^2+y^2)/y} \xi^s e^{-2\pi i r x} \mathrm{d}x \frac{\mathrm{d}\xi}{\xi}. \end{aligned}$$

For $r=0$ the above expression becomes

$$\begin{aligned} \int_0^\infty \int_{-\infty}^{\infty} e^{-\pi\xi(x^2+y^2)/y} \xi^s \mathrm{d}x \frac{\mathrm{d}\xi}{\xi} &= \int_0^\infty \sqrt{\frac{y}{\xi}} e^{-\pi\xi y} \xi^s \frac{\mathrm{d}\xi}{\xi} \\ &= \pi^{-s+\frac{1}{2}} y^{1-s} \Gamma(1-\frac{1}{2}). \end{aligned}$$

For $r \neq 0$ we obtain, using the change of variables $\xi \mapsto \frac{\xi}{|r|}$,

$$\begin{aligned} \int_0^\infty \int_{-\infty}^{\infty} e^{-\pi\xi(x^2+y^2)/y} \xi^s e^{-2\pi i r x} \mathrm{d}x \frac{\mathrm{d}\xi}{\xi} &= \int_0^\infty \sqrt{\frac{y}{\xi}} e^{-y\pi r^2/\xi} e^{-\pi\xi y} \xi^s \frac{\mathrm{d}\xi}{\xi} \\ &= \sqrt{y}|r|^{s-\frac{1}{2}} \int_0^\infty \xi^{t-\frac{1}{2}} e^{-y\pi|r|(1/\xi+\xi)} \frac{\mathrm{d}\xi}{\xi} \\ &= 2|r|^{s-\frac{1}{2}} \sqrt{y} K_{s-\frac{1}{2}}(2\pi|r|y). \end{aligned}$$

Here we used the following integral representation of the modified Bessel function $K_\nu(z)$

$$K_\nu(z) = \frac{1}{2} \int_0^\infty e^{-\frac{z}{2}(t+t^{-1})} t^{-s-1} \mathrm{d}t.$$

□

Lemma 7. For $z = x + iy \in \mathbb{H}$ and $\mathrm{Re}(s) > \frac{1}{2}$ we have

$$\sum_{n\in\mathbb{Z}} \frac{1}{|z+n|^{2s}} = \frac{\sqrt{\pi}\Gamma(s-\frac{1}{2})}{\Gamma(s)} y^{1-2s} + \frac{2\pi^s}{\Gamma(s)} y^{\frac{1}{2}-s} \sum_{r\neq 0} |r|^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|r|y)\, e^{2\pi irx}. \tag{9}$$

Proof. Recall the Poisson summation formula

$$\sum_{n\in\mathbb{Z}} \varphi(x+n) = \sum_{n\in\mathbb{Z}} \widehat{\varphi}(n) e^{2\pi inx},$$

where φ is a continuous function that decays rapidly at infinity and where $\widehat{\varphi}(n) = \int_{-\infty}^{\infty} \varphi(x) e^{-2\pi inx} \mathrm{d}x$ is the Fourier transform. We apply this formula to $\varphi(x) = |x+iy|^{-2s}$, where $z = x+iy \in \mathbb{H}$ and $\mathrm{Re}(s) > \frac{1}{2}$. The Poisson summation formula gives

$$\sum_{n\in\mathbb{Z}} \frac{1}{|z+n|^{2s}} = \sum_{n\in\mathbb{Z}} \left(\int_{-\infty}^{\infty} \frac{e^{-2\pi inx}}{(x^2+y^2)^s} \mathrm{d}x \right) e^{2\pi inx}.$$

Using Lemma 6 we get the result. □

We would like to adapt the left hand side of Equation 9 to the setting where the sum is twisted by a primitive Dirichlet character χ mod q. For this we need the twisted variant of the Poisson summation formula.

Lemma 8. Let χ be a primitive Dirichlet character mod q. Then

$$\sum_{n\in\mathbb{Z}} \chi(n)\varphi(x+n) = \frac{\tau(\chi)}{q} \sum_{n\in\mathbb{Z}} \overline{\chi(n)} \widehat{\varphi}(n/q) e^{2\pi inx/q},$$

where $\tau(\chi) = \sum_{n \bmod q} \chi(n) e^{2\pi in/q}$ is the Gauss sum attached to χ.

Proof. One can prove the identity

$$\chi(n) = \frac{1}{\tau(\overline{\chi})} \sum_{m \bmod q} \overline{\chi(m)} e^{2\pi inm/q} = \frac{\chi(-1)\tau(\chi)}{q} \sum_{m \bmod q} \overline{\chi(m)} e^{2\pi inm/q}.$$

Let us now consider $\sum_{n\in\mathbb{Z}} \chi(n)\varphi(n) = \sum_{n\in\mathbb{Z}} \varphi_1(n)$, where

$$\varphi_1(x) = \frac{\chi(-1)\tau(\chi)}{q} \sum_{m \bmod q} \overline{\chi(m)} e^{2\pi ixm/q} \varphi(x).$$

The Fourier transform of $\varphi_1(x)$ is

$$\begin{aligned} \widehat{\varphi_1}(\xi) &= \frac{\chi(-1)\tau(\chi)}{q} \sum_{m \bmod q} \overline{\chi(m)} \int_{-\infty}^{\infty} \varphi(x) e^{-2\pi ix(\xi - \frac{m}{q})} \mathrm{d}x \\ &= \frac{\chi(-1)\tau(\chi)}{q} \sum_{m \bmod q} \overline{\chi(m)} \widehat{\varphi}\left(\xi - \frac{m}{q}\right). \end{aligned}$$

We apply Poisson summation formula and thus $\sum_{n\in\mathbb{Z}} \varphi_1(n)$ equals

$$\frac{\chi(-1)\tau(\chi)}{q} \sum_{m \bmod q} \sum_{n=-\infty}^{\infty} \overline{\chi(m)} \widehat{\varphi}\left(\frac{nq-m}{q}\right).$$

We can write $\overline{\chi(m)} = \chi(-1)\overline{\chi(nq-m)}$. When n runs through $\mathbb{Z}$ and m through $\mathbb{Z}/q\mathbb{Z}$, the terms $nq-m$ run uniquely through $\mathbb{Z}$. Thus we have shown

$$\sum_{n\in\mathbb{Z}} \chi(n)\varphi(n) = \frac{\tau(\chi)}{q} \sum_{n\in\mathbb{Z}} \overline{\chi(n)} \widehat{\varphi}(n/q).$$

Replacing $\varphi(n)$ by $\varphi(n+x)$ replaces $\widehat{\varphi}(\xi)$ by $\widehat{\varphi}(\xi) e^{2\pi i\xi x}$. This completes the proof. □

We can now determine the Fourier expansion of $E^*(z,s,\chi)$.

Theorem 9. *The function $E^*(z,s,\chi)$ has the Fourier expansion*

$$\begin{aligned}E^*(z,s,\chi) &= L(2s,\chi)y^s \\ &\quad + \tau(\chi)q^{-2s}\sqrt{y}\frac{2\pi^s}{\Gamma(s)}\sum_{k\neq 0}|k|^{\frac{1}{2}-s}\sigma_{2s-1}(|k|,\overline{\chi})K_{s-\frac{1}{2}}(2\pi|k|y)e^{2\pi ikx},\end{aligned}$$

where $\sigma_s(k,\overline{\chi}) = \sum_{d|k}\overline{\chi(d)}d^s$.

Proof. We split up the sum (7) into the terms with $m=0$ and those with $m\neq 0$. We also use the evenness of χ to combine each positive summand with its negative. We obtain

$$E^*(z,s,\chi) = L(2s,\chi)y^s + \sum_{m=1}^{\infty}\sum_{n\in\mathbb{Z}}\chi(n)\frac{y^s}{|mNq^2z+n|^{2s}}.$$

Substituting into this the twisted variant of the formula of Lemma 7 gives (note that $\chi(0)=0$)

$$\begin{aligned}E^*(z,s,\chi) &= L(2s,\chi)y^s \\ &\quad + \frac{\tau(\chi)}{q}y^s\frac{2\pi^s}{\Gamma(s)}\sum_{m=1}^{\infty}\sum_{n\neq 0}\overline{\chi(n)}\left(\frac{|n|}{q}\right)^{s-\frac{1}{2}}\left(ymNq^2\right)^{\frac{1}{2}-s}K_{s-\frac{1}{2}}(2\pi|n|ymNq)e^{2\pi inxmNq}.\end{aligned}$$

Summing $m\in Nq\mathbb{Z}_{\geq 1}$ and $n\in\mathbb{Z}_{\geq 1}$ is the same as summing their product k over $\mathbb{Z}_{\geq 1}$ and summing for each k over the pairs (n,mNq) such that $Nqmn=k$. Accordingly we can write

$$\begin{aligned}E^*(z,s,\chi) &= L(2s,\chi)y^s \\ &\quad + \tau(\chi)q^{-2s}\sqrt{y}\frac{2\pi^s}{\Gamma(s)}\sum_{m=1}^{\infty}\sum_{n\neq 0}\overline{\chi(n)}|n|^{s-\frac{1}{2}}(mNq)^{\frac{1}{2}-s}K_{s-\frac{1}{2}}(2\pi|n|ymNq)e^{2\pi inxmNq} \\ &= L(2s,\chi)y^s \\ &\quad + \tau(\chi)q^{-2s}\sqrt{y}\frac{2\pi^s}{\Gamma(s)}\sum_{k\neq 0}|k|^{\frac{1}{2}-s}\sigma_{2s-1}(|k|,\overline{\chi})K_{s-\frac{1}{2}}(2\pi|k|y)e^{2\pi ikx},\end{aligned}$$

where $\sigma_s(k,\chi) = \sum_{d|k}\chi(d)d^s$ is the twisted divisor function. □

A consequence of the above Fourier expansion is the meromorphic continuation in s of the twisted Eisenstein series. To derive a functional equation for the twisted Eisenstein series we calculate the Fourier expansion of $E^*\left(\frac{-1}{q^2Nz},s,\chi\right)$.

Lemma 10. The function $E^*\left(\frac{-1}{q^2Nz},s,\chi\right)$ has the Fourier expansion

$$\begin{aligned}E^*\left(\frac{-1}{q^2Nz},s,\chi\right) &= \frac{1}{q^{2s}N^s}L(2s-1,\chi)\frac{\sqrt{\pi}\Gamma(s-\frac{1}{2})}{\Gamma(s)}y^{1-s} \\ &\quad + \frac{\sqrt{y}}{q^{2s}N^s}\frac{2\pi^s}{\Gamma(s)}\sum_{r\neq 0}|r|^{s-\frac{1}{2}}\sigma_{1-2s}(r,\chi)K_{s-\frac{1}{2}}(2\pi|r|y)e^{2\pi irx}.\end{aligned}$$

Proof. We will use the fact that the matrix $\omega = \begin{pmatrix}0 & -1\\ Nq^2 & 0\end{pmatrix}$ normalizes the group $\Gamma_0(Nq^2)$. Indeed, we have

$$\omega\begin{pmatrix}a & b\\ c & d\end{pmatrix}\omega^{-1} = \begin{pmatrix}d & -\frac{c}{Nq^2}\\ -bNq^2 & a\end{pmatrix}.$$

We now compute

$$
\begin{aligned}
E^*\left(\frac{-1}{q^2Nz}, s, \chi\right) &= \frac{L(2s,\chi)}{2} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(Nq^2)} \chi(d)\,\mathrm{Im}(\gamma\omega z)^s \\
&= \frac{L(2s,\chi)}{2} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(Nq^2)} \chi(\omega\gamma\omega^{-1})\,\mathrm{Im}(\omega\gamma z)^s \\
&= \frac{y^s}{q^{2s}N^s} \sum_{a=1}^{\infty}\sum_{b\in\mathbb{Z}} \frac{\chi(a)}{|az+b|^{2s}} \\
&= \frac{y^s}{q^{2s}N^s} \sum_{a=1}^{\infty} \frac{\chi(a)}{a^{2s}} \sum_{d=1}^{a}\sum_{m\in\mathbb{Z}} \left|m+z+\frac{d}{a}\right|^{-2s} \\
&= \frac{1}{q^{2s}N^s} L(2s-1,\chi) \frac{\sqrt{\pi}\Gamma(s-\frac{1}{2})}{\Gamma(s)} y^{1-s} \\
&+ \frac{\sqrt{y}}{q^{2s}N^s}\frac{2\pi^s}{\Gamma(s)} \sum_{r\neq 0} |r|^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|r|y) e^{2\pi i r x} \sum_{a=1}^{\infty}\frac{\chi(a)}{a^{2s}} \sum_{d=1}^{a} e^{2\pi i r\frac{d}{a}}.
\end{aligned}
$$

Because

$$
\sum_{d=1}^{a} e^{2\pi i r \frac{d}{a}} = \begin{cases} a & \text{if } a|r \\ 0 & \text{else} \end{cases}
$$

we have

$$
\sum_{a=1}^{\infty} \frac{\chi(a)}{a^{2s}} \sum_{d=1}^{a} e^{2\pi i r \frac{d}{a}} = \sum_{a|r} \frac{\chi(a)}{a^{2s-1}} = \sigma_{1-2s}(r,\chi).
$$

Hence we obtain the claimed Fourier expansion. □

The functional equation for the twisted Eisenstein series takes a simple form when we define

$$
E^{\#}(z,s,\chi) := \pi^{-s}\Gamma(s)E^*(z,s,\chi) = \pi^{-s}\Gamma(s)L(2s,\chi)E(z,s,\chi).
$$

Theorem 11. *The function $E^{\#}(z,s,\chi)$ satisfies the functional equation*

$$
E^{\#}(z,s,\chi) = \frac{\tau(\chi)}{\sqrt{q}} N^{1-s} q^{\frac{5}{2}-4s} E^{\#}\left(\frac{-1}{q^2Nz}, 1-s, \overline{\chi}\right).
$$

Proof. This follows from the above Fourier expansions, together with the functional equation $\xi(s,\chi) = \frac{\tau(\chi)}{\sqrt{q}}\xi(1-s,\overline{\chi})$ for the completed Dirichlet L-series $\xi(s,\chi) = \left(\frac{q}{\pi}\right)^{\frac{s}{2}}\Gamma(s/2)L(s,\chi)$, and the symmetry $K_\nu(z) = K_{-\nu}(z)$ of the K-Bessel function.

□

2.3 Hecke operators for $L^2(\Gamma_0(N)\setminus\mathbb{H})$

Let $\Gamma_0(N) = \left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \Gamma \,|\, c \equiv 0 \pmod N\right\}$. The Hecke operator $T_n : L^2(\Gamma_0(N)\setminus\mathbb{H}) \to L^2(\Gamma_0(N)\setminus\mathbb{H})$ is defined by

$$
(T_nf)(z) = \frac{1}{\sqrt{n}} \sum_{ad=n}\sum_{b=0}^{d-1} f\left(\frac{az+b}{d}\right).
$$

Given two positive integers n and m, the Hecke operators T_n and T_m commute. In fact, we have the multilicative rule

$$
T_mT_n = \sum_{d|(m,n)} T_{mn/d^2}.
$$

Let $f(z) \in \mathcal{C}(\Gamma_0(N) \setminus \mathbb{H})$ be a Maass cusp form with eigenvalue $1/4 + \nu^2$. It has a Fourier expansion of the type

$$f(z) = \sum_{m \neq 0} a(m) y^{1/2} K_\nu(2\pi|m|y) e(mx).$$

The action of T_n on $f(z)$ is

$$(T_n f)(z) = \sum_{m \neq 0} t_n(m) y^{1/2} K_\nu(2\pi|m|y) e(mx),$$

where $t_n(m) = \sum_{d|(m,n)} a(mn/d^2)$.

The Hecke operator T_n, if $(n, N) = 1$, is self-adjoint in $L^2(\Gamma_0(N) \setminus \mathbb{H})$, that is

$$\langle T_n f, g \rangle = \langle f, T_n g \rangle, \quad \text{if } (n, N) = 1.$$

Therefore, for each n, in the space $\mathcal{C}(\Gamma_0(N) \setminus \mathbb{H})$ of cusp forms an orthogonal basis $\{u_j\}$ can be chosen which consists of simultaneous eigenfunctions for T_n, that is,

$$T_n u_j = \lambda_j(n) u_j.$$

The Fourier coefficients $a_j(m)$ are proportional to the Hecke eigenvalues $\lambda_j(m)$. More precisely,

$$\lambda_j(m) a_j(1) = a_j(m).$$

2.4 Rankin-Selberg convolution for $\Gamma_0(N)$

3 Other things

3.1 Rankin-Selberg method

Let f be a function on $\Gamma \setminus \mathbb{H}$, of sufficient rapid decay. Then the scalar product of f with an Eisenstein series is equal to an integral transform of the constant term in the Fourier expansion of f. More precisely, for $\mathrm{Re}(s)$ sufficiently large,

$$\begin{aligned}
\langle f, E(\cdot, s) \rangle &= \int_{\Gamma\setminus\mathbb{H}} f(z) \sum_{\gamma \in \Gamma_\infty \setminus \Gamma} \mathrm{Im}(\gamma z)^s \mathrm{d}\mu(z) \\
&= \int_{\Gamma_\infty \setminus \mathbb{H}} f(z) \, \mathrm{Im}(z)^s \mathrm{d}\mu(z) \\
&= \int_0^\infty \left(\int_0^1 f(x + iy) \mathrm{d}x \right) y^{s-2} \mathrm{d}y.
\end{aligned}$$

3.1.1 Example

Let $\{f(z)\}$ be a orthonormal basis of cuspidal Maass forms for the discrete spectrum. These have Fourier expansions of the type

$$f_j(z) = \sum_{m \neq 0} a_j(m) y^{1/2} K_{\nu_j}(2\pi|m|y) e(mx).$$

Recall that $f_j(z)$ is an eigenfunction for the Laplacian with eigenvalue $1/4 - \nu_j^2$. The constant term $A_j(0)$ of $|f_j(z)|^2 = f_j(z)\overline{f_j(z)}$ is given by

$$A_j(0) = y \sum_{m \neq 0} |a_j(m)|^2 K_{\nu_j}(2\pi|m|y)^2.$$

Integrating $|f_j(z)|^2$ against an Eisenstein series gives, for $\mathrm{Re}(s) > 1$,

$$\begin{aligned}
\int_{\Gamma\setminus\mathbb{H}} |f_j(z)|^2 E(z, s) \mathrm{d}\mu(z) &= \int_0^\infty y^{s-1} \sum_{m \neq 0} |a_j(m)|^2 K_{\nu_j}(2\pi|m|y)^2 \mathrm{d}y \\
&= \sum_{m \neq 0} |a_j(m)|^2 \int_0^\infty y^{s-1} K_{\nu_j}(2\pi|m|y)^2 \mathrm{d}y.
\end{aligned}$$

The Mellin transform of $K_{\nu_j}(2\pi|m|y)^2$ is given by

$$\int_0^\infty y^{s-1}K_{\nu_j}(2\pi|m|y)^2\mathrm{d}y = \frac{1}{8}(\pi|n|)^{-s}\Gamma(s)^{-1}\Gamma(s/2)^2\Gamma(s/2+\nu_j)\Gamma(s/2-\nu_j).$$

For each j we define the zeta-function $R_{f_j}(s)$ by

$$R_{f_j}(s) = \frac{\Gamma(s/2)^2}{8\pi^s\Gamma(s)}\Gamma(s/2+\nu_j)\Gamma(s/2-\nu_j)\sum_{m\neq 0}\frac{|a_j(m)|^2}{|n|^s}.$$

Thus $\int_{\Gamma\backslash\mathbb{H}}|f_j(z)|^2E(z,s)\mathrm{d}\mu(z) = R_{f_j}(s)$. The analytic properties of R_{f_j} (meromorphic continuation, functional equation, position of poles) are inherited from the corresponding properties of the Eisenstein series $E(z,s)$.