Contents

1 Dirichlet characters

Let m be a positive integer. A Dirichlet character of modulus m is a function $\chi:\mathbb{Z}\to\mathbb{C}$ such that

- $\chi(ab)=\chi(a)\chi(b)$ for all $a,b\in\mathbb{Z}$,
- $\chi(a)=0$ if and only if $(a,m)>1$,
- If $a\equiv b \pmod m$, then $\chi(a)=\chi(b)$.

Let χ_1 and χ_2 be Dirichlet characters of modulus m_1 and m_2, respectively, with $m_1|m_2$. If $\chi_2(a)=\chi_1(a)$ for $a\in(\mathbb{Z}/m_2\mathbb{Z})^\times$, then χ_2 is said to be induced by χ_1. A Dirichlet character is called primitive if it is not induced by any Dirichlet character other than itself. A Dirichlet character induced by the identity function is called principal. We denote a principal Dirichlet character by χ_0. By definition a principal Dirichlet character mod m is

$$\chi_0(a)=\begin{cases}1 & \text{if } (a,m)=1\\ 0 & \text{else}\end{cases}. \tag{1}$$

If p is prime, then every nonprincipal Dirichlet character of modulus p is primitive. We call a Dirichlet character even if $\chi(-1)=1$, odd if $\chi(-1)=-1$.

Lemma 1. *Let χ be a nonprincipal Dirichlet character modulo a prime q. Then we have*

$$\sum_{a=1}^{q}\chi(a)=0.$$

Proof. Because χ is not principal, there is an integer b such that $\chi(b)\neq\{0,1\}$. Furthermore, the sum may be restricted to the terms with $(a,q)=1, 1\leq a\leq q$. Multiplication by b is a bijection $(\mathbb{Z}/q\mathbb{Z})^\times\to(\mathbb{Z}/q\mathbb{Z})^\times$. Therefore we have

$$\chi(b)\sum_{a=1}^{q}\chi(a)=\sum_{a\in(\mathbb{Z}/q\mathbb{Z})^\times}\chi(ab)=\sum_{c\in(\mathbb{Z}/q\mathbb{Z})^\times}\chi(c),$$

which implies $\sum_{a=1}^{q} \chi(a) = 0$. ◻

The sum of the n-th roots of unity is zero. We will investigate more closely the twist of this sum by a Dirichlet character mod q, a so-called Gauss sum. More precisely, the Gauss sum $\tau(\chi)$ attached to a primitive Dirichlet character χ mod q is

$$\tau(\chi) = \sum_{a \bmod q} \chi(a) e^{2\pi i a/q}.$$

Lemma 2. *Let χ be a nonprincipal Dirichlet character modulo a prime q. Then for all $n \in \mathbb{Z}$ we have*

$$\tau(\chi)\overline{\chi(n)} = \sum_{a=1}^{q} \chi(a) e^{2\pi i n a/q}.$$

Proof. See the second half of the proof of Lemma 4. ◻

Lemma 3. *Let χ be a primitive Dirichlet character modulo a prime q. Then*

$$|\tau(\chi)| = \sqrt{q}.$$

Proof. By Lemma 2 we have

$$\begin{aligned} |\tau(\chi)|^2 &= \sum_{a=1}^{q} \overline{\chi(a)} e^{-2\pi i a/q} \tau(\chi) \\ &= \sum_{a=1}^{q} e^{-2\pi i a/q} \left(\sum_{b=1}^{q} \chi(b) e^{2\pi i a b/q} \right) \\ &= \sum_{b=1}^{q} \chi(b) \left(\sum_{a=1}^{q} e^{2\pi i a(b-1)/q} \right). \end{aligned}$$

If $b = 1$, the inner sum equals q. If $b \neq 1$, the inner sum is zero. Therefore we obtain

$$|\tau(\chi)|^2 = \chi(1) q = q.$$

◻

We will also need the following variant of Lemma 2.

Lemma 4. *Let χ be a nonprincipal Dirichlet character modulo a prime q, and let $n, m \in \mathbb{Z}$. Then*

$$\sum_{a=1}^{nq^2} \chi(a) e^{2\pi i m a/nq^2} = \begin{cases} qn\tau(\chi)\overline{\chi}(l) & \text{if } m = lnq \text{ for some } l \in \mathbb{Z} \\ 0 & \text{else} \end{cases}. \tag{2}$$

Proof. Every a in the above sum is of the form $a = a_1 + tq$ with $1 \leq a_1 \leq q$ and $0 \leq t < nq$. Thus we can write

$$\sum_{a=1}^{nq^2} \chi(a) e^{2\pi i m a/nq^2} = \sum_{a_1=1}^{q} \chi(a_1) e^{\frac{2\pi i m a_1}{nq^2}} \sum_{t=0}^{nq-1} e^{\frac{2\pi i m t}{nq}}.$$

The sum over t is zero unless $m = lnq$ for some $l \in \mathbb{Z}$ in which case the sum is nq. Therefore

$$\begin{aligned} \sum_{a=1}^{nq^2} \chi(a) e^{2\pi i m a/nq^2} &= nq \sum_{\substack{a \bmod q \\ m = lnq}} \chi(a) e^{\frac{2\pi i m a}{nq^2}} \\ &= nq \sum_{a \bmod q} \chi(a) e^{\frac{2\pi i l a}{q}} \end{aligned}$$

If $(l, q) = 1$, then $a \mapsto al$ permutes the residues mod q. In this case we get

$$nq \sum_{a \bmod q} \chi(a) e^{\frac{2\pi i l a}{q}} = nq\tau(\chi)\overline{\chi(l)}.$$

Now suppose that $(l, q) > 1$. Then $\chi(l) = 0$ and we have to show that the left side of Equation 2 vanishes. For this let $l' \in \mathbb{Z}$ be such that $ql' = l$. Then we have

$$\begin{aligned} nq \sum_{a \bmod q} \chi(a) e^{\frac{2\pi i l a}{q}} &= nq \sum_{a \bmod q} \chi(a) e^{2\pi i l' a} \\ &= nq \sum_{a \bmod q} \chi(a). \end{aligned}$$

The last sum is zero by Lemma 1. This completes the proof of the lemma.

$\square$

2 $SL(2,\mathbb{Z})$

2.1 Eisenstein series and the spectral decomposition of $L^2(\Gamma \setminus \mathbb{H})$

Let $\Gamma = \mathrm{SL}(2,\mathbb{Z})$. This group has only a cusp at infinity. The stabilizer of the cusp ∞ in Γ is

$$\Gamma_\infty = \left\{ \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \mid m \in \mathbb{Z} \right\}.$$

The Eisenstein series associated to this cusp, defined on $\mathbb{H} \times \mathbb{C}$, is defined by

$$E(z,s) := E_\infty(z,s) = \frac{1}{2} \sum_{\gamma \in \Gamma_\infty \setminus \Gamma} \mathrm{Im}(\gamma z)^s. \tag{3}$$

Notice that $\mathrm{Im}(z)$ is Γ_∞-invariant, and the Eisenstein series $E(z,s)$ defines an automorphic function with respect to Γ, that is, it satisfies $E(\gamma z, s) = E(z,s)$ for all $\gamma \in \Gamma$. For $\mathrm{Re}(s) > 1$, this series converges absolutely and uniformly on compact sets. Because $\Delta y^s = s(1-s)y^s$ and because Δ commutes with the Γ-action, the Eisenstein series is also an eigenfunction of the Laplacian

$$\Delta E(z,s) = s(1-s)E(z,s).$$

Automorphic functions which are eigenfunctions of the Laplace operator are called Maass forms. The Fourier expansion of $E(z,s)$ is given by

$$E(z,s) = y^s + \phi(s)y^{1-s} + \frac{2\pi^s}{\Gamma(s)\zeta(2s)} y^{1/2} \sum_{n \neq 0} \sigma_{1-2s}(n)|n|^{s-\frac{1}{2}} K_{s-1/2}(2\pi|n|y)e^{2\pi i n x}, \tag{4}$$

where

$$\phi(s) = \pi^{1/2} \frac{\Gamma(s-1/2)}{\Gamma(s)} \frac{\zeta(2s-1)}{\zeta(2s)}.$$

The modified Bessel functions K_ν are exponentially decaying functions. In particular, $E(z,s) = y^s + \phi(s)y^{1-s} + O(e^{-cy})$, for some constant $c > 0$. For a function to be in $L^2(\Gamma \setminus \mathbb{H})$ its growth need to be $O(y^{1/2})$. From the constant term of the Fourier expansion we see that the Eisenstein series is not in $L^2(\Gamma \setminus \mathbb{H})$. But for $\mathrm{Re}(s) = \frac{1}{2}$ the Eisenstein series is "almost square integrable", and this suggests to work on the line $\mathrm{Re}(s) = \frac{1}{2}$. We now have to address two issues: We want to work with square integrable functions on $\Gamma \setminus \mathbb{H}$, and we need to meromorphically continue $E(z,s)$ to the line $\mathrm{Re}(s) = \frac{1}{2}$. The meromorphic continuation of $E(z,s)$ follows from the Fourier expansion. In the half-plane $\mathrm{Re}(s) \geq \frac{1}{2}$ there is only a simple pole at $s = 1$ with residue $\frac{3}{\pi}$. The Eisenstein series enjoys the functional equation

$$E(z, 1-s) = \phi(1-s)E(z,s).$$

For a smooth, compactly supported function ψ on $\mathbb{R}^{>0}$, the incomplete Eisenstein series is

$$E(z|\psi) = \sum_{\gamma \in \Gamma_\infty \setminus \Gamma} \psi(\mathrm{Im}(\gamma z)).$$

The incomplete Eisenstein series $E(z|\psi)$ lies in $C_c^\infty(\Gamma \setminus \mathbb{H}) \subset L^2(\Gamma \setminus \mathbb{H})$, but it is not an eigenfunction of Δ. By Mellin inversion it can however be represented as an integral of Eisenstein series

$$E(z|\psi) = \frac{1}{2\pi i} \int_{\mathrm{Re}(s)=\sigma} E(z,s)\widehat{\psi}(s)\mathrm{d}s, \tag{5}$$

for $\sigma > 1$ and where

$$\widehat{\psi}(s) = \int_0^\infty \psi(y)y^{-s-1}\mathrm{d}y,$$

is the Mellin transform of ψ.

We denote by $\mathcal{E}(\Gamma \setminus \mathbb{H}) \subset L^2(\Gamma \setminus \mathbb{H})$ the space spanned by the incomplete Eisenstein series $E(z|\psi)$. The inner product of a function $f \in L^2(\Gamma \setminus \mathbb{H})$ with an incomplete Eisenstein series $E(z|\psi)$ is

$$\langle f, E(\cdot|\psi)\rangle = \int_0^\infty f_0(y)\overline{\psi}(y)y^{-2}\mathrm{d}y,$$

where f_0 is the constant term in the Fourier expansion of f. If f is orthogonal to $\mathcal{E}(\Gamma \setminus \mathbb{H})$, then the above integral is zero for all smooth functions ψ of compact support in $(0, \infty)$. Thus the orthogonal complement $\mathcal{C} = \mathcal{E}^\perp$ consists of functions whose constant term in the Fourier series is zero. The Laplace operator Δ has discrete spectrum in $\mathcal{C}$ and $\mathcal{C}$ is spanned by cusp forms. For an orthonormal basis of cusp forms $\{u_j\}$ every $f \in \mathcal{C}(\Gamma \setminus \mathbb{H})$ has the spectral expansion

$$f(z) = \sum_{j\geq 1}\langle f, u_j\rangle u_j(z).$$

The spectrum of Δ in $\mathcal{E}(\Gamma\setminus\mathbb{H})$ turns out to consist of a continuous part spanned by the Eisenstein series $E(z, \frac{1}{2} + ir)$, and the zero eigenvalue corresponding to the constant function u_0.

For $\mathrm{Re}(s) \geq \frac{1}{2}$, the Eisenstein series has only a simple pole at $s = 1$. The Eisenstein series are holomorphic on the line $\mathrm{Re}(s) = \frac{1}{2}$ and are of polynomial growth in vertical strips $-\epsilon \leq \mathrm{Re}(s) \leq 1 + \epsilon$. One can then shift the integration in Equation 5 to the line $\mathrm{Re}(s) = \frac{1}{2}$, thereby picking up the residue of $E(z, s)$ at the pole $s = 1$. As a result we obtain

$$E(z|\psi) = \widehat{\psi}(1)\mathrm{Res}_{s=1}(E(z, s)) + \frac{1}{2\pi i}\int_{\mathrm{Re}(s)=\frac{1}{2}} E(z, s)\widehat{\psi}(s)\mathrm{d}s.$$

The term $\widehat{\psi}(1)$ can be written as $\frac{\langle E(\cdot|\psi), u_0\rangle}{\langle u_0, u_0\rangle}$. We still need the projection of $E(z|\psi)$ onto $E(z, s)$. The functional equation along with

$$\langle E(\cdot|\psi), E(\cdot, 1/2 + ir)\rangle = \widehat{\psi}(1/2 + ir) + \phi(1/2 - ir)\widehat{\psi}(1/2 - ir)$$

yields the spectral decomposition of $E(z|\psi)$ onto the Eisenstein series

$$\frac{1}{2\pi i}\int_{\mathrm{Re}(s)=\frac{1}{2}} E(z, s)\widehat{\psi}(s)\mathrm{d}s = \frac{1}{4\pi i}\int_{\mathrm{Re}(s)=\frac{1}{2}} \langle E(\cdot|\psi), E(\cdot, s)\rangle E(z, s)\mathrm{d}s.$$

In conclusion, for an orthonormal basis of cusp forms $\{u_j\}$, every $f \in L^2(\Gamma \setminus \mathbb{H})$ has the spectral expansion

$$f(z) = \sum_j\langle f, u_j\rangle u_j(z) + \frac{1}{4\pi i}\int_{\mathrm{Re}(s)=\frac{1}{2}} \langle f(\cdot), E(\cdot, s)\rangle E(z, s)\mathrm{d}s.$$

2.2 Basic properties of Maass forms

Let $\nu \in \mathbb{C}$. A Maass form of type ν for $\mathrm{SL}(2, \mathrm{Z})$ is a non-zero function on $\mathbb{H}$ which satisfies

- $f(\gamma z) = f(z)$ for all $\gamma \in \mathrm{SL}(2, \mathbb{Z})$ and $z \in \mathbb{H}$;
- $\Delta f = \nu(1 - \nu)f$;
- $f(x + iy) = O(y^M)$ for some $M > 0$ as $y \to \infty$.

If in addition

$$\int_0^1 f(x + iy)\mathrm{d}x = 0,$$

we call f a cuspidal Maass form or a Maass cusp form. It is not obvious that there should exists non-constant Maass cusp forms. But in fact one can prove that there are infinitely many Maass cusp forms, see for example Chapter 4 in [1] or Chapter 4 in [2].

2.2.1 The Fourier expansion of Maass forms

2.2.2 Even and Odd Maass forms

We define the operator $T_{-1} : L^2(\mathrm{SL}(2, \mathbb{Z}) \setminus \mathbb{H}) \to L^2(\mathrm{SL}(2, \mathbb{Z}) \setminus \mathbb{H})$ by

$$(T_{-1}f)(x + iy) = f(-x + iy).$$

2.3 Hecke operators for $L^2(\mathrm{SL}(2,\mathbb{Z})\setminus\mathbb{H})$

For $n \geq 1$, the Hecke operator $T_n : L^2(\mathrm{SL}(2,\mathbb{Z})\setminus\mathbb{H}) \to L^2(\mathrm{SL}(2,\mathbb{Z})\setminus\mathbb{H})$ is defined by

$$(T_n f)(z) = \frac{1}{\sqrt{n}} \sum_{ad=n} \sum_{b=0}^{d-1} f\left(\frac{az+b}{d}\right).$$

The Hecke operators T_n $(n = 1, 2, \dots)$ are self-adjoint in $L^2(\mathrm{SL}(2,\mathbb{Z})\setminus\mathbb{H})$, that is

$$\langle T_n f, g\rangle = \langle f, T_n g\rangle,$$

for all $f, g \in L^2(\mathrm{SL}(2,\mathbb{Z})\setminus\mathbb{H})$. Given two positive integers n and m, the Hecke operators T_n and T_m commute. In fact, we have the multilicative rule

$$T_m T_n = \sum_{d|(m,n)} T_{mn/d^2}.$$

Furthermore, the Hecke operators commute with T_{-1} and the Laplacian Δ, and T_{-1} and Δ also commute. It follows that the space $L^2(\mathrm{SL}(2,\mathbb{Z})\setminus\mathbb{H})$ can be simultaneously diagonalized by the operators $\{\Delta, T_n |\, n = -1, 1, 2, \dots\}$. Consequently, we can consider even or odd Maass forms that are simultaneous eigenfunctions for $\{T_n |\, n = 1, 2, \dots\}$. For such a Maass form, the following multiplicative relations of the Fourier coefficients hold.

Lemma 5. *Let*

$$f(z) = \sum_{n\neq 0} a(n)\sqrt{y} K_{\nu-\frac{1}{2}}(2\pi|n|y)e^{2\pi i n x}$$

be a cuspidal Maass form of type ν for $SL(2,\mathbb{Z})$ which is an eigenfunction of all the Hecke operators. Assume that f is normalized such that $a(1) = 1$. Then

$$T_n f = a(n) f, \qquad \forall n = 1, 2, \dots$$

Furthermore, we have the multiplicative relations of the Fourier coefficients

$$\begin{aligned} a(m)a(n) &= a(mn), \quad \text{if } (m,n) = 1,\\ a(m)a(n) &= \sum_{d|(m,n)} a\left(\frac{mn}{d^2}\right),\\ a(p^{k+1}) &= a(p)a(p^k) - a(p^{k-1}), \end{aligned}$$

for all primes p and all $k \geq 1$.

Proof. ... □

2.4 L-functions

2.4.1 L-functions associated to Maass forms

Let

$$f(z) = \sum_{n\neq 0} a(n)\sqrt{y} K_{\nu_f}(2\pi|n|y)e^{2\pi i n x} \tag{6}$$

be a cuspidal Maass form for $SL(2,\mathbb{Z})$.

Lemma 6. *The coefficients $a(n)$ in the Fourier expansion of $f(z)$ satisfy*

$$a(n) = O(\sqrt{|n|}).$$

Proof. Because f is a cusp form, it is bounded as $\mathrm{Im}(z) \to \infty$. Thus

$$\left| a(n)\sqrt{y}K_{\nu_f}(2\pi|n|y)\right| = \left|\int_0^1 f(x+iy)e^{-2\pi i n x}\mathrm{d}x\right| \leq \int_0^1 |f(x+iy)|\mathrm{d}x \leq C,$$

for some constant C (that depends on f). If we choose $y = \frac{1}{|n|}$, the lemma is proved. □

For $\mathrm{Re}(s) \geq \frac{3}{2}$ we define the L-function $L_f(s)$ associated to $f(z)$ by the absolutely convergent series

$$L_f(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}. \tag{7}$$

Lemma 7. *Let $f(z)$ be a cuspidal Maass form for $SL(2,\mathbb{Z})$ of parity ϵ. Then its L-function $L_f(s)$ can be meromorphically continued to all $\mathbb{C}$ and it satisfies the functional equation*

$$\Lambda_f(s) = \pi^{-s}\Gamma\left(\frac{s+\epsilon+\nu_f}{2}\right)\Gamma\left(\frac{s+\epsilon-\nu_f}{2}\right)L_f(s) = (-1)^{\epsilon}\Lambda_f(1-s).$$

Proof. The idea of the proof is to set $x = 0$ and take the Mellin transform of $f(z)$. We first assume that f is even. For sufficiently large $\mathrm{Re}(s)$ we have

$$\begin{aligned}
\int_0^{\infty} f(iy)y^s \frac{\mathrm{d}y}{y} &= 2\int_0^{\infty} \sum_{n=1}^{\infty} a(n)\sqrt{y}K_{\nu_f}(2\pi ny)y^s \frac{\mathrm{d}y}{y} \\
&= \frac{2}{(2\pi)^{s+\frac{1}{2}}} L_f(s+\frac{1}{2}) \int_0^{\infty} K_{\nu_f}(y)y^{s+\frac{1}{2}} \frac{\mathrm{d}y}{y} \\
&= \frac{1}{2\pi^{s+\frac{1}{2}}} L_f(s+\frac{1}{2})\Gamma\left(\frac{s+\frac{1}{2}+\nu_f}{2}\right)\Gamma\left(\frac{s+\frac{1}{2}-\nu_f}{2}\right),
\end{aligned}$$

where we used the Mellin transform of the K-Bessel function (see page 127 in [3])

$$\int_0^{\infty} K_{\nu}(y)y^s \frac{\mathrm{d}y}{y} = 2^{s-2}\Gamma\left(\frac{s+\nu}{2}\right)\Gamma\left(\frac{s-\nu}{2}\right).$$

The integrals converge absolutely and this gives the meromorphic continuation. Because f is automorphic, we have

$$f(iy) = f(S(iy)) = f(i/y).$$

It follows that

$$\begin{aligned}
\int_0^{\infty} f(iy)y^s \frac{\mathrm{d}y}{y} &= \int_0^{1} f(i/y)y^s \frac{\mathrm{d}y}{y} + \int_1^{\infty} f(iy)y^s \frac{\mathrm{d}y}{y} \\
&= \int_1^{\infty} f(iy)\left(y^s + y^{-s}\right)\frac{\mathrm{d}y}{y}.
\end{aligned}$$

The above integral is invariant under $s \mapsto -s$ and from this the functional equation follows immediately.

If f is odd, the above calculation does not quite work, because $\sum_{n\neq 0} a(n)|n|^{-s} = 0$ and so $\int_0^{\infty} f(iy)y^s \frac{\mathrm{d}y}{y} = 0$. However, we can deduce the functional equation in the same way as above once we replace in the above calculation $f(iy)$ by

$$\frac{\partial}{\partial x} f(z)\bigg|_{x=0} = 4\pi i \sum_{n=1}^{\infty} a(n)n\sqrt{y}K_{\nu_f}(2\pi ny).$$

□

Lemma 8. *Let f be a Maass cusp form as in Equation (6) which is an eigenfunction of all the Hecke operators and is normalized so that $a(1) = 1$. Then the L-function $L_f(s)$ admits the Euler product*

$$L_f(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} = \prod_p \left(1 - a(p)p^{-s} + p^{-2s}\right)^{-1},$$

where the product runs over all primes p.

Proof. It follows from the multiplicativity of the Fourier coefficients (Lemma 5) that

$$\sum_{n=1}^{\infty} \frac{a(n)}{n^s} = \prod_p \left(\sum_{l=0}^{\infty} \frac{a(p^l)}{p^{ls}} \right).$$

For a fixed prime p we have

$$\begin{aligned}
&\sum_{l=0}^{\infty} a(p^l) p^{-ls} \left(1 - a(p)p^{-s} + p^{-2s}\right) \\
&\qquad = \sum_{l=0}^{\infty} a(p^l)p^{-ls} - \sum_{l=1}^{\infty} a(p^{l-1})a(p)p^{-ls} + \sum_{l=2}^{\infty} a(p^{l-2})p^{-ls} \\
&\qquad = 1 + \sum_{l=2}^{\infty} \left(a(p^l) - a(p^{l-1})a(p) + a(p^{l-2})\right) p^{-ls} = 1,
\end{aligned}$$

where the last equality follows from Lemma 5. This shows that

$$\sum_{l=0}^{\infty} \frac{a(p^l)}{p^{ls}} = \left(1 - a(p)p^{-s} + p^{-2s}\right)^{-1}.$$

□

2.4.2 L-functions associated to Eisenstein series

By formula (4), the non-constant term in the Fourier expansion of the Eisenstein series $E(z, v)$ is given by

$$\frac{2\pi^v}{\Gamma(v)\zeta(2v)} y^{1/2} \sum_{n \neq 0} \sigma_{1-2v}(n) |n|^{v-\frac{1}{2}} K_{v-1/2}(2\pi|n|y) e^{2\pi i n x}.$$

We define the L-function $L_{E_v}(s)$ associated to the Eisenstein series $E(z, v)$ to be

$$L_{E_v}(s) = \sum_{n=1}^{\infty} \frac{\sigma_{1-2v}(n)\, n^{v-\frac{1}{2}}}{n^s}.$$

Lemma 9. *The L-function $L_{E_v}(s)$ is simply a product of shifted Riemann zeta functions*

$$L_{E_v}(s) = \zeta(s + v - 1/2)\, \zeta(s - v + 1/2).$$

Furthermore, we have the functional equation

$$G_{E_v}(s) = G_{E_v}(1 - s),$$

where

$$G_{E_v}(s) = \pi^{-s} \Gamma\left(\frac{s + v - \frac{1}{2}}{2}\right) \Gamma\left(\frac{s - v + \frac{1}{2}}{2}\right) \zeta(s + v - 1/2)\, \zeta(s - v + 1/2).$$

Proof. We calculate

$$\begin{aligned}
\sum_{n=1}^{\infty} \sigma_{1-2v}(n)\, n^{v-\frac{1}{2}-s} &= \sum_{n=1}^{\infty} n^{v-\frac{1}{2}-s} \sum_{d|n} d^{1-2v} \\
&= \sum_{n=1}^{\infty} \sum_{d=1}^{\infty} (dn)^{v-\frac{1}{2}-s} d^{1-2v} \\
&= \sum_{n=1}^{\infty} n^{v-\frac{1}{2}-s} \sum_{d=1}^{\infty} d^{\frac{1}{2}-v-s} \\
&= \zeta(s - v + 1/2)\, \zeta(s + v - 1/2).
\end{aligned}$$

Now consider

$$
\begin{aligned}
G_{E_v}(s) = &\ \pi^{-\frac{s+v-\frac{1}{2}}{2}} \Gamma\left(\frac{s+v-\frac{1}{2}}{2}\right) \zeta(s+v-1/2) \\
&\times \pi^{-\frac{s-v+\frac{1}{2}}{2}} \Gamma\left(\frac{s-v+\frac{1}{2}}{2}\right) \zeta(s-v+1/2).
\end{aligned}
$$

The functional equation $G_{E_v}(s) = G_{E_v}(1-s)$ follows directly from the functional equation

$$
\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{-\frac{1-s}{2}} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s) \tag{8}
$$

of the Riemann zeta function.

□

As expected, the functional equation of $L_{E_v}(s)$ matches the functional equation of a Maass form of even parity (as in Lemma 7). In fact, the functional equation of a Maass form of type ν and of even parity is identical to the functional equation of $L_{E_\nu}(s)$. The reason is that the K-Bessel functions in both Fourier expansions coincide and the proof of the functional equation in Lemma 7 only uses the analytical properties of the K-Bessel function, and does not depend on the arithmetic Fourier coefficients. This gives a method of obtaining functional equations for Maass forms from studying the functional equation of Eisenstein series.

Let χ be an even primitive Dirichlet character mod a prime q. The twisted L-function associated to the Eisenstein series $E(z, v)$ is

$$
L_{E_v}(s, \chi) = L(s+v-1/2, \chi)\, L(s-v+1/2, \chi),
$$

with the Dirichlet L-function $L(s, \chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s}$. The function $L(s, \chi)$ satisfies the functional equation (reference)

$$
\Lambda(s, \chi) = \left(\frac{q}{\pi}\right)^{\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) L(s, \chi) = \frac{\tau(\chi)}{\sqrt{q}}\ \Lambda(1-s, \overline{\chi}). \tag{9}
$$

From this it follows that the L-function $L_{E_v}(s, \chi)$ satisfies the functional equation

$$
\begin{aligned}
\Lambda_{E_v}(s, \chi) &= \left(\frac{q}{\pi}\right)^{s} \Gamma\left(\frac{s+v-1/2}{2}\right) \Gamma\left(\frac{s-v+1/2}{2}\right) L_{E_v}(s, \chi) \\
&= \frac{\tau(\chi)}{\sqrt{q}} \Lambda_{E_v}(1-s, \overline{\chi}).
\end{aligned}
$$

By the remark after Lemma 9, the twisted L-function of any even Maass form of type ν satisfies the functional equation $\Lambda_{E_v}(s, \chi) = \frac{\tau(\chi)}{\sqrt{q}} \Lambda_{E_v}(1-s, \overline{\chi})$.

2.5 Rankin-Selberg convolution for $\mathrm{SL}(2, \mathbb{Z})$

Let

$$
f(z) = \sum_{n \neq 0} a(n) \sqrt{y} K_{\nu_f}(2\pi |n| y) e^{2\pi i n x}, \tag{10}
$$

$$
g(z) = \sum_{n \neq 0} b(n) \sqrt{y} K_{\nu_g}(2\pi |n| y) e^{2\pi i n x}, \tag{11}
$$

be cuspidal Maass forms for $SL(2, \mathbb{Z})$. Recall that $f(z)$ (resp. $g(z)$) is an eigenfunction for the Laplacian with eigenvalue $1/4 - \nu_f$ (resp. $1/4 - \nu_g$). For sufficiently large $\mathrm{Re}(s)$ we define the Rankin-Selberg convolution $L_{f \times g}(s)$ as the absolutely convergent series

$$
L_{f \times g}(s) = \zeta(2s) \sum_{n=1}^{\infty} \frac{a(n)\overline{b(n)}}{n^s}.
$$

We will prove that $L_{f \times g}$ can be expressed as an inner product of $f\overline{g}$ with an Eisenstein series. This construction gives the meromorphic continuation and functional equation for $L_{f \times g}$.

Theorem 10. *Let $f(z)$ and $g(z)$ be cuspidal Maass forms as in Equation* 10 *and* 11. *Then $L_{f\times g}$ can be meromorphically continued to all $s\in\mathbb{C}$ with at most a simple pole at $s=1$. Furthermore, we have the functional equation*

$$L^*_{f\times g}(s)=\pi^{-2s}G_{\nu_f,\nu_g}(s)L_{f\times g}(s)=L^*_{f\times g}(1-s),$$

where $G_{\nu_f,\nu_g}(s)=\prod\Gamma\left(\frac{s\pm\nu_f\pm\nu_g}{2}\right)$.

Proof. Let $E(z,s)$ be the non-holomorphic Eisenstein series as defined in Equation 3. For sufficiently large $\mathrm{Re}(s)$, we have

$$\begin{aligned}
\zeta(2s)\langle f\overline{g},E(\cdot,\overline{s})\rangle &= \zeta(2s)\int_{SL(2,\mathbb{Z})\backslash\mathbb{H}} f(z)\overline{g(z)E(z,\overline{s})}\mathrm{d}\mu(z)\\
&= \zeta(2s)\sum_{\gamma\in\Gamma_\infty\backslash SL(2,\mathbb{Z}}\int_{SL(2,\mathbb{Z})\backslash\mathbb{H}} f(\gamma z)\overline{g(\gamma z)}\,\mathrm{Im}(\gamma z)^s\mathrm{d}\mu(\gamma z)\\
&= \zeta(2s)\int_{\Gamma_\infty\backslash\mathbb{H}} f(z)\overline{g(z)}\,\mathrm{Im}(z)^s\mathrm{d}\mu(z)\\
&= \zeta(2s)\int_0^\infty\int_0^1 f(z)\overline{g(z)}y^{s-2}\mathrm{d}x\mathrm{d}y\\
&= \zeta(2s)\sum_{n,m\neq 0}a(n)\overline{b(m)}\int_0^\infty K_{\nu_f}(2\pi|n|y)K_{\nu_g}(2\pi|m|y)y^{s-1}\int_0^1 e^{2\pi i(n-m)x}\mathrm{d}x\mathrm{d}y\\
&= \zeta(2s)\sum_{n\neq 0}a(n)\overline{b(n)}\int_0^\infty K_{\nu_f}(2\pi|n|y)K_{\nu_g}(2\pi|n|y)y^s\frac{\mathrm{d}y}{y}\\
&= \frac{\zeta(2s)}{(2\pi)^s}\sum_{n\neq 0}\frac{a(n)\overline{b(n)}}{n^s}\int_0^\infty K_{\nu_f}(y)K_{\nu_g}(y)y^s\frac{\mathrm{d}y}{y}\\
&= (2\pi)^{-s}\,L_{f\times g}(s)\int_0^\infty K_{\nu_f}(y)K_{\nu_g}(y)y^s\frac{\mathrm{d}y}{y}.
\end{aligned}$$

The Mellin transform of $K_\nu(y)K_{\nu'}(y)$ is given by (see page 145 in [3])

$$\int_0^\infty K_\nu(y)K_{\nu'}(y)y^s\frac{\mathrm{d}y}{y}=\frac{2^{s-3}}{\Gamma(s)}\prod\Gamma\left(\frac{s\pm\nu\pm\nu'}{2}\right), \tag{12}$$

which is valid for $\mathrm{Re}(s)>|\mathrm{Re}(\nu)|+|\mathrm{Re}(\nu')|$.

From the calculation it follows that the convolution function $L_{f\times g}$ inherits the analytical properties of the Eisenstein series $E(z,s)$. This means that $L_{f\times g}$ can be meromorphically continued on $\mathbb{C}$. Because the Eisenstein series has a simple pole at $s=1$ and the Gamma function no zeros, it follows that $L_{f\times g}$ has a simple pole at $s=1$ if and only if $\langle f,g\rangle\neq 0$. The functional equation follows from the functional equation of the Eisenstein series

$$E^*(z,s)=\pi^{-s}\Gamma(s)\zeta(2s)E(z,s)=E^*(z,1-s).$$

□

Lemma 11. *Let $\alpha_i,\beta_i\in\mathbb{C}$ for $i=1,2$. Then for $x\in\mathbb{C}$, $|x|$ sufficiently small, we have*

$$\prod_{i=1}^2\prod_{j=1}^2(1-\alpha_i\beta_jx)^{-1}=\sum_{k=0}^\infty S_k(\alpha_1,\alpha_2)S_k(\beta_1,\beta_2)x^k(1-\alpha_1\alpha_2\beta_1\beta_2x^2)^{-1},$$

where $S_k(x_1,x_2)$ is the Schur polynomial

$$S_k(x_1,x_2)=\frac{\det\begin{pmatrix}x_1^{k+1}&x_2^{k+1}\\1&1\end{pmatrix}}{\det\begin{pmatrix}x_1&x_2\\1&1\end{pmatrix}}=\frac{x_1^{k+1}-x_2^{k+1}}{x_1-x_2}.$$

Proof. This is proved on page 67 in [4] by evaluating the determinant of the matrix

$$\left(\frac{1}{1-\alpha_i\beta_j x}\right)_{1\le i,j\le 2}$$

in two different ways. □

Theorem 12. *Let $f(z)$ and $g(z)$ be cuspidal Maass forms as in Equation* 10 *and* 11*. Assume that $L_f(s)$ and $L_g(s)$ have Euler products*

$$L_f(s) = \prod_p \prod_{i=1}^{2} \left(1 - \frac{\alpha_{i,p}}{p^s}\right)^{-1}, \quad L_g(s) = \prod_p \prod_{j=1}^{2} \left(1 - \frac{\beta_{j,p}}{p^s}\right)^{-1}.$$

Then $L_{f\times g}(s)$ admits the Euler product

$$L_{f\times g}(s) = \prod_p \prod_{i=1}^{2} \prod_{j=1}^{2} \left(1 - \frac{\alpha_{i,p}\,\overline{\beta}_{j,p}}{p^s}\right)^{-1}.$$

Proof. By assumption we have

$$L_{f\times g}(s) = \prod_p \frac{\sum_{k=1}^{\infty} a(p^k)\overline{b(p^k)}p^{-ks}}{(1-p^{-2s})}.$$

In view of Lemma 11, after choosing $x = p^{-s}$, it suffices to show that

$$a(p^k) = S_k(\alpha_{1,p}, \alpha_{2,p}), \quad b(p^k) = S_k(\beta_{1,p}, \beta_{2,p}).$$

The above equalities are obtained inductively from the relations

$$\begin{aligned} 1 &= \alpha_{1,p}\,\alpha_{2,p}, \\ a(p) &= \alpha_{1,p} + \alpha_{2,p}, \\ a(p^{k+1}) &= a(p)a(p^k) - a(p^{k-1}). \end{aligned}$$

□

Following the remark after Lemma 9, the functional equation in Theorem 10 is identical to the functional equation of the Rankin-Selberg L-function of two Eisenstein series E_v and E_w of type v and w, respectively. We now verify this claim. By Lemma 9, the L-functions $L_{E_v}(s)$ and $L_{E_w}(s)$ are given by

$$\begin{aligned} L_{E_v}(s) &= \prod_{i=1}^{2} \zeta(s - \lambda_i(v)) \\ L_{E_w}(s) &= \prod_{i=1}^{2} \zeta(s - \lambda_i(w)), \end{aligned}$$

where $\lambda_1(v) = v - \frac{1}{2}$ and $\lambda_2(v) = -v + \frac{1}{2}$. According to Theorem 12, the Rankin-Selberg convolution of E_v and E_w is then simply

$$L_{E_v\times E_w}(s) = \prod_{i=1}^{2} \prod_{j=1}^{2} \zeta(s - \lambda_i(v) - \overline{\lambda_j(w)}).$$

It follows from the functional equation (8) for the Riemann Zeta function that the functional equation of $L_{E_v\times E_w}(s)$ is indeed the same as the one stated in Theorem 10. We also note that a corollary of the Rankin-Selberg convolution $L_{E_v\times E_w}(s)$ is the Ramanujan identity

$$\sum_{n=1}^{\infty} \frac{\sigma_a(s)\sigma_b(s)}{n^s} = \frac{\zeta(s)\zeta(s-a)\zeta(s-b)\zeta(s-a-b)}{\zeta(2s-a-b)}.$$

3 $\Gamma_0(N)$

Let $\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) \,|\, c \equiv 0 \pmod{N} \right\}$.

3.1 Twisted Maass Forms

Let

$$f(z) = \sum_{n\neq 0} a(n) y^{1/2} K_\nu(2\pi|n|y) e^{2\pi i n x}$$

be a cuspidal Maass form for $\mathrm{SL}(2,\mathbb{Z})$. Let χ be a primitive Dirichlet character mod a prime q. The twist of f by χ is defined by

$$f_\chi(z) = \sum_{n\neq 0} a(n)\chi(n) y^{1/2} K_\nu(2\pi|n|y) e^{2\pi i n x}. \tag{13}$$

Lemma 13. *The twist of a cuspidal Maass form for $\Gamma_0(N)$ is an automorphic form for $\Gamma_0(q^2N)$ with character χ^2. That is, for all $\gamma \in \Gamma_0(q^2N)$, we have*

$$f_\chi(\gamma z) = \chi(\gamma)^2 f_\chi(z).$$

Proof. By Lemma 3, the Gauss sum $\tau(\overline{\chi})$ is not zero. Then Lemma 2 allows us to write

$$f_\chi(z) = \frac{1}{\tau(\overline{\chi})} \sum_{a=1}^{q} \overline{\chi(a)} f(z + \frac{a}{q}).$$

Now let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(q^2N)$. Then

$$\begin{pmatrix} 1 & a/q \\ 0 & 1 \end{pmatrix} \gamma \in \Gamma_0(N) \begin{pmatrix} 1 & d^2a/q \\ 0 & 1 \end{pmatrix}.$$

Thus for all $z \in \mathbb{H}$, the point $\gamma z + \frac{a}{q}$ lies in the $\Gamma_0(N)$-orbit of $z + \frac{d^2 a}{q}$. It follows that

$$\begin{aligned}
f_\chi(\gamma z) &= \frac{1}{\tau(\overline{\chi})} \sum_{a=1}^{q} \overline{\chi(a)} f(\gamma z + \frac{a}{q}) \\
&= \frac{1}{\tau(\overline{\chi})} \sum_{a=1}^{q} \overline{\chi(a)} f(z + \frac{d^2 a}{q}) \\
&= \frac{1}{\tau(\overline{\chi})} \sum_{a=1}^{q} \overline{\chi(ad^2)} \chi(d)^2 f(z + \frac{d^2 a}{q}) \\
&= \chi(d)^2 f_\chi(z).
\end{aligned}$$

□

3.2 Twisted Eisenstein series

Let q be a prime. In this section we assume that χ is an even, non-principal (and thus primitive) Dirichlet character mod q, and that $(q, N) = 1$ (at the moment the last condition is not used.). For $\mathrm{Re}(s) > 1$ we define the twisted Eisenstein series by the absolutely convergent series

$$E(z, s, \chi) = \frac{1}{2} \sum_{\gamma} \chi(d) \,\mathrm{Im}(\gamma z)^s,$$

where the sum goes over a set of coset representatives $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\infty \setminus \Gamma_0(Nq^2)$. We will sometimes write $\chi(\gamma)$ to denote $\chi(d)$.

The Eisenstein series $E(z,s,\chi)$ satisfies the automorphic relation $E(\gamma z,s,\chi)=\overline{\chi(\gamma)}E(z,s,\chi)$ for all $\gamma\in\Gamma_0(Nq^2)$. In particular, the function $E(z,s,\chi)$ is invariant under $z\mapsto z+1$. Hence it has a Fourier expansion (our assumption that χ is even simplifies the calculation of the Fourier expansion). In order to determine its Fourier expansion we first establish a few identities.

The cosets $\Gamma_\infty\setminus\Gamma_0(Nq^2)$ are determined by the bottom row of a representative

$$\Gamma_\infty\begin{pmatrix}a&b\\c&d\end{pmatrix}=\left\{\begin{pmatrix}s&t\\c&d\end{pmatrix}\,\middle|\, sd-ct=1, Nq^2|c\right\}.$$

A pair of coprime integers (c,d), subject to $Nq^2|c$, uniquely determines such a coset. To sum over $(m,n)\in\mathbb{Z}\setminus\{0,0\}$ is the same as to sum over all positive integers M and all pairs (c,d) of coprime integers by taking $(m,n)=(Mc,Md)$. As a consequence, we can write

$$\begin{aligned}E^*(z,s,\chi)&:=L(2s,\chi)E(z,s,\chi)\\&=\frac{L(2s,\chi)}{2}\sum_{\gamma\in\Gamma_\infty\setminus\Gamma_0(Nq^2)}\chi(d)\operatorname{Im}(\gamma z)^s\\&=\frac{L(2s,\chi)}{2}\sum_{\substack{(c,d)=1\\Nq^2|c}}\chi(d)\frac{y^s}{|cz+d|^{2s}}\\&=\frac{1}{2}\sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq0}}\chi(n)\frac{y^s}{|mNq^2z+n|^{2s}}.\end{aligned}\tag{14}$$

Each summand $\chi(n)\frac{y^s}{|mNq^2z+n|^{2s}}$ in Equation 14 is hit by a summand $\chi(\tilde n)\frac{y^s}{|\tilde mq^2z+\tilde n|^{2s}}$ in Equation 14 when specialized to $N=1$ (choose $\tilde m=Nm$ and $\tilde n=n$). But not all terms in the latter sum appear in the former sum. However, the proof of Theorem 17 shows that these terms do not contribute to the sum (without the twisting by χ this would not be true). As a consequence, the Eisenstein series $E^*(z,s,\chi)$ does in fact not depend on N.

Lemma 14. *If* $\operatorname{Re}(s)>1/2$ *and* $r\in\mathbb{R}$, *then*

$$\int_{-\infty}^{\infty}\frac{e^{-2\pi irx}}{(x^2+y^2)^s}\mathrm{d}x=\frac{\pi^s}{y^s\Gamma(s)}\begin{cases}\pi^{-s+\frac12}\Gamma(s-\frac12)y^{1-s} & \text{if } r=0\\ 2|r|^{s-\frac12}\sqrt{y}K_{s-\frac12}(2\pi|r|y) & \text{if } r\neq 0\end{cases}.\tag{15}$$

Proof. Recall the integral representation of the Gamma function

$$\Gamma(s)=\int_0^\infty t^{s-1}e^{-t}\mathrm{d}t.$$

Thus we have

$$\begin{aligned}\frac{y^s}{\pi^s}\Gamma(s)\int_{-\infty}^{\infty}\frac{e^{-2\pi irx}}{(x^2+y^2)^s}\mathrm{d}x&=\int_0^\infty\int_{-\infty}^\infty e^{-t}\left(\frac{ty}{\pi(x^2+y^2)}\right)^s e^{-2\pi irx}\mathrm{d}x\frac{\mathrm{d}t}{t}\\&=\int_0^\infty\int_{-\infty}^\infty e^{-\pi\xi(x^2+y^2)/y}\xi^s e^{-2\pi irx}\mathrm{d}x\frac{\mathrm{d}\xi}{\xi}.\end{aligned}$$

For $r=0$ the above expression becomes

$$\begin{aligned}\int_0^\infty\int_{-\infty}^\infty e^{-\pi\xi(x^2+y^2)/y}\xi^s\mathrm{d}x\frac{\mathrm{d}\xi}{\xi}&=\int_0^\infty\sqrt{\frac{y}{\xi}}e^{-\pi\xi y}\xi^s\frac{\mathrm{d}\xi}{\xi}\\&=\pi^{-s+\frac12}y^{1-s}\Gamma(s-\frac12).\end{aligned}$$

For $r\neq0$ we obtain, using the change of variables $\xi\mapsto\frac{\xi}{|r|}$,

$$\begin{aligned}\int_0^\infty\int_{-\infty}^\infty e^{-\pi\xi(x^2+y^2)/y}\xi^s e^{-2\pi irx}\mathrm{d}x\frac{\mathrm{d}\xi}{\xi}&=\int_0^\infty\sqrt{\frac{y}{\xi}}e^{-y\pi r^2/\xi}e^{-\pi\xi y}\xi^s\frac{\mathrm{d}\xi}{\xi}\\&=\sqrt{y}|r|^{s-\frac12}\int_0^\infty\xi^{s-\frac12}e^{-y\pi|r|(1/\xi+\xi)}\frac{\mathrm{d}\xi}{\xi}\\&=2|r|^{s-\frac12}\sqrt{y}K_{s-\frac12}(2\pi|r|y).\end{aligned}$$

Here we used the following integral representation of the modified Bessel function of the second kind $K_\nu(x)$ (see [5] page 182)

$$K_\nu(x) = \frac{1}{2}\int_0^\infty e^{-\frac{x}{2}(t+t^{-1})} t^{-\nu-1} \mathrm{d}t.$$

$\square$

Lemma 15. *For $z = x + iy \in \mathbb{H}$ and $\mathrm{Re}(s) > \frac{1}{2}$ we have*

$$\sum_{n\in\mathbb{Z}} \frac{1}{|z+n|^{2s}} = \frac{\sqrt{\pi}\Gamma(s-\frac{1}{2})}{\Gamma(s)} y^{1-2s} + \frac{2\pi^s}{\Gamma(s)} y^{\frac{1}{2}-s} \sum_{r\neq 0} |r|^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|r|y)\, e^{2\pi i r x}. \tag{16}$$

Proof. Recall the Poisson summation formula

$$\sum_{n\in\mathbb{Z}} \varphi(x+n) = \sum_{n\in\mathbb{Z}} \widehat{\varphi}(n) e^{2\pi i n x},$$

where φ is a continuous function that decays sufficiently rapidly at infinity (for example, $|f(x)| < |x|^{-c}$ with $c > 1$) and where $\widehat{\varphi}(n) = \int_{-\infty}^{\infty} \varphi(x) e^{-2\pi i n x} \mathrm{d}x$ is the Fourier transform. We apply this formula to $\varphi(x) = |x+iy|^{-2s}$, where $z = x+iy \in \mathbb{H}$ and $\mathrm{Re}(s) > \frac{1}{2}$. The Poisson summation formula gives

$$\sum_{n\in\mathbb{Z}} \frac{1}{|z+n|^{2s}} = \sum_{n\in\mathbb{Z}} \left(\int_{-\infty}^{\infty} \frac{e^{-2\pi i n x}}{(x^2+y^2)^s} \mathrm{d}x \right) e^{2\pi i n x}.$$

Using Lemma 14 we get the result. $\square$

We would like to adapt the left hand side of Equation 16 to the setting where the sum is twisted by a primitive Dirichlet character χ mod q. For this we need the twisted variant of the Poisson summation formula.

Lemma 16. *Let φ be a function that satisfies the conditions of the Poisson summation formula. Let χ be a primitive Dirichlet character mod q. Then*

$$\sum_{n\in\mathbb{Z}} \chi(n)\varphi(x+n) = \frac{\tau(\chi)}{q} \sum_{n\in\mathbb{Z}} \overline{\chi(n)} \widehat{\varphi}(n/q) e^{2\pi i n x/q}.$$

Proof. From Lemma 2 we know that

$$\chi(n) = \frac{1}{\tau(\overline{\chi})} \sum_{m \bmod q} \overline{\chi(m)} e^{2\pi i n m/q} = \frac{\chi(-1)\tau(\chi)}{q} \sum_{m \bmod q} \overline{\chi(m)} e^{2\pi i n m/q}.$$

Let us now consider $\sum_{n\in\mathbb{Z}} \chi(n)\varphi(n) = \sum_{n\in\mathbb{Z}} \varphi_1(n)$, where

$$\varphi_1(x) = \frac{\chi(-1)\tau(\chi)}{q} \sum_{m \bmod q} \overline{\chi(m)} e^{2\pi i x m/q} \varphi(x).$$

The Fourier transform of $\varphi_1(x)$ is

$$\begin{aligned}\widehat{\varphi_1}(\xi) &= \frac{\chi(-1)\tau(\chi)}{q} \sum_{m \bmod q} \overline{\chi(m)} \int_{-\infty}^{\infty} \varphi(x) e^{-2\pi i x (\xi - \frac{m}{q})} \mathrm{d}x \\ &= \frac{\chi(-1)\tau(\chi)}{q} \sum_{m \bmod q} \overline{\chi(m)} \widehat{\varphi}\left(\xi - \frac{m}{q}\right).\end{aligned}$$

We apply Poisson summation formula and thus $\sum_{n\in\mathbb{Z}} \varphi_1(n)$ equals

$$\frac{\chi(-1)\tau(\chi)}{q} \sum_{m \bmod q} \sum_{n=-\infty}^{\infty} \overline{\chi(m)} \widehat{\varphi}\left(\frac{nq-m}{q}\right).$$

We can write $\overline{\chi(m)} = \chi(-1)\overline{\chi(nq-m)}$. When n runs through $\mathbb{Z}$ and m through $\mathbb{Z}/q\mathbb{Z}$, the terms $nq - m$ run uniquely through $\mathbb{Z}$. Thus we have shown

$$\sum_{n\in\mathbb{Z}} \chi(n)\varphi(n) = \frac{\tau(\chi)}{q} \sum_{n\in\mathbb{Z}} \overline{\chi(n)}\widehat{\varphi}(n/q).$$

Replacing $\varphi(n)$ by $\varphi(n+x)$ replaces $\widehat{\varphi}(\xi)$ by $\widehat{\varphi}(\xi)e^{2\pi i\xi x}$. This completes the proof. $\square$

We can now determine the Fourier expansion of $E^*(z,s,\chi)$.

Theorem 17. *The function $E^*(z,s,\chi)$ has the Fourier expansion*

$$\begin{aligned} E^*(z,s,\chi) &= L(2s,\chi)y^s \\ &\quad + \tau(\chi)q^{-2s}\sqrt{y}\frac{2\pi^s}{\Gamma(s)} \sum_{k\neq 0} |k|^{\frac{1}{2}-s}\sigma_{2s-1}(|k|,\overline{\chi})K_{s-\frac{1}{2}}(2\pi|k|y)e^{2\pi ikx}, \end{aligned}$$

where $\sigma_s(k,\overline{\chi}) = \sum_{d|k} \overline{\chi(d)}d^s$.

First Proof. We split up the sum (14) into the terms with $m = 0$ and those with $m \neq 0$. The assumption that χ is even allows us to combine each positive summand with its negative. We obtain

$$E^*(z,s,\chi) = L(2s,\chi)y^s + \sum_{m=1}^{\infty}\sum_{n\in\mathbb{Z}} \chi(n)\frac{y^s}{|mNq^2z+n|^{2s}}.$$

Substituting into this the twisted variant of the formula of Lemma 15 gives (note that $\chi(0) = 0$)

$$\begin{aligned} E^*(z,s,\chi) &= L(2s,\chi)y^s \\ &\quad + \frac{\tau(\chi)}{q}y^s\frac{2\pi^s}{\Gamma(s)} \sum_{m=1}^{\infty}\sum_{n\neq 0} \overline{\chi(n)} \left(\frac{|n|}{q}\right)^{s-\frac{1}{2}} \left(ymNq^2\right)^{\frac{1}{2}-s} K_{s-\frac{1}{2}}(2\pi|n|ymNq)e^{2\pi inxmNq}. \end{aligned}$$

Summing $m \in Nq\mathbb{Z}_{\geq 1}$ and $n \in \mathbb{Z}_{\geq 1}$ is the same as summing their product k over $\mathbb{Z}_{\geq 1}$ and summing for each k over the pairs (n, mNq) such that $Nqmn = k$. Accordingly we can write

$$\begin{aligned} E^*(z,s,\chi) &= L(2s,\chi)y^s \\ &\quad + \tau(\chi)q^{-2s}\sqrt{y}\frac{2\pi^s}{\Gamma(s)} \sum_{m=1}^{\infty}\sum_{n\neq 0} \overline{\chi(n)}|n|^{s-\frac{1}{2}}(mNq)^{\frac{1}{2}-s} K_{s-\frac{1}{2}}(2\pi|n|ymNq)e^{2\pi inxmNq} \\ &= L(2s,\chi)y^s \\ &\quad + \tau(\chi)q^{-2s}\sqrt{y}\frac{2\pi^s}{\Gamma(s)} \sum_{k\neq 0} |k|^{\frac{1}{2}-s}\sigma_{2s-1}(|k|,\overline{\chi})K_{s-\frac{1}{2}}(2\pi|k|y)e^{2\pi ikx}, \end{aligned}$$

where $\sigma_s(k,\chi) = \sum_{d|k} \chi(d)d^s$ is the twisted divisor function. $\square$

Second Proof. Again we start with

$$E^*(z,s,\chi) = L(2s,\chi)y^s + \sum_{m=1}^{\infty}\sum_{n\in\mathbb{Z}} \chi(n)\frac{y^s}{|mNq^2z+n|^{2s}}.$$

Using the substitution $n = tmNq^2 + r$ and Lemma 15 gives

$$\begin{aligned}
E^*(z,s,\chi) &= L(2s,\chi)y^s \\
&\quad + \frac{y^s}{q^{4s}N^{2s}} \sum_{m=1}^{\infty} |m|^{-2s} \sum_{r=1}^{mNq^2} \chi(r) \sum_{t\in\mathbb{Z}} \left| t + z + \frac{r}{mNq^2} \right|^{-2s} \\
&= L(2s,\chi)y^s + \frac{y^s}{q^{4s}N^{2s}} \sum_{m=1}^{\infty} |m|^{-2s} \sum_{r=1}^{mNq^2} \chi(r) \\
&\quad \times \left[\frac{\sqrt{\pi}\Gamma(s-\frac{1}{2})}{\Gamma(s)} y^{1-2s} + \frac{2\pi^s}{\Gamma(s)} y^{\frac{1}{2}-s} \sum_{t\neq 0} |t|^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|t|y)\, e^{2\pi i t\left(x+\frac{r}{mNq^2}\right)} \right] \\
&= L(2s,\chi)y^s + \frac{2\pi^s\sqrt{y}}{q^{4s}N^{2s}\Gamma(s)} \sum_{t\neq 0} |t|^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|t|y) e^{2\pi i t x} \sum_{m=1}^{\infty} |m|^{-2s} \sum_{r=1}^{mNq^2} \chi(r) e^{\frac{2\pi i t r}{mNq^2}}.
\end{aligned}$$

In the last equality we used Lemma 1 according to which $\sum_{r=1}^{mNq^2} \chi(r) = \sum_{r=1}^{q} \chi(r) = 0$. By Lemma 4 the last sum equals

$$\sum_{r=1}^{mNq^2} \chi(r) e^{\frac{2\pi i t r}{mNq^2}} = \begin{cases} mNq\tau(\chi)\overline{\chi}(l) & \text{if } t = mqNl \text{ for some } l\in\mathbb{Z} \\ 0 & \text{else} \end{cases}.$$

Therefore we get

$$\begin{aligned}
E^*(z,s,\chi) &= L(2s,\chi)y^s + \frac{2\pi^s\tau(\chi)\sqrt{y}}{q^{4s}N^{2s}\Gamma(s)} \sum_{t\neq 0} |t|^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|t|y) e^{2\pi i t x} \sum_{\substack{m\geq 1\\ t=mqNl}} |m|^{-2s} mNq\overline{\chi}(l) \\
&= L(2s,\chi)y^s + \frac{2\pi^s\tau(\chi)\sqrt{y}}{q^{4s}N^{2s}\Gamma(s)} \sum_{t\neq 0} |t|^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|t|y) e^{2\pi i t x} \sum_{l|t} \left| \frac{t}{qNl} \right|^{-2s} \frac{t}{l}\overline{\chi}(l) \\
&= L(2s,\chi)y^s + \frac{2\pi^s\tau(\chi)\sqrt{y}}{q^{2s}\Gamma(s)} \sum_{t\neq 0} |t|^{\frac{1}{2}-s} K_{s-\frac{1}{2}}(2\pi|t|y) e^{2\pi i t x} \sum_{l|t} \overline{\chi}(l) l^{2s-1} \\
&= L(2s,\chi)y^s + \frac{2\pi^s\tau(\chi)\sqrt{y}}{q^{2s}\Gamma(s)} \sum_{t\neq 0} |t|^{\frac{1}{2}-s} \sigma_{2s-1}(|t|,\overline{\chi}) K_{s-\frac{1}{2}}(2\pi|t|y) e^{2\pi i t x}.
\end{aligned}$$

□

A consequence of the above Fourier expansion is the meromorphic continuation in s of the twisted Eisenstein series $E(z,s,\chi)$. We would also like a functional equation for the twisted Eisenstein series. Simply replacing s by $1-s$ in $E(z,s,\chi)$ will not provide us with a functional equation since y^s and y^{1-s} in the constant term will not match. Therefore we also need to appropriately modify the argument z in $E(z,s,\chi)$. For this we consider the Eisenstein series $E(\omega z,s,\chi)$, where $\omega = \begin{pmatrix} 0 & -1 \\ Nq^2 & 0 \end{pmatrix} \in \mathrm{GL}(2,\mathbb{Z})$. Again the automorphic relation $E(\gamma\omega z,s,\chi) = \overline{\chi(\gamma)}E(\omega z,s,\chi)$ holds and consequently $E(\omega z,s,\chi)$ admits a Fourier expansion.

Lemma 18. *The function $E^*(\omega z,s,\chi)$ has the Fourier expansion*

$$\begin{aligned}
E^*(\omega z,s,\chi) &= \frac{1}{q^{2s}N^s} L(2s-1,\chi) \frac{\sqrt{\pi}\Gamma(s-\frac{1}{2})}{\Gamma(s)} y^{1-s} \\
&\quad + \frac{\sqrt{y}}{q^{2s}N^s} \frac{2\pi^s}{\Gamma(s)} \sum_{r\neq 0} |r|^{s-\frac{1}{2}} \sigma_{1-2s}(r,\chi) K_{s-\frac{1}{2}}(2\pi|r|y) e^{2\pi i r x}.
\end{aligned}$$

Proof. We will use the fact that the matrix $\omega = \begin{pmatrix} 0 & -1 \\ Nq^2 & 0 \end{pmatrix}$ normalizes the group $\Gamma_0(Nq^2)$. Indeed, we have

$$\omega \begin{pmatrix} a & b \\ c & d \end{pmatrix} \omega^{-1} = \begin{pmatrix} d & -\frac{c}{Nq^2} \\ -bNq^2 & a \end{pmatrix}.$$

We now compute

$$
\begin{aligned}
E^*(\omega z, s, \chi) &= \frac{L(2s,\chi)}{2} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(Nq^2)} \chi(\gamma)\,\mathrm{Im}(\gamma\omega z)^s \\
&= \frac{L(2s,\chi)}{2} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(Nq^2)} \chi(\omega\gamma\omega^{-1})\,\mathrm{Im}(\omega\gamma z)^s \\
&= \frac{y^s}{q^{2s}N^s} \sum_{a=1}^{\infty}\sum_{b\in\mathbb{Z}} \frac{\chi(a)}{|az+b|^{2s}} \\
&= \frac{y^s}{q^{2s}N^s} \sum_{a=1}^{\infty} \frac{\chi(a)}{a^{2s}} \sum_{d=1}^{a}\sum_{m\in\mathbb{Z}} \left| m + z + \frac{d}{a}\right|^{-2s} \\
&= \frac{1}{q^{2s}N^s} L(2s-1,\chi) \frac{\sqrt{\pi}\Gamma(s-\frac{1}{2})}{\Gamma(s)} y^{1-s} \\
&+ \frac{\sqrt{y}}{q^{2s}N^s}\frac{2\pi^s}{\Gamma(s)} \sum_{r\neq 0} |r|^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|r|y) e^{2\pi i r x} \sum_{a=1}^{\infty} \frac{\chi(a)}{a^{2s}} \sum_{d=1}^{a} e^{2\pi i r\frac{d}{a}}.
\end{aligned}
$$

Because

$$
\sum_{d=1}^{a} e^{2\pi i r\frac{d}{a}} = \begin{cases} a & \text{if } a|r \\ 0 & \text{else} \end{cases}
$$

we have

$$
\sum_{a=1}^{\infty} \frac{\chi(a)}{a^{2s}} \sum_{d=1}^{a} e^{2\pi i r\frac{d}{a}} = \sum_{a|r} \frac{\chi(a)}{a^{2s-1}} = \sigma_{1-2s}(r,\chi).
$$

Hence we obtain the claimed Fourier expansion. □

The functional equation for the twisted Eisenstein series takes a simple form when we define

$$
E^{\#}(z,s,\chi) := \pi^{-s}\Gamma(s)E^*(z,s,\chi) = \pi^{-s}\Gamma(s)L(2s,\chi)E(z,s,\chi).
$$

Theorem 19. *The function $E^{\#}(z,s,\chi)$ satisfies the functional equation*

$$
\begin{aligned}
E^{\#}(z,s,\chi) &= \frac{\tau(\chi)}{\sqrt{q}} N^{1-s} q^{\frac{5}{2}-4s} E^{\#}(\omega z, 1-s, \overline{\chi}) \\
&= \frac{\tau(\chi)}{\sqrt{q}} N^{1-s} q^{\frac{5}{2}-4s} E^{\#}\left(\frac{-1}{q^2 N z}, 1-s, \overline{\chi}\right).
\end{aligned}
$$

Proof. This follows from the above Fourier expansions, the functional equation (9) for the Dirichlet L-series, and the symmetry $K_\nu(z) = K_{-\nu}(z)$ of the K-Bessel function.

□

3.3 Rankin-Selberg convolution for $\Gamma_0(N)$

Let

$$
f(z) = \sum_{n\neq 0} a(n)\sqrt{y} K_{\nu_f}(2\pi|n|y) e^{2\pi i n x}, \tag{17}
$$

$$
g(z) = \sum_{n\neq 0} b(n)\sqrt{y} K_{\nu_g}(2\pi|n|y) e^{2\pi i n x}, \tag{18}
$$

be cuspidal Maass forms for $\Gamma_0(N)$ of the same parity ϵ. Again we fix an even, primitive Dirichlet character χ mod a prime q. We recall that the twist $f_\chi(z)$, defined by Equation (13), is automorphic for $\Gamma_0(q^2N)$ with character χ^2. We define the Rankin-Selberg L-function $L_{f_\chi \times g}(s)$ by

$$L_{f_\chi \times g}(s) = \sum_{n=1}^{\infty} \chi(n) \frac{a(n)b(n)}{n^s}.$$

Because the Fourier coefficients $a(n), b(n)$ satisfy the bound $O(\sqrt{|n|})$ (Lemma 6), the above series converges absolutely for $\mathrm{Re}(s) > 2$. By the same methods used to prove Theorem 10, we will show that the L-function $L_{f_\chi \times g}(s)$ admits a meromorphic continuation to $s \in \mathbb{C}$ and a functional equation. Key is the following lemma, which constructs the L-function $L_{f_\chi \times g}(s)$ as an inner product of $f_\chi(z)E(z, s, \chi^2)$ with $\overline{g(z)}$.

Lemma 20. *For* $\mathrm{Re}(s)$ *sufficiently large we have*

$$\int_{\Gamma_0(Nq^2)\backslash\mathbb{H}} E(z, s, \chi^2) f_\chi(z) g(z) \mathrm{d}\mu(z) = \frac{(-1)^\epsilon}{2^3} \frac{1}{\Gamma(s)\pi^s} \left(\prod \Gamma\left(\frac{s \pm \nu \pm \nu'}{2} \right) \right) L_{f_\chi \times g}(s).$$

Proof. By folding/unfolding it follows that

$$\begin{aligned}
\int_{\Gamma_0(Nq^2)\backslash\mathbb{H}} E(z, s, \chi^2) f_\chi(z) g(z) \mathrm{d}\mu(z) &= \frac{1}{2} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(Nq^2)} \int_{\Gamma_0(Nq^2)\backslash\mathbb{H}} \chi(\gamma)^2 \operatorname{Im}(\gamma z)^s f_\chi(z) g(z) \mathrm{d}\mu(z) \\
&= \frac{1}{2} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(Nq^2)} \int_{\Gamma_0(Nq^2)\backslash\mathbb{H}} \operatorname{Im}(\gamma z)^s f_\chi(\gamma z) g(\gamma z) \mathrm{d}\mu(\gamma z) \\
&= \frac{1}{2} \int_{\Gamma_\infty\backslash\mathbb{H}} \operatorname{Im}(z)^s f_\chi(z) g(z) \mathrm{d}\mu(z) \\
&= \frac{1}{2} \int_0^\infty \int_0^1 y^{s-2} f_\chi(x+iy) g(x+iy) \mathrm{d}x \mathrm{d}y.
\end{aligned}$$

For the second equality we used Lemma 13. Upon inserting the Fourier expansions, the integral over the x-coordinates becomes

$$\begin{aligned}
\int_0^1 f_\chi(x+iy) g(x+iy) \mathrm{d}x &= y \sum_{n \neq 0} \chi(n) a(n) b(-n) K_\nu(2\pi|n|y) K_{\nu'}(2\pi|n|y) \\
&= (-1)^\epsilon y \sum_{n \neq 0} \chi(n) a(n) b(n) K_\nu(2\pi|n|y) K_{\nu'}(2\pi|n|y).
\end{aligned}$$

Then formula (12) for the Mellin transform of $K_\nu K_{\nu'}$ gives

$$\begin{aligned}
\int_{\Gamma_0(Nq^2)\backslash\mathbb{H}} E(z, s, \chi^2) f_\chi(z) g(z) \mathrm{d}\mu(z) &= \frac{(-1)^\epsilon}{2^4} \frac{1}{\Gamma(s)\pi^s} \left(\prod \Gamma\left(\frac{s \pm \nu \pm \nu'}{2} \right) \right) \sum_{n \neq 0} \chi(n) \frac{a(n)b(n)}{|n|^s} \\
&= \frac{(-1)^\epsilon}{2^3} \frac{1}{\Gamma(s)\pi^s} \left(\prod \Gamma\left(\frac{s \pm \nu \pm \nu'}{2} \right) \right) \sum_{n=1}^{\infty} \chi(n) \frac{a(n)b(n)}{n^s},
\end{aligned}$$

where in the last line we used that f and g are of the same parity. $\square$

4 The Luo-Rudnik-Sarnak Theorem

4.1 SL(2,Z)

4.2 Congruence subgroups

5 References

1. Bergeron, N. *The Spectrum of Hyperbolic Surfaces* (Springer, 2016).
2. Goldfeld, D. *Automorphic Forms and L-Functions for the Group GL(n,R)* (Cambridge University Press, 2006).
3. Bateman, H. & Erdélyi, A. *Tables of Integral Transforms (Volume II)* (McGraw Hill, New York, 1954).
4. Macdonald, I. *Symmetric Functions and Hall Polynomials* (Oxford University Press, 1979).
5. Watson, G. N. *A Treatise on the Theory of Bessel Functions* (Cambridge University Press, 1966).