L-functions and the Selberg eigenvalue conjecture

Semester Paper

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1 Outline

Let $\Gamma < \mathrm{SL}(2, \mathbb{Z})$ be a congruence subgroup. The Selberg eigenvalue conjecture asserts that the smallest non-zero eigenvalue of the hyperbolic Laplacian on $L^2(\Gamma \setminus \mathbb{H})$ is greater than or equal to $\frac{1}{4}$. The best current lower bound towards the generalized Selberg eigenvalue conjecture was obtained by Luo-Rudnick-Sarnack in ([1], [2]). In this semester paper we illustrate the method of proof of the Luo-Rudnick-Sarnack theorem first for $\mathrm{SL}(2, \mathbb{Z})$ and then adapt the proof to the setting of Hecke congruence subgroups. The proof is based on the study of L-functions that are associated with Maass forms for Γ. These L-functions encode spectral data and moreover possess analytical and functional properties. Proving results about these L-functions allows one then to deduce properties of the spectrum.

In the first section we collect background on Dirichlet characters and Gauss sums that is used throughout the text. In the second section we study Maass forms for $\mathrm{SL}(2, \mathbb{Z})$ and prove that various L-functions attached to them enjoy meromorphic continuation, functional equations and in privileged cases an Euler product expansion. Of particular importance are Rankin-Selberg convolution L-functions that are constructed as an integral transform of Eisenstein series. The Luo-Rudnick-Sarnack theorem is then proved by showing that there is a family of twisted Rankin-Selberg L-functions that do not vanish at a given value. In the last section we translate the results to the more general setting of Hecke congruence subgroups.

2 Dirichlet characters and Gauss sums

Let G be a finite group. A character χ on G is a group homomorphism $\chi : G \to \mathbb{C}^\times$, where $\mathbb{C}^\times$ is the group of units of $\mathbb{C}$. Let m be a positive integer. A Dirichlet character of modulus m is a character on the multiplicative group $G = (\mathbb{Z}/m\mathbb{Z})^\times$ of the integers modulo m. Equivalently, a Dirichlet character of modulus m is a function $\chi : \mathbb{Z} \to \mathbb{C}$ such that

- $\chi(ab) = \chi(a)\chi(b)$ for all $a, b \in \mathbb{Z}$,
- $\chi(a) = 0$ if and only if $(a, m) > 1$,
- If $a \equiv b \pmod{m}$, then $\chi(a) = \chi(b)$.

Let χ_1 and χ_2 be Dirichlet characters of modulus m_1 and m_2, respectively, with $m_1|m_2$. If $\chi_2(a) = \chi_1(a)$ for all $a \in (\mathbb{Z}/m_2\mathbb{Z})^\times$, then χ_2 is said to be induced by χ_1. A Dirichlet character is called primitive if it is not induced by any Dirichlet character other than itself. A Dirichlet character induced by the identity function is called principal. We denote a principal Dirichlet character by χ_0. By definition a principal Dirichlet character mod m is

$$\chi_0(a) = \begin{cases} 1 & \text{if } (a, m) = 1 \\ 0 & \text{else} \end{cases}. \tag{1}$$

If q is prime, then every nonprincipal Dirichlet character of modulus q is primitive. We call a Dirichlet character even if $\chi(-1) = 1$, odd if $\chi(-1) = -1$.

Lemma 1. *Let χ be a nonprincipal Dirichlet character modulo a prime q. Then we have*

$$\sum_{a=1}^{q} \chi(a) = 0.$$

Proof. Because χ is not principal, there is an integer b such that $\chi(b) \neq \{0, 1\}$. Furthermore, the sum may be restricted to the terms with $(a, q) = 1, 1 \leq a \leq q$. Multiplication by b is a bijection $(\mathbb{Z}/q\mathbb{Z})^\times \to (\mathbb{Z}/q\mathbb{Z})^\times$. Therefore we have

$$\chi(b) \sum_{a=1}^{q} \chi(a) = \sum_{a \in (\mathbb{Z}/q\mathbb{Z})^\times} \chi(ab) = \sum_{c \in (\mathbb{Z}/q\mathbb{Z})^\times} \chi(c),$$

which implies $\sum_{a=1}^{q} \chi(a) = 0$. □

The sum of the n-th roots of unity is zero. We will investigate more closely the twist of this sum by a Dirichlet character mod q, a so-called Gauss sum. More precisely, the Gauss sum $\tau(\chi)$ attached to a primitive Dirichlet character χ mod q is

$$\tau(\chi) = \sum_{a \bmod q} \chi(a) e^{2\pi i a/q}.$$

Lemma 2. *Let χ be a nonprincipal Dirichlet character modulo a prime q. Then for all $n \in \mathbb{Z}$ we have*

$$\tau(\chi)\overline{\chi(n)} = \sum_{a=1}^{q} \chi(a) e^{2\pi i n a/q}. \tag{2}$$

In particular,

$$\tau(\overline{\chi}) = \chi(-1)\overline{\tau(\chi)}.$$

Proof. If $(n, q) = 1$, then $a \mapsto an$ permutes the residues mod q. In this case we get

$$\sum_{a \bmod q} \chi(a) e^{\frac{2\pi i n a}{q}} = \tau(\chi)\overline{\chi(n)}.$$

Now suppose that $(n,q)>1$. Then $\chi(n)=0$ and we have to show that the right side of Equation 2 vanishes. For this let $n'\in\mathbb{Z}$ be such that $qn'=n$. Then we have

$$\begin{aligned}\sum_{a \bmod q}\chi(a)e^{\frac{2\pi ina}{q}} &= \sum_{a \bmod q}\chi(a)e^{2\pi in'a}\\ &= \sum_{a \bmod q}\chi(a).\end{aligned}$$

The last sum is zero by Lemma 1. This completes the proof of the lemma. □

Lemma 3. *Let χ be a primitive Dirichlet character modulo a prime q. Then*

$$|\tau(\chi)|=\sqrt{q}.$$

Proof. By Lemma 2 we have

$$\begin{aligned}|\tau(\chi)|^2 &= \sum_{a=1}^{q}\overline{\chi(a)}e^{-2\pi ia/q}\tau(\chi)\\ &= \sum_{a=1}^{q}e^{-2\pi ia/q}\left(\sum_{b=1}^{q}\chi(b)e^{2\pi iab/q}\right)\\ &= \sum_{b=1}^{q}\chi(b)\left(\sum_{a=1}^{q}e^{2\pi ia(b-1)/q}\right).\end{aligned}$$

If $b=1$, the inner sum equals q. If $b\neq 1$, the inner sum is zero. Therefore we obtain

$$|\tau(\chi)|^2=\chi(1)q=q.$$

□

We will also need the following variant of Lemma 2.

Lemma 4. *Let χ be a nonprincipal Dirichlet character modulo a prime q, and let $n,m\in\mathbb{Z}$. Then*

$$\sum_{a=1}^{nq^2}\chi(a)e^{2\pi ima/nq^2}=\begin{cases}qn\tau(\chi)\overline{\chi}(l) & \text{if } m=lnq \text{ for some } l\in\mathbb{Z}\\ 0 & \text{else}\end{cases}.$$

Proof. Every a in the above sum is of the form $a=a_1+tq$ with $1\le a_1\le q$ and $0\le t<nq$. Thus we can write

$$\sum_{a=1}^{nq^2}\chi(a)e^{2\pi ima/nq^2}=\sum_{a_1=1}^{q}\chi(a_1)e^{\frac{2\pi ima_1}{nq^2}}\sum_{t=0}^{nq-1}e^{\frac{2\pi imt}{nq}}.$$

The sum over t is zero unless $m=lnq$ for some $l\in\mathbb{Z}$ in which case the sum is nq. Therefore

$$\begin{aligned}\sum_{a=1}^{nq^2}\chi(a)e^{2\pi ima/nq^2} &= nq\sum_{\substack{a \bmod q\\ m=lnq}}\chi(a)e^{\frac{2\pi ima}{nq^2}}\\ &= nq\sum_{a \bmod q}\chi(a)e^{\frac{2\pi ila}{q}}\end{aligned}$$

We now conclude the proof using Lemma 2. □

The set of characters on a finite group G forms an abelian group $\hat{G}$ under the operations

$$\chi\chi'(g) := \chi(g)\chi'(g); \quad \chi^{-1}(g) := \chi(g)^{-1},$$

and the identity is the trivial character which sends every $g \in G$ to $1 \in \mathbb{C}^\times$.

Lemma 5. *Let G be a finite abelian group. Then the group of characters $\hat{G}$ on G is isomorphic to the group G. In particular, the number of Dirichlet characters of modulus q is given by $\varphi(q) = |(\mathbb{Z}/q\mathbb{Z})^\times|$.*

Proof. A finite abelian group G is isomorphic to

$$G \cong \mathbb{Z}/n_1\mathbb{Z} \times \cdots \times \mathbb{Z}/n_k\mathbb{Z}.$$

Let $g_1, \ldots, g_k$ be generators of G of orders $n_1, \ldots, n_k$. Every $g \in G$ can be uniquely written as $g = g_1^{r_1} \cdots g_k^{r_k}$ with $r_i \in [0, n_k - 1]$. For each g_i we pick a primitive n_i-th root of unity $\mu_i \in \mathbb{C}$. We consider the homomorphism from G to $\hat{G}$ that sends $g = g_1^{r_1} \cdots g_k^{r_k}$ to

$$(g_1^{s_1}, \ldots, g_k^{s_k}) \mapsto \mu_1^{r_1 s_1} \cdots \mu_k^{r_k s_k}.$$

Different choices of the roots of unity give rise to distinct characters, hence the above homomorphism is bijective. □

We have the following orthogonality relations.

Lemma 6. *Let $a \in \mathbb{Z}$. Then*

$$\sum_\chi \chi(a) = \begin{cases} \varphi(q) & \text{if } a \equiv 1 \pmod q \\ 0 & \text{else} \end{cases},$$

where the sum runs over over all Dirichlet characters mod q.

Proof. If $a \equiv 1 \pmod q$, we get

$$\sum_\chi \chi(a) = \sum_\chi \chi(1) = \sum_\chi 1 = \varphi(q)$$

by Lemma 5. The lemma also holds for $(a, q) > 1$ because in that case $\chi(a)$ is zero. Now suppose that $a \not\equiv 1 \pmod q$ and $(a, q) = 1$. By the proof of Lemma 5 there exists a Dirichlet character χ_1 mod q with $\chi_1(a) \neq 1$. Because the characters mod q for a group, if χ runs through all characters mod q, then so does $\chi_1\chi$. Thus we have

$$\chi_1(a) \sum_\chi \chi(a) = \sum_\chi (\chi_1\chi)(a) = \sum_\chi \chi(a),$$

which implies $\sum_\chi \chi(a) = 0$. □

We now restrict the sum in Lemma 6 to non-trivial even Dirichlet characters.

Lemma 7. *Let $a \in \mathbb{Z}$. The sum of $\chi(a)$ over all non-trivial even Dirichlet characters mod q is*

$$\sum_{\substack{\chi \pmod q \\ \chi \neq \chi_0,\, \chi(-1)=1}} \chi(a) = \begin{cases} 0 & \text{if } a \equiv 0 \pmod q \\ \frac{\varphi(q)}{2} - 1 & \text{if } a \equiv \pm 1 \pmod q \\ -1 & \text{else} \end{cases}.$$

Proof. If $a \equiv 0 \pmod q$, the lemma holds. There are $\varphi(q)/2$ even Dirichlet characters mod q, one of which is the trivial character, and therefore the the lemma holds for $a \equiv \pm 1 \pmod q$. For the other choices of $a \in \mathbb{Z}$, we may choose an even Dirichlet character χ_1 mod q with $\chi_1(a) \neq 1$. Because the even characters mod q for a group, with the same argument as in the proof of Lemma 6 we conclude that $\sum_\chi \chi(a) = 0$, where the sum runs over all even Dirichlet characters mod q. The contribution of the trivial character to this sum is 1. This completes the proof of the lemma. □

We will also make use of the following identity which expresses the average of $\overline{\chi(m)}\tau(\chi)^n$ in terms of hyper-Kloosterman sums.

Lemma 8. *Let $n \geq 1$, q be prime and $m \in \mathbb{Z}$ be such that $(q,m) = 1$. Then we have*

$$\sum_{\substack{\chi \ (mod\ q) \\ \chi \neq \chi_0,\, \chi(-1)=1}} \overline{\chi(m)}\tau(\chi)^n = \frac{q-1}{2}\left(K_n(m,q) + K_n(-m,q)\right) - (-1)^n,$$

where the sum runs over even primitive Dirichlet characters mod q and where $K_n(m,q)$ denotes the hyper-Kloosterman sum

$$K_n(m,q) = \sum_{\substack{x_1,\ldots,x_n \ mod\ q \\ x_1\cdots x_n \equiv m \ mod\ q}} e^{2\pi i\left(\frac{x_1+\cdots+x_n}{q}\right)}.$$

Proof. Writing out the Gauss sums, interchanging the order of summation and using Lemma 7 gives

$$\begin{aligned}
\sum_{\substack{\chi \ (\text{mod}\ q) \\ \chi \neq \chi_0,\, \chi(-1)=1}} \overline{\chi(m)}\tau(\chi)^n &= \sum_{x_1,\ldots,x_n \ \text{mod}\ q} e^{2\pi i\left(\frac{x_1+\cdots+x_n}{q}\right)} \sum_{\substack{\chi \ (\text{mod}\ q) \\ \chi \neq \chi_0,\, \chi(-1)=1}} \chi(x_1\cdots x_n m^{-1}) \\
&= \left(\frac{q-1}{2} - 1\right) \sum_{\substack{x_1,\ldots,x_n \ \text{mod}\ q \\ x_1\cdots x_n \equiv \pm m \ \text{mod}\ q}} e^{2\pi i\left(\frac{x_1+\cdots+x_n}{q}\right)} - \sum_{\substack{x_1,\ldots,x_n \not\equiv 0 \ \text{mod}\ q \\ x_1\cdots x_n \not\equiv \pm m \ \text{mod}\ q}} e^{2\pi i\left(\frac{x_1+\cdots+x_n}{q}\right)}.
\end{aligned}$$

If we write $z = x_1^{-1}\cdots x_{n-1}^{-1}m$, we obtain for the second summand

$$\sum_{\substack{x_1,\ldots,x_n \not\equiv 0 \ \text{mod}\ q \\ x_1\cdots x_n \not\equiv \pm m \ \text{mod}\ q}} e^{2\pi i\left(\frac{x_1+\cdots+x_n}{q}\right)} = \sum_{x_1,\ldots,x_{n-1} \not\equiv 0 \ \text{mod}\ q} e^{2\pi i\left(\frac{x_1+\cdots+x_{n-1}}{q}\right)} \sum_{\substack{x_n \not\equiv 0 \ \text{mod}\ q \\ x_n \not\equiv \pm z \ \text{mod}\ q}} e^{2\pi i x_n/q}. \tag{3}$$

Furthermore we have

$$\begin{aligned}
\sum_{\substack{x_n \not\equiv 0 \ \text{mod}\ q \\ x_n \not\equiv \pm z \ \text{mod}\ q}} e^{2\pi i x_n/q} &= \left(\sum_{x_n \ \text{mod}\ q} e^{2\pi i x_n/q} - \sum_{\substack{x_n \equiv 0 \ \text{mod}\ q \\ x_n \equiv \pm z \ \text{mod}\ q}} e^{2\pi i x_n/q} \right) \\
&= \left(\sum_{\substack{x_n \ \text{mod}\ q \\ x_n \not\equiv 0 \ \text{mod}\ q}} e^{2\pi i x_n/q} - \sum_{x_n \equiv \pm z \ \text{mod}\ q} e^{2\pi i x_n/q} \right) \\
&= \left(-1 - \sum_{\substack{x_1,\ldots,x_n \ \text{mod}\ q \\ x_1\cdots x_n \equiv \pm m \ \text{mod}\ q}} e^{2\pi i x_n/q} \right).
\end{aligned}$$

Therefore (3) equals

$$\sum_{\substack{x_1,\ldots,x_n \not\equiv 0 \ \text{mod}\ q \\ x_1\cdots x_n \not\equiv \pm m \ \text{mod}\ q}} e^{2\pi i\left(\frac{x_1+\cdots+x_n}{q}\right)} = (-1)^n - \sum_{\substack{x_1,\ldots,x_n \ \text{mod}\ q \\ x_1\cdots x_n \equiv \pm m \ \text{mod}\ q}} e^{2\pi i\left(\frac{x_1+\cdots+x_n}{q}\right)}.$$

The identity of the lemma now follows immediately. $\square$

3 The case of $\mathrm{SL}(2,\mathbb{Z})$

3.1 Maass forms for $\mathrm{SL}(2,\mathbb{Z})$

The hyperbolic Laplacian Δ defined by $\Delta = -y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)$ acts on functions on $\mathbb{H}$ and is invariant under the action of $\mathrm{SL}(2,\mathbb{R})$.

Let $\nu \in \mathbb{C}$. A Maass form of type ν for $\mathrm{SL}(2,\mathbb{Z})$ is a non-zero function on $\mathbb{H}$ which satisfies

- $f(\gamma z) = f(z)$ for all $\gamma \in \mathrm{SL}(2,\mathbb{Z})$ and $z \in \mathbb{H}$;
- $\Delta f = \nu(1-\nu)f$;
- $f(x+iy) = O(y^M)$ for some $M > 0$ as $y \to \infty$.

If in addition

$$\int_0^1 f(x+iy)\mathrm{d}x = 0,$$

we call f a cuspidal Maass form or a Maass cusp form. One can prove that there are infinitely many Maass cusp forms, see for example Chapter 4 in [3] or Chapter 4 in [4].

3.1.1 The Fourier expansion of Maass forms

Let f be a Maass form of type ν for $\mathrm{SL}(2,\mathbb{Z})$. Since the element $\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$ is in $\mathrm{SL}(2,\mathbb{Z})$, the automorphic condition for f implies that $f(z+1) = f(z)$. Thus any Maass form has a Fourier expansion

$$f(z) = \sum_{n\in\mathbb{Z}} A(n,y)e^{2\pi i n x}.$$

If f is a Maass cusp form, then $A(0,y) = 0$.

Lemma 9. *Let f be a Maass cusp form of type ν for $\mathrm{SL}(2,\mathbb{Z})$. Then we have the Fourier expansion*

$$f(z) = \sum_{n\neq 0} a(n)\sqrt{y}K_{\nu-\frac{1}{2}}(2\pi|n|y)e^{2\pi i n x}$$

for some coefficients $a(n) \in \mathbb{C}$ and where $K_\nu(x)$ is the modified Bessel function of the second kind.

Proof. The automorphic condition $\Delta f = \nu(1-\nu)f$ implies that

$$-y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\left(A(n,y)e^{2\pi i n x}\right) = \nu(1-\nu)A(n,y)e^{2\pi i n x}.$$

It follows that $A(n,y)$ satisfies the differential equation

$$\frac{\mathrm{d}^2 A(n,y)}{\mathrm{d}y^2} - \left(4\pi^2 n^2 - \frac{\nu(1-\nu)}{y^2}\right)A(n,y) = 0.$$

This differential equation admits two linearly independent solutions over $\mathbb{C}$, namely $\sqrt{2\pi|n|y}K_{\nu-\frac{1}{2}}(2\pi|n|y)$ and $\sqrt{2\pi|n|y}I_{\nu-\frac{1}{2}}(2\pi|n|y)$, where K_ν and I_ν are the modified Bessel functions. Of these, only K_ν is of rapid decay at ∞ and thus $A(n,y)$ must be a scalar multiple of $\sqrt{y}K_{\nu-\frac{1}{2}}(2\pi|n|y)$. For a reference for these facts on Bessel functions see *e.g.* 9.6 in [5]. $\square$

The Selberg conjecture for a Maass forms of type ν for $\mathrm{SL}(2,\mathbb{Z})$ asserts that $\nu(1-\nu)$ is greater than or equal to $1/4$. The next lemma proves the Selberg conjecture for Maass forms for $\mathrm{SL}(2,\mathbb{Z})$ with the much better lower bound $3\pi^2/2 > 1/4$.

Lemma 10. *Let f be a Maass cusp form of type ν for $\mathrm{SL}(2,\mathbb{Z})$. Then $\nu(1-\nu) \geq 3\pi^2/2$.*

Proof. Let

$$f(z) = \sum_{n \neq 0} a(n) \sqrt{y} K_{\nu - \frac{1}{2}}(2\pi |n| y) e^{2\pi i n x}$$

be a Maass cusp form of type ν for $\mathrm{SL}(2, \mathbb{Z})$. It is an eigenfunction of the Laplace operator Δ with eigenvalue $\lambda = \nu(1-\nu)$. Let $\mathcal{F}$ denote the standard fundamental domain for the action of $\mathrm{SL}(2, \mathbb{Z})$ on $\mathbb{H}$ and let $\mathcal{F}^* = S(\mathcal{F})$, where $S(z) = -1/z$. Note that $\mathcal{F} \cup \mathcal{F}^* \supset \{z \in \mathbb{H} \mid |x| \leq \frac{1}{2}, y \geq \frac{\sqrt{3}}{2}\}$. Using that $\{e^{2\pi i n x}\}_{n \in \mathbb{Z}}$ is a complete orthogonal system for $L^2([-1/2, 1/2])$, we can estimate

$$\begin{aligned}
2\lambda \langle f, f \rangle = 2 \langle \nabla f, \nabla f \rangle &= \int_{\mathcal{F} \cup \mathcal{F}^*} \left(\left(\frac{\partial f}{\partial x} \right)^2 + \left(\frac{\partial f}{\partial y} \right)^2 \right) \mathrm{d}x \mathrm{d}y \\
&\geq \int_{\frac{\sqrt{3}}{2}}^{\infty} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\frac{\partial f}{\partial x} \right)^2 \mathrm{d}x \mathrm{d}y \\
&= 4\pi^2 \int_{\frac{\sqrt{3}}{2}}^{\infty} \sum_{n \neq 0} n^2 \left| a(n) \sqrt{y} K_{\nu - \frac{1}{2}}(2\pi |n| y) \right|^2 \mathrm{d}y \\
&\geq 4\pi^2 \frac{3}{4} \int_{\frac{\sqrt{3}}{2}}^{\infty} \sum_{n \neq 0} \left| a(n) \sqrt{y} K_{\nu - \frac{1}{2}}(2\pi |n| y) \right|^2 \frac{\mathrm{d}y}{y^2} \\
&= 3\pi^2 \int_{\frac{\sqrt{3}}{2}}^{\infty} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| f(z) \right|^2 \frac{\mathrm{d}x \mathrm{d}y}{y^2} \\
&\geq 3\pi^2 \int_{\mathcal{F}} \left| f(z) \right|^2 \mathrm{d}\mu(z) = 3\pi^2 \langle f, f \rangle.
\end{aligned}$$

It follows that $\lambda \geq 3\pi^2/2$. □

3.1.2 Even and Odd Maass forms

We define the operator $T_{-1} : L^2(\mathrm{SL}(2, \mathbb{Z}) \setminus \mathbb{H}) \to L^2(\mathrm{SL}(2, \mathbb{Z}) \setminus \mathbb{H})$ by

$$(T_{-1} f)(x + iy) = f(-x + iy).$$

Since the Laplacian is invariant unter $x \mapsto -x$, the operator T_{-1} maps Maass forms of type ν to Maass forms of type ν. Since $(T_{-1})^2 = 1$, the eigenvalues of T_{-1} are ± 1. A Maass form f is called even if $T_{-1} f = f$, and odd if $T_{-1} f = -f$. If f is even, its Fourier coefficients satisfy $a(n) = a(-n)$, while they satisfy $a(n) = -a(-n)$ if f is odd. To unify notation, we write $a(n) = (-1)^{\epsilon} a(-n)$, where $\epsilon = 0$ if f is even and $\epsilon = 1$ if f is odd.

3.1.3 The non-holomorphic Eisenstein series

In this section we will introduce a (non-cuspidal) Maass form for $\mathrm{SL}(2, \mathbb{Z})$. The group $\mathrm{SL}(2, \mathbb{Z})$ has only a cusp at infinity. The stabilizer of this cups in $\mathrm{SL}(2, \mathbb{Z})$ is

$$\Gamma_\infty = \left\{ \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \mid m \in \mathbb{Z} \right\}.$$

Notice that $\mathrm{Im}(z)$ is Γ_∞-invariant. The Eisenstein series associated to this cusp, defined on $\mathbb{H} \times \mathbb{C}$, is defined by

$$E(z, s) := E_\infty(z, s) = \frac{1}{2} \sum_{\gamma \in \Gamma_\infty \setminus \mathrm{SL}(2, \mathbb{Z})} \mathrm{Im}(\gamma z)^s. \tag{4}$$

For $\mathrm{Re}(s) > 1$, one can show that this series converges absolutely and uniformly on compact sets. The Eisenstein series $E(z, s)$ defines an automorphic function with respect to $\mathrm{SL}(2, \mathbb{Z})$, that is, it satisfies $E(\gamma z, s) = E(z, s)$ for all $\gamma \in \mathrm{SL}(2, \mathbb{Z})$. Because $\Delta y^s = s(1-s) y^s$ and because Δ commutes with the $\mathrm{SL}(2, \mathbb{Z})$-action, the Eisenstein series is also an eigenfunction of the Laplacian

$$\Delta E(z, s) = s(1-s) E(z, s).$$

The Eisenstein series $E(z,s)$ is in fact a Maass form. This will come as a consequence of the Fourier expansion of $E(z,s)$.

We first rewrite the Eisenstein series in a form more convenient for explicit computations. The cosets $\Gamma_\infty \setminus \mathrm{SL}(2,\mathbb{Z})$ are determined by the bottom row of a representative

$$\Gamma_\infty \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \left\{ \begin{pmatrix} s & t \\ c & d \end{pmatrix} \middle| \, sd - ct = 1 \right\}.$$

A pair of coprime integers (c,d) uniquely determines such a coset. To sum over $(m,n) \in \mathbb{Z}^2 \setminus \{0,0\}$ is the same as to sum over all positive integers M and all pairs (c,d) of coprime integers by taking $(m,n) = (Mc, Md)$. As a consequence, we can write

$$\begin{aligned} E(z,s) &= \frac{1}{2} \sum_{\gamma \in \Gamma_\infty \setminus \mathrm{SL}(2,\mathbb{Z})} \mathrm{Im}(\gamma z)^s \\ &= \frac{1}{2} \sum_{(c,d)=1} \frac{y^s}{|cz+d|^{2s}} \\ &= \frac{1}{2\zeta(2s)} \sum_{\substack{m,n \in \mathbb{Z} \\ (m,n) \neq 0}} \frac{y^s}{|mz+n|^{2s}}. \end{aligned} \tag{5}$$

For the calculation of the Fourier expansion of $E(z,s)$ we need to following two identities.

Lemma 11. *If* $\mathrm{Re}(s) > 1/2$ *and* $r \in \mathbb{R}$, *then*

$$\int_{-\infty}^{\infty} \frac{e^{-2\pi i r x}}{(x^2+y^2)^s} \mathrm{d}x = \frac{\pi^s}{y^s \Gamma(s)} \begin{cases} \pi^{-s+\frac{1}{2}} \Gamma(s - \frac{1}{2}) y^{1-s} & \text{if } r = 0 \\ 2|r|^{s-\frac{1}{2}} \sqrt{y} K_{s-\frac{1}{2}}(2\pi |r| y) & \text{if } r \neq 0 \end{cases}. \tag{6}$$

Proof. Recall the integral representation of the Gamma function

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} \mathrm{d}t.$$

Thus we have

$$\begin{aligned} \frac{y^s}{\pi^s} \Gamma(s) \int_{-\infty}^{\infty} \frac{e^{-2\pi i r x}}{(x^2+y^2)^s} \mathrm{d}x &= \int_0^\infty \int_{-\infty}^{\infty} e^{-t} \left(\frac{ty}{\pi(x^2+y^2)} \right)^s e^{-2\pi i r x} \mathrm{d}x \frac{\mathrm{d}t}{t} \\ &= \int_0^\infty \int_{-\infty}^{\infty} e^{-\pi \xi (x^2+y^2)/y} \xi^s e^{-2\pi i r x} \mathrm{d}x \frac{\mathrm{d}\xi}{\xi}. \end{aligned}$$

For $r = 0$ the above expression becomes

$$\begin{aligned} \int_0^\infty \int_{-\infty}^{\infty} e^{-\pi \xi (x^2+y^2)/y} \xi^s \mathrm{d}x \frac{\mathrm{d}\xi}{\xi} &= \int_0^\infty \sqrt{\frac{y}{\xi}} e^{-\pi \xi y} \xi^s \frac{\mathrm{d}\xi}{\xi} \\ &= \pi^{-s+\frac{1}{2}} y^{1-s} \Gamma(s - \frac{1}{2}). \end{aligned}$$

For $r \neq 0$ we obtain, using the change of variables $\xi \mapsto \frac{\xi}{|r|}$,

$$\begin{aligned} \int_0^\infty \int_{-\infty}^{\infty} e^{-\pi \xi (x^2+y^2)/y} \xi^s e^{-2\pi i r x} \mathrm{d}x \frac{\mathrm{d}\xi}{\xi} &= \int_0^\infty \sqrt{\frac{y}{\xi}} e^{-y\pi r^2/\xi} e^{-\pi \xi y} \xi^s \frac{\mathrm{d}\xi}{\xi} \\ &= \sqrt{y} |r|^{s-\frac{1}{2}} \int_0^\infty \xi^{s-\frac{1}{2}} e^{-y\pi |r| (1/\xi + \xi)} \frac{\mathrm{d}\xi}{\xi} \\ &= 2|r|^{s-\frac{1}{2}} \sqrt{y} K_{s-\frac{1}{2}}(2\pi |r| y). \end{aligned}$$

Here we used the following integral representation of the modified Bessel function of the second kind $K_\nu(x)$ (see [6] page 182)

$$K_\nu(x) = \frac{1}{2} \int_0^\infty e^{-\frac{x}{2}(t+t^{-1})} t^{-\nu-1} \mathrm{d}t.$$

□

Lemma 12. *For $z = x + iy \in \mathbb{H}$ and $\mathrm{Re}(s) > \frac{1}{2}$ we have*

$$\sum_{n\in\mathbb{Z}} \frac{1}{|z+n|^{2s}} = \frac{\sqrt{\pi}\Gamma(s-\frac{1}{2})}{\Gamma(s)} y^{1-2s} + \frac{2\pi^s}{\Gamma(s)} y^{\frac{1}{2}-s} \sum_{r\neq 0} |r|^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|r|y)\, e^{2\pi irx}. \tag{7}$$

Proof. Recall the Poisson summation formula

$$\sum_{n\in\mathbb{Z}} \varphi(x+n) = \sum_{n\in\mathbb{Z}} \widehat{\varphi}(n) e^{2\pi inx},$$

where φ is a continuous function that decays sufficiently rapidly at infinity (for example, $|f(x)| < |x|^{-c}$ with $c > 1$) and where $\widehat{\varphi}(n) = \int_{-\infty}^{\infty} \varphi(x)e^{-2\pi inx}\mathrm{d}x$ is the Fourier transform. We apply this formula to $\varphi(x) = |x+iy|^{-2s}$, where $z = x+iy \in \mathbb{H}$ and $\mathrm{Re}(s) > \frac{1}{2}$. The Poisson summation formula gives

$$\sum_{n\in\mathbb{Z}} \frac{1}{|z+n|^{2s}} = \sum_{n\in\mathbb{Z}} \left(\int_{-\infty}^{\infty} \frac{e^{-2\pi inx}}{(x^2+y^2)^s} \mathrm{d}x \right) e^{2\pi inx}.$$

Using Lemma 11 we get the result. □

Lemma 13. *The Fourier expansion of $E(z,s)$ is given by*

$$E(z,s) = y^s + \phi(s)y^{1-s} + \frac{2\pi^s}{\Gamma(s)\zeta(2s)} y^{1/2} \sum_{n\neq 0} \sigma_{1-2s}(n)|n|^{s-\frac{1}{2}} K_{s-1/2}(2\pi|n|y)e^{2\pi inx}, \tag{8}$$

where

$$\begin{aligned}
\phi(s) &= \pi^{1/2} \frac{\Gamma(s-1/2)}{\Gamma(s)} \frac{\zeta(2s-1)}{\zeta(2s)}, \\
\sigma_s(n) &= \sum_{\substack{d|n \\ d>0}} d^s.
\end{aligned}$$

Proof. We split up the sum (5) into the terms with $m = 0$ and those with $m \neq 0$ and combine each positive summand with its negative. We obtain

$$\zeta(2s)E(z,s) = \zeta(2s)y^s + \sum_{m=1}^{\infty} \sum_{n\in\mathbb{Z}} \frac{y^s}{|mz+n|^{2s}}.$$

Using the substitution $n = tm + r$ and Lemma 12 gives

$$\begin{aligned}
\zeta(2s)E(z,s) &= \zeta(2s)y^s + y^s \sum_{m=1}^{\infty} |m|^{-2s} \sum_{r=1}^{m} \sum_{t\in\mathbb{Z}} \left| t + z + \frac{r}{m} \right|^{-2s} \\
&= \zeta(2s)y^s + y^s \sum_{m=1}^{\infty} |m|^{-2s} \\
&\quad \times \left[\frac{\sqrt{\pi}\Gamma(s-\frac{1}{2})}{\Gamma(s)} m y^{1-2s} + \frac{2\pi^s}{\Gamma(s)} y^{\frac{1}{2}-s} \sum_{t\neq 0} |t|^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|t|y) \sum_{r=1}^{m} e^{2\pi it\left(x+\frac{r}{m}\right)} \right] \\
&= \zeta(2s)y^s + \frac{\sqrt{\pi}\Gamma(s-\frac{1}{2})\zeta(2s-1)}{\Gamma(s)} y^{1-s} \\
&\quad + \frac{2\pi^s\sqrt{y}}{\Gamma(s)} \sum_{t\neq 0} |t|^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|t|y) e^{2\pi itx} \sum_{m=1}^{\infty} |m|^{-2s} \sum_{r=1}^{m} e^{\frac{2\pi itr}{m}}.
\end{aligned}$$

The sum $\sum_{r=1}^{m} e^{\frac{2\pi i t r}{m}}$ is zero unless $t = lm$ for some $l \in \mathbb{Z}$ in which case the sum is m. Therefore we get

$$\begin{aligned}
\zeta(2s)E(z,s) &= \zeta(2s)y^s + \frac{\sqrt{\pi}\Gamma(s-\frac{1}{2})\zeta(2s-1)}{\Gamma(s)}y^{1-s} + \frac{2\pi^s\sqrt{y}}{\Gamma(s)}\sum_{t\neq 0}|t|^{s-\frac{1}{2}}K_{s-\frac{1}{2}}(2\pi|t|y)e^{2\pi i t x}\sum_{\substack{m\geq 1\\ t=ml}}|m|^{-2s+1} \\
&= \zeta(2s)y^s + \frac{\sqrt{\pi}\Gamma(s-\frac{1}{2})\zeta(2s-1)}{\Gamma(s)}y^{1-s} + \frac{2\pi^s\sqrt{y}}{\Gamma(s)}\sum_{t\neq 0}|t|^{s-\frac{1}{2}}K_{s-\frac{1}{2}}(2\pi|t|y)e^{2\pi i t x}\sum_{l|t}\left|\frac{t}{l}\right|^{-2s+1} \\
&= \zeta(2s)y^s + \frac{\sqrt{\pi}\Gamma(s-\frac{1}{2})\zeta(2s-1)}{\Gamma(s)}y^{1-s} + \frac{2\pi^s\sqrt{y}}{\Gamma(s)}\sum_{t\neq 0}\sigma_{1-2s}(t)|t|^{s-\frac{1}{2}}K_{s-\frac{1}{2}}(2\pi|t|y)e^{2\pi i t x}.
\end{aligned}$$

□

A consequence of the Fourier expansion is the meromorphic continuation of $E(z,s)$ to $s \in \mathbb{C}$. From the Fourier expansion it is also easy to deduce the functional equation

$$E(z,s) = \frac{1}{\phi(1-s)}E(z,1-s).$$

3.1.4 Twisted Maass forms

Let

$$f(z) = \sum_{n\neq 0} a(n)y^{1/2}K_{\nu-\frac{1}{2}}(2\pi|n|y)e^{2\pi i n x}$$

be a cuspidal Maass form of type ν for $\mathrm{SL}(2,\mathbb{Z})$. Let χ be a primitive Dirichlet character modulo a prime q. The twist of f by χ is defined by

$$f_\chi(z) = \sum_{n\neq 0} a(n)\chi(n)y^{1/2}K_{\nu-\frac{1}{2}}(2\pi|n|y)e^{2\pi i n x}. \tag{9}$$

The twist of a $\mathrm{SL}(2,\mathbb{Z})$-Maass form by a Dirichlet character χ of modulus q is not automorphic for $\mathrm{SL}(2,\mathbb{Z})$ anymore. Instead it is automorphic with nebentypus χ^2 for the Hecke congruence subgroup

$$\Gamma_0(q^2) = \left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) \,|\, c \equiv 0 \pmod{q^2}\right\}.$$

More generally, for twists of $\Gamma_0(N)$-Maass forms we have the following result.

Lemma 14. *The twist of a cuspidal Maass form for $\Gamma_0(N)$ is an automorphic form for $\Gamma_0(q^2N)$ with nebentypus χ^2. That is, for all $\gamma \in \Gamma_0(q^2N)$, we have*

$$f_\chi(\gamma z) = \chi(\gamma)^2 f_\chi(z).$$

Proof. By Lemma 3, the Gauss sum $\tau(\overline{\chi})$ is not zero. Then Lemma 2 allows us to write

$$f_\chi(z) = \frac{1}{\tau(\overline{\chi})}\sum_{l=1}^{q}\overline{\chi(l)}f(z+\frac{l}{q}).$$

Now let $\gamma = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \Gamma_0(q^2N)$. Then

$$\begin{pmatrix} 1 & l/q \\ 0 & 1\end{pmatrix}\gamma \in \Gamma_0(N)\begin{pmatrix} 1 & d^2l/q \\ 0 & 1\end{pmatrix}.$$

Thus for all $z \in \mathbb{H}$, the point $\gamma z + \frac{l}{q}$ lies in the $\Gamma_0(N)$-orbit of $z + \frac{d^2 l}{q}$. It follows that

$$\begin{aligned} f_\chi(\gamma z) &= \frac{1}{\tau(\overline{\chi})} \sum_{l=1}^{q} \overline{\chi(l)} f(\gamma z + \frac{l}{q}) \\ &= \frac{1}{\tau(\overline{\chi})} \sum_{l=1}^{q} \overline{\chi(l)} f(z + \frac{d^2 l}{q}) \\ &= \frac{1}{\tau(\overline{\chi})} \sum_{l=1}^{q} \overline{\chi(ld^2)} \chi(d)^2 f(z + \frac{d^2 l}{q}) \\ &= \chi(d)^2 f_\chi(z). \end{aligned}$$

□

3.1.5 Eisenstein series of level Nq^2 and nebentypus χ

The Rankin-Selberg method, see Section 3.4, consists in constructing L-functions as integral transforms of Maass forms and Eisenstein series. For this method to work for twisted Maass forms, we need Eisenstein series that have a nebentypus and that are automorphic for congruence subgroups. In this section, we will define such Eisenstein series and prove their analytic continuation and functional equation.

Let q be a prime and let N be a positive integer. In this section we assume that χ is an even, non-principal (and thus primitive) Dirichlet character mod q. For $\mathrm{Re}(s) > 1$ we define the Eisenstein series $E(z, s, \chi)$ of level Nq^2 and with nebentypus χ by the absolutely convergent series

$$E(z, s, \chi) = \frac{1}{2} \sum_{\gamma} \chi(d) \, \mathrm{Im}(\gamma z)^s,$$

where the sum goes over a set of coset representatives $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\infty \setminus \Gamma_0(Nq^2)$. We will sometimes write $\chi(\gamma)$ to denote $\chi(d)$.

The Eisenstein series $E(z, s, \chi)$ satisfies indeed the automorphic relation $E(\gamma z, s, \chi) = \overline{\chi(\gamma)} E(z, s, \chi)$ for all $\gamma \in \Gamma_0(Nq^2)$. It follows in particular that the function $E(z, s, \chi)$ is invariant under $z \mapsto z + 1$. Hence it has a Fourier expansion in the x-variable (our assumption that χ is even simplifies the calculation of the Fourier expansion). We determine the Fourier expansion of $E(z, s, \chi)$ by adapting the calculation we did in Section 3.1.3 for the non-holomorphic Eisenstein series $E(z, s)$.

The cosets $\Gamma_\infty \setminus \Gamma_0(Nq^2)$ are determined by the bottom row of a representative

$$\Gamma_\infty \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \left\{ \begin{pmatrix} s & t \\ c & d \end{pmatrix} \middle| \, sd - ct = 1, Nq^2 | c \right\}.$$

A pair of coprime integers (c, d), subject to $Nq^2 | c$, uniquely determines such a coset. To sum over $(m, n) \in \mathbb{Z}^2 \setminus \{0, 0\}$ is the same as to sum over all positive integers M and all pairs (c, d) of coprime integers by taking $(m, n) = (Mc, Md)$. As a consequence, we can write

$$\begin{aligned} E^*(z, s, \chi) &:= L(2s, \chi) E(z, s, \chi) \\ &= \frac{L(2s, \chi)}{2} \sum_{\gamma \in \Gamma_\infty \setminus \Gamma_0(Nq^2)} \chi(d) \, \mathrm{Im}(\gamma z)^s \\ &= \frac{L(2s, \chi)}{2} \sum_{\substack{(c,d)=1 \\ Nq^2 | c}} \chi(d) \frac{y^s}{|cz + d|^{2s}} \\ &= \frac{1}{2} \sum_{\substack{m,n \in \mathbb{Z} \\ (m,n) \neq 0}} \chi(n) \frac{y^s}{|mNq^2 z + n|^{2s}}, \end{aligned} \tag{10}$$

where we have introduced the Dirichlet L-function $L(s,\chi) = \sum_{n=1}^{\infty} \chi(n)n^{-s}$. We would like to adapt the left hand side of Equation 7 to the setting where the sum is twisted by a primitive Dirichlet character χ mod q. For this we need the twisted variant of the Poisson summation formula.

Lemma 15. *Let φ be a function that satisfies the conditions of the Poisson summation formula. Let χ be a primitive Dirichlet character mod q. Then*

$$\sum_{n\in\mathbb{Z}} \chi(n)\varphi(x+n) = \frac{\tau(\chi)}{q} \sum_{n\in\mathbb{Z}} \overline{\chi(n)}\widehat{\varphi}(n/q)e^{2\pi i n x/q}.$$

Proof. From Lemma 2 we know that

$$\chi(n) = \frac{1}{\tau(\overline{\chi})} \sum_{m \bmod q} \overline{\chi(m)} e^{2\pi i n m/q} = \frac{\chi(-1)\tau(\chi)}{q} \sum_{m \bmod q} \overline{\chi(m)} e^{2\pi i n m/q}.$$

Let us now consider $\sum_{n\in\mathbb{Z}} \chi(n)\varphi(n) = \sum_{n\in\mathbb{Z}} \varphi_1(n)$, where

$$\varphi_1(x) = \frac{\chi(-1)\tau(\chi)}{q} \sum_{m \bmod q} \overline{\chi(m)} e^{2\pi i x m/q}\varphi(x).$$

The Fourier transform of $\varphi_1(x)$ is

$$\begin{aligned}
\widehat{\varphi_1}(\xi) &= \frac{\chi(-1)\tau(\chi)}{q} \sum_{m \bmod q} \overline{\chi(m)} \int_{-\infty}^{\infty} \varphi(x) e^{-2\pi i x\left(\xi - \frac{m}{q}\right)} \mathrm{d}x \\
&= \frac{\chi(-1)\tau(\chi)}{q} \sum_{m \bmod q} \overline{\chi(m)}\widehat{\varphi}\left(\xi - \frac{m}{q}\right).
\end{aligned}$$

We apply Poisson summation formula and thus $\sum_{n\in\mathbb{Z}} \varphi_1(n)$ equals

$$\frac{\chi(-1)\tau(\chi)}{q} \sum_{m \bmod q} \sum_{n=-\infty}^{\infty} \overline{\chi(m)}\widehat{\varphi}\left(\frac{nq-m}{q}\right).$$

We can write $\overline{\chi(m)} = \chi(-1)\overline{\chi(nq-m)}$. When n runs through $\mathbb{Z}$ and m through $\mathbb{Z}/q\mathbb{Z}$, the terms $nq-m$ run uniquely through $\mathbb{Z}$. Thus we have shown

$$\sum_{n\in\mathbb{Z}} \chi(n)\varphi(n) = \frac{\tau(\chi)}{q} \sum_{n\in\mathbb{Z}} \overline{\chi(n)}\widehat{\varphi}(n/q). \tag{11}$$

Replacing $\varphi(n)$ by $\varphi(n+x)$ replaces $\widehat{\varphi}(\xi)$ by $\widehat{\varphi}(\xi)e^{2\pi i\xi x}$. This completes the proof. ☐

We can now determine the Fourier expansion of $E^*(z,s,\chi)$. We present two calculations, one making directly use of the twisted Poisson summation formula and the other not.

Theorem 16. *The function $E^*(z,s,\chi)$ has the Fourier expansion*

$$E^*(z,s,\chi) = L(2s,\chi)y^s + \frac{2\pi^s\tau(\chi)\sqrt{y}}{q^{4s-1}N^{2s-1}\Gamma(s)} \sum_{t\neq 0} |t|^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|t|y)e^{2\pi i t x} \sum_{qNl|t} \left|\frac{t}{qNl}\right|^{1-2s} \overline{\chi}(l).$$

First Proof. We split up the sum (10) into the terms with $m=0$ and those with $m\neq 0$. The assumption that χ is even allows us to combine each positive summand with its negative. We obtain

$$E^*(z,s,\chi) = L(2s,\chi)y^s + \sum_{m=1}^{\infty} \sum_{n\in\mathbb{Z}} \chi(n) \frac{y^s}{|mNq^2z+n|^{2s}}.$$

Using the twisted Poisson summation formula (15) in the proof of Lemma 12 allows us to deduce

$$E^*(z,s,\chi) = L(2s,\chi)y^s + \frac{\tau(\chi)}{q} y^s \frac{2\pi^s}{\Gamma(s)} \sum_{m=1}^{\infty} \sum_{n\neq 0} \overline{\chi(n)} \left(\frac{|n|}{q}\right)^{s-\frac{1}{2}} \left(ymNq^2\right)^{\frac{1}{2}-s} K_{s-\frac{1}{2}}(2\pi|n|ymNq) e^{2\pi i nxmNq}.$$

Summing $m \in Nq\mathbb{Z}_{\geq 1}$ and $n \in \mathbb{Z}_{\geq 1}$ is the same as summing their product k over $\mathbb{Z}_{\geq 1}$ and summing for each k over the pairs (n, mNq) such that $Nqmn = k$. Accordingly we can write

$$\begin{aligned}
E^*(z,s,\chi) &= L(2s,\chi)y^s \\
&\quad + \tau(\chi) q^{-2s} \sqrt{y} \frac{2\pi^s}{\Gamma(s)} \sum_{m=1}^{\infty} \sum_{n\neq 0} \overline{\chi(n)} |n|^{s-\frac{1}{2}} (mNq)^{\frac{1}{2}-s} K_{s-\frac{1}{2}}(2\pi|n|ymNq) e^{2\pi i nxmNq} \\
&= L(2s,\chi)y^s \\
&\quad + \frac{\tau(\chi)}{q^{4s-1}N^{2s-1}} \sqrt{y} \frac{2\pi^s}{\Gamma(s)} \sum_{k\neq 0} |k|^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|k|y) e^{2\pi i kx} \sum_{\substack{m\geq 1 \\ k=mqNn}} |m|^{1-2s} \overline{\chi(n)} \\
&= L(2s,\chi)y^s \\
&\quad + \frac{\tau(\chi)}{q^{4s-1}N^{2s-1}} \sqrt{y} \frac{2\pi^s}{\Gamma(s)} \sum_{k\neq 0} |k|^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|k|y) e^{2\pi i kx} \sum_{Nqn|k} \left|\frac{k}{Nqn}\right|^{1-2s} \overline{\chi(n)}.
\end{aligned}$$

□

Second Proof. Again we start with

$$E^*(z,s,\chi) = L(2s,\chi)y^s + \sum_{m=1}^{\infty} \sum_{n\in\mathbb{Z}} \chi(n) \frac{y^s}{|mNq^2 z + n|^{2s}}.$$

Using the substitution $n = tmNq^2 + r$ and Lemma 12 gives

$$\begin{aligned}
E^*(z,s,\chi) &= L(2s,\chi)y^s \\
&\quad + \frac{y^s}{q^{4s}N^{2s}} \sum_{m=1}^{\infty} |m|^{-2s} \sum_{r=1}^{mNq^2} \chi(r) \sum_{t\in\mathbb{Z}} \left| t + z + \frac{r}{mNq^2} \right|^{-2s} \\
&= L(2s,\chi)y^s + \frac{y^s}{q^{4s}N^{2s}} \sum_{m=1}^{\infty} |m|^{-2s} \sum_{r=1}^{mNq^2} \chi(r) \\
&\quad \times \left[\frac{\sqrt{\pi}\Gamma(s-\frac{1}{2})}{\Gamma(s)} y^{1-2s} + \frac{2\pi^s}{\Gamma(s)} y^{\frac{1}{2}-s} \sum_{t\neq 0} |t|^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|t|y)\, e^{2\pi i t\left(x + \frac{r}{mNq^2}\right)} \right] \\
&= L(2s,\chi)y^s + \frac{2\pi^s\sqrt{y}}{q^{4s}N^{2s}\Gamma(s)} \sum_{t\neq 0} |t|^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|t|y) e^{2\pi i tx} \sum_{m=1}^{\infty} |m|^{-2s} \sum_{r=1}^{mNq^2} \chi(r) e^{\frac{2\pi i tr}{mNq^2}}.
\end{aligned}$$

In the last equality we used Lemma 1 according to which $\sum_{r=1}^{mNq^2} \chi(r) = \sum_{r=1}^{q} \chi(r) = 0$. By Lemma 4 the last sum equals

$$\sum_{r=1}^{mNq^2} \chi(r) e^{\frac{2\pi i tr}{mNq^2}} = \begin{cases} mNq\tau(\chi)\overline{\chi}(l) & \text{if } t = mqNl \text{ for some } l \in \mathbb{Z} \\ 0 & \text{else} \end{cases}.$$

Therefore we get

$$\begin{aligned}
E^*(z,s,\chi) &= L(2s,\chi)y^s + \frac{2\pi^s\tau(\chi)\sqrt{y}}{q^{4s}N^{2s}\Gamma(s)} \sum_{t\neq 0} |t|^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|t|y) e^{2\pi i tx} \sum_{\substack{m\geq 1 \\ t=mqNl}} |m|^{-2s} mNq\overline{\chi}(l) \\
&= L(2s,\chi)y^s + \frac{2\pi^s\tau(\chi)\sqrt{y}}{q^{4s-1}N^{2s-1}\Gamma(s)} \sum_{t\neq 0} |t|^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|t|y) e^{2\pi i tx} \sum_{qNl|t} \left|\frac{t}{qNl}\right|^{1-2s} \overline{\chi}(l).
\end{aligned}$$

□

Because $\Gamma(s)$ has no zeros and $L(2s,\chi)$ is entire, the above Fourier expansion shows that the function $s \mapsto E^*(z,s,\chi)$ is entire. We would also like to obtain a functional equation for the twisted Eisenstein series $E(z,s,\chi)$. To this end, we introduce the completed Eisenstein series

$$E^{\#}(z,s,\chi) := \pi^{-s}\Gamma(s)E^*(z,s,\chi) = \pi^{-s}\Gamma(s)L(2s,\chi)E(z,s,\chi). \tag{12}$$

The zeros of $L(2s,\chi)$ cancel with the poles of $\Gamma(s)$, hence $s \mapsto E^{\#}(z,s,\chi)$ is also an entire function.

Theorem 17. *The completed Eisenstein series $E^{\#}(z,s,\chi)$ satisfies the functional equation*

$$E^{\#}(z,s,\chi) = \frac{\tau(\chi)}{\sqrt{q}} N^{1-2s} q^{\frac{5}{2}-5s} E^{\#}\left(\frac{-1}{q^3N^2z}, 1-s, \overline{\chi}\right). \tag{13}$$

Proof. The idea of the proof is to express the completed Eisenstein series $E^{\#}(z,s,\chi)$ as the Mellin transform of the twisted theta series

$$\Theta_\chi(t,z) = \sum_{(m,n)\in\mathbb{Z}^2} \chi(n) e^{-\pi|mz+n|^2 t/y},$$

where $t > 0$. This theta series itself satisfies a functional equation.

Step 1. *The twisted theta series $\Theta_\chi(t,z)$ satisfies the functional equation*

$$\Theta_\chi(t,z) = \frac{\tau(\chi)}{qt}\Theta_{\overline{\chi}}(1/qt, -q/z).$$

The Fourier transform of

$$f_{z,t}(m,n) = e^{-\pi|mz+n|^2 t/y}$$

is given by

$$\widehat{f_{z,t}}(m,n) = \frac{1}{t} e^{-\pi\left((yn)^2+(m-xn)\right)^2/ty}.$$

Hence the twisted Poisson summation formula (11) applied to $f_{z,t}(m,n)$ implies that

$$\begin{aligned}
\Theta_\chi(t,z) &= \frac{\tau(\chi)}{qt} \sum_{(m,n)\in\mathbb{Z}^2} \overline{\chi(n)} e^{-\pi\left((yn)^2+(qm-xn)\right)^2/tq^2y} \\
&= \frac{\tau(\chi)}{qt} \sum_{(m,n)\in\mathbb{Z}^2} \overline{\chi(n)} e^{-\pi|nz-qm|^2/tq^2y} \\
&= \frac{\tau(\chi)}{qt} \sum_{(m,n)\in\mathbb{Z}^2} \overline{\chi(n)} e^{-\pi|n-qmz^{-1}|^2|z|^2/tq^2y} \\
&= \frac{\tau(\chi)}{qt}\Theta_{\overline{\chi}}(1/qt, -q/z),
\end{aligned}$$

where in the last line we used that the imaginary part of $-q/z$ is $qy/|z|^2$.

Step 2. *We deduce the functional equation (13).*

Step 1. followed by the change of variables $t \mapsto N^{-s}q^{-5}t^{-1}$ gives

$$\begin{aligned}
E^{\#}(z,s,\chi) &= \frac{1}{2}\int_0^\infty \Theta_\chi(Nq^2t, Nq^2z)t^s \frac{\mathrm{d}t}{t} \\
&= \frac{\tau(\chi)}{Nq^3}\frac{1}{2}\int_0^\infty \Theta_{\overline{\chi}}(1/Nq^3t, -1/Nqz)t^{s-1}\frac{\mathrm{d}t}{t} \\
&= \tau(\chi)N^{1-2s}q^{2-5s}\frac{1}{2}\int_0^\infty \Theta_{\overline{\chi}}(Nq^2t, Nq^2\left(-1/N^2q^3z\right))t^{1-s}\frac{\mathrm{d}t}{t} \\
&= \tau(\chi)N^{1-2s}q^{2-5s}E^{\#}\left(-1/N^2q^3z, 1-s, \overline{\chi}\right).
\end{aligned}$$

□

3.2 Hecke operators for $L^2(\mathrm{SL}(2,\mathbb{Z}) \setminus \mathbb{H})$

For $n \geq 1$, the Hecke operator $T_n : L^2(\mathrm{SL}(2,\mathbb{Z}) \setminus \mathbb{H}) \to L^2(\mathrm{SL}(2,\mathbb{Z}) \setminus \mathbb{H})$ is defined by

$$(T_n f)(z) = \frac{1}{\sqrt{n}} \sum_{ad=n} \sum_{b=0}^{d-1} f\left(\frac{az+b}{d}\right). \tag{14}$$

The Hecke operators T_n $(n = 1, 2, \dots)$ are self-adjoint in $L^2(\mathrm{SL}(2,\mathbb{Z}) \setminus \mathbb{H})$, that is

$$\langle T_n f, g\rangle = \langle f, T_n g\rangle,$$

for all $f, g \in L^2(\mathrm{SL}(2,\mathbb{Z}) \setminus \mathbb{H})$. Given two positive integers n and m, the Hecke operators T_n and T_m commute. In fact, we have the multilicative rule

$$T_m T_n = \sum_{d|(m,n)} T_{mn/d^2}.$$

Furthermore, the Hecke operators commute with T_{-1} and the Laplacian Δ, and T_{-1} and Δ also commute. It follows that the space $L^2(\mathrm{SL}(2,\mathbb{Z}) \setminus \mathbb{H})$ can be simultaneously diagonalized by the operators $\{\Delta, T_n | \, n = -1, 1, 2, \dots\}$. Consequently, we can consider even or odd Maass forms that are simultaneous eigenfunctions for $\{T_n | \, n = 1, 2, \dots\}$. Such an eigenform f thus satisfies

$$T_n f = \lambda_n f,$$

for all n and for some $\lambda_n \in \mathbb{R}$. The importance of such eigenforms lies in the following multiplicative relations of their Fourier coefficients.

Lemma 18. *Let*

$$f(z) = \sum_{M \neq 0} a(M) \sqrt{y} K_{\nu - \frac{1}{2}}(2\pi |M| y) e^{2\pi i M x}$$

be a cuspidal Maass form of type ν for $SL(2,\mathbb{Z})$ which is an eigenfunction of all the Hecke operators. If $a(1) = 0$, then f vanishes. Assume that $f \neq 0$, and that it is normalized so that $a(1) = 1$. Then

$$T_n f = a(n) f, \qquad \forall n = 1, 2, \dots$$

Furthermore, we have the multiplicative relations of the Fourier coefficients

$$\begin{aligned}
a(m)a(n) &= a(mn), \qquad \text{if } (m,n) = 1,\\
a(m)a(n) &= \sum_{d|(m,n)} a\left(\frac{mn}{d^2}\right),\\
a(p^{k+1}) &= a(p)a(p^k) - a(p^{k-1}),
\end{aligned}$$

for all primes p and all $k \geq 1$.

Proof. It follows from (14) that

$$T_n f(z) = \sum_{M \neq 0} a(M) \sum_{ad=n} \sqrt{\frac{ay}{nd}} K_{\nu - \frac{1}{2}}(2\pi |M| ay/d) \sum_{0 \leq b < d} e^{2\pi i M \frac{ax+b}{d}}.$$

The last sum over b is zero unless $d|M$, in which case the sum is d. So if we write $M = dm'$, and then $m = m'a$, we obtain

$$\begin{aligned}
T_n f(z) &= \sum_{m' \neq 0} a(dm') \sum_{ad=n} \sqrt{\frac{ady}{n}} K_{\nu - \frac{1}{2}}(2\pi |m'| ay) e^{2\pi i m' ax}\\
&= \sum_{m \neq 0} \sum_{\substack{ad=n\\ a|m}} a\left(\frac{dm}{a}\right) \sqrt{y} K_{\nu - \frac{1}{2}}(2\pi |m| y) e^{2\pi i m x}\\
&= \sum_{m \neq 0} \sum_{\substack{a|n\\ a|m}} a\left(\frac{nm}{a^2}\right) \sqrt{y} K_{\nu - \frac{1}{2}}(2\pi |m| y) e^{2\pi i m x}.
\end{aligned}$$

Because $T_n f = \lambda_n f$, it follows that

$$\lambda_n a(m) = \sum_{d|(m,n)} a\left(\frac{nm}{d^2}\right). \tag{15}$$

For $m = 1$, Equation (15) yields

$$a(n) = \lambda_n a(1).$$

Thus if $a(1) = 0$, the Maass form f vanishes identically. If $f \neq 0$, so that $a(1) = 1$, then we see that $a(n) = \lambda_n$. The relations of the Fourier coefficients of Lemma 18 now follow immediately from the identity (15). □

3.3 L-functions

3.3.1 L-functions associated to Maass forms

Let

$$f(z) = \sum_{n\neq 0} a(n)\sqrt{y}K_{\nu-\frac{1}{2}}(2\pi|n|y)e^{2\pi inx} \tag{16}$$

be an even or odd cuspidal Maass form for $SL(2,\mathbb{Z})$.

Lemma 19. *The coefficients $a(n)$ in the Fourier expansion of $f(z)$ satisfy*

$$a(n) = O(\sqrt{|n|}).$$

Proof. Because f is a cusp form, it is bounded on $\mathbb{H}$. Thus

$$\left|a(n)\sqrt{y}K_{\nu-\frac{1}{2}}(2\pi|n|y)\right| = \left|\int_0^1 f(x+iy)e^{-2\pi inx}\mathrm{d}x\right| \leq \int_0^1 |f(x+iy)|\mathrm{d}x \leq C,$$

for some constant C (that depends on f). If we choose $y = \frac{1}{|n|}$, the stated bound is proved. □

The coefficients $a(n)$ behave like a constant on average.

Lemma 20. *For Y sufficiently large we have*

$$\sum_{1\leq n\leq Y} |a(n)|^2 \ll Y.$$

Proof. Parseval's identity applied to a Maass cusp form f asserts that

$$\sum_{n\neq 0}|a(n)\sqrt{y}K_{\nu-\frac{1}{2}}(2\pi|n|y)|^2 = \int_0^1 |f(x+iy)|^2\mathrm{d}x.$$

It follows that

$$2\sum_{1\leq n\leq Y}|a(n)|^2 y|K_{\nu-\frac{1}{2}}(2\pi|n|y)|^2 \leq \int_0^1 |f(x+iy)|^2\mathrm{d}x.$$

From this we conclude that for any positive real number A we have

$$2\sum_{1\leq n\leq Y}|a(n)|^2\int_A^\infty |K_{\nu-\frac{1}{2}}(2\pi|n|y)|^2\frac{\mathrm{d}y}{y} \leq \int_A^\infty\int_0^1 |f(x+iy)|^2\mathrm{d}x\frac{\mathrm{d}y}{y^2}.$$

Because f is bounded on $\mathbb{H}$, it follows that

$$\int_A^\infty\int_0^1 |f(x+iy)|^2\mathrm{d}x\frac{\mathrm{d}y}{y^2} \ll \int_A^\infty \frac{\mathrm{d}y}{y^2} = \frac{1}{A}.$$

Choosing $A = \frac{1}{Y}$ finishes the proof of the lemma. □

For $\mathrm{Re}(s) > \frac{3}{2}$, we define the L-function $L_f(s)$ associated to a Maass cusp form $f(z)$ by the absolutely convergent series

$$L_f(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}. \tag{17}$$

Lemma 21. *Let $f(z)$ be a cuspidal Maass form for $SL(2,\mathbb{Z})$ of parity ϵ. Then its L-function $L_f(s)$ can be meromorphically continued to all $\mathbb{C}$ and it satisfies the functional equation*

$$\Lambda_f(s) = \pi^{-s}\Gamma\left(\frac{s+\epsilon+\nu-\frac{1}{2}}{2}\right)\Gamma\left(\frac{s+\epsilon-\nu+\frac{1}{2}}{2}\right)L_f(s) = (-1)^{\epsilon}\Lambda_f(1-s).$$

Proof. The idea of the proof is to set $x = 0$ and take the Mellin transform of $f(z)$. We first assume that f is even. For sufficiently large $\mathrm{Re}(s)$ we have

$$\begin{aligned}
\int_0^{\infty} f(iy)y^s\frac{\mathrm{d}y}{y} &= 2\int_0^{\infty}\sum_{n=1}^{\infty} a(n)\sqrt{y}K_{\nu-\frac{1}{2}}(2\pi ny)y^s\frac{\mathrm{d}y}{y} \\
&= \frac{2}{(2\pi)^{s+\frac{1}{2}}}L_f(s+\frac{1}{2})\int_0^{\infty} K_{\nu-\frac{1}{2}}(y)y^{s+\frac{1}{2}}\frac{\mathrm{d}y}{y} \\
&= \frac{1}{2\pi^{s+\frac{1}{2}}}L_f(s+\frac{1}{2})\Gamma\left(\frac{s+\nu}{2}\right)\Gamma\left(\frac{s+1-\nu}{2}\right),
\end{aligned}$$

where we used the Mellin transform of the K-Bessel function (see page 127 in [7])

$$\int_0^{\infty} K_{\nu}(y)y^s\frac{\mathrm{d}y}{y} = 2^{s-2}\Gamma\left(\frac{s+\nu}{2}\right)\Gamma\left(\frac{s-\nu}{2}\right).$$

The meromorphic continuation of $L_f(s)$ follows. Because f is automorphic, we have

$$f(iy) = f(S(iy)) = f(i/y).$$

It follows that

$$\begin{aligned}
\int_0^{\infty} f(iy)y^s\frac{\mathrm{d}y}{y} &= \int_0^{1} f(i/y)y^s\frac{\mathrm{d}y}{y} + \int_1^{\infty} f(iy)y^s\frac{\mathrm{d}y}{y} \\
&= \int_1^{\infty} f(iy)\left(y^s+y^{-s}\right)\frac{\mathrm{d}y}{y}.
\end{aligned}$$

The above integral is invariant under $s \mapsto -s$ and from this the functional equation follows immediately.

If f is odd, the above calculation does not quite work, because $\sum_{n\neq 0} a(n)|n|^{-s} = 0$ and so $\int_0^{\infty} f(iy)y^s\frac{\mathrm{d}y}{y} = 0$. However, we can deduce the functional equation in the same way as above once we replace in the above calculation $f(iy)$ by

$$\left.\frac{\partial}{\partial x}f(z)\right|_{x=0} = 4\pi i\sum_{n=1}^{\infty} a(n)n\sqrt{y}K_{\nu-\frac{1}{2}}(2\pi ny).$$

□

Lemma 22. *Let f be a Maass cusp form as in Equation (16) which is an eigenfunction of all the Hecke operators and is normalized so that $a(1) = 1$. Then the L-function $L_f(s) = \sum_{n=1}^{\infty}\frac{a(n)}{n^s}$ admits the Euler product*

$$L_f(s) = \prod_p \left(1 - a(p)p^{-s} + p^{-2s}\right)^{-1},$$

where the product runs over all primes p.

Proof. It follows from the multiplicativity of the Fourier coefficients (Lemma 18) that

$$\sum_{n=1}^{\infty} \frac{a(n)}{n^s} = \prod_p \left(\sum_{l=0}^{\infty} \frac{a(p^l)}{p^{ls}} \right).$$

For a fixed prime p we have

$$\begin{aligned}
&\sum_{l=0}^{\infty} a(p^l) p^{-ls} \left(1 - a(p)p^{-s} + p^{-2s}\right) \\
&\qquad = \sum_{l=0}^{\infty} a(p^l)p^{-ls} - \sum_{l=1}^{\infty} a(p^{l-1})a(p)p^{-ls} + \sum_{l=2}^{\infty} a(p^{l-2})p^{-ls} \\
&\qquad = 1 + \sum_{l=2}^{\infty} \left(a(p^l) - a(p^{l-1})a(p) + a(p^{l-2})\right) p^{-ls} = 1,
\end{aligned}$$

where the last equality follows from Lemma 18. This shows that

$$\sum_{l=0}^{\infty} \frac{a(p^l)}{p^{ls}} = \left(1 - a(p)p^{-s} + p^{-2s}\right)^{-1}.$$

□

3.3.2 L-functions associated to Eisenstein series

By formula (8), the non-constant term in the Fourier expansion of the Eisenstein series $E(z,v)$ is given by

$$\frac{2\pi^v}{\Gamma(v)\zeta(2v)} y^{1/2} \sum_{n \neq 0} \sigma_{1-2v}(n) |n|^{v-\frac{1}{2}} K_{v-1/2}(2\pi|n|y) e^{2\pi i n x}.$$

We define the L-function $L_{E_v}(s)$ associated to the Eisenstein series $E(z,v)$ to be

$$L_{E_v}(s) = \sum_{n=1}^{\infty} \frac{\sigma_{1-2v}(n)\, n^{v-\frac{1}{2}}}{n^s}.$$

Lemma 23. *The L-function $L_{E_v}(s)$ is simply a product of shifted Riemann zeta functions*

$$L_{E_v}(s) = \zeta(s+v-1/2)\,\zeta(s-v+1/2).$$

Furthermore, we have the functional equation

$$G_{E_v}(s) = G_{E_v}(1-s),$$

where

$$G_{E_v}(s) = \pi^{-s} \Gamma\left(\frac{s+v-\frac{1}{2}}{2}\right) \Gamma\left(\frac{s-v+\frac{1}{2}}{2}\right) \zeta(s+v-1/2)\,\zeta(s-v+1/2).$$

Proof. We calculate

$$\begin{aligned}
\sum_{n=1}^{\infty} \sigma_{1-2v}(n)\, n^{v-\frac{1}{2}-s} &= \sum_{n=1}^{\infty} n^{v-\frac{1}{2}-s} \sum_{d|n} d^{1-2v} \\
&= \sum_{n=1}^{\infty} \sum_{d=1}^{\infty} (dn)^{v-\frac{1}{2}-s} d^{1-2v} \\
&= \sum_{n=1}^{\infty} n^{v-\frac{1}{2}-s} \sum_{d=1}^{\infty} d^{\frac{1}{2}-v-s} \\
&= \zeta(s-v+1/2)\,\zeta(s+v-1/2).
\end{aligned}$$

Now consider

$$G_{E_v}(s) = \pi^{-\frac{s+v-\frac{1}{2}}{2}}\Gamma\left(\frac{s+v-\frac{1}{2}}{2}\right)\zeta(s+v-1/2)$$
$$\times\, \pi^{-\frac{s-v+\frac{1}{2}}{2}}\Gamma\left(\frac{s-v+\frac{1}{2}}{2}\right)\zeta(s-v+1/2).$$

The functional equation $G_{E_v}(s) = G_{E_v}(1-s)$ follows directly from the functional equation

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s) \tag{18}$$

of the Riemann zeta function.

□

As expected, the functional equation of $L_{E_v}(s)$ matches the functional equation of a Maass form of even parity (as in Lemma 21). In fact, the functional equation of a Maass form of type ν and of even parity is identical to the functional equation of $L_{E_\nu}(s)$. The reason is that the K-Bessel functions in both Fourier expansions coincide and the proof of the functional equation in Lemma 21 only uses the analytical properties of the K-Bessel function, and does not depend on the arithmetic Fourier coefficients. This gives a method of obtaining functional equations for Maass forms from studying the functional equation of Eisenstein series. We will use this observation several times.

3.3.3 Twisted L-functions associated to Maass forms

Let

$$f(z) = \sum_{n\neq 0} a(n) y^{1/2} K_{\nu-\frac{1}{2}}(2\pi|n|y)e^{2\pi i n x}$$

be a cuspidal Maass form of type ν for $\mathrm{SL}(2,\mathbb{Z})$. Let χ be an even primitive Dirichlet character mod a prime q. The twist of f by χ is defined by

$$f_\chi(z) = \sum_{n\neq 0} a(n)\chi(n) y^{1/2} K_{\nu-\frac{1}{2}}(2\pi|n|y)e^{2\pi i n x}, \tag{19}$$

and the associated L-function by

$$L_{f_\chi}(s) = \sum_{n=1}^{\infty} \frac{\chi(n)a(n)}{n^s}. \tag{20}$$

Following the remark after Lemma 23, the functional equation of the twisted L-function $L_{f_\chi}(s)$ satisfies the functional equation of the twisted L-function $L_{E_v}(s,\chi)$ associated to the Eisenstein series $E(z,v)$. We now determine the functional equation for $L_{E_v}(s,\chi)$.

Including χ in the proof of Lemma 23 shows that the twisted L-function $L_{E_v}(s,\chi)$ is

$$L_{E_v}(s,\chi) = L(s+v-1/2,\chi)\, L(s-v+1/2,\chi),$$

with the Dirichlet L-function $L(s,\chi) = \sum_{n=1}^{\infty}\chi(n)n^{-s}$. The Dirichlet L-function $L(s,\chi)$ satisfies the functional equation

$$\Lambda(s,\chi) = \left(\frac{q}{\pi}\right)^{\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)L(s,\chi) = \frac{\tau(\chi)}{\sqrt{q}}\,\Lambda(1-s,\overline{\chi}). \tag{21}$$

From this it follows that the L-function $L_{E_v}(s,\chi)$ satisfies the functional equation

$$\Lambda_{E_v}(s,\chi) = \left(\frac{q}{\pi}\right)^{s}\Gamma\left(\frac{s+v-1/2}{2}\right)\Gamma\left(\frac{s-v+1/2}{2}\right)L_{E_v}(s,\chi)$$
$$= \left(\frac{\tau(\chi)}{\sqrt{q}}\right)^{2}\Lambda_{E_v}(1-s,\overline{\chi}).$$

3.4 Rankin-Selberg convolution

The Dirichlet series associated to two Maass cusp forms has as its coefficients the product of the Fourier coefficients of the two cusp forms. The Rankin-Selberg method seeks to explicitly represent such convolution functions by integrating Maass forms against Eisenstein series. The Eisenstein series have themselves functional and analytical properties, and those are then inherited by the integral transforms. Notably, the Rankin-Selberg method allows to prove the meromorphic continuation and functional equation of convolution L-functions.

3.4.1 Rankin-Selberg convolution of two $\mathrm{SL}(2,\mathbb{Z})$-Maass forms

Let

$$f(z) = \sum_{n\neq 0} a(n)\sqrt{y}K_{\nu_f-\frac{1}{2}}(2\pi|n|y)e^{2\pi inx}, \tag{22}$$

$$g(z) = \sum_{n\neq 0} b(n)\sqrt{y}K_{\nu_g-\frac{1}{2}}(2\pi|n|y)e^{2\pi inx}, \tag{23}$$

be cuspidal Maass forms for $SL(2,\mathbb{Z})$ of the same parity. For $\mathrm{Re}(s) > 2$ we define the Rankin-Selberg convolution $L_{f\times g}(s)$ as the absolutely convergent series

$$L_{f\times g}(s) = \zeta(2s)\sum_{n=1}^{\infty}\frac{a(n)\overline{b(n)}}{n^s}.$$

We will prove that $L_{f\times g}$ can be expressed as an inner product of $f\overline{g}$ with an Eisenstein series. This construction establishes the meromorphic continuation and functional equation for the convolution function $L_{f\times g}$.

Theorem 24. *Let $f(z)$ and $g(z)$ be cuspidal Maass forms of the same parity as in Equation* 22 *and* 23*. Then $L_{f\times g}(s)$ can be meromorphically continued to all $s\in\mathbb{C}$ with at most a simple pole at $s=1$. Furthermore, we have the functional equation*

$$\Lambda_{f\times g}(s) = \pi^{-2s}G_{\nu_f,\nu_g}(s)L_{f\times g}(s) = \Lambda_{f\times g}(1-s),$$

where

$$G_{\nu_f,\nu_g}(s) = \Gamma\left(\frac{s+\nu_f+\nu_g-1}{2}\right)\Gamma\left(\frac{s-\nu_f+\nu_g}{2}\right)\Gamma\left(\frac{s+\nu_f-\nu_g}{2}\right)\Gamma\left(\frac{s-\nu_f-\nu_g+1}{2}\right). \tag{24}$$

Proof. Let $E(z,s)$ be the non-holomorphic Eisenstein series as defined in Equation 4. For sufficiently large $\mathrm{Re}(s)$, it follows from folding/unfolding that

$$\begin{aligned}
\zeta(2s)\langle f\overline{g}, E(\cdot,\overline{s})\rangle &= \frac{\zeta(2s)}{2}\int_{SL(2,\mathbb{Z})\backslash\mathbb{H}} f(z)\overline{g(z)E(z,\overline{s})}\mathrm{d}\mu(z)\\
&= \frac{\zeta(2s)}{2}\sum_{\gamma\in\Gamma_\infty\backslash SL(2,\mathbb{Z}}\int_{SL(2,\mathbb{Z})\backslash\mathbb{H}} f(\gamma z)\overline{g(\gamma z)}\,\mathrm{Im}(\gamma z)^s\mathrm{d}\mu(\gamma z)\\
&= \frac{\zeta(2s)}{2}\int_{\Gamma_\infty\backslash\mathbb{H}} f(z)\overline{g(z)}\,\mathrm{Im}(z)^s\mathrm{d}\mu(z)\\
&= \frac{\zeta(2s)}{2}\int_0^\infty\int_0^1 f(z)\overline{g(z)}y^{s-2}\mathrm{d}x\mathrm{d}y\\
&= \frac{\zeta(2s)}{2}\sum_{n,m\neq 0} a(n)\overline{b(m)}\int_0^\infty K_{\nu_f-\frac{1}{2}}(2\pi|n|y)K_{\nu_g-\frac{1}{2}}(2\pi|m|y)y^{s-1}\int_0^1 e^{2\pi i(n-m)x}\mathrm{d}x\mathrm{d}y\\
&= \frac{\zeta(2s)}{2}\sum_{n\neq 0} a(n)\overline{b(n)}\int_0^\infty K_{\nu_f-\frac{1}{2}}(2\pi|n|y)K_{\nu_g-\frac{1}{2}}(2\pi|n|y)y^s\frac{\mathrm{d}y}{y}\\
&= \frac{\zeta(2s)}{2(2\pi)^s}\sum_{n\neq 0}\frac{a(n)\overline{b(n)}}{|n|^s}\int_0^\infty K_{\nu_f-\frac{1}{2}}(y)K_{\nu_g-\frac{1}{2}}(y)y^s\frac{\mathrm{d}y}{y}\\
&= \frac{1}{(2\pi)^s}L_{f\times g}(s)\int_0^\infty K_{\nu_f-\frac{1}{2}}(y)K_{\nu_g-\frac{1}{2}}(y)y^s\frac{\mathrm{d}y}{y}.
\end{aligned}$$

In the last line we used that f and g are of the same parity. The Mellin transform of $K_\nu(y)K_{\nu'}(y)$ is given by (see page 145 in [7])

$$\int_0^\infty K_\nu(y)K_{\nu'}(y)y^s\frac{\mathrm{d}y}{y} = \frac{2^{s-3}}{\Gamma(s)}\prod\Gamma\left(\frac{s\pm\nu\pm\nu'}{2}\right), \tag{25}$$

which is valid for $\mathrm{Re}(s) > |\mathrm{Re}(\nu)| + |\mathrm{Re}(\nu')|$.

From the calculation it follows that the convolution function $L_{f\times g}(s)$ inherits the analytical properties of the Eisenstein series $E(z,s)$. This means that $L_{f\times g}(s)$ can be meromorphically continued on $\mathbb{C}$. Because the Eisenstein series has a simple pole at $s=1$ and the Gamma function no zeros, it follows that $L_{f\times g}(s)$ has a simple pole at $s=1$ if and only if $\langle f,g\rangle \neq 0$. The functional equation follows immediately from the functional equation of the Eisenstein series

$$E^*(z,s) = \pi^{-s}\Gamma(s)\zeta(2s)E(z,s) = E^*(z,1-s).$$

$\square$

3.4.2 Euler product of Rankin-Selberg L-functions

If the L-functions $L_f(s)$ and $L_g(s)$ admit Euler products, then the Rankin-Selberg convolution $L_{f\times g}(s)$ admits an Euler product as well. The proof is based on Cauchy's identity.

Lemma 25. *Let $\alpha_i,\beta_i \in \mathbb{C}$ for $i=1,2$. Then for $x\in\mathbb{C}$, $|x|$ sufficiently small, we have*

$$\prod_{i=1}^{2}\prod_{j=1}^{2}(1-\alpha_i\beta_j x)^{-1} = \sum_{k=0}^{\infty} S_k(\alpha_1,\alpha_2)S_k(\beta_1,\beta_2)x^k(1-\alpha_1\alpha_2\beta_1\beta_2x^2)^{-1},$$

where $S_k(x_1,x_2)$ is the Schur polynomial

$$S_k(x_1,x_2) = \frac{\det\begin{pmatrix}x_1^{k+1} & x_2^{k+1}\\ 1 & 1\end{pmatrix}}{\det\begin{pmatrix}x_1 & x_2\\ 1 & 1\end{pmatrix}} = \frac{x_1^{k+1}-x_2^{k+1}}{x_1-x_2}.$$

Proof. This is proved on page 67 in [8] by evaluating the determinant of the matrix

$$\left(\frac{1}{1-\alpha_i\beta_j x}\right)_{1\le i,j\le 2}$$

in two different ways. $\square$

Theorem 26. *Let $f(z)$ and $g(z)$ be cuspidal Maass forms as in Equation* 22 *and* 23. *Assume that $L_f(s)$ and $L_g(s)$ have Euler products*

$$L_f(s) = \prod_p\prod_{i=1}^{2}\left(1-\frac{\alpha_{i,p}}{p^s}\right)^{-1}, \quad L_g(s) = \prod_p\prod_{j=1}^{2}\left(1-\frac{\beta_{j,p}}{p^s}\right)^{-1}.$$

Then $L_{f\times g}(s)$ admits the Euler product

$$L_{f\times g}(s) = \prod_p\prod_{i=1}^{2}\prod_{j=1}^{2}\left(1-\frac{\alpha_{i,p}\overline{\beta}_{j,p}}{p^s}\right)^{-1}.$$

Proof. By assumption we have

$$L_{f\times g}(s) = \prod_p \frac{\sum_{k=1}^\infty a(p^k)\overline{b(p^k)}p^{-ks}}{(1-p^{-2s})}.$$

In view of Lemma 25, after choosing $x = p^{-s}$, it suffices to show that

$$a(p^k) = S_k(\alpha_{1,p}, \alpha_{2,p}), \quad b(p^k) = S_k(\beta_{1,p}, \beta_{2,p}).$$

The above equalities are obtained inductively from the relations

$$\begin{aligned} 1 &= \alpha_{1,p}\,\alpha_{2,p}, \\ a(p) &= \alpha_{1,p} + \alpha_{2,p}, \\ a(p^{k+1}) &= a(p)a(p^k) - a(p^{k-1}). \end{aligned}$$

□

Following the remark after Lemma 23, the functional equation in Theorem 24 is identical to the functional equation of the Rankin-Selberg L-function of two Eisenstein series E_v and E_w of type v and w, respectively. We now verify this observation. By Lemma 23, the L-functions $L_{E_v}(s)$ and $L_{E_w}(s)$ are given by

$$\begin{aligned} L_{E_v}(s) &= \prod_{i=1}^{2} \zeta(s - \mu_i(v)) \\ L_{E_w}(s) &= \prod_{i=1}^{2} \zeta(s - \mu_i(w)), \end{aligned}$$

where $\mu_1(v) = v - \frac{1}{2}$ and $\mu_2(v) = -v + \frac{1}{2}$. From the Euler product

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}$$

and from Theorem 26 we deduce that the Rankin-Selberg convolution of E_v and E_w is

$$\begin{aligned} L_{E_v \times E_w}(s) &= \prod_p \prod_{i=1}^{2} \prod_{j=1}^{2} \left(1 - \frac{p^{\mu_i(v) + \overline{\mu_j(w)}}}{p^s}\right)^{-1} \\ &= \prod_{i=1}^{2} \prod_{j=1}^{2} \zeta(s - \mu_i(v) - \overline{\mu_j(w)}). \end{aligned}$$

It follows from the functional equation (18) for the Riemann Zeta function that the functional equation of $L_{E_v \times E_w}(s)$ is indeed the same as the one stated in Theorem 24. We also note that a corollary of the Rankin-Selberg convolution $L_{E_v \times E_w}(s)$ is the Ramanujan identity

$$\sum_{n=1}^{\infty} \frac{\sigma_a(n)\sigma_b(n)}{n^s} = \frac{\zeta(s)\zeta(s-a)\zeta(s-b)\zeta(s-a-b)}{\zeta(2s-a-b)}.$$

3.4.3 Rankin-Selberg convolution of twisted Maass forms

Let

$$f(z) = \sum_{n \neq 0} a(n) \sqrt{y} K_{\nu_f - \frac{1}{2}}(2\pi |n| y) e^{2\pi i n x}, \tag{26}$$

$$g(z) = \sum_{n \neq 0} b(n) \sqrt{y} K_{\nu_g - \frac{1}{2}}(2\pi |n| y) e^{2\pi i n x}, \tag{27}$$

be cuspidal Maass for $SL(2, \mathbb{Z})$ of the same parity ϵ. We fix an even, primitive Dirichlet character χ of prime modulus q. We have shown in Lemma 14 that the twist $f_\chi(z)$ is automorphic for $\Gamma_0(q^2)$ with nebentypus χ^2. We define the Rankin-Selberg L-function $L_{f_\chi \times g}(s)$ by

$$L_{f_\chi \times g}(s) = L(2s, \chi^2) \sum_{n=1}^{\infty} \chi(n) \frac{a(n)b(n)}{n^s}.$$

Because the Fourier coefficients $a(n), b(n)$ satisfy the bound $O(\sqrt{|n|})$ (Lemma 19), the above series converges absolutely for $\mathrm{Re}(s) > 2$. We will by the same method as in Theorem 24 show that this function admits an analytic continuation to $s \in \mathbb{C}$. That is, we construct $L_{f_\chi \times g}(s)$ as an inner product of $f_\chi(z)E^{\#}(z, s, \chi^2)$ with $\overline{g(z)}$.

Theorem 27. *For* $\mathrm{Re}(s)$ *sufficiently large we have*

$$\int_{\Gamma_0(q^2)\backslash\mathbb{H}} E^{\#}(z, s, \chi^2) f_\chi(z) g(z) \mathrm{d}\mu(z) = \frac{(-1)^\epsilon}{2^3} \frac{1}{\pi^{2s}} G_{\nu_f,\nu_g}(s) L_{f_\chi \times g}(s),$$

where $G_{\nu_f,\nu_g}(s)$ *is given by Equation 24. In particular, the function* $L_{f_\chi \times g}(s)$ *is entire.*

Proof. Let $E^{\#}(z, s, \chi^2)$ be the completed Eisenstein series of level q^2 and nebentypus χ^2, as defined in Equation 12. By folding/unfolding it follows that

$$\begin{aligned}
\int_{\Gamma_0(q^2)\backslash\mathbb{H}} E^{\#}(z, s, \chi^2) f_\chi(z) g(z) \mathrm{d}\mu(z) &= \frac{L(2s, \chi^2)\Gamma(s)}{2\pi^s} \sum_{\gamma \in \Gamma_\infty\backslash\Gamma_0(q^2)} \int_{\Gamma_0(q^2)\backslash\mathbb{H}} \chi(\gamma)^2 \mathrm{Im}(\gamma z)^s f_\chi(z) g(z) \mathrm{d}\mu(z) \\
&= \frac{L(2s, \chi^2)\Gamma(s)}{2\pi^s} \sum_{\gamma \in \Gamma_\infty\backslash\Gamma_0(q^2)} \int_{\Gamma_0(q^2)\backslash\mathbb{H}} \mathrm{Im}(\gamma z)^s f_\chi(\gamma z) g(\gamma z) \mathrm{d}\mu(\gamma z) \\
&= \frac{L(2s, \chi^2)\Gamma(s)}{2\pi^s} \int_{\Gamma_\infty\backslash\mathbb{H}} \mathrm{Im}(z)^s f_\chi(z) g(z) \mathrm{d}\mu(z) \\
&= \frac{L(2s, \chi^2)\Gamma(s)}{2\pi^s} \int_0^\infty \int_0^1 y^{s-2} f_\chi(x+iy) g(x+iy) \mathrm{d}x\mathrm{d}y.
\end{aligned}$$

For the second equality we used Lemma 14. After inserting the Fourier expansions, the integral over the x-coordinate becomes

$$\begin{aligned}
\int_0^1 f_\chi(x+iy) g(x+iy) \mathrm{d}x &= y \sum_{n\neq 0} \chi(n) a(n) b(-n) K_{\nu_f - \frac{1}{2}}(2\pi|n|y) K_{\nu_g - \frac{1}{2}}(2\pi|n|y) \\
&= (-1)^\epsilon y \sum_{n\neq 0} \chi(n) a(n) b(n) K_{\nu_f - \frac{1}{2}}(2\pi|n|y) K_{\nu_g - \frac{1}{2}}(2\pi|n|y).
\end{aligned}$$

Then formula (25) for the Mellin transform of $K_\nu K_{\nu'}$ gives

$$\begin{aligned}
\int_{\Gamma_0(q^2)\backslash\mathbb{H}} E^{\#}(z, s, \chi^2) f_\chi(z) g(z) \mathrm{d}\mu(z) &= \frac{(-1)^\epsilon}{2^4} \frac{L(2s, \chi^2)}{\pi^{2s}} G_{\nu_f,\nu_g}(s) \sum_{n\neq 0} \chi(n) \frac{a(n)b(n)}{|n|^s} \\
&= \frac{(-1)^\epsilon}{2^3} \frac{1}{\pi^{2s}} G_{\nu_f,\nu_g}(s) L_{f_\chi \times g}(s),
\end{aligned}$$

where in the last line we used that f and g are of the same parity. Because the completed Eisenstein series $E^{\#}(z, s, \chi^2)$ is entire and the Gamma function has no zeros, it follows that the function $s \mapsto L_{f_\chi \times g}(s)$ is an entire function. □

3.5 The Luo-Rudnick-Sarnak Theorem for $\mathrm{SL}(2,\mathbb{Z})$

Let

$$f(z) = \sum_{n\neq 0} a(n)\sqrt{y}K_{\nu-\frac{1}{2}}(2\pi|n|y)e^{2\pi i n x}$$

be a Maass cusp form for $\mathrm{SL}(2,\mathbb{Z})$ of type ν so that $a(1) = 1$ and so that $f(z)$ is an eigenfunction of the Hecke operators. We have shown in Lemma 26 that the associated L-function $L_f(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$ has the Euler product

$$\begin{aligned} L_f(s) &= \prod_p \left(1 - a(p)p^{-s} + p^{-2s}\right)^{-1}, \\ &= \prod_p \prod_{i=1}^{2} \left(1 - \frac{\alpha_{i,p}}{p^s}\right)^{-1}, \end{aligned} \tag{28}$$

where for each prime p the $\alpha_{i,p} \in \mathbb{C}$ satisfy

$$\begin{aligned} 1 &= \alpha_{1,p}\,\alpha_{2,p}, \\ a(p) &= \alpha_{1,p} + \alpha_{2,p}. \end{aligned}$$

Here is the Ramanujan conjecture for this setting.

Conjecture 28. *Lef f be a Maass cusp form as above. Then the $\alpha_{i,p}$ in Equation 28 satisfy*

$$|\alpha_{i,p}| = 1,$$

for all primes p and $i = 1, 2$.

Another important conjecture is the Selberg eigenvalue conjecture.

Conjecture 29. *Lef f be a Maass cusp form of type $\nu \in \mathbb{C}$ as above. Then*

$$\nu(1-\nu) \geq \frac{1}{4},$$

or equivalently,

$$\mathrm{Re}(\nu) = \frac{1}{2},$$

or equivalently,

$$\mathrm{Re}(\mu_i(\nu)) = 0,$$

for $i = 1, 2$ and where $\mu_1(\nu) = \nu - \frac{1}{2}$ and $\mu_2(\nu) = -\nu + \frac{1}{2}$.

These two conjectures can in fact be treated on an equal footing. The Selberg eigenvalue conjecture is not hard to prove for $\mathrm{SL}(2,\mathbb{Z})$. In fact, in Lemma 10 we proved the much better lower bound $\nu(1-\nu) \geq 3\pi^2/2$. The above conjectures can be generalized to $\mathrm{SL}(n,\mathbb{Z})$ for $n \geq 2$ and to congruence subgroups and in this setting the conjectures are not proved. The current best bound for both the Ramanujan conjecture and the Selberg eigenvalue conjecture for Maass forms for $\mathrm{SL}(n,\mathbb{Z})$ was obtained by Luo, Rudnick and Sarnak ([1], [2]). In this section we will illustrate their method specialized to Maass forms for $\mathrm{SL}(2,\mathbb{Z})$. We will present their proof towards the Selberg eigenvalue conjecture. The proof towards the Ramanujan conjecture is based on the same argument, but requires more care, and we will not go into details.

3.5.1 The Selberg eigenvalue conjecture

Theorem 30. *Lef f be a Maass cusp form of type ν for $\mathrm{SL}(2,\mathbb{Z})$ as above. Then, using the above notation, we have*

$$\mathrm{Re}(\mu_i(\nu)) \leq \frac{1}{2} - \frac{1}{5}, \tag{29}$$

for $i = 1, 2$.

The proof of Theorem 30 is based on the Rankin-Selberg convolution of f with its twist f_χ by a primitive Dirichlet character modulo a prime q. According to Theorem 26, this L-function has the Euler product

$$L_{f_\chi \times f}(s) = \prod_p \prod_{i=1}^{2} \prod_{j=1}^{2} \left(1 - \frac{\alpha_{i,p}\, \overline{\alpha}_{j,p}\, \chi(p)}{p^s}\right)^{-1}. \tag{30}$$

According to Section 3.3.2, the functional equation of $L_{f_\chi \times f}(s)$ can be deduced from the functional equation for the Eisenstein L-functions $L_{E_v}(s, \chi)$ and $L_{E_v}(s)$. We recall from Section 3.3.2 and 3.3.3 that these L-functions take the simple form

$$\begin{aligned} L_{E_v}(s) &= \prod_{i=1}^{2} \zeta(s - \mu_i(\nu)), \\ L_{E_v}(s, \chi) &= \prod_{j=1}^{2} L(s - \mu_j(\nu), \chi). \end{aligned}$$

From the Euler products

$$\begin{aligned} \zeta(s) &= \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}, \\ L(s, \chi) &= \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}, \end{aligned}$$

and from Theorem 26 we deduce that the Rankin-Selberg convolution of E_ν and $E_{\nu,\chi}$ is

$$\begin{aligned} L_{E_{\nu,\chi} \times E_\nu}(s) &= \prod_p \prod_{i=1}^{2} \prod_{j=1}^{2} \left(1 - \frac{\chi(p) p^{\mu_i(\nu) + \overline{\mu_j(\nu)}}}{p^s}\right)^{-1} \\ &= \prod_{i=1}^{2} \prod_{j=1}^{2} L(s - \mu_i(\nu) - \overline{\mu_j(\nu)}, \chi). \end{aligned}$$

The functional equation (21) of the Dirichlet L-series gives us now the functional equation

$$\begin{aligned} \Lambda_{E_{\nu,\chi} \times E_\nu}(s) &= \prod_{i=1}^{2} \prod_{j=1}^{2} \left(\frac{q}{\pi}\right)^{\frac{s - \mu_i(\nu) - \overline{\mu_j(\nu)}}{2}} \Gamma\left(\frac{s - \mu_i(\nu) - \overline{\mu_j(\nu)}}{2}\right) L_{E_{\nu,\chi} \times E_\nu}(s) \\ &= \left(\frac{q}{\pi}\right)^{2s} \prod_{i=1}^{2} \prod_{j=1}^{2} \Gamma\left(\frac{s - \mu_i(\nu) - \overline{\mu_j(\nu)}}{2}\right) L_{E_{\nu,\chi} \times E_\nu}(s) \\ &= \left(\frac{\tau(\chi)}{\sqrt{q}}\right)^4 \Lambda_{E_v}(1 - s, \overline{\chi}), \end{aligned} \tag{31}$$

which, as we have just mentioned, is also the functional equation satisfied by $L_{f_\chi \times f}(s)$. Furthermore, we proved in Theorem 27 that $\Lambda_{f_\chi \times f}(s)$ is an entire function.

The idea of the proof for the bound (29) in Theorem 30 is to balance poles of the Gamma factors in $\Lambda_{f_\chi \times f}(s)$ with zeros of $L_{f_\chi \times f}(s)$. Key is the following theorem which gives us zero-free regions for a collection of L-functions $L_{f_\chi \times f}(s)$.

Theorem 31. *Let f be a Maass form as above. Then for any real number $\beta > 1 - \frac{2}{5}$, there are even primitive Dirichlet characters χ such that*

$$L_{f_\chi \times f}(\beta) \neq 0.$$

Before proving Theorem 31, let us show that it implies the bound (29) in Theorem 30.

Proof of Theorem 30. We argue by contradiction. Let f be a Maass cusp form as in Theorem 30 and assume that for some $1 \leq i \leq 2$ we have $\mathrm{Re}(\mu_i(\nu)) = \beta$ for some $\beta > \frac{1}{2} - \frac{1}{5}$. Then by Theorem 31 there is an even primitive Dirichlet character χ such that

$$L_{f_\chi \times f}(2\beta) \neq 0.$$

On the other hand, the Gamma factor

$$\prod_{i=1}^{2} \Gamma\left(\frac{s - 2\,\mathrm{Re}(\mu_i(\nu))}{2}\right)$$

in $\Lambda_{f_\chi \times f}(s)$ has a pole at $s = 2\beta$. Because $\Lambda_{f_\chi \times f}(s)$ is entire, we obtain a contradiction.

□

Proof of Theorem 31. According to Lemma 19, we have the bound $|\alpha_{i,p}| \ll \sqrt{p}$ for $i = 1, 2$. Therefore, for $s \geq 1$, the Euler factor $\left(1 - \frac{\alpha_{i,p}\,\overline{\alpha}_{j,p}\,\chi(p)}{p^s}\right)^{-1}$ in Equation (30) is non-zero for every p, thus also $L_{f_\chi \times f}(\beta) \neq 0$ for $\beta \geq 1$. We can thus assume $1 - \frac{2}{5} < \beta < 1$, and the theorem is a consequence of the following lemma. □

Lemma 32. *Let $\beta \in (1 - \frac{2}{5}, 1)$. Then for Q sufficiently large we have*

$$\sum_{q \bowtie Q} \sum_{\chi} L_{f_\chi \times f}(\beta) \gtrsim Q^2 / \log(Q) \neq 0, \tag{32}$$

where q is prime, where the notations $q \bowtie Q$ means $Q \leq q \leq 2Q$ and where the inner sum is taken over non-trivial even primitive Dirichlet characters χ mod q.

Proof. We write the Dirichlet series of $L_{f_\chi \times f}(s)$ as

$$L_{f_\chi \times f}(s) = \sum_{n=1}^{\infty} \chi(n) \frac{b(n)}{n^s}.$$

By Lemma 20, this Dirichlet series converges absolutely for $\mathrm{Re}(s) > 1$. Therefore $\beta \in (1 - \frac{2}{5}, 1)$ does not lie in the region of absolute convergence of the Dirichlet series of $L_{f_\chi \times f}(s)$. The method of approximate functional equations allows us to compute values of L-functions inside regions where the Dirichlet series of the L-function fails to converge. The first step of the proof is thus the derivation of an approximate function equation for $L_{f_\chi \times f}(\beta)$.

Step 1. *Set $\beta_0 = 2 \max_{1 \leq i \leq 2} \mathrm{Re}(\mu_i(\nu))$. There are two real-valued functions h_1 and h_2 that satisfy*

$$\begin{aligned} &h_1(y), h_2(y) = O_A(y^{-A}) \quad \text{as } y \to \infty, \\ &h_1(y) = 1 + O_A(y^A) \quad \text{as } y \to 0, \\ &h_2(y) \ll 1 + y^{1-\beta_0-\beta-\epsilon} \quad \text{as } y \to 0, \end{aligned}$$

for all $A \geq 1$ and all $\epsilon > 0$ and such that for any $Y > 1$ we have the approximate functional equation

$$\begin{aligned} L_{f_\chi \times f}(\beta) = &\sum_{n=1}^{\infty} \frac{b(n)}{n^\beta} \chi(n)\, h_1\left(\frac{n}{Y}\right) \\ &- \frac{\tau(\chi)^4 q^{-4\beta}}{\pi^{2-4\beta}} \sum_{n=1}^{\infty} \frac{b(n)}{n^{1-\beta}} \overline{\chi(n)}\, h_2\left(\frac{nY\pi^4}{q^4}\right). \end{aligned} \tag{33}$$

Let $h : (0, \infty) \to \mathbb{R}$ be a non-negative smooth, compactly supported function such that $\int_0^\infty h(y) \frac{\mathrm{d}y}{y} = 1$. Let

$$\tilde{h}(s) = \int_0^\infty h(y) y^s \frac{\mathrm{d}y}{y}$$

be the Mellin transform of h, which is holomorphic, bounded on vertical strips and satisfies $\tilde{h}(0) = 1$. We define

$$h_1(y) = \frac{1}{2\pi i} \int_{\mathrm{Re}(s)=2} \tilde{h}(s) y^{-s} \frac{\mathrm{d}s}{s}.$$

By a shift of the line of integration to the right (left), we get the stated behaviour as $y \to \infty$ ($y \to 0$). Next, for $y > 0$, we define

$$h_2(y) = \frac{1}{2\pi i} \int_{\mathrm{Re}(s)=1} \tilde{h}(-s) G(-s+\beta) y^{-s} \frac{\mathrm{d}s}{s}, \tag{34}$$

where

$$G(s) = \frac{\prod_{i=1}^{2} \prod_{j=1}^{2} \Gamma\left(\frac{1-s-\mu_i(\nu)-\overline{\mu_j(\nu)}}{2}\right)}{\prod_{i=1}^{2} \prod_{j=1}^{2} \Gamma\left(\frac{s-\mu_i(\nu)-\overline{\mu_j(\nu)}}{2}\right)}.$$

By Stirling's formula, $G(s)$ has at most polynomial growth in vertical strips. Therefore, by shifting the contour in Equation (34) to the right, we see that $h_2(y)$ decays rapidly as $y \to \infty$. For the behaviour as $y \to 0$, we shift the integration line to the left. Picking up the residues, gives the bound $h_2(y) \ll 1 + y^{1-\beta_0-\beta-\epsilon}$ as $y \to 0$.

Now we come to the derivation of an approximate functional equation. To this end, for $Y > 1$, consider the integral

$$\begin{aligned} \frac{1}{2\pi i} \int_{\mathrm{Re}(s)=2} \tilde{h}(s) L_{f_\chi \times f}(s+\beta) Y^s \frac{\mathrm{d}s}{s} &= \sum_{n=1}^{\infty} \frac{b(n)\chi(n)}{n^\beta} \frac{1}{2\pi i} \int_{\mathrm{Re}(s)=2} \tilde{h}(s) \left(\frac{Y}{n}\right)^s \frac{\mathrm{d}s}{s} \\ &= \sum_{n=1}^{\infty} \frac{b(n)\chi(n)}{n^\beta} h_1\left(\frac{n}{Y}\right). \end{aligned} \tag{35}$$

Since $L_{f_\chi \times f}(s)$ is bounded in vertical strips, we may shift the integration line in the above integral to $\mathrm{Re}(s) = -1$, thereby picking up a pole at $s = 0$, and we obtain

$$\begin{aligned} &\frac{1}{2\pi i} \int_{\mathrm{Re}(s)=2} \tilde{h}(s) L_{f_\chi \times f}(s+\beta) Y^s \frac{\mathrm{d}s}{s} \\ &\qquad = L_{f_\chi \times f}(\beta) + \frac{1}{2\pi i} \int_{\mathrm{Re}(s)=-1} \tilde{h}(s) L_{f_\chi \times f}(s+\beta) Y^s \frac{\mathrm{d}s}{s} \\ &\qquad = L_{f_\chi \times f}(\beta) + \frac{1}{2\pi i} \left(\frac{\tau(\chi)}{\sqrt{q}}\right)^4 \left(\frac{q}{\pi}\right)^{2-4\beta} \int_{\mathrm{Re}(s)=-1} \tilde{h}(s) G(s+\beta) L_{f_{\overline{\chi}} \times f}(1-s-\beta) \left(\frac{\pi^4 Y}{q^4}\right)^s \frac{\mathrm{d}s}{s}, \end{aligned}$$

where in the last step we applied the functional equation (31). On changing variables $s \mapsto -s$ the last expression equals

$$L_{f_\chi \times f}(\beta) + \frac{1}{2\pi i} \left(\frac{\tau(\chi)}{\sqrt{q}}\right)^4 \left(\frac{q}{\pi}\right)^{2-4\beta} \int_{\mathrm{Re}(s)=1} \tilde{h}(-s) G(-s+\beta) L_{f_{\overline{\chi}} \times f}(1+s-\beta) \left(\frac{\pi^4 Y}{q^4}\right)^{-s} \frac{\mathrm{d}s}{s},$$

which equals

$$L_{f_\chi \times f}(\beta) + \left(\frac{\tau(\chi)}{\sqrt{q}}\right)^4 \left(\frac{q}{\pi}\right)^{2-4\beta} \sum_{n=1}^{\infty} \overline{\chi}(n) \frac{b(n)}{n^{1-\beta}} h_2\left(\frac{nY\pi^4}{q^4}\right). \tag{36}$$

Combining (35) and (36) gives the approximate functional equation (33).

Step 2. *We average the approximate functional equation over even primitive Dirichlet characters to deduce (32).*

According to the approximate functional equation (33), we can decompose $\sum_{q\bowtie Q}\sum_{\chi} L_{f_\chi\times f}(\beta)$ into $T_1 - T_2$, where

$$T_1 = \sum_{q\bowtie Q}\sum_{\chi}\sum_{n=1}^{\infty}\frac{b(n)}{n^\beta}\chi(n)\,h_1\left(\frac{n}{Y}\right),$$
$$T_2 = \sum_{q\bowtie Q}\sum_{\chi}\frac{\tau(\chi)^4 q^{-4\beta}}{\pi^{2-4\beta}}\sum_{n=1}^{\infty}\frac{b(n)}{n^{1-\beta}}\overline{\chi(n)}\,h_2\left(\frac{nY\pi^4}{q^4}\right).$$

By Lemma 7, we can write T_1 as

$$T_1 = \sum_{q\bowtie Q}\frac{q-1}{2}\sum_{n\equiv\pm1 \pmod q}\frac{b(n)}{n^\beta}h_1\left(\frac{n}{Y}\right) - \sum_{q\bowtie Q}\sum_{(n,q)=1}\frac{b(n)}{n^\beta}h_1\left(\frac{n}{Y}\right).$$

All terms in the first sum are non-negative. We thus get a lower bound on the first sum by considering only the contribution from the $n=1$ term, which is

$$\sum_{q\bowtie Q}\frac{q-1}{2}h_1\left(\frac{1}{Y}\right) = \sum_{q\bowtie Q}\frac{q-1}{2},$$

as $Y\to\infty$. On the other hand, we can estimate

$$\sum_{q\bowtie Q}\sum_{(n,q)=1}\frac{b(n)}{n^\beta}h_1\left(\frac{n}{Y}\right) \ll \sum_{q\bowtie Q}\sum_{n=1}^{\infty}\frac{|b(n)|}{n^\beta}\left|h_1\left(\frac{n}{Y}\right)\right| \ll Q\sum_{n=1}^{\infty}\frac{|b(n)|}{n^\beta}\left|h_1\left(\frac{n}{Y}\right)\right|.$$

As $Y\to\infty$, we deduce from the growth conditions on h_1 and from Lemma 20 that

$$\begin{aligned}\sum_{n=1}^{\infty} n^{-\beta}\,|b(n)|\left|h_1\left(\frac{n}{Y}\right)\right| &\ll \sum_{i\geq1, i\leq Y}\ \sum_{i<n\leq 2i} n^{-\beta}|b(n)| \\ &\ll \sum_{i\geq 1, i\leq Y} i^{1-\beta} \ll Y^{1-\beta}.\end{aligned}$$

Thus a lower bound for T_1 is

$$T_1 \geq \sum_{q\bowtie Q}\frac{q-1}{2} - O(Q\,Y^{1-\beta}). \tag{37}$$

We now consider the term T_2. It contains the sum

$$\sum_{\substack{\chi \pmod q,\\ \chi\neq\chi_0,\,\chi(-1)=1}}\overline{\chi(n)}\tau(\chi)^4$$

which is zero if $(q,n)>1$ and otherwise equals by Lemma 8

$$\frac{q-1}{2}\left(K_4(n,q)+K_4(-n,q)\right)-1.$$

It is a consequence of Deligne's proof of the Weil conjectures ([Sommes trig.] in [9]) that the hyper-Kloosterman sum $K_d(n,q)$ is bounded by

$$K_d(n,q)\ll q^{\frac{d-1}{2}}.$$

Hence if $(q,n)=1$ we get

$$\sum_{\substack{\chi \pmod q,\\ \chi\neq\chi_0,\,\chi(-1)=1}}\overline{\chi(n)}\tau(\chi)^4 \ll q^{\frac{5}{2}}.$$

We can now bound the contribution of T_2. We have

$$
\begin{aligned}
T_2 &= \sum_{q \bowtie Q} \sum_{\chi} \frac{\tau(\chi)^4 q^{-4\beta}}{\pi^{2-4\beta}} \sum_{n=1}^{\infty} \frac{|b(n)|}{n^{1-\beta}} \overline{\chi(n)}\, h_2\left(\frac{nY\pi^4}{q^4}\right) \\
&\ll \sum_{q \bowtie Q} \frac{q^{\frac{5}{2}} q^{-4\beta}}{\pi^{2-4\beta}} \sum_{(n,q)=1} \frac{|b(n)|}{n^{1-\beta}} \left| h_2\left(\frac{nY\pi^4}{q^4}\right) \right| \\
&\ll \sum_{q \bowtie Q} \frac{q^{\frac{5}{2}} q^{-4\beta}}{\pi^{2-4\beta}} \int_1^{\infty} \left| h_2\left(\frac{rY\pi^4}{q^4}\right) \right| r^{\beta} \frac{\mathrm{d}r}{r} \\
&= \sum_{q \bowtie Q} \frac{q^{\frac{5}{2}}}{\pi^2 Y^{\beta}} \int_{\frac{Y\pi^4}{q^4}}^{\infty} |h_2(r)| r^{\beta} \frac{\mathrm{d}r}{r} \\
&\ll \sum_{q \bowtie Q} \frac{q^{\frac{5}{2}}}{Y^{\beta}} \\
&\ll Q^{1+\frac{5}{2}} Y^{-\beta}.
\end{aligned}
\tag{38}
$$

From the upper bounds (37) and (38) it follows that

$$
\begin{aligned}
\sum_{q \bowtie Q} \sum_{\chi} L_{f_\chi \times f}(\beta) &\geq \sum_{q \bowtie Q} \frac{q-1}{2} - O(QY^{1-\beta} + Q^{1+\frac{5}{2}} Y^{-\beta}) \\
&= \frac{1}{2} \sum_{q \bowtie Q} q - O(QY^{1-\beta} + Q^{1+\frac{5}{2}} Y^{-\beta}) + O(Q).
\end{aligned}
$$

Choosing $Y = Q^{\frac{5}{2}}$ results in

$$
\sum_{q \bowtie Q} \sum_{\chi} L_{f_\chi \times f}(\beta) \geq \frac{1}{2} \sum_{q \bowtie Q} q + O(Q^{1+\frac{5}{2}(1-\beta)}) + O(Q).
$$

The prime number theorem yields

$$
\sum_{q \bowtie Q} q \sim \frac{(2Q)^2}{2\log(2Q)} - \frac{Q^2}{2\log(Q)} \sim \frac{3}{2} \frac{Q^2}{\log(Q)}.
$$

For $\beta \in (1 - \frac{2}{5}, 1)$ and Q sufficiently large, the term $\sum_{q \bowtie Q} q \sim Q^2/\log(Q)$ dominates the O-terms. In particular, the average $\sum_{q \bowtie Q} \sum_{\chi} L_{f_\chi \times f}(\beta)$ is non-zero and therefore $L_{f_\chi \times f}(\beta)$ is non-zero for some χ. This completes the proof of the lemma.

□

3.5.2 The Ramanujan conjecture

Theorem 33. *Lef f be a Maass cusp form of type ν for* $\mathrm{SL}(2,\mathbb{Z})$ *as in the beginning of this section. Then we have*

$$|\alpha_{i,p}| \leq p^{\frac{1}{2}-\frac{1}{5}}, \tag{39}$$

for all primes p and for $i = 1, 2$.

Let us fix a prime p_0. For the proof of the bound (29) towards the Selberg eigenvalue conjecture we considered the Rankin-Selberg L-function $L_{f_\chi \times f}(s)$. We now remove from its Euler product expansion (30) the p_0-th factor. This results in the modified Rankin-Selberg L-function

$$L^{p_0}_{f_\chi \times f}(s) = L_{f_\chi \times f}(s) \prod_{i=1}^{2} \prod_{j=1}^{2} \left(1 - \frac{\alpha_{i,p_0}\, \overline{\alpha}_{j,p_0}\, \chi(p_0)}{p_0^s}\right). \tag{40}$$

Theorem 31 also holds for the modified Rankin-Selberg L-function $L^{p_0}_{f_\chi \times f}(s)$.

Theorem 34. *Let f be a Maass form as in Theorem 33. Then for any real number $\beta > 1 - \frac{2}{5}$, there are even primitive Dirichlet characters χ with $\chi(p_0) = 1$ such that*

$$L^{p_0}_{f_\chi \times f}(\beta) \neq 0.$$

We refer to [2] for a proof of Theorem 34. Similarly as in the case $p_0 = 1$, we now show that Theorem 34 implies Theorem 33.

Proof of Theorem 33. We argue by contradiction. Let f be a Maass cusp form as in Theorem 33 and assume that for some $1 \leq i \leq 2$ we have $|\alpha_{i,p_0}|^2 = p_0^\beta$ with $\beta > 1 - \frac{2}{5}$. It follows that

$$\prod_{i=1}^{2} \left(1 - \frac{|\alpha_{i,p_0}|^2}{p_0^s}\right)^{-1}$$

has a pole at $s = \beta$. Furthermore, by Theorem 34 there is an even primitive Dirichlet character χ with $\chi(p_0) = 1$ such that

$$L^{p_0}_{f_\chi \times f}(\beta) \neq 0.$$

Using (40) we thus deduce that

$$L_{f_\chi \times f}(s) = L^{p_0}_{f_\chi \times f}(s) \prod_{i=1}^{2} \prod_{j=1}^{2} \left(1 - \frac{\alpha_{i,p_0}\, \overline{\alpha}_{j,p_0}}{p_0^s}\right)^{-1}$$

has a pole at $s = \beta$. But this is a contradiction because

$$\Lambda_{f_\chi \times f}(s) = \left(\frac{q}{\pi}\right)^{2s} \prod_{i=1}^{2} \prod_{j=1}^{2} \Gamma\left(\frac{s - \mu_i(\nu) - \overline{\mu_j(\nu)}}{2}\right) L_{f_\chi \times f}(s)$$

is entire. □

4 The case of $\Gamma_0(N)$

Throughout this section we fix a positive integer N, which we assume to be square-free. Let

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}) \,|\, c \equiv 0 \pmod N \right\}.$$

Denote the space of Maass cusp forms of level N by $\mathcal{C}(\Gamma_0(N) \setminus \mathbb{H})$.

4.1 Hecke operators and Atkin-Lehner theory for $\Gamma_0(N)$

The Hecke operator T_n on $\mathcal{C}(\Gamma_0(N) \setminus \mathbb{H})$ is defined by the formula

$$(T_n f)(z) = \frac{1}{\sqrt{n}} \sum_{\substack{ad=n \\ (a,N)=1}} \sum_{b=0}^{d-1} f\left(\frac{az+b}{d}\right).$$

There is an orthonormal basis of $\mathcal{C}(\Gamma_0(N) \setminus \mathbb{H})$ which consists of eigenfunctions of all the Hecke operators T_n, for $(n, N) = 1$. Such an eigencuspform f thus satisfies

$$T_n f = \lambda(n) f, \quad \text{if} (n, N) = 1. \tag{41}$$

We wish to have a basis of eigenfunctions of all the Hecke operators without exceptions. This becomes possibly after putting aside certain Maass cusp forms that come from lower levels. This theory of newforms was developed originally for modular forms, but the results also apply in the context of Maass forms, see *e.g.* [10] for a review.

The space of newforms $\mathcal{C}(\Gamma_0(N) \setminus \mathbb{H})^{\text{new}}$ is the orthogonal complement of the (old) subspace generated by all forms $f(dz)$, where $d|N$ and where $f \in \mathcal{C}(\Gamma_0(N') \setminus \mathbb{H})$ for some $N' < N$ such that $N'd|N$. For f such a newform, Equation (41) holds for all n. Furthermore, the first coefficient in the Fourier expansion of f does not vanish, so f may be normalized by setting $a(1) = 1$. In that case, for all n, we have $a(n) = \lambda(n)$.

An important role plays the Atkin-Lehner operator W_p on $\mathcal{C}(\Gamma_0(N) \setminus \mathbb{H})$. Let p be a prime divisor of N satisfying $(p, N/p) = 1$. Let a, b, c, t be any integers such that

$$W_p = \begin{pmatrix} pa & b \\ Nc & pt \end{pmatrix} \tag{42}$$

has determinant p. The matrix W_p normalizes the group $\Gamma_0(N)$, and induces a linear operator on the space of cusp forms $\mathcal{C}(\Gamma_0(N) \setminus \mathbb{H})$ through $W_p f(z) := f(W_p z)$.

If f is a Maass cusp eigenform, then it is also an eigenfunction of the operator W_p with eigenvalue ξ_p a complex number of modulus one, that is,

$$W_p f = \xi_p f, \quad \text{and } |\xi_p| = 1.$$

The eigenvalue ξ_p will appear in the functional equation for L-functions that are associated to f.

4.2 Twisted Maass Forms for $\Gamma_0(N)$

Let

$$f(z) = \sum_{n \neq 0} a(n) y^{1/2} K_{\nu - \frac{1}{2}}(2\pi |n| y) e^{2\pi i n x}$$

be a cuspidal Maass form for $\Gamma_0(N)$ of type ν. Let χ be a primitive Dirichlet character mod a prime q. The twist of f by χ is defined by

$$f_\chi(z) = \sum_{n \neq 0} a(n) \chi(n) y^{1/2} K_{\nu - \frac{1}{2}}(2\pi |n| y) e^{2\pi i n x}. \tag{43}$$

In Lemma 14 we proved that the twist $f_\chi(z)$ is automorphic for $\Gamma_0(q^2 N)$ with nebentypus χ^2. Let W_p be the Atkin-Lehner operator on $\mathcal{C}(\Gamma_0(Nq^2) \setminus \mathbb{H})$.

Lemma 35. *The twist f_χ of a newform $f \in \mathcal{C}(\Gamma_0(N) \setminus \mathbb{H})^{\text{new}}$ is an eigenfunction of W_p with eigenvalue $\chi(p)\xi_p$, that is,*

$$W_p f_\chi = \chi(p)\xi_p f_\chi.$$

Proof. The proof is similar as the proof of Lemma 14. We write

$$f_\chi(z) = \frac{1}{\tau(\overline{\chi})} \sum_{l=1}^{q} \overline{\chi(l)} f(z + \frac{l}{q}).$$

Moreover we have

$$\begin{pmatrix} 1 & l/q \\ 0 & 1 \end{pmatrix} W_p \in \Gamma_0(N) W_p \begin{pmatrix} 1 & lp/q \\ 0 & 1 \end{pmatrix}.$$

Thus for all $z \in \mathbb{H}$, the point $W_p(z) + \frac{l}{q}$ lies in the $\Gamma_0(N)$-orbit of $W_p \begin{pmatrix} 1 & lp/q \\ 0 & 1 \end{pmatrix} z = W_p(z + lp/q)$. It then follows that

$$\begin{aligned} W_p f_\chi(z) &= \frac{1}{\tau(\overline{\chi})} \sum_{l=1}^{q} \overline{\chi(l)}\, f(W_p(z) + \frac{l}{q}) \\ &= \frac{\chi(p)}{\tau(\overline{\chi})} \sum_{l=1}^{q} \overline{\chi(lp)}\, W_p f(z + \frac{lp}{q}) \\ &= \chi(p)\xi_p \frac{1}{\tau(\overline{\chi})} \sum_{l=1}^{q} \overline{\chi(lp)}\, f(z + \frac{lp}{q}) \\ &= \chi(p)\xi_p f_\chi(z). \end{aligned}$$

□

4.3 Rankin-Selberg convolution for $\Gamma_0(N)$

Let

$$f(z) = \sum_{n \neq 0} a(n) \sqrt{y} K_{\nu_f - \frac{1}{2}}(2\pi |n| y) e^{2\pi i n x}, \tag{44}$$

$$g(z) = \sum_{n \neq 0} b(n) \sqrt{y} K_{\nu_g - \frac{1}{2}}(2\pi |n| y) e^{2\pi i n x}, \tag{45}$$

be cuspidal Maass newforms for $\Gamma_0(N)$ of the same parity ϵ. We fix an even, primitive Dirichlet character χ of prime modulus q, and such that $(q, N) = 1$. The character χ induces an imprimitive character mod Nq^2 which we denote by χ'. It is given by

$$\chi'(a) = \begin{cases} \chi(a) & \text{if } (a, N) = 1 \\ 0 & \text{else} \end{cases}.$$

We recall that the twist $f_\chi(z)$, defined by Equation (43), is automorphic for $\Gamma_0(q^2 N)$ with nebentypus χ^2. We define the Rankin-Selberg L-function $L_{f_\chi \times g}(s)$ by

$$L_{f_\chi \times g}(s) = L(2s, \chi'^2) \sum_{n=1}^{\infty} \chi(n) \frac{a(n) b(n)}{n^s}.$$

Because the Fourier coefficients $a(n), b(n)$ satisfy the bound $O(\sqrt{|n|})$ (Lemma 19), the above series converges absolutely for $\mathrm{Re}(s) > 2$. We will show this function admits a meromorphic continuation to $s \in \mathbb{C}$ and a functional equation. Key is the following lemma, which constructs the series

$$\sum_{n=1}^{\infty} \chi(n) \frac{a(n) b(n)}{n^s}$$

as an integral transform of the Eisenstein series $E(z, s, \chi^2)$ of level Nq^2 and nebentypus χ^2.

Lemma 36. *For* $\mathrm{Re}(s)$ *sufficiently large we have*

$$\int_{\Gamma_0(Nq^2)\backslash\mathbb{H}} E(z,s,\chi^2) f_\chi(z) g(z) \mathrm{d}\mu(z) = \frac{(-1)^\epsilon}{2^3} \frac{1}{\Gamma(s)\pi^s} G_{\nu_f,\nu_g}(s) \sum_{n=1}^\infty \chi(n) \frac{a(n)b(n)}{n^s}.$$

Proof. By folding/unfolding it follows that

$$\begin{aligned}\int_{\Gamma_0(Nq^2)\backslash\mathbb{H}} E(z,s,\chi^2) f_\chi(z) g(z) \mathrm{d}\mu(z) &= \frac{1}{2} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(Nq^2)} \int_{\Gamma_0(Nq^2)\backslash\mathbb{H}} \chi(\gamma)^2 \,\mathrm{Im}(\gamma z)^s f_\chi(z) g(z) \mathrm{d}\mu(z) \\ &= \frac{1}{2} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(Nq^2)} \int_{\Gamma_0(Nq^2)\backslash\mathbb{H}} \mathrm{Im}(\gamma z)^s f_\chi(\gamma z) g(\gamma z) \mathrm{d}\mu(\gamma z) \\ &= \frac{1}{2} \int_{\Gamma_\infty\backslash\mathbb{H}} \mathrm{Im}(z)^s f_\chi(z) g(z) \mathrm{d}\mu(z) \\ &= \frac{1}{2} \int_0^\infty \int_0^1 y^{s-2} f_\chi(x+iy) g(x+iy) \mathrm{d}x\mathrm{d}y.\end{aligned}$$

For the second equality we used Lemma 14. After inserting the Fourier expansions, the integral over the x-coordinate becomes

$$\begin{aligned}\int_0^1 f_\chi(x+iy) g(x+iy) \mathrm{d}x &= y \sum_{n\neq 0} \chi(n) a(n) b(-n) K_{\nu_f-\frac{1}{2}}(2\pi|n|y) K_{\nu_g-\frac{1}{2}}(2\pi|n|y) \\ &= (-1)^\epsilon y \sum_{n\neq 0} \chi(n) a(n) b(n) K_{\nu_f-\frac{1}{2}}(2\pi|n|y) K_{\nu_g-\frac{1}{2}}(2\pi|n|y).\end{aligned}$$

Then formula (25) for the Mellin transform of $K_\nu K_{\nu'}$ gives

$$\begin{aligned}\int_{\Gamma_0(Nq^2)\backslash\mathbb{H}} E(z,s,\chi^2) f_\chi(z) g(z) \mathrm{d}\mu(z) &= \frac{(-1)^\epsilon}{2^4} \frac{1}{\Gamma(s)\pi^s} G_{\nu_f,\nu_g}(s) \sum_{n\neq 0} \chi(n) \frac{a(n)b(n)}{|n|^s} \\ &= \frac{(-1)^\epsilon}{2^3} \frac{1}{\Gamma(s)\pi^s} G_{\nu_f,\nu_g}(s) \sum_{n=1}^\infty \chi(n) \frac{a(n)b(n)}{n^s},\end{aligned}$$

where in the last line we used that f and g are of the same parity. ☐

To express the L-function $L_{f_\chi\times g}(s)$ as a Rankin-Selberg convolution, we still need to deal with the Dirichlet L-series $L(2s,\chi'^2)$. To this end, we note that

$$\begin{aligned}\pi^{-s}\Gamma(s) L(2s,\chi') E(z,s,\chi) &= \pi^{-s}\Gamma(s) \frac{1}{2} \sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq 0}} \chi'(n) \frac{y^s}{|mNq^2z+n|^{2s}} \\ &= \pi^{-s}\Gamma(s) \frac{1}{2} \sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq 0\\(n,N)=1}} \chi(n) \frac{y^s}{|mNq^2z+n|^{2s}} \\ &= \pi^{-s}\Gamma(s) \frac{1}{2} \sum_{d|N} \mu(d)\chi(d) \sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq 0}} \chi(n) \frac{y^s}{|mNq^2z+dn|^{2s}} \\ &= \pi^{-s}\Gamma(s) \frac{1}{2} \sum_{d|N} \mu(d)\chi(d) d^{-s} \sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq 0}} \chi(n) \frac{(y/d)^s}{|mNq^2z/d+n|^{2s}} \\ &= \sum_{d|N} \mu(d)\chi(d) d^{-s} E^{\#}\left(z/d,s,\chi\right),\end{aligned}$$

where $\mu(n)$ is the Möbius function. We can now write the L-function $L_{f_\chi\times g}(s)$ as the Rankin-Selberg convolution

$$\frac{1}{4\pi^{2s}} G_{\nu_f,\nu_g}(s) L_{f_\chi\times g}(s) = (-1)^\epsilon \sum_{d|N} \mu(d)\chi^2(d) d^{-s} \int_{\Gamma_0(Nq^2)\backslash\mathbb{H}} E^{\#}(z/d,s,\chi^2) f_\chi(z) g(z) \mathrm{d}\mu(z). \tag{46}$$

Lemma 37. *Let d be a divisor of N and p a prime dividing N but not d. Then*

$$\int_{\Gamma_0(Nq^2)\backslash\mathbb{H}} E^{\#}(z/dp,s,\chi^2)f_\chi(z)g(z)\mathrm{d}\mu(z) = \overline{\chi(p)}\xi_p^{-1}\eta_p^{-1}\int_{\Gamma_0(Nq^2)\backslash\mathbb{H}} E^{\#}(z/d,s,\chi^2)f_\chi(z)g(z)\mathrm{d}\mu(z).$$

Proof. This is proved in Section 7.4 of [3]. □

Using Lemma 37 we can write for a divisor d of N

$$\int_{\Gamma_0(Nq^2)\backslash\mathbb{H}} E^{\#}(z/d,s,\chi^2)f_\chi(z)g(z)\mathrm{d}\mu(z) = \chi(N/d)\prod_{p|(N/d)}\xi_p\eta_p\int_{\Gamma_0(Nq^2)\backslash\mathbb{H}} E^{\#}(z/N,s,\chi^2)f_\chi(z)g(z)\mathrm{d}\mu(z) \tag{47}$$

On the other hand, we can write

$$\begin{aligned}\sum_{d|N}\mu(d)\chi(d)d^{-s}\prod_{p|d}\xi_p^{-1}\eta_p^{-1} &= \sum_{d|N}\mu(d)\prod_{p|d}\chi(p)\xi_p^{-1}\eta_p^{-1}p^{-s}\\ &= \prod_{p|N}(1-\chi(p)\xi_p^{-1}\eta_p^{-1}p^{-s}).\end{aligned} \tag{48}$$

We define the completed Rankin-Selberg L-function to be

$$\Lambda_{f_\chi\times g}(s) = \left(\frac{q}{\pi}\right)^{2s} G_{\nu_f,\nu_g}(s)\left(\prod_{p|N}(1-\chi(p)\xi_p^{-1}\eta_p^{-1}p^{-s})^{-1}\right) L_{f_\chi\times g}(s). \tag{49}$$

Combining Equation (46)-(49) allows us to rewrite $\Lambda_{f_\chi\times g}(s)$ in the form

$$\Lambda_{f_\chi\times g}(s) = (-1)^\epsilon 4q^{2s}\chi(N)\left(\prod_{p|N}\xi_p\eta_p\right)\int_{\Gamma_0(Nq^2)\backslash\mathbb{H}} E^{\#}(z/N,s,\chi^2)f_\chi(z)g(z)\mathrm{d}\mu(z). \tag{50}$$

Because $s\mapsto E^{\#}(z/N,s,\chi^2)$ is entire (see Section 3.1.5) it follows that the convolution function $\Lambda_{f_\chi\times g}(s)$ is entire as well. The functional equation of the Eisenstein series $E^{\#}(z/N,s,\chi^2)$ allows to establish a functional equation for the completed Rankin-Selberg L-function $\Lambda_{f_\chi\times g}(s)$.

Theorem 38. *The function $\Lambda_{f_\chi\times g}(s)$ satisfies the functional equation*

$$\Lambda_{f_\chi\times g}(s) = w(f,g)\chi^2(N)\left(\frac{\tau(\chi)}{\sqrt{q}}\right)^4 N^{1-2s}\Lambda_{\tilde{f}_{\overline{\chi}}\times\tilde{g}}(1-s), \tag{51}$$

where $w(f,g)$ is a complex number of modulus 1.

Let I denote the integral on the RHS of (50). To prove Theorem 38, we apply the functional equation (13) of the Eisenstein series $E^{\#}$ to I. This results in

$$\begin{aligned}I &= \tau(\chi^2)N^{1-2s}q^{2-5s}\int_{\Gamma_0(Nq^2)\backslash\mathbb{H}} E^{\#}(-1/Nq^3z,1-s,\overline{\chi}^2)f_\chi(z)g(z)\mathrm{d}\mu(z)\\ &= \tau(\chi^2)N^{1-2s}q^{2-5s}J,\end{aligned}$$

where

$$J = \int_{\Gamma_0(Nq^2)\backslash\mathbb{H}} E^{\#}(z/q,1-s,\overline{\chi}^2)f_\chi(-1/Nq^2z)g(-1/Nq^2z)\mathrm{d}\mu(z).$$

Here we made the change of variables $z\mapsto\begin{pmatrix}0 & -1/(q\sqrt{N})\\ q\sqrt{N} & 0\end{pmatrix}z$.

The rest of the proof of the functional equation of $\Lambda_{f_\chi\times g}(s)$ consists in deriving a managable expression for J. We will not present the calculation here and simply state the final formula.

Lemma 39. *We have*

$$J = q^{s-1}\chi(N)\left(\frac{\tau(\chi)}{\sqrt{q}}\right)^4 \tau(\overline{\chi}^2)\int_{\Gamma_0(Nq^2)\backslash\mathbb{H}} \tilde{f}_{\overline{\chi}}(z)\tilde{g}(z)E^{\#}(z,1-s,\overline{\chi}^2)\mathrm{d}\mu(z). \tag{52}$$

Proof. This is proved in Section 7.4 of [3]. □

Proof of Theorem 38. Applying Lemma 37 inductively gives

$$\begin{aligned}
&\int_{\Gamma_0(Nq^2)\backslash\mathbb{H}} \tilde{f}_{\overline{\chi}}(z)\tilde{g}(z)E^{\#}(z,1-s,\overline{\chi}^2)\mathrm{d}\mu(z)\\
&\qquad = \overline{\chi(N)}\left(\prod_{p|N}\overline{\xi}_p\overline{\eta}_p\right)\int_{\Gamma_0(Nq^2)\backslash\mathbb{H}} \tilde{f}_{\overline{\chi}}(z)\tilde{g}(z)E^{\#}(z/N,1-s,\overline{\chi}^2)\mathrm{d}\mu(z).
\end{aligned}$$

This last identity together with (50) and (52) imply that

$$\begin{aligned}
\Lambda_{f_\chi\times g}(s) &= (-1)^{\epsilon}4q^{2s}\chi(N)\left(\prod_{p|N}\xi_p\eta_p\right)\tau(\chi^2)N^{1-2s}q^{2-5s}J\\
&= \frac{1}{q}\chi^2(N)\tau(\chi^2)\tau(\overline{\chi}^2)N^{1-2s}\left(\frac{\tau(\chi)}{\sqrt{q}}\right)^4\left(\prod_{p|N}\overline{\xi}_p\overline{\eta}_p\right)\Lambda_{\tilde{f}_{\overline{\chi}}\times\tilde{g}}(1-s)\\
&= w(f,g)\chi^2(N)N^{1-2s}\left(\frac{\tau(\chi)}{\sqrt{q}}\right)^4\Lambda_{\tilde{f}_{\overline{\chi}}\times\tilde{g}}(1-s).
\end{aligned}$$

Here, using Lemma 3,

$$w(f,g) = \frac{1}{q}\tau(\chi^2)\tau(\overline{\chi}^2)\left(\prod_{p|N}\overline{\xi}_p\overline{\eta}_p\right) = \prod_{p|N}\overline{\xi}_p\overline{\eta}_p$$

is a complex number of modulus one. This completes the proof of the theorem. □

In addition, the convolution function $L_{f_\chi\times g}(s)$ admits an Euler product. The Euler factor at a prime p that does not divide N is given by

$$\prod_{i=1}^{2}\prod_{j=1}^{2}\left(1-\frac{\chi(p)\alpha_{i,p}\,\overline{\beta}_{j,p}}{p^s}\right)^{-1}.$$

This is proved in the same way as in Lemma 26. If $p|N$, then it holds that $a(p)a(m) = a(pm)$. The Euler factor at a prime p that divides N is then given by

$$\begin{aligned}
\left(1-\chi(p)^2p^{-2s}\right)^{-1}\sum_{k=1}^{\infty}\chi(p^k)a(p^k)b(p^k)p^{-ks} &= \left(1-\chi(p)^2p^{-2s}\right)^{-1}\sum_{k=1}^{\infty}(\chi(p)a(p)b(p))^kp^{-ks}\\
&= \left(1-\chi(p)^2p^{-2s}\right)^{-1}\left(1-\frac{\chi(p)a(p)b(p)}{p^s}\right)^{-1}.
\end{aligned}$$

4.4 The Luo-Rudnick-Sarnak Theorem for $\Gamma_0(N)$

Let

$$f(z) = \sum_{n \neq 0} a(n) \sqrt{y} K_{\nu - \frac{1}{2}}(2\pi |n| y) e^{2\pi i n x} \tag{53}$$

be a Maass cusp form for $\Gamma_0(N)$ of type ν. The Selberg eigenvalue conjecture asserts that the eigenvalue $\nu(1-\nu) \geq \frac{1}{4}$, or equivalently that $\mathrm{Re}(\mu) = 0$, where $\mu = \nu - \frac{1}{2}$. In this section we will prove the Luo-Rudnick-Sarnak Theorem for $\Gamma_0(N)$, providing the best current lower bound towards the Selberg eigenvalue conjecture.

Theorem 40. *Lef f be a Maass cusp form for $\Gamma_0(N)$ as above. Then we have*

$$\mathrm{Re}(\mu) \leq \frac{1}{2} - \frac{1}{5}. \tag{54}$$

The proof of Theorem 40 consists in translating the proof of Theorem 30 into the $\Gamma_0(N)$ setting. Without loss of generality we may assume f to be a Maass cusp newform. We fix an even, primitive Dirichlet character χ mod a prime q, with $(q, N) = 1$. In Section 4.3 we have shown that the completed Rankin-Selberg L-function

$$\begin{aligned} \Lambda_{f_\chi \times \overline{f}}(s) = \left(\frac{q}{\pi}\right)^{2s} \Gamma\left(\frac{s + 2\,\mathrm{Re}(\mu)}{2}\right) \Gamma\left(\frac{s - 2\,\mathrm{Re}(\mu)}{2}\right) \Gamma\left(\frac{s + 2i\,\mathrm{Im}(\mu)}{2}\right) \\ \times \Gamma\left(\frac{s + 2i\,\mathrm{Im}(\mu)}{2}\right) \left(\prod_{p|N} (1 - \chi(p)|\xi_p|^{-2} p^{-s})^{-1}\right) L_{f_\chi \times \overline{f}}(s) \end{aligned} \tag{55}$$

is entire. Key to prove Theorem 30 was the existence of Rankin-Selberg L-functions $L_{f_\chi \times \overline{f}}(s)$ that have no zeros in predescribed regions. This result also holds in the context of $\Gamma_0(N)$-Maass newforms.

Theorem 41. *Let f be a Maass form as above. Then for any real number $\beta > 1 - \frac{2}{5}$, there are even primitive Dirichlet characters χ such that*

$$L_{f_\chi \times \overline{f}}(\beta) \neq 0.$$

Exactly as in the $\mathrm{SL}(2, \mathbb{Z})$ setting, one shows that Theorem 41 implies Theorem 40.

Proof of Theorem 40. We argue by contradiction. Let f be a Maass cusp form as in Theorem 40 and assume that we have $\mathrm{Re}(\mu) = \beta$ for some $\beta > \frac{1}{2} - \frac{1}{5}$. Then by Theorem 41 there is an even primitive Dirichlet character χ such that

$$L_{f_\chi \times \overline{f}}(2\beta) \neq 0.$$

On the other hand, the Gamma factor

$$\Gamma\left(\frac{s - 2\,\mathrm{Re}(\mu)}{2}\right)$$

in $\Lambda_{f_\chi \times \overline{f}}(s)$ has a pole at $s = 2\beta$. Because $\Lambda_{f_\chi \times \overline{f}}(s)$ is entire, we obtain a contradiction. □

The proof of Theorem 41 also goes along the same lines as the proof of Theorem 31. In that proof, one simply has to replace the functional equation (31), valid for the $\mathrm{SL}(2, \mathbb{Z})$ setting, with the corresponding functional equation (51) that holds in the $\Gamma_0(N)$ setting.

Proof of Theorem 41. According to Lemma 19, we have the bound $|\alpha_{i,p}| \ll \sqrt{p}$ for $i = 1, 2$. Therefore, for $s \geq 1$, the Euler factors $\left(1 - \frac{\alpha_{i,p}\, \overline{\alpha}_{j,p}\, \chi(p)}{p^s}\right)^{-1}$ and $\left(1 - \frac{\chi(p) a(p) b(p)}{p^s}\right)^{-1}$ in $L_{f_\chi \times \overline{f}}$ are non-zero for every p, thus also $L_{f_\chi \times \overline{f}}(\beta) \neq 0$ for $\beta \geq 1$. We can thus assume $1 - \frac{2}{5} < \beta < 1$, and the theorem is a consequence of the following lemma. □

Lemma 42. *Let $\beta \in (1-\frac{2}{5}, 1)$. Then for Q sufficiently large we have*

$$\sum_{q \bowtie Q} \sum_{\chi} L_{f_\chi \times f}(\beta) \gtrsim Q^2/\log(Q) \neq 0, \tag{56}$$

where q is prime, where the notations $q \bowtie Q$ means $Q \leq q \leq 2Q$ and where the inner sum is taken over non-trivial even primitive Dirichlet characters χ mod q.

Proof. We write the Dirichlet series of $L_{f_\chi \times f}(s)$ as

$$L_{f_\chi \times f}(s) = \sum_{n=1}^{\infty} \chi(n) \frac{b(n)}{n^s}.$$

By Lemma 20, this Dirichlet series converges absolutely for $\mathrm{Re}(s) > 1$. Therefore $\beta \in (1-\frac{2}{5}, 1)$ does not lie in the region of absolute convergence of the Dirichlet series of $L_{f_\chi \times f}(s)$. The method of approximate functional equations allows us to compute values of L-functions inside regions where the Dirichlet series of the L-function fails to converge. The first step of the proof is thus the derivation of an approximate function equation for $L_{f_\chi \times f}(\beta)$.

Step 1. *Set $\beta_0 = 2\,\mathrm{Re}(\mu)$. There are two real-valued functions h_1 and h_2 that satisfy*

$$\begin{aligned} &h_1(y), h_2(y) = O_A(y^{-A}) \quad \text{as } y \to \infty, \\ &h_1(y) = 1 + O_A(y^A) \quad \text{as } y \to 0, \\ &h_2(y) \ll 1 + y^{1-\beta_0-\beta-\epsilon} \quad \text{as } y \to 0, \end{aligned}$$

for all $A \geq 1$ and all $\epsilon > 0$ and such that for any $Y > 1$ we have the approximate functional equation

$$\begin{aligned} L_{f_\chi \times f}(\beta) = &\sum_{n=1}^{\infty} \frac{b(n)}{n^\beta} \chi(n)\, h_1\left(\frac{n}{Y}\right) \\ &- w(f, \overline{f}) \chi^2(N) \frac{\tau(\chi)^4 q^{-4\beta}}{N^{2\beta-1} \pi^{2-4\beta}} \sum_{n=1}^{\infty} \frac{b(n)}{n^{1-\beta}} \overline{\chi(n)}\, h_2\left(\frac{nY\pi^4}{N^2 q^4}\right). \end{aligned} \tag{57}$$

This is proved by simply replacing in the proof of Lemma 32 the functional equation (31) by the functional equation (51).

Step 2. *We average the approximate functional equation over even primitive Dirichlet characters to deduce (56).*

According to the approximate functional equation (57), we can decompose $\sum_{q \bowtie Q} \sum_{\chi} L_{f_\chi \times f}(\beta)$ into $T_1 - T_2$, where

$$\begin{aligned} T_1 &= \sum_{q \bowtie Q} \sum_{\chi} \sum_{n=1}^{\infty} \frac{b(n)}{n^\beta} \chi(n)\, h_1\left(\frac{n}{Y}\right), \\ T_2 &= \sum_{q \bowtie Q} \sum_{\chi} w(f, \overline{f}) \frac{\tau(\chi)^4 q^{-4\beta}}{N^{2\beta-1} \pi^{2-4\beta}} \sum_{n=1}^{\infty} \frac{b(n)}{n^{1-\beta}} \chi^2(N) \overline{\chi(n)}\, h_2\left(\frac{nY\pi^4}{N^2 q^4}\right). \end{aligned}$$

By the proof of Lemma 32 a lower bound for T_1 is given by

$$T_1 \geq \sum_{q \bowtie Q} \frac{q-1}{2} - O(Q\, Y^{1-\beta}). \tag{58}$$

We now consider the term T_2. It contains the sum

$$\sum_{\substack{\chi \pmod q, \\ \chi \neq \chi_0,\, \chi(-1)=1}} \chi^2(N) \overline{\chi(n)} \tau(\chi)^4$$

which is zero if $(q, nN^{-2}) > 1$ and otherwise equals by Lemma 8

$$\frac{q-1}{2}\left(K_4(nN^{-2},q)+K_4(-nN^{-2},q)\right)-1.$$

Again the hyper-Kloosterman sum $K_d(n,q)$ is bounded by

$$K_d(n,q) \ll q^{\frac{d-1}{2}}.$$

Hence if $(q, nN^{-2}) = 1$ we get

$$\sum_{\substack{\chi \pmod q,\\ \chi\neq\chi_0,\, \chi(-1)=1}} \chi^2(N)\overline{\chi(n)}\tau(\chi)^4 \ll q^{\frac{5}{2}}.$$

We can now bound the contribution of T_2. We have

$$\begin{aligned}
T_2 &= \sum_{q\bowtie Q}\sum_{\chi} w(f,\overline{f})\frac{\tau(\chi)^4 q^{-4\beta}}{N^{2\beta-1}\pi^{2-4\beta}}\sum_{n=1}^{\infty}\frac{b(n)}{n^{1-\beta}}\chi^2(N)\overline{\chi(n)}\,h_2\left(\frac{nY\pi^4}{N^2q^4}\right)\\
&\ll \sum_{q\bowtie Q} q^{\frac{5}{2}}q^{-4\beta}\sum_{(q,nN^{-2})=1}\frac{|b(n)|}{n^{1-\beta}}\left|h_2\left(\frac{nY\pi^4}{N^2q^4}\right)\right|\\
&\ll \sum_{q\bowtie Q} q^{\frac{5}{2}}q^{-4\beta}\int_1^{\infty}\left|h_2\left(\frac{rY\pi^4}{N^2q^4}\right)\right| r^{\beta}\frac{\mathrm{d}r}{r}\\
&\ll \sum_{q\bowtie Q}\frac{q^{\frac{5}{2}}}{Y^{\beta}}\int_{\frac{Y\pi^4}{N^2q^4}}^{\infty}|h_2\left(r\right)| r^{\beta}\frac{\mathrm{d}r}{r}\\
&\ll \sum_{q\bowtie Q}\frac{q^{\frac{5}{2}}}{Y^{\beta}}\\
&\ll Q^{1+\frac{5}{2}}Y^{-\beta}. \qquad (59)
\end{aligned}$$

From the upper bounds (58) and (59) it follows that

$$\sum_{q\bowtie Q}\sum_{\chi} L_{f_\chi\times f}(\beta) \geq \frac{1}{2}\sum_{q\bowtie Q} q - O(QY^{1-\beta}+Q^{1+\frac{5}{2}}Y^{-\beta}) + O(Q).$$

Exactly as in the proof of Lemma 32 we now conclude that the average $\sum_{q\bowtie Q}\sum_{\chi} L_{f_\chi\times f}(\beta)$ is non-zero and therefore $L_{f_\chi\times f}(\beta)$ is non-zero for some χ.

□

5 References

1. Luo, W., Rudnick, Z. & Sarnak, P. On Selberg's eigenvalue conjecture. *Geom. Funct. Anal.* **5,** 387–401. (1995).
2. Luo, W., Rudnick, Z. & Sarnak, P. On the generalized Ramanujan conjecture for GL(n). *Proceedings of Symposia in Pure Mathematics* **66.2** (1999).
3. Bergeron, N. *The Spectrum of Hyperbolic Surfaces* (Springer, 2016).
4. Goldfeld, D. *Automorphic Forms and L-Functions for the Group GL(n,R)* (Cambridge University Press, 2006).
5. Abramowitz, M. & Stegun, I. A. *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables* (Dover Publications, 1972).
6. Watson, G. N. *A Treatise on the Theory of Bessel Functions* (Cambridge University Press, 1966).
7. Bateman, H. & Erdélyi, A. *Tables of Integral Transforms (Volume II)* (McGraw Hill, New York, 1954).
8. Macdonald, I. *Symmetric Functions and Hall Polynomials* (Oxford University Press, 1979).
9. Deligne, P. *SGA* $4\frac{1}{2}$*, Lecture Notes in Math. 569,* (Springer-Verlag, 1977).
10. Kowalski, E., Michel, P. & VanderKam, J. Rankin-Selberg L-functions in the level aspect. *Duke Mathematical Journal* **114,** 123 –191. `https://doi.org/10.1215/S0012-7094-02-11416-1` (2002).