Contents

1 $SL(2,\mathbb{Z})$

1.1 Eisenstein series and the spectral decomposition of $L^2(\Gamma\setminus\mathbb{H})$

Let $\Gamma=\mathrm{SL}(2,\mathbb{Z})$. This group has only a cusp at infinity. The stabilizer of the cusp ∞ in Γ is

$$\Gamma_\infty=\left\{\begin{pmatrix}1 & m\\ 0 & 1\end{pmatrix}\mid m\in\mathbb{Z}\right\}.$$

The Eisenstein series associated to this cusp, defined on $\mathbb{H}\times\mathbb{C}$, is defined by

$$E(z,s):=E_\infty(z,s)=\sum_{\gamma\in\Gamma_\infty\setminus\Gamma}\mathrm{Im}(\gamma z)^s. \tag{1}$$

Notice that $\mathrm{Im}(z)$ is Γ_∞-invariant, and the Eisenstein series $E(z,s)$ defines an automorphic function with respect to Γ, that is, it satisfies $E(\gamma z,s)=E(z,s)$ for all $\gamma\in\Gamma$. For $\mathrm{Re}(s)>1$, this series converges absolutely and uniformly on compact sets. Because $\Delta y^s=s(1-s)y^s$ and because Δ commutes with the Γ-action, the Eisenstein series is also an eigenfunction of the Laplacian

$$\Delta E(z,s)=s(1-s)E(z,s).$$

Automorphic functions which are eigenfunctions of the Laplace operator are called Maass forms. The Fourier expansion of $E(z,s)$ is given by

$$E(z,s)=y^s+\phi(s)y^{1-s}+2\sum_{m\neq 0}a_m y^{1/2}K_{s-1/2}(2\pi|m|y)e(mx),$$

where

$$\phi(s)=\pi^{1/2}\frac{\Gamma(s-1/2)}{\Gamma(s)}\frac{\zeta(2s-1)}{\zeta(2s)},$$
$$a_m=\frac{\pi^s}{\Gamma(s)\zeta(2s)|n|^{1/2}}\sum_{ab=|n|}\left(\frac{a}{b}\right)^{s-\frac{1}{2}}.$$

The modified Bessel functions K_ν are exponentially decaying functions. In particular, $E(z,s)=y^s+\phi(s)y^{1-s}+O(e^{-cy})$, for some constant $c>0$. For a function to be in $L^2(\Gamma\setminus\mathbb{H})$ its growth need to be $O(y^{1/2})$. From the constant term of the Fourier expansion we see that the Eisenstein series is not in $L^2(\Gamma\setminus\mathbb{H})$. But for $\mathrm{Re}(s)=\frac{1}{2}$ the Eisenstein series is "almost square integrable", and this suggests to work on the line $\mathrm{Re}(s)=\frac{1}{2}$. We now have to address two issues: We want to work with square integrable functions on $\Gamma\setminus\mathbb{H}$, and we need to meromorphically continue $E(z,s)$ to the line $\mathrm{Re}(s)=\frac{1}{2}$. The meromorphic continuation of $E(z,s)$ follows from the Fourier expansion. In the half-plane $\mathrm{Re}(s) \geq \frac{1}{2}$ there is only a simple pole at $s = 1$ with residue $\frac{3}{\pi}$. The Eisenstein series enjoys the functional equation

$$E(z, 1-s) = \phi(1-s)E(z, s).$$

For a smooth, compactly supported function ψ on $\mathbb{R}^{>0}$, the incomplete Eisenstein series is

$$E(z|\psi) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \psi(\mathrm{Im}(\gamma z)).$$

The incomplete Eisenstein series $E(z|\psi)$ lies in $C_c^\infty(\Gamma \setminus \mathbb{H}) \subset L^2(\Gamma \setminus \mathbb{H})$, but it is not an eigenfunction of Δ. By Mellin inversion it can however be represented as an integral of Eisenstein series

$$E(z|\psi) = \frac{1}{2\pi i} \int_{(\sigma)} E(z, s)\widehat{\psi}(s)\mathrm{d}s, \tag{2}$$

where $\sigma > 1$ and

$$\widehat{\psi}(s) = \int_0^\infty \psi(y) y^{-s-1} \mathrm{d}y.$$

We denote by $\mathcal{E}(\Gamma \setminus \mathbb{H}) \subset L^2(\Gamma \setminus \mathbb{H})$ the space of incomplete Eisenstein series $E(z|\psi)$. The inner product of a function $f \in L^2(\Gamma \setminus \mathbb{H})$ with an incomplete Eisenstein series $E(z|\psi)$ is

$$\langle f, E(\cdot|\psi)\rangle = \int_0^\infty f_0(y)\overline{\psi}(y)y^{-2}\mathrm{d}y,$$

where f_0 is the constant term in the Fourier expansion of f. If f is orthogonal to $\mathcal{E}(\Gamma \setminus \mathbb{H})$, then the above integral is zero for all smooth functions ψ of compact support in $(0, \infty)$. Thus the orthogonal complement $\mathcal{C} = \mathcal{E}^\perp$ consists of functions whose constant term in the Fourier series is zero. The Laplace operator Δ has discrete spectrum in $\mathcal{C}$ and $\mathcal{C}$ is spanned by cusp forms. For an orthonormal basis of cusp forms $\{u_j\}$ every $f \in \mathcal{C}(\Gamma \setminus \mathbb{H})$ has the spectral expansion

$$f(z) = \sum_j \langle f, u_j\rangle u_j(z).$$

The spectrum of Δ in $\mathcal{E}(\Gamma \setminus \mathbb{H})$ turns out to consist of a continuous part spanned by the Eisenstein series $E(z, \frac{1}{2} + ir)$, and the zero eigenvalue corresponding to the constant function u_0.

For $\mathrm{Re}(s) \geq \frac{1}{2}$, the Eisenstein series has only a simple pole at $s = 1$. The Eisenstein series are holomorphic on the line $\mathrm{Re}(s) = \frac{1}{2}$ and are of polynomial growth in vertical strips $-\epsilon \leq \mathrm{Re}(s) \leq 1 + \epsilon$. One can then shift the integration in Equation 2 to the line $\mathrm{Re}(s) = \frac{1}{2}$, thereby picking up the residue of $E(z, s)$ at the pole $s = 1$. As a result we obtain

$$E(z|\psi) = \widehat{\psi}(1)\mathrm{Res}_{s=1}(E(z, s)) + \frac{1}{2\pi i} \int_{(1/2)} E(z, s)\widehat{\psi}(s)\mathrm{d}s.$$

The term $\widehat{\psi}(1)$ can be written as $\frac{\langle E(\cdot|\psi), u_0\rangle}{\langle u_0, u_0\rangle}$. We still need the projection of $E(z|\psi)$ onto $E(z, s)$. The functional equation along with

$$\langle E(\cdot|\psi), E(\cdot, 1/2 + ir)\rangle = \widehat{\psi}(1/2 + ir) + \phi(1/2 - ir)\widehat{\psi}(1/2 - ir)$$

yields the spectral decomposition of $E(z|\psi)$ onto the Eisenstein series

$$\frac{1}{2\pi i} \int_{(1/2)} E(z, s)\widehat{\psi}(s)\mathrm{d}s = \frac{1}{4\pi i} \int_{(1/2)} \langle E(\cdot|\psi), E(\cdot, s)\rangle E(z, s)\mathrm{d}s.$$

In conclusion, for an orthonormal basis of cusp forms $\{u_j\}$, every $f \in L^2(\Gamma \setminus \mathbb{H})$ has the spectral expansion

$$f(z) = \sum_j \langle f, u_j\rangle u_j(z) + \frac{1}{4\pi i} \int_{(1/2)} \langle E(\cdot|\psi), E(\cdot, s)\rangle E(z, s)\mathrm{d}s.$$

1.2 Hecke operators for $L^2(\mathrm{SL}(2,\mathbb{Z}) \setminus \mathbb{H})$

1.3 L-functions

Let

$$f(z) = \sum_{n \neq 0} a(n)\sqrt{y}K_{\nu_f}(2\pi|n|y)e^{2\pi inx} \tag{3}$$

be a cuspidal Maass form for $SL(2,\mathbb{Z})$.

Lemma 1. The coefficients $a(n)$ in the Fourier expansion of $f(z)$ satisfy

$$a(n) = O(\sqrt{|n|}).$$

Proof. Because f is a cusp form, it is bounded as $\mathrm{Im}(z) \to \infty$. Thus

$$\left| a(n)\sqrt{y}K_{\nu_f}(2\pi|n|y) \right| = \left| \int_0^1 f(x+iy)e^{-2\pi inx}\mathrm{d}x \right| \leq \int_0^1 |f(x+iy)|\mathrm{d}x \leq C,$$

for some constant C (that depends on f). If we choose $y = \frac{1}{|n|}$, the lemma is proved. □

For $\mathrm{Re}(s) \geq \frac{3}{2}$ we define the L-function $L_f(s)$ associated to $f(z)$ by the absolutely convergent series

$$L_f(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}. \tag{4}$$

Lemma 2. Let $f(z)$ be a cuspidal Maass form for $SL(2,\mathbb{Z})$. Then its L-function $L_f(s)$ can be meromorphically continued to all $\mathbb{C}$ and it satisfies the functional equation

$$\Lambda_f(s) = \pi^{-2}\Gamma\left(\frac{s+\epsilon+\nu_f}{2}\right)\Gamma\left(\frac{s+\epsilon-\nu_f}{2}\right)L_f(s) = (-1)^{\epsilon}\Lambda_f(1-s).$$

Proof. ... □

1.4 Rankin-Selberg convolution for $\mathrm{SL}(2,\mathbb{Z})$

Let

$$f(z) = \sum_{n \neq 0} a(n)\sqrt{y}K_{\nu_f}(2\pi|n|y)e^{2\pi inx}, \tag{5}$$

$$g(z) = \sum_{n \neq 0} b(n)\sqrt{y}K_{\nu_g}(2\pi|n|y)e^{2\pi inx}, \tag{6}$$

be cuspidal Maass forms for $SL(2,\mathbb{Z})$. Recall that $f(z)$ (resp. $g(z)$) is an eigenfunction for the Laplacian with eigenvalue $1/4-\nu_f$ (resp. $1/4-\nu_g$). For sufficiently large $\mathrm{Re}(s)$ we define the convolution function as the absolutely convergent series

$$L_{f\times g}(s) = \zeta(2s)\sum_{n=1}^{\infty} \frac{a(n)\overline{b(n)}}{n^s}.$$

We will prove that $L_{f\times g}$ can be expressed as an inner product of $f\overline{g}$ with an Eisenstein series. This construction gives the meromorphic continuation and functional equation for $L_{f\times g}$.

Theorem 3. *Let $f(z)$ and $g(z)$ be cuspidal Maass forms as in Equation* 5 *and* 6*. Then $L_{f\times g}$ can be meromorphically continued to all $s \in \mathbb{C}$ with at most a simple pole at $s=1$. Furthermore, we have the functional equation*

$$L^*_{f\times g}(s) = \pi^{-2s}G_{\nu_f,\nu_g}(s)L_{f\times g}(s) = L^*_{f\times g}(1-s),$$

where $G_{\nu_f,\nu_g}(s) = \prod \Gamma\left(\frac{s\pm\nu_f\pm\nu_g}{2}\right)$.

Proof. Let $E(z,s)$ be the non-holomorphic Eisenstein series as defined in Equation 1. For sufficiently large $\mathrm{Re}(s)$, we have

$$\begin{aligned}
\zeta(2s)\langle f\overline{g}, E(\cdot,\overline{s})\rangle &= \zeta(2s)\int_{SL(2,\mathbb{Z})\backslash\mathbb{H}} f(z)\overline{g(z)E(z,\overline{s})}\mathrm{d}\mu(z)\\
&= \zeta(2s)\sum_{\gamma\in\Gamma_\infty\backslash SL(2,\mathbb{Z}}\int_{SL(2,\mathbb{Z})\backslash\mathbb{H}} f(\gamma z)\overline{g(\gamma z)}\,\mathrm{Im}(\gamma z)^s\mathrm{d}\mu(\gamma z)\\
&= \zeta(2s)\int_{\Gamma_\infty\backslash\mathbb{H}} f(z)\overline{g(z)}\,\mathrm{Im}(z)^s\mathrm{d}\mu(z)\\
&= \zeta(2s)\int_0^\infty\int_0^1 f(z)\overline{g(z)}y^{s-2}\mathrm{d}x\mathrm{d}y\\
&= \zeta(2s)\sum_{n,m\neq 0} a(n)\overline{b(m)}\int_0^\infty K_{\nu_f}(2\pi|n|y)K_{\nu_g}(2\pi|m|y)y^{s-1}\int_0^1 e^{2\pi i(n-m)x}\mathrm{d}x\mathrm{d}y\\
&= \zeta(2s)\sum_{n\neq 0} a(n)\overline{b(n)}\int_0^\infty K_{\nu_f}(2\pi|n|y)K_{\nu_g}(2\pi|n|y)y^s\frac{\mathrm{d}y}{y}\\
&= \frac{\zeta(2s)}{(2\pi)^s}\sum_{n\neq 0}\frac{a(n)\overline{b(n)}}{n^s}\int_0^\infty K_{\nu_f}(y)K_{\nu_g}(y)y^s\frac{\mathrm{d}y}{y}\\
&= (2\pi)^{-s}\,L_{f\times g}(s)\int_0^\infty K_{\nu_f}(y)K_{\nu_g}(y)y^s\frac{\mathrm{d}y}{y}.
\end{aligned}$$

The Mellin transform of $K_\nu K_{\nu'}$ is given by

$$\int_0^\infty K_\nu(y)K_{\nu'}(y)y^s\frac{\mathrm{d}y}{y} = \frac{2^{s-3}}{\Gamma(s)}\prod\Gamma\left(\frac{s\pm\nu\pm\nu'}{2}\right).$$

From the calculation it follows that the convolution function $L_{f\times g}$ inherits the analytical properties of the Eisenstein series $E(z,s)$. This means that $L_{f\times g}$ can be meromorphically continued on $\mathbb{C}$. Because the Eisenstein series has a simple pole at $s=1$ and the Gamma function no zeros, it follows that $L_{f\times g}$ has a simple pole at $s=1$ if and only if $\langle f,g\rangle\neq 0$. The functional equation follows from the functional equation of the Eisenstein series

$$E^*(z,s) = \pi^{-s}\Gamma(s)\zeta(2s)E(z,s) = E^*(z,1-s).$$

□

Lemma 4. Let $\alpha_i,\beta_i\in\mathbb{C}$ for $i=1,2$. Then

$$\prod_{i=1}^2\prod_{j=1}^2(1-\alpha_i\beta_j x)^{-1} = \sum_{k=0}^\infty S_k(\alpha_1,\alpha_2)S_k(\beta_1,\beta_2)x^k(1-\alpha_1\alpha_2\beta_1\beta_2x^2)^{-1},$$

where $S_k(x_1,x_2)$ is the Schur polynomial

$$S_k(x_1,x_2) = \frac{\det\begin{pmatrix} x_1^{k+1} & x_2^{k+1}\\ 1 & 1\end{pmatrix}}{\det\begin{pmatrix} x_1 & x_2\\ 1 & 1\end{pmatrix}} = \frac{x_1^{k+1}-x_2^{k+1}}{x_1-x_2}.$$

Proof. □

Theorem 5. *Let $f(z)$ and $g(z)$ be cuspidal Maass forms as in Equation* 5 *and* 6*. Assume that L_f and L_g have Euler products*

$$L_f(s) = \prod_p\prod_{i=1}^2\left(1-\frac{\alpha_{i,p}}{p^s}\right)^{-1},\quad L_g(s) = \prod_p\prod_{j=1}^2\left(1-\frac{\beta_{j,p}}{p^s}\right)^{-1}.$$

Then $L_{f\times g}(s)$ admits the Euler product

$$L_{f\times g}(s) = \prod_p \prod_{i=1}^{2} \prod_{j=1}^{2} \left(1 - \frac{\alpha_{i,p}\,\overline{\beta}_{j,p}}{p^s}\right)^{-1}.$$

Proof. By assumption we have

$$L_{f\times g}(s) = \prod_p \frac{\sum_{k=1}^{\infty} a(p^k)\overline{b(p^k)}p^{-ks}}{(1-p^{-2s})}.$$

In view of Lemma 4, after choosing $x = p^{-s}$, it suffices to show that

$$a(p^k) = S_k(\alpha_{1,p}, \alpha_{2,p}), \quad b(p^k) = S_k(\beta_{1,p}, \beta_{2,p}).$$

The above equalities are obtained inductively from the relations

$$\begin{aligned} 1 &= \alpha_{1,p}\,\alpha_{2,p}, \\ a(p) &= \alpha_{1,p} + \alpha_{2,p}, \\ a(p^{k+1}) &= a(p)a(p^k) - a(p^{k-1}). \end{aligned}$$

□

2 $\Gamma_0(N)$

Let $\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) \mid c \equiv 0 \pmod N \right\}$.

2.1 Dirichlet characters

Let m be a positive integer. A Dirichlet character of modulus m is a function $\chi : \mathbb{Z} \to \mathbb{C}$ such that

- $\chi(ab) = \chi(a)\chi(b)$ for all $a, b \in \mathbb{Z}$,
- $\chi(a) = 0$ if and only if $(a, m) > 1$,
- If $a \equiv b \pmod m$, then $\chi(a) = \chi(b)$.

Let χ_1 and χ_2 be Dirichlet characters of modulus m_1 and m_2, respectively, with $m_1|m_2$. If $\chi_2(a) = \chi_1(a)$ for $a \in (\mathbb{Z}/m_2\mathbb{Z})^\times$, then χ_2 is said to be induced by χ_1. A Dirichlet character is called primitive if it is not induced by any Dirichlet character other than itself. A Dirichlet character induced by the identity function is called principal. We denote a principal Dirichlet character by χ_0. By definition a principal Dirichlet character mod m is

$$\chi_0(a) = \begin{cases} 1 & \text{if } (a,m) = 1 \\ 0 & \text{else} \end{cases}. \tag{7}$$

If p is prime, then every nonprincipal Dirichlet character of modulus p is primitive.

Lemma 6. Let χ be a nonprincipal Dirichlet character modulo a prime q. Then we have

$$\sum_{a=1}^{q} \chi(a) = 0.$$

Proof. Because χ is not principal, there is an integer b such that $\chi(b) \neq \{0,1\}$. Furthermore, the sum may be restricted to the terms with $(a,q) = 1, 1 \leq a \leq q$. Multiplication by b is a bijection $(\mathbb{Z}/q\mathbb{Z})^\times \to (\mathbb{Z}/q\mathbb{Z})^\times$. Therefore we have

$$\chi(b)\sum_{a=1}^{q}\chi(a) = \sum_{a\in(\mathbb{Z}/q\mathbb{Z})^\times} \chi(ab) = \sum_{c\in(\mathbb{Z}/q\mathbb{Z})^\times} \chi(c),$$

which implies $\sum_{a=1}^{q} \chi(a) = 0$. □

The Gauss sum $\tau(\chi)$ attached to a primitive Dirichlet character χ mod q is

$$\tau(\chi) = \sum_{a \bmod q} \chi(a) e^{2\pi i a/q}.$$

Lemma 7. Let χ be a nonprincipal Dirichlet character modulo a prime q. Then for all $n \in \mathbb{Z}$ we have

$$\tau(\chi)\overline{\chi(n)} = \sum_{a=1}^{q} \chi(a) e^{2\pi i n a/q}. \tag{8}$$

Proof. The proof is similar to the proof of Lemma 8. $\square$

We will also need the following variant of Lemma 7.

Lemma 8. Let χ be a nonprincipal Dirichlet charactermodulo a prime q, and let $n, m \in \mathbb{Z}$. Then

$$\sum_{a=1}^{nq^2} \chi(a) e^{2\pi i m a/nq^2} = \begin{cases} qn\tau(\chi)\overline{\chi}(l) & \text{if } m = lnq \text{ for some } l \in \mathbb{Z} \\ 0 & \text{else} \end{cases}. \tag{9}$$

Proof. Every a in the above sum is of the form $a = a_1 + tq$ with $1 \le a_1 \le q$ and $0 \le t < nq$. Thus we can write

$$\sum_{a=1}^{nq^2} \chi(a) e^{2\pi i m a/nq^2} = \sum_{a_1=1}^{q} \chi(a_1) e^{\frac{2\pi i m a_1}{nq^2}} \sum_{t=0}^{nq-1} e^{\frac{2\pi i m t}{nq}}.$$

The sum over t is zero unless $m = lnq$ for some $l \in \mathbb{Z}$ in which case the sum is nq. Therefore

$$\begin{aligned} \sum_{a=1}^{nq^2} \chi(a) e^{2\pi i m a/nq^2} &= nq \sum_{\substack{a \bmod q \\ m = lnq}} \chi(a) e^{\frac{2\pi i m a}{nq^2}} \\ &= nq \sum_{a \bmod q} \chi(a) e^{\frac{2\pi i l a}{q}} \end{aligned}$$

If $(l, q) = 1$, then $a \mapsto al$ permutes the residues mod q. In this case we get

$$nq \sum_{a \bmod q} \chi(a) e^{\frac{2\pi i l a}{q}} = nq\tau(\chi)\overline{\chi(l)}.$$

Now suppose that $(l, q) > 1$. Then $\chi(l) = 0$ and we have to show that the left side of Equation 9 vanishes. For this let $l' \in \mathbb{Z}$ be such that $ql' = l$. Then we have

$$\begin{aligned} nq \sum_{a \bmod q} \chi(a) e^{\frac{2\pi i l a}{q}} &= nq \sum_{a \bmod q} \chi(a) e^{2\pi i l' a} \\ &= nq \sum_{a \bmod q} \chi(a). \end{aligned}$$

The last sum is zero by Lemma 6. This completes the proof of the lemma.

$\square$

2.2 Twisted Eisenstein series

Let q be a prime. In this section we assume that χ is an even ($\chi(-1) = 1$), non-principal (and thus primitive) Dirichlet character mod q, and that $(q, N) = 1$. For $\mathrm{Re}(s) > 1$ we define the twisted Eisenstein series by the absolutely convergent series

$$E(z, s, \chi) = \frac{1}{2} \sum_{\gamma} \chi(d) \,\mathrm{Im}(\gamma z)^s,$$

where the sum goes over a set of coset representatives $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\infty \setminus \Gamma_0(Nq^2)$.

The Eisenstein series $E(z, s, \chi)$ satisfies $E(\gamma z, s, \chi) = \chi(\gamma) E(z, s, \chi)$ for all $\gamma \in \Gamma_0(Nq^2)$. In particular, the function $E(z, s, \chi)$ is invariant under $z \mapsto z+1$. Hence it has a Fourier expansion. In order to determine its Fourier expansion we first establish a few identities.

We associate to a pair of coprime integers (c, d), subject to $Nq^2|c$, the set of all matrices in $\Gamma_0(Nq^2)$ whose bottom row is (c, d). Such a matrix represents a unique coset $\Gamma_\infty \setminus \Gamma_0(Nq^2)$. A pair $(m, n) \in \mathbb{Z} \setminus \{0, 0\}$ can be uniquely written as (Mc, Md) for $(c, d) = 1$ and with $M = \gcd(c, d) > 0$. As a consequence, we can write

$$\begin{aligned} E^*(z, s, \chi) &:= L(2s, \chi) E(z, s, \chi) \\ &= \frac{L(2s, \chi)}{2} \sum_{\gamma \in \Gamma_\infty \setminus \Gamma_0(Nq^2)} \chi(d) \,\mathrm{Im}(\gamma z)^s \\ &= \frac{L(2s, \chi)}{2} \sum_{\substack{(c,d)=1 \\ Nq^2|c}} \chi(d) \frac{y^s}{|cz+d|^{2s}} \\ &= \frac{1}{2} \sum_{\substack{m,n \in \mathbb{Z} \\ (m,n) \neq 0}} \chi(n) \frac{y^s}{|mNq^2 z + n|^{2s}}, \end{aligned} \tag{10}$$

where $L(s, \chi)$ is the Dirichlet series $L(s, \chi) = \sum_{n=1}^\infty \chi(n) n^{-s}$.

Lemma 9. If $\mathrm{Re}(s) > 1/2$ and $r \in \mathbb{R}$, then

$$\int_{-\infty}^{\infty} \frac{e^{-2\pi i r x}}{(x^2 + y^2)^s} \mathrm{d}x = \frac{\pi^s}{y^s \Gamma(s)} \begin{cases} \pi^{-s+\frac{1}{2}} \Gamma(s - \frac{1}{2}) y^{1-s} & \text{if } r = 0 \\ 2|r|^{s-\frac{1}{2}} \sqrt{y} K_{s-\frac{1}{2}}(2\pi |r| y) & \text{if } r \neq 0 \end{cases}. \tag{11}$$

Proof. Recall the integral representation of the Gamma function

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} \mathrm{d}t.$$

Thus we have

$$\begin{aligned} \frac{y^s}{\pi^s} \Gamma(s) \int_{-\infty}^{\infty} \frac{e^{-2\pi i r x}}{(x^2 + y^2)^s} \mathrm{d}x &= \int_0^\infty \int_{-\infty}^\infty e^{-t} \left(\frac{ty}{\pi(x^2 + y^2)} \right)^s e^{-2\pi i r x} \mathrm{d}x \frac{\mathrm{d}t}{t} \\ &= \int_0^\infty \int_{-\infty}^\infty e^{-\pi \xi (x^2 + y^2)/y} \xi^s e^{-2\pi i r x} \mathrm{d}x \frac{\mathrm{d}\xi}{\xi}. \end{aligned}$$

For $r = 0$ the above expression becomes

$$\begin{aligned} \int_0^\infty \int_{-\infty}^\infty e^{-\pi \xi (x^2+y^2)/y} \xi^s \mathrm{d}x \frac{\mathrm{d}\xi}{\xi} &= \int_0^\infty \sqrt{\frac{y}{\xi}} e^{-\pi \xi y} \xi^s \frac{\mathrm{d}\xi}{\xi} \\ &= \pi^{-s+\frac{1}{2}} y^{1-s} \Gamma(s - \frac{1}{2}). \end{aligned}$$

For $r \neq 0$ we obtain, using the change of variables $\xi \mapsto \frac{\xi}{|r|}$,

$$\begin{aligned}\int_0^\infty \int_{-\infty}^\infty e^{-\pi\xi(x^2+y^2)/y}\xi^s e^{-2\pi irx}\mathrm{d}x\frac{\mathrm{d}\xi}{\xi} &= \int_0^\infty \sqrt{\frac{y}{\xi}}e^{-y\pi r^2/\xi}e^{-\pi\xi y}\xi^s\frac{\mathrm{d}\xi}{\xi} \\ &= \sqrt{y}|r|^{s-\frac{1}{2}}\int_0^\infty \xi^{t-\frac{1}{2}}e^{-y\pi|r|(1/\xi+\xi)}\frac{\mathrm{d}\xi}{\xi} \\ &= 2|r|^{s-\frac{1}{2}}\sqrt{y}K_{s-\frac{1}{2}}(2\pi|r|y).\end{aligned}$$

Here we used the following integral representation of the modified Bessel function $K_\nu(z)$

$$K_\nu(z) = \frac{1}{2}\int_0^\infty e^{-\frac{z}{2}(t+t^{-1})}t^{-s-1}\mathrm{d}t.$$

□

Lemma 10. For $z = x+iy \in \mathbb{H}$ and $\mathrm{Re}(s) > \frac{1}{2}$ we have

$$\sum_{n\in\mathbb{Z}} \frac{1}{|z+n|^{2s}} = \frac{\sqrt{\pi}\Gamma(s-\frac{1}{2})}{\Gamma(s)}y^{1-2s} + \frac{2\pi^s}{\Gamma(s)}y^{\frac{1}{2}-s}\sum_{r\neq 0}|r|^{s-\frac{1}{2}}K_{s-\frac{1}{2}}(2\pi|r|y)\,e^{2\pi irx}. \tag{12}$$

Proof. Recall the Poisson summation formula

$$\sum_{n\in\mathbb{Z}}\varphi(x+n) = \sum_{n\in\mathbb{Z}}\widehat{\varphi}(n)e^{2\pi inx},$$

where φ is a continuous function that decays rapidly at infinity and where $\widehat{\varphi}(n) = \int_{-\infty}^\infty \varphi(x)e^{-2\pi inx}\mathrm{d}x$ is the Fourier transform. We apply this formula to $\varphi(x) = |x+iy|^{-2s}$, where $z = x+iy \in \mathbb{H}$ and $\mathrm{Re}(s) > \frac{1}{2}$. The Poisson summation formula gives

$$\sum_{n\in\mathbb{Z}}\frac{1}{|z+n|^{2s}} = \sum_{n\in\mathbb{Z}}\left(\int_{-\infty}^\infty \frac{e^{-2\pi inx}}{(x^2+y^2)^s}\mathrm{d}x\right)e^{2\pi inx}.$$

Using Lemma 9 we get the result. □

We would like to adapt the left hand side of Equation 12 to the setting where the sum is twisted by a primitive Dirichlet character χ mod q. For this we need the twisted variant of the Poisson summation formula.

Lemma 11. Let φ be a function that satisfies the conditions of the Poisson summation formula. Let χ be a primitive Dirichlet character mod q. Then

$$\sum_{n\in\mathbb{Z}}\chi(n)\varphi(x+n) = \frac{\tau(\chi)}{q}\sum_{n\in\mathbb{Z}}\overline{\chi(n)}\widehat{\varphi}(n/q)e^{2\pi inx/q}.$$

Proof. From Lemma 7 we know that

$$\chi(n) = \frac{1}{\tau(\overline{\chi})}\sum_{m \bmod q}\overline{\chi(m)}e^{2\pi inm/q} = \frac{\chi(-1)\tau(\chi)}{q}\sum_{m \bmod q}\overline{\chi(m)}e^{2\pi inm/q}.$$

Let us now consider $\sum_{n\in\mathbb{Z}}\chi(n)\varphi(n) = \sum_{n\in\mathbb{Z}}\varphi_1(n)$, where

$$\varphi_1(x) = \frac{\chi(-1)\tau(\chi)}{q}\sum_{m \bmod q}\overline{\chi(m)}e^{2\pi ixm/q}\varphi(x).$$

The Fourier transform of $\varphi_1(x)$ is

$$\begin{aligned}\widehat{\varphi_1}(\xi) &= \frac{\chi(-1)\tau(\chi)}{q}\sum_{m \bmod q}\overline{\chi(m)}\int_{-\infty}^\infty \varphi(x)e^{-2\pi ix(\xi-\frac{m}{q})}\mathrm{d}x \\ &= \frac{\chi(-1)\tau(\chi)}{q}\sum_{m \bmod q}\overline{\chi(m)}\widehat{\varphi}\left(\xi-\frac{m}{q}\right).\end{aligned}$$

We apply Poisson summation formula and thus $\sum_{n\in\mathbb{Z}} \varphi_1(n)$ equals

$$\frac{\chi(-1)\tau(\chi)}{q} \sum_{m \bmod q} \sum_{n=-\infty}^{\infty} \overline{\chi(m)} \widehat{\varphi}\left(\frac{nq-m}{q}\right).$$

We can write $\overline{\chi(m)} = \chi(-1)\overline{\chi(nq-m)}$. When n runs through $\mathbb{Z}$ and m through $\mathbb{Z}/q\mathbb{Z}$, the terms $nq-m$ run uniquely through $\mathbb{Z}$. Thus we have shown

$$\sum_{n\in\mathbb{Z}} \chi(n)\varphi(n) = \frac{\tau(\chi)}{q} \sum_{n\in\mathbb{Z}} \overline{\chi(n)} \widehat{\varphi}(n/q).$$

Replacing $\varphi(n)$ by $\varphi(n+x)$ replaces $\widehat{\varphi}(\xi)$ by $\widehat{\varphi}(\xi)e^{2\pi i\xi x}$. This completes the proof. □

We can now determine the Fourier expansion of $E^*(z,s,\chi)$.

Theorem 12. *The function $E^*(z,s,\chi)$ has the Fourier expansion*

$$\begin{aligned} E^*(z,s,\chi) &= L(2s,\chi)y^s \\ &\quad + \tau(\chi)q^{-2s}\sqrt{y}\frac{2\pi^s}{\Gamma(s)} \sum_{k\neq 0} |k|^{\frac{1}{2}-s} \sigma_{2s-1}(|k|,\overline{\chi}) K_{s-\frac{1}{2}}(2\pi|k|y)e^{2\pi ikx}, \end{aligned}$$

where $\sigma_s(k,\overline{\chi}) = \sum_{d|k} \overline{\chi(d)} d^s$.

First Proof. We split up the sum (10) into the terms with $m=0$ and those with $m\neq 0$. We also use the evenness of χ to combine each positive summand with its negative. We obtain

$$E^*(z,s,\chi) = L(2s,\chi)y^s + \sum_{m=1}^{\infty} \sum_{n\in\mathbb{Z}} \chi(n) \frac{y^s}{|mNq^2z+n|^{2s}}.$$

Substituting into this the twisted variant of the formula of Lemma 10 gives (note that $\chi(0)=0$)

$$\begin{aligned} E^*(z,s,\chi) &= L(2s,\chi)y^s \\ &\quad + \frac{\tau(\chi)}{q} y^s \frac{2\pi^s}{\Gamma(s)} \sum_{m=1}^{\infty} \sum_{n\neq 0} \overline{\chi(n)} \left(\frac{|n|}{q}\right)^{s-\frac{1}{2}} \left(ymNq^2\right)^{\frac{1}{2}-s} K_{s-\frac{1}{2}}(2\pi|n|ymNq)e^{2\pi inxmNq}. \end{aligned}$$

Summing $m \in Nq\mathbb{Z}_{\geq 1}$ and $n \in \mathbb{Z}_{\geq 1}$ is the same as summing their product k over $\mathbb{Z}_{\geq 1}$ and summing for each k over the pairs (n, mNq) such that $Nqmn = k$. Accordingly we can write

$$\begin{aligned} E^*(z,s,\chi) &= L(2s,\chi)y^s \\ &\quad + \tau(\chi)q^{-2s}\sqrt{y}\frac{2\pi^s}{\Gamma(s)} \sum_{m=1}^{\infty} \sum_{n\neq 0} \overline{\chi(n)} |n|^{s-\frac{1}{2}} (mNq)^{\frac{1}{2}-s} K_{s-\frac{1}{2}}(2\pi|n|ymNq)e^{2\pi inxmNq} \\ &= L(2s,\chi)y^s \\ &\quad + \tau(\chi)q^{-2s}\sqrt{y}\frac{2\pi^s}{\Gamma(s)} \sum_{k\neq 0} |k|^{\frac{1}{2}-s} \sigma_{2s-1}(|k|,\overline{\chi}) K_{s-\frac{1}{2}}(2\pi|k|y)e^{2\pi ikx}, \end{aligned}$$

where $\sigma_s(k,\chi) = \sum_{d|k} \chi(d)d^s$ is the twisted divisor function. □

Second Proof. Again we start with

$$E^*(z,s,\chi) = L(2s,\chi)y^s + \sum_{m=1}^{\infty} \sum_{n\in\mathbb{Z}} \chi(n) \frac{y^s}{|mNq^2z+n|^{2s}}.$$

Using the substitution $n = tmNq^2 + r$ and Lemma 10 gives

$$\begin{aligned}
E^*(z,s,\chi) &= L(2s,\chi)y^s \\
&\quad + \frac{y^s}{q^{4s}N^{2s}} \sum_{m=1}^{\infty} |m|^{-2s} \sum_{r=1}^{mNq^2} \chi(r) \sum_{t\in\mathbb{Z}} \left| t + z + \frac{r}{mNq^2} \right|^{-2s} \\
&= L(2s,\chi)y^s + \frac{y^s}{q^{4s}N^{2s}} \sum_{m=1}^{\infty} |m|^{-2s} \sum_{r=1}^{mNq^2} \chi(r) \\
&\quad \times \left[\frac{\sqrt{\pi}\Gamma(s-\frac{1}{2})}{\Gamma(s)}\, y^{1-2s} + \frac{2\pi^s}{\Gamma(s)} y^{\frac{1}{2}-s} \sum_{t\neq 0} |t|^{s-\frac{1}{2}}\, K_{s-\frac{1}{2}}(2\pi|t|y)\, e^{2\pi i t\left(x + \frac{r}{mNq^2}\right)} \right] \\
&= L(2s,\chi)y^s + \frac{2\pi^s\sqrt{y}}{q^{4s}N^{2s}\Gamma(s)} \sum_{t\neq 0} |t|^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|t|y) e^{2\pi i t x} \sum_{m=1}^{\infty} |m|^{-2s} \sum_{r=1}^{mNq^2} \chi(r) e^{\frac{2\pi i t r}{mNq^2}}.
\end{aligned}$$

In the last equality we used Lemma 6 according to which $\sum_{r=1}^{mNq^2} \chi(r) = \sum_{r=1}^{q} \chi(r) = 0$. By Lemma 8 the last sum equals

$$\sum_{r=1}^{mNq^2} \chi(r) e^{\frac{2\pi i t r}{mNq^2}} = \begin{cases} mNq\tau(\chi)\overline{\chi}(l) & \text{if } t = mqNl \text{ for some } l \in \mathbb{Z} \\ 0 & \text{else} \end{cases}.$$

Therefore we get

$$\begin{aligned}
E^*(z,s,\chi) &= L(2s,\chi)y^s + \frac{2\pi^s\tau(\chi)\sqrt{y}}{q^{4s}N^{2s}\Gamma(s)} \sum_{t\neq 0} |t|^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|t|y) e^{2\pi i t x} \sum_{\substack{m\geq 1\\ t = mqNl}} |m|^{-2s} mNq\overline{\chi}(l) \\
&= L(2s,\chi)y^s + \frac{2\pi^s\tau(\chi)\sqrt{y}}{q^{4s}N^{2s}\Gamma(s)} \sum_{t\neq 0} |t|^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|t|y) e^{2\pi i t x} \sum_{l|t} \left| \frac{t}{qNl} \right|^{-2s} \frac{t}{l}\overline{\chi}(l) \\
&= L(2s,\chi)y^s + \frac{2\pi^s\tau(\chi)\sqrt{y}}{q^{2s}\Gamma(s)} \sum_{t\neq 0} |t|^{\frac{1}{2}-s} K_{s-\frac{1}{2}}(2\pi|t|y) e^{2\pi i t x} \sum_{l|t} \overline{\chi}(l) l^{2s-1} \\
&= L(2s,\chi)y^s + \frac{2\pi^s\tau(\chi)\sqrt{y}}{q^{2s}\Gamma(s)} \sum_{t\neq 0} |t|^{\frac{1}{2}-s} \sigma_{2s-1}(|t|,\overline{\chi}) K_{s-\frac{1}{2}}(2\pi|t|y) e^{2\pi i t x}.
\end{aligned}$$

□

A consequence of the above Fourier expansion is the meromorphic continuation in s of the twisted Eisenstein series. To derive a functional equation for the twisted Eisenstein series we calculate the Fourier expansion of $E^*\left(\frac{-1}{q^2Nz}, s, \chi\right)$.

Lemma 13. The function $E^*\left(\frac{-1}{q^2Nz}, s, \chi\right)$ has the Fourier expansion

$$\begin{aligned}
E^*\left(\frac{-1}{q^2Nz}, s, \chi\right) &= \frac{1}{q^{2s}N^s} L(2s-1,\chi) \frac{\sqrt{\pi}\Gamma(s-\frac{1}{2})}{\Gamma(s)} y^{1-s} \\
&\quad + \frac{\sqrt{y}}{q^{2s}N^s} \frac{2\pi^s}{\Gamma(s)} \sum_{r\neq 0} |r|^{s-\frac{1}{2}} \sigma_{1-2s}(r,\chi) K_{s-\frac{1}{2}}(2\pi|r|y) e^{2\pi i r x}.
\end{aligned}$$

Proof. We will use the fact that the matrix $\omega = \begin{pmatrix} 0 & -1 \\ Nq^2 & 0 \end{pmatrix}$ normalizes the group $\Gamma_0(Nq^2)$. Indeed, we have

$$\omega \begin{pmatrix} a & b \\ c & d \end{pmatrix} \omega^{-1} = \begin{pmatrix} d & -\frac{c}{Nq^2} \\ -bNq^2 & a \end{pmatrix}.$$

We now compute

$$\begin{aligned}
E^*\left(\frac{-1}{q^2Nz}, s, \chi\right) &= \frac{L(2s,\chi)}{2} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(Nq^2)} \chi(d)\,\mathrm{Im}(\gamma\omega z)^s \\
&= \frac{L(2s,\chi)}{2} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(Nq^2)} \chi(\omega\gamma\omega^{-1})\,\mathrm{Im}(\omega\gamma z)^s \\
&= \frac{y^s}{q^{2s}N^s} \sum_{a=1}^{\infty}\sum_{b\in\mathbb{Z}} \frac{\chi(a)}{|az+b|^{2s}} \\
&= \frac{y^s}{q^{2s}N^s} \sum_{a=1}^{\infty} \frac{\chi(a)}{a^{2s}} \sum_{d=1}^{a}\sum_{m\in\mathbb{Z}} \left| m+z+\frac{d}{a}\right|^{-2s} \\
&= \frac{1}{q^{2s}N^s} L(2s-1,\chi)\frac{\sqrt{\pi}\Gamma(s-\frac{1}{2})}{\Gamma(s)} y^{1-s} \\
&+ \frac{\sqrt{y}}{q^{2s}N^s}\frac{2\pi^s}{\Gamma(s)} \sum_{r\neq 0} |r|^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|r|y)e^{2\pi irx} \sum_{a=1}^{\infty}\frac{\chi(a)}{a^{2s}} \sum_{d=1}^{a} e^{2\pi ir\frac{d}{a}}.
\end{aligned}$$

Because

$$\sum_{d=1}^{a} e^{2\pi ir\frac{d}{a}} = \begin{cases} a & \text{if } a|r \\ 0 & \text{else} \end{cases}$$

we have

$$\sum_{a=1}^{\infty}\frac{\chi(a)}{a^{2s}} \sum_{d=1}^{a} e^{2\pi ir\frac{d}{a}} = \sum_{a|r}\frac{\chi(a)}{a^{2s-1}} = \sigma_{1-2s}(r,\chi).$$

Hence we obtain the claimed Fourier expansion. □

The functional equation for the twisted Eisenstein series takes a simple form when we define

$$E^{\#}(z,s,\chi) := \pi^{-s}\Gamma(s)E^*(z,s,\chi) = \pi^{-s}\Gamma(s)L(2s,\chi)E(z,s,\chi).$$

Theorem 14. *The function $E^{\#}(z,s,\chi)$ satisfies the functional equation*

$$E^{\#}(z,s,\chi) = \frac{\tau(\chi)}{\sqrt{q}} N^{1-s} q^{\frac{5}{2}-4s} E^{\#}\left(\frac{-1}{q^2Nz}, 1-s, \overline{\chi}\right).$$

Proof. This follows from the above Fourier expansions, together with the functional equation $\xi(s,\chi) = \frac{\tau(\chi)}{\sqrt{q}}\xi(1-s,\overline{\chi})$ for the completed Dirichlet L-series $\xi(s,\chi) = \left(\frac{q}{\pi}\right)^{\frac{s}{2}}\Gamma(s/2)L(s,\chi)$, and the symmetry $K_\nu(z) = K_{-\nu}(z)$ of the K-Bessel function.

□

2.3 Hecke operators for $L^2(\Gamma_0(N)\setminus\mathbb{H})$

2.4 Rankin-Selberg convolution for $\Gamma_0(N)$

3 Other things

3.1 Rankin-Selberg method

Let f be a function on $\Gamma\setminus\mathbb{H}$, of sufficient rapid decay. Then the scalar product of f with an Eisenstein series is equal to an integral transform of the constant term in the Fourier expansion of f. More precisely, for $\mathrm{Re}(s)$ sufficiently large,

$$
\begin{aligned}
\langle f, E(\cdot, s) &= \int_{\Gamma\backslash\mathbb{H}} f(z) \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \mathrm{Im}(\gamma z)^s \mathrm{d}\mu(z) \\
&= \int_{\Gamma_\infty\backslash\mathbb{H}} f(z)\,\mathrm{Im}(z)^s \mathrm{d}\mu(z) \\
&= \int_0^\infty \left(\int_0^1 f(x+iy)\mathrm{d}x \right) y^{s-2}\mathrm{d}y.
\end{aligned}
$$

3.1.1 Example

Let $\{f(z)\}$ be a orthonormal basis of cuspidal Maass forms for the discrete spectrum. These have Fourier expansions of the type

$$f_j(z) = \sum_{m\neq 0} a_j(m) y^{1/2} K_{\nu_j}(2\pi|m|y)e(mx).$$

Recall that $f_j(z)$ is an eigenfunction for the Laplacian with eigenvalue $1/4 - \nu_j^2$. The constant term $A_j(0)$ of $|f_j(z)|^2 = f_j(z)\overline{f_j(z)}$ is given by

$$A_j(0) = y \sum_{m\neq 0} |a_j(m)|^2 K_{\nu_j}(2\pi|m|y)^2.$$

Integrating $|f_j(z)|^2$ against an Eisenstein series gives, for $\mathrm{Re}(s) > 1$,

$$
\begin{aligned}
\int_{\Gamma\backslash\mathbb{H}} |f_j(z)|^2 E(z,s)\mathrm{d}\mu(z) &= \int_0^\infty y^{s-1} \sum_{m\neq 0} |a_j(m)|^2 K_{\nu_j}(2\pi|m|y)^2 \mathrm{d}y \\
&= \sum_{m\neq 0} |a_j(m)|^2 \int_0^\infty y^{s-1} K_{\nu_j}(2\pi|m|y)^2 \mathrm{d}y.
\end{aligned}
$$

The Mellin transform of $K_{\nu_j}(2\pi|m|y)^2$ is given by

$$\int_0^\infty y^{s-1} K_{\nu_j}(2\pi|m|y)^2 \mathrm{d}y = \frac{1}{8}(\pi|n|)^{-s}\Gamma(s)^{-1}\Gamma(s/2)^2\Gamma(s/2+\nu_j)\Gamma(s/2-\nu_j).$$

For each j we define the zeta-function $R_{f_j}(s)$ by

$$R_{f_j}(s) = \frac{\Gamma(s/2)^2}{8\pi^s\Gamma(s)}\Gamma(s/2+\nu_j)\Gamma(s/2-\nu_j) \sum_{m\neq 0} \frac{|a_j(m)|^2}{|n|^s}.$$

Thus $\int_{\Gamma\backslash\mathbb{H}} |f_j(z)|^2 E(z,s)\mathrm{d}\mu(z) = R_{f_j}(s)$. The analytic properties of R_{f_j} (meromorphic continuation, functional equation, position of poles) are inherited from the corresponding properties of the Eisenstein series $E(z,s)$.