Contents

1 Dirichlet characters

Let m be a positive integer. A Dirichlet character of modulus m is a function $\chi : \mathbb{Z} \to \mathbb{C}$ such that

- $\chi(ab) = \chi(a)\chi(b)$ for all $a, b \in \mathbb{Z}$,
- $\chi(a) = 0$ if and only if $(a, m) > 1$,
- If $a \equiv b \pmod{m}$, then $\chi(a) = \chi(b)$.

Let χ_1 and χ_2 be Dirichlet characters of modulus m_1 and m_2, respectively, with $m_1|m_2$. If $\chi_2(a) = \chi_1(a)$ for $a \in (\mathbb{Z}/m_2\mathbb{Z})^\times$, then χ_2 is said to be induced by χ_1. A Dirichlet character is called primitive if it is not induced by any Dirichlet character other than itself. A Dirichlet character induced by the identity function is called principal. We denote a principal Dirichlet character by χ_0. By definition a principal Dirichlet character mod m is

$$\chi_0(a) = \begin{cases} 1 & \text{if } (a,m) = 1 \\ 0 & \text{else} \end{cases} . \tag{1}$$

If p is prime, then every nonprincipal Dirichlet character of modulus p is primitive. We call a Dirichlet character even if $\chi(-1) = 1$, odd if $\chi(-1) = -1$.

Lemma 1. *Let χ be a nonprincipal Dirichlet character modulo a prime q. Then we have*

$$\sum_{a=1}^{q} \chi(a) = 0.$$

Proof. Because χ is not principal, there is an integer b such that $\chi(b) \neq \{0, 1\}$. Furthermore, the sum may be restricted to the terms with $(a, q) = 1, 1 \leq a \leq q$. Multiplication by b is a bijection $(\mathbb{Z}/q\mathbb{Z})^\times \to (\mathbb{Z}/q\mathbb{Z})^\times$. Therefore we have

$$\chi(b) \sum_{a=1}^{q} \chi(a) = \sum_{a \in (\mathbb{Z}/q\mathbb{Z})^\times} \chi(ab) = \sum_{c \in (\mathbb{Z}/q\mathbb{Z})^\times} \chi(c),$$

which implies $\sum_{a=1}^{q} \chi(a) = 0$. □

The sum of the n-th roots of unity is zero. We will investigate more closely the twist of this sum by a Dirichlet character mod q, a so-called Gauss sum. More precisely, the Gauss sum $\tau(\chi)$ attached to a primitive Dirichlet character χ mod q is

$$\tau(\chi) = \sum_{a \bmod q} \chi(a) e^{2\pi i a/q}.$$

Lemma 2. *Let χ be a nonprincipal Dirichlet character modulo a prime q. Then for all $n \in \mathbb{Z}$ we have*

$$\tau(\chi)\overline{\chi(n)} = \sum_{a=1}^{q} \chi(a) e^{2\pi i n a/q}.$$

Proof. See the second half of the proof of Lemma 4. □

Lemma 3. *Let χ be a primitive Dirichlet character modulo a prime q. Then*

$$|\tau(\chi)| = \sqrt{q}.$$

Proof. By Lemma 2 we have

$$\begin{aligned} |\tau(\chi)|^2 &= \sum_{a=1}^{q} \overline{\chi(a)} e^{-2\pi i a/q} \tau(\chi) \\ &= \sum_{a=1}^{q} e^{-2\pi i a/q} \left(\sum_{b=1}^{q} \chi(b) e^{2\pi i a b/q} \right) \\ &= \sum_{b=1}^{q} \chi(b) \left(\sum_{a=1}^{q} e^{2\pi i a(b-1)/q} \right). \end{aligned}$$

If $b = 1$, the inner sum equals q. If $b \neq 1$, the inner sum is zero. Therefore we obtain

$$|\tau(\chi)|^2 = \chi(1) q = q.$$

□

We will also need the following variant of Lemma 2.

Lemma 4. *Let χ be a nonprincipal Dirichlet character modulo a prime q, and let $n, m \in \mathbb{Z}$. Then*

$$\sum_{a=1}^{nq^2} \chi(a) e^{2\pi i m a/nq^2} = \begin{cases} qn\tau(\chi)\overline{\chi}(l) & \text{if } m = lnq \text{ for some } l \in \mathbb{Z} \\ 0 & \text{else} \end{cases}. \tag{2}$$

Proof. Every a in the above sum is of the form $a = a_1 + tq$ with $1 \leq a_1 \leq q$ and $0 \leq t < nq$. Thus we can write

$$\sum_{a=1}^{nq^2} \chi(a) e^{2\pi i m a/nq^2} = \sum_{a_1=1}^{q} \chi(a_1) e^{\frac{2\pi i m a_1}{nq^2}} \sum_{t=0}^{nq-1} e^{\frac{2\pi i m t}{nq}}.$$

The sum over t is zero unless $m = lnq$ for some $l \in \mathbb{Z}$ in which case the sum is nq. Therefore

$$\begin{aligned} \sum_{a=1}^{nq^2} \chi(a) e^{2\pi i m a/nq^2} &= nq \sum_{\substack{a \bmod q \\ m = lnq}} \chi(a) e^{\frac{2\pi i m a}{nq^2}} \\ &= nq \sum_{a \bmod q} \chi(a) e^{\frac{2\pi i l a}{q}} \end{aligned}$$

If $(l,q)=1$, then $a \mapsto al$ permutes the residues mod q. In this case we get

$$nq \sum_{a \bmod q} \chi(a) e^{\frac{2\pi i l a}{q}} = nq\tau(\chi)\overline{\chi(l)}.$$

Now suppose that $(l,q)>1$. Then $\chi(l)=0$ and we have to show that the left side of Equation 2 vanishes. For this let $l' \in \mathbb{Z}$ be such that $ql' = l$. Then we have

$$\begin{aligned} nq \sum_{a \bmod q} \chi(a) e^{\frac{2\pi i l a}{q}} &= nq \sum_{a \bmod q} \chi(a) e^{2\pi i l' a} \\ &= nq \sum_{a \bmod q} \chi(a). \end{aligned}$$

The last sum is zero by Lemma 1. This completes the proof of the lemma.

$\square$

2 $SL(2,\mathbb{Z})$

2.1 Eisenstein series and the spectral decomposition of $L^2(\Gamma \setminus \mathbb{H})$

Let $\Gamma = \mathrm{SL}(2,\mathbb{Z})$. This group has only a cusp at infinity. The stabilizer of the cusp ∞ in Γ is

$$\Gamma_\infty = \left\{ \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \mid m \in \mathbb{Z} \right\}.$$

The Eisenstein series associated to this cusp, defined on $\mathbb{H} \times \mathbb{C}$, is defined by

$$E(z,s) := E_\infty(z,s) = \frac{1}{2} \sum_{\gamma \in \Gamma_\infty \setminus \Gamma} \mathrm{Im}(\gamma z)^s. \tag{3}$$

Notice that $\mathrm{Im}(z)$ is Γ_∞-invariant, and the Eisenstein series $E(z,s)$ defines an automorphic function with respect to Γ, that is, it satisfies $E(\gamma z, s) = E(z,s)$ for all $\gamma \in \Gamma$. For $\mathrm{Re}(s) > 1$, this series converges absolutely and uniformly on compact sets. Because $\Delta y^s = s(1-s)y^s$ and because Δ commutes with the Γ-action, the Eisenstein series is also an eigenfunction of the Laplacian

$$\Delta E(z,s) = s(1-s)E(z,s).$$

Automorphic functions which are eigenfunctions of the Laplace operator are called Maass forms. The Fourier expansion of $E(z,s)$ is given by

$$E(z,s) = y^s + \phi(s)y^{1-s} + \frac{2\pi^s}{\Gamma(s)\zeta(2s)} y^{1/2} \sum_{n \neq 0} \sigma_{1-2s}(n)|n|^{s-\frac{1}{2}} K_{s-1/2}(2\pi|n|y)e^{2\pi i n x}, \tag{4}$$

where

$$\phi(s) = \pi^{1/2} \frac{\Gamma(s-1/2)}{\Gamma(s)} \frac{\zeta(2s-1)}{\zeta(2s)}.$$

The modified Bessel functions K_ν are exponentially decaying functions. In particular, $E(z,s) = y^s + \phi(s)y^{1-s} + O(e^{-cy})$, for some constant $c > 0$. For a function to be in $L^2(\Gamma \setminus \mathbb{H})$ its growth need to be $O(y^{1/2})$. From the constant term of the Fourier expansion we see that the Eisenstein series is not in $L^2(\Gamma \setminus \mathbb{H})$. But for $\mathrm{Re}(s) = \frac{1}{2}$ the Eisenstein series is "almost square integrable", and this suggests to work on the line $\mathrm{Re}(s) = \frac{1}{2}$. We now have to address two issues: We want to work with square integrable functions on $\Gamma \setminus \mathbb{H}$, and we need to meromorphically continue $E(z,s)$ to the line $\mathrm{Re}(s) = \frac{1}{2}$. The meromorphic continuation of $E(z,s)$ follows from the Fourier expansion. In the half-plane $\mathrm{Re}(s) \geq \frac{1}{2}$ there is only a simple pole at $s = 1$ with residue $\frac{3}{\pi}$. The Eisenstein series enjoys the functional equation

$$E(z, 1-s) = \phi(1-s)E(z,s).$$

For a smooth, compactly supported function ψ on $\mathbb{R}^{>0}$, the incomplete Eisenstein series is

$$E(z|\psi) = \sum_{\gamma \in \Gamma_\infty \setminus \Gamma} \psi(\mathrm{Im}(\gamma z)).$$

The incomplete Eisenstein series $E(z|\psi)$ lies in $C_c^\infty(\Gamma \setminus \mathbb{H}) \subset L^2(\Gamma \setminus \mathbb{H})$, but it is not an eigenfunction of Δ. By Mellin inversion it can however be represented as an integral of Eisenstein series

$$E(z|\psi) = \frac{1}{2\pi i} \int_{\mathrm{Re}(s)=\sigma} E(z,s)\widehat{\psi}(s)\mathrm{d}s, \tag{5}$$

for $\sigma > 1$ and where

$$\widehat{\psi}(s) = \int_0^\infty \psi(y) y^{-s-1} \mathrm{d}y,$$

is the Mellin transform of ψ.

We denote by $\mathcal{E}(\Gamma \setminus \mathbb{H}) \subset L^2(\Gamma \setminus \mathbb{H})$ the space spanned by the incomplete Eisenstein series $E(z|\psi)$. The inner product of a function $f \in L^2(\Gamma \setminus \mathbb{H})$ with an incomplete Eisenstein series $E(z|\psi)$ is

$$\langle f, E(\cdot|\psi)\rangle = \int_0^\infty f_0(y)\overline{\psi}(y)y^{-2}\mathrm{d}y,$$

where f_0 is the constant term in the Fourier expansion of f. If f is orthogonal to $\mathcal{E}(\Gamma \setminus \mathbb{H})$, then the above integral is zero for all smooth functions ψ of compact support in $(0, \infty)$. Thus the orthogonal complement $\mathcal{C} = \mathcal{E}^\perp$ consists of functions whose constant term in the Fourier series is zero. The Laplace operator Δ has discrete spectrum in $\mathcal{C}$ and $\mathcal{C}$ is spanned by cusp forms. For an orthonormal basis of cusp forms $\{u_j\}$ every $f \in \mathcal{C}(\Gamma \setminus \mathbb{H})$ has the spectral expansion

$$f(z) = \sum_{j\geq 1}\langle f, u_j\rangle u_j(z).$$

The spectrum of Δ in $\mathcal{E}(\Gamma\setminus\mathbb{H})$ turns out to consist of a continuous part spanned by the Eisenstein series $E(z, \frac{1}{2} + ir)$, and the zero eigenvalue corresponding to the constant function u_0.

For $\mathrm{Re}(s) \geq \frac{1}{2}$, the Eisenstein series has only a simple pole at $s = 1$. The Eisenstein series are holomorphic on the line $\mathrm{Re}(s) = \frac{1}{2}$ and are of polynomial growth in vertical strips $-\epsilon \leq \mathrm{Re}(s) \leq 1 + \epsilon$. One can then shift the integration in Equation 5 to the line $\mathrm{Re}(s) = \frac{1}{2}$, thereby picking up the residue of $E(z, s)$ at the pole $s = 1$. As a result we obtain

$$E(z|\psi) = \widehat{\psi}(1)\mathrm{Res}_{s=1}(E(z, s)) + \frac{1}{2\pi i}\int_{\mathrm{Re}(s)=\frac{1}{2}} E(z, s)\widehat{\psi}(s)\mathrm{d}s.$$

The term $\widehat{\psi}(1)$ can be written as $\frac{\langle E(\cdot|\psi),u_0\rangle}{\langle u_0,u_0\rangle}$. We still need the projection of $E(z|\psi)$ onto $E(z, s)$. The functional equation along with

$$\langle E(\cdot|\psi), E(\cdot, 1/2 + ir)\rangle = \widehat{\psi}(1/2 + ir) + \phi(1/2 - ir)\widehat{\psi}(1/2 - ir)$$

yields the spectral decomposition of $E(z|\psi)$ onto the Eisenstein series

$$\frac{1}{2\pi i}\int_{\mathrm{Re}(s)=\frac{1}{2}} E(z, s)\widehat{\psi}(s)\mathrm{d}s = \frac{1}{4\pi i}\int_{\mathrm{Re}(s)=\frac{1}{2}}\langle E(\cdot|\psi), E(\cdot, s)\rangle E(z, s)\mathrm{d}s.$$

In conclusion, for an orthonormal basis of cusp forms $\{u_j\}$, every $f \in L^2(\Gamma \setminus \mathbb{H})$ has the spectral expansion

$$f(z) = \sum_j\langle f, u_j\rangle u_j(z) + \frac{1}{4\pi i}\int_{\mathrm{Re}(s)=\frac{1}{2}}\langle f(\cdot), E(\cdot, s)\rangle E(z, s)\mathrm{d}s.$$

2.2 Basic properties of Maass forms

- Fourier expansion

- Even / Odd.

2.3 Hecke operators for $L^2(\mathrm{SL}(2, \mathbb{Z}) \setminus \mathbb{H})$

2.4 L-functions

2.4.1 L-functions associated to Maass forms

Let

$$f(z) = \sum_{n\neq 0} a(n)\sqrt{y}K_{\nu_f}(2\pi|n|y)e^{2\pi inx} \tag{6}$$

be a cuspidal Maass form for $SL(2, \mathbb{Z})$.

Lemma 5. *The coefficients $a(n)$ in the Fourier expansion of $f(z)$ satisfy*

$$a(n) = O(\sqrt{|n|}).$$

Proof. Because f is a cusp form, it is bounded as $\mathrm{Im}(z) \to \infty$. Thus

$$\left|a(n)\sqrt{y}K_{\nu_f}(2\pi|n|y)\right| = \left|\int_0^1 f(x+iy)e^{-2\pi inx}\mathrm{d}x\right| \le \int_0^1 |f(x+iy)|\mathrm{d}x \le C,$$

for some constant C (that depends on f). If we choose $y = \frac{1}{|n|}$, the lemma is proved. $\square$

For $\mathrm{Re}(s) \ge \frac{3}{2}$ we define the L-function $L_f(s)$ associated to $f(z)$ by the absolutely convergent series

$$L_f(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}. \tag{7}$$

Lemma 6. *Let $f(z)$ be a cuspidal Maass form for $SL(2,\mathbb{Z})$ of parity ϵ. Then its L-function $L_f(s)$ can be meromorphically continued to all $\mathbb{C}$ and it satisfies the functional equation*

$$\Lambda_f(s) = \pi^{-s}\Gamma\left(\frac{s+\epsilon+\nu_f}{2}\right)\Gamma\left(\frac{s+\epsilon-\nu_f}{2}\right)L_f(s) = (-1)^{\epsilon}\Lambda_f(1-s).$$

Proof. ... $\square$

Lemma 7. *Euler product*

Proof. ... $\square$

2.4.2 L-functions associated to Eisenstein series

By formula (4), the non-constant term in the Fourier expansion of the Eisenstein series $E(z,v)$ is given by

$$\frac{2\pi^v}{\Gamma(v)\zeta(2v)}y^{1/2}\sum_{n\neq 0}\sigma_{1-2v}(n)|n|^{v-\frac{1}{2}}K_{v-1/2}(2\pi|n|y)e^{2\pi inx}.$$

We define the L-function $L_{E_v}(s)$ associated to the Eisenstein series $E(z,v)$ to be

$$L_{E_v}(s) = \sum_{n=1}^{\infty}\frac{\sigma_{1-2v}(n)\,n^{v-\frac{1}{2}}}{n^s}.$$

Lemma 8. *The L-function $L_{E_v}(s)$ is simply a product of shifted Riemann zeta functions*

$$L_{E_v}(s) = \zeta(s+v-1/2)\,\zeta(s-v+1/2).$$

Furthermore, we have the functional equation

$$G_{E_v}(s) = G_{E_v}(1-s),$$

where

$$G_{E_v}(s) = \pi^{-s}\Gamma\left(\frac{s+v-\frac{1}{2}}{2}\right)\Gamma\left(\frac{s-v+\frac{1}{2}}{2}\right)\zeta(s+v-1/2)\,\zeta(s-v+1/2).$$

Proof. We calculate

$$\begin{aligned}\sum_{n=1}^{\infty}\sigma_{1-2v}(n)\,n^{v-\frac{1}{2}-s} &= \sum_{n=1}^{\infty}n^{v-\frac{1}{2}-s}\sum_{d|n}d^{1-2v}\\ &= \sum_{n=1}^{\infty}\sum_{d=1}^{\infty}(dn)^{v-\frac{1}{2}-s}d^{1-2v}\\ &= \sum_{n=1}^{\infty}n^{v-\frac{1}{2}-s}\sum_{d=1}^{\infty}d^{\frac{1}{2}-v-s}\\ &= \zeta(s-v+1/2)\,\zeta(s+v-1/2).\end{aligned}$$

Now consider

$$G_{E_v}(s) = \pi^{-\frac{s+v-\frac{1}{2}}{2}} \Gamma\left(\frac{s+v-\frac{1}{2}}{2}\right) \zeta(s+v-1/2)$$
$$\times \pi^{-\frac{s-v+\frac{1}{2}}{2}} \Gamma\left(\frac{s-v+\frac{1}{2}}{2}\right) \zeta(s-v+1/2).$$

The functional equation $G_{E_v}(s) = G_{E_v}(1-s)$ follows directly from the functional equation

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{-\frac{1-s}{2}} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s)$$

of the Riemann zeta function.

□

As expected, the functional equation of $L_{E_v}(s)$ matches the functional equation of a Maass form of even parity (as in Lemma 6). In fact, the functional equation of a Maass form of type ν and of even parity is identical to the functional equation of $L_{E_\nu}(s)$ (proof, explanation). This gives a method of obtaining functional equations for Maass forms.

Let χ be an even primitive Dirichlet character mod a prime q. The twisted L-function associated to the Eisenstein series $E(z,v)$ is

$$L_{E_v}(s,\chi) = L(s+v-1/2,\chi)\, L(s-v+1/2,\chi),$$

with the Dirichlet L-function $L(s,\chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s}$. The function $L(s,\chi)$ satisfies the functional equation (reference)

$$\Lambda(s,\chi) = \left(\frac{q}{\pi}\right)^{\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) L(s,\chi) = \frac{\tau(\chi)}{\sqrt{q}}\, \Lambda(1-s,\overline{\chi}). \tag{8}$$

It follows that the L-function $L_{E_v}(s,\chi)$ satisfies the functional equation

$$\Gamma_{E_v}(s,\chi) = \left(\frac{q}{\pi}\right)^{s} \Gamma\left(\frac{s+v-1/2}{2}\right) \Gamma\left(\frac{s-v+1/2}{2}\right) L_{E_v}(s,\chi)$$
$$=$$

2.5 Rankin-Selberg convolution for $\mathrm{SL}(2,\mathbb{Z})$

Let

$$f(z) = \sum_{n \neq 0} a(n) \sqrt{y} K_{\nu_f}(2\pi|n|y) e^{2\pi i n x}, \tag{9}$$
$$g(z) = \sum_{n \neq 0} b(n) \sqrt{y} K_{\nu_g}(2\pi|n|y) e^{2\pi i n x}, \tag{10}$$

be cuspidal Maass forms for $SL(2,\mathbb{Z})$. Recall that $f(z)$ (resp. $g(z)$) is an eigenfunction for the Laplacian with eigenvalue $1/4 - \nu_f$ (resp. $1/4 - \nu_g$). For sufficiently large $\mathrm{Re}(s)$ we define the Rankin-Selberg convolution $L_{f\times g}(s)$ as the absolutely convergent series

$$L_{f\times g}(s) = \zeta(2s) \sum_{n=1}^{\infty} \frac{a(n)\overline{b(n)}}{n^s}.$$

We will prove that $L_{f\times g}$ can be expressed as an inner product of $f\overline{g}$ with an Eisenstein series. This construction gives the meromorphic continuation and functional equation for $L_{f\times g}$.

Theorem 9. *Let $f(z)$ and $g(z)$ be cuspidal Maass forms as in Equation 16 and 17. Then $L_{f\times g}$ can be meromorphically continued to all $s \in \mathbb{C}$ with at most a simple pole at $s = 1$. Furthermore, we have the functional equation*

$$L^*_{f\times g}(s) = \pi^{-2s} G_{\nu_f,\nu_g}(s) L_{f\times g}(s) = L^*_{f\times g}(1-s),$$

where $G_{\nu_f,\nu_g}(s) = \prod \Gamma\left(\frac{s\pm\nu_f\pm\nu_g}{2}\right)$.

Proof. Let $E(z,s)$ be the non-holomorphic Eisenstein series as defined in Equation 3. For sufficiently large $\mathrm{Re}(s)$, we have

$$\begin{aligned}
\zeta(2s)\langle f\overline{g}, E(\cdot,\overline{s})\rangle &= \zeta(2s)\int_{SL(2,\mathbb{Z})\backslash\mathbb{H}} f(z)\overline{g(z)\overline{E(z,\overline{s})}}\mathrm{d}\mu(z) \\
&= \zeta(2s)\sum_{\gamma\in\Gamma_\infty\backslash SL(2,\mathbb{Z}}\int_{SL(2,\mathbb{Z})\backslash\mathbb{H}} f(\gamma z)\overline{g(\gamma z)}\,\mathrm{Im}(\gamma z)^s\mathrm{d}\mu(\gamma z) \\
&= \zeta(2s)\int_{\Gamma_\infty\backslash\mathbb{H}} f(z)\overline{g(z)}\,\mathrm{Im}(z)^s\mathrm{d}\mu(z) \\
&= \zeta(2s)\int_0^\infty\int_0^1 f(z)\overline{g(z)}y^{s-2}\mathrm{d}x\mathrm{d}y \\
&= \zeta(2s)\sum_{n,m\neq 0} a(n)\overline{b(m)}\int_0^\infty K_{\nu_f}(2\pi|n|y)K_{\nu_g}(2\pi|m|y)y^{s-1}\int_0^1 e^{2\pi i(n-m)x}\mathrm{d}x\mathrm{d}y \\
&= \zeta(2s)\sum_{n\neq 0} a(n)\overline{b(n)}\int_0^\infty K_{\nu_f}(2\pi|n|y)K_{\nu_g}(2\pi|n|y)y^s\frac{\mathrm{d}y}{y} \\
&= \frac{\zeta(2s)}{(2\pi)^s}\sum_{n\neq 0}\frac{a(n)\overline{b(n)}}{n^s}\int_0^\infty K_{\nu_f}(y)K_{\nu_g}(y)y^s\frac{\mathrm{d}y}{y} \\
&= (2\pi)^{-s}\,L_{f\times g}(s)\int_0^\infty K_{\nu_f}(y)K_{\nu_g}(y)y^s\frac{\mathrm{d}y}{y}.
\end{aligned}$$

The Mellin transform of $K_\nu(y)K_{\nu'}(y)$ is given by (see page 145 in [1])

$$\int_0^\infty K_\nu(y)K_{\nu'}(y)y^s\frac{\mathrm{d}y}{y} = \frac{2^{s-3}}{\Gamma(s)}\prod\Gamma\left(\frac{s\pm\nu\pm\nu'}{2}\right), \tag{11}$$

which is valid for $\mathrm{Re}(s) > |\mathrm{Re}(\nu)| + |\mathrm{Re}(\nu')|$.

From the calculation it follows that the convolution function $L_{f\times g}$ inherits the analytical properties of the Eisenstein series $E(z,s)$. This means that $L_{f\times g}$ can be meromorphically continued on $\mathbb{C}$. Because the Eisenstein series has a simple pole at $s=1$ and the Gamma function no zeros, it follows that $L_{f\times g}$ has a simple pole at $s=1$ if and only if $\langle f,g\rangle \neq 0$. The functional equation follows from the functional equation of the Eisenstein series

$$E^*(z,s) = \pi^{-s}\Gamma(s)\zeta(2s)E(z,s) = E^*(z,1-s).$$

□

Lemma 10. *Let $\alpha_i,\beta_i\in\mathbb{C}$ for $i=1,2$. Then for $x\in\mathbb{C}$, $|x|$ sufficiently small, we have*

$$\prod_{i=1}^2\prod_{j=1}^2(1-\alpha_i\beta_j x)^{-1} = \sum_{k=0}^\infty S_k(\alpha_1,\alpha_2)S_k(\beta_1,\beta_2)x^k(1-\alpha_1\alpha_2\beta_1\beta_2x^2)^{-1},$$

where $S_k(x_1,x_2)$ is the Schur polynomial

$$S_k(x_1,x_2) = \frac{\det\begin{pmatrix} x_1^{k+1} & x_2^{k+1} \\ 1 & 1\end{pmatrix}}{\det\begin{pmatrix} x_1 & x_2 \\ 1 & 1\end{pmatrix}} = \frac{x_1^{k+1}-x_2^{k+1}}{x_1-x_2}.$$

Proof. This is proved on page 67 in [2] by evaluating the determinant of the matrix

$$\left(\frac{1}{1-\alpha_i\beta_j x}\right)_{1\le i,j\le 2}$$

in two different ways. □

Theorem 11. *Let $f(z)$ and $g(z)$ be cuspidal Maass forms as in Equation* 16 *and* 17*. Assume that $L_f(s)$ and $L_g(s)$ have Euler products*

$$L_f(s) = \prod_p \prod_{i=1}^{2} \left(1 - \frac{\alpha_{i,p}}{p^s}\right)^{-1}, \quad L_g(s) = \prod_p \prod_{j=1}^{2} \left(1 - \frac{\beta_{j,p}}{p^s}\right)^{-1}.$$

Then $L_{f\times g}(s)$ admits the Euler product

$$L_{f\times g}(s) = \prod_p \prod_{i=1}^{2} \prod_{j=1}^{2} \left(1 - \frac{\alpha_{i,p}\,\overline{\beta}_{j,p}}{p^s}\right)^{-1}.$$

Proof. By assumption we have

$$L_{f\times g}(s) = \prod_p \frac{\sum_{k=1}^{\infty} a(p^k)\overline{b(p^k)}p^{-ks}}{(1-p^{-2s})}.$$

In view of Lemma 10, after choosing $x = p^{-s}$, it suffices to show that

$$a(p^k) = S_k(\alpha_{1,p}, \alpha_{2,p}), \quad b(p^k) = S_k(\beta_{1,p}, \beta_{2,p}).$$

The above equalities are obtained inductively from the relations

$$\begin{aligned} 1 &= \alpha_{1,p}\,\alpha_{2,p}, \\ a(p) &= \alpha_{1,p} + \alpha_{2,p}, \\ a(p^{k+1}) &= a(p)a(p^k) - a(p^{k-1}). \end{aligned}$$

$\square$

3 $\Gamma_0(N)$

Let $\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) \,|\, c \equiv 0 \pmod{N} \right\}$.

3.1 Twisted Maass Forms

Let

$$f(z) = \sum_{n \neq 0} a(n) y^{1/2} K_\nu(2\pi|n|y) e^{2\pi i n x}$$

be a cuspidal Maass form for $\mathrm{SL}(2,\mathbb{Z})$. Let χ be a primitive Dirichlet character mod a prime q. The twist of f by χ is defined by

$$f_\chi(z) = \sum_{n \neq 0} a(n)\chi(n) y^{1/2} K_\nu(2\pi|n|y) e^{2\pi i n x}. \tag{12}$$

Lemma 12. *The twist of a cuspidal Maass form for $\Gamma_0(N)$ is an automorphic form for $\Gamma_0(q^2N)$ with character χ^2. That is, for all $\gamma \in \Gamma_0(q^2N)$, we have*

$$f_\chi(\gamma z) = \chi(\gamma)^2 f_\chi(z).$$

Proof. By Lemma 3, the Gauss sum $\tau(\overline{\chi})$ is not zero. Then Lemma 2 allows us to write

$$f_\chi(z) = \frac{1}{\tau(\overline{\chi})} \sum_{a=1}^{q} \overline{\chi(a)} f(z + \frac{a}{q}).$$

Assume that $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(q^2N)$. Then

$$\begin{pmatrix} 1 & a/q \\ 0 & 1 \end{pmatrix} \gamma \in \Gamma_0(N) \begin{pmatrix} 1 & d^2a/q \\ 0 & 1 \end{pmatrix}.$$

Thus for all $z \in \mathbb{H}$, the point $\gamma z + \frac{a}{q}$ lies in the $\Gamma_0(N)$-orbit of $z + \frac{d^2a}{q}$. It follows that

$$\begin{aligned} f_\chi(\gamma z) &= \frac{1}{\tau(\overline{\chi})} \sum_{a=1}^{q} \overline{\chi(a)} f(\gamma z + \frac{a}{q}) \\ &= \frac{1}{\tau(\overline{\chi})} \sum_{a=1}^{q} \overline{\chi(a)} f(z + \frac{d^2a}{q}) \\ &= \frac{1}{\tau(\overline{\chi})} \sum_{a=1}^{q} \overline{\chi(ad^2)} \chi(d)^2 f(z + \frac{d^2a}{q}) \\ &= \chi(d)^2 f_\chi(z). \end{aligned}$$

□

3.2 Twisted Eisenstein series

Let q be a prime. In this section we assume that χ is an even, non-principal (and thus primitive) Dirichlet character mod q, and that $(q,N) = 1$ (at the moment the last condition is not used.). For $\mathrm{Re}(s) > 1$ we define the twisted Eisenstein series by the absolutely convergent series

$$E(z,s,\chi) = \frac{1}{2} \sum_{\gamma} \chi(d)\, \mathrm{Im}(\gamma z)^s,$$

where the sum goes over a set of coset representatives $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\infty \setminus \Gamma_0(Nq^2)$. We will sometimes write $\chi(\gamma)$ to denote $\chi(d)$.

The Eisenstein series $E(z,s,\chi)$ satisfies the automorphic relation $E(\gamma z,s,\chi) = \overline{\chi(\gamma)}E(z,s,\chi)$ for all $\gamma \in \Gamma_0(Nq^2)$. In particular, the function $E(z,s,\chi)$ is invariant under $z \mapsto z+1$. Hence it has a Fourier expansion (our assumption that χ is even simplifies the calculation of the Fourier expansion). In order to determine its Fourier expansion we first establish a few identities.

The cosets $\Gamma_\infty \setminus \Gamma_0(Nq^2)$ are determined by the bottom row of a representative

$$\Gamma_\infty \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \left\{ \begin{pmatrix} s & t \\ c & d \end{pmatrix} \middle| sd - ct = 1, Nq^2 | c \right\}.$$

A pair of coprime integers (c,d), subject to $Nq^2|c$, uniquely determines such a coset. To sum over $(m,n) \in \mathbb{Z} \setminus \{0,0\}$ is the same as to sum over all positive integers M and all pairs (c,d) of coprime integers by taking $(m,n) = (Mc, Md)$. As a consequence, we can write

$$\begin{aligned} E^*(z,s,\chi) &:= L(2s,\chi)E(z,s,\chi) \\ &= \frac{L(2s,\chi)}{2} \sum_{\gamma \in \Gamma_\infty \setminus \Gamma_0(Nq^2)} \chi(d) \operatorname{Im}(\gamma z)^s \\ &= \frac{L(2s,\chi)}{2} \sum_{\substack{(c,d)=1 \\ Nq^2|c}} \chi(d) \frac{y^s}{|cz+d|^{2s}} \\ &= \frac{1}{2} \sum_{\substack{m,n \in \mathbb{Z} \\ (m,n) \neq 0}} \chi(n) \frac{y^s}{|mNq^2 z + n|^{2s}}. \end{aligned} \tag{13}$$

Each summand $\chi(n)\frac{y^s}{|mNq^2z+n|^{2s}}$ in Equation 13 is hit by a summand $\chi(\tilde{n})\frac{y^s}{|\tilde{m}q^2z+\tilde{n}|^{2s}}$ in Equation 13 when specialized to $N=1$ (choose $\tilde{m} = Nm$ and $\tilde{n} = n$). But not all terms in the latter sum appear in the former sum. However, the proof of Theorem 16 shows that these terms do not contribute to the sum (without the twisting by χ this would not be true). As a consequence, the Eisenstein series $E^*(z,s,\chi)$ does in fact not depend on N.

Lemma 13. *If* $\operatorname{Re}(s) > 1/2$ *and* $r \in \mathbb{R}$, *then*

$$\int_{-\infty}^{\infty} \frac{e^{-2\pi i r x}}{(x^2+y^2)^s} \mathrm{d}x = \frac{\pi^s}{y^s \Gamma(s)} \begin{cases} \pi^{-s+\frac{1}{2}} \Gamma(s-\frac{1}{2}) y^{1-s} & \textit{if } r = 0 \\ 2|r|^{s-\frac{1}{2}} \sqrt{y} K_{s-\frac{1}{2}}(2\pi |r| y) & \textit{if } r \neq 0 \end{cases}. \tag{14}$$

Proof. Recall the integral representation of the Gamma function

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} \mathrm{d}t.$$

Thus we have

$$\begin{aligned} \frac{y^s}{\pi^s}\Gamma(s) \int_{-\infty}^{\infty} \frac{e^{-2\pi i r x}}{(x^2+y^2)^s} \mathrm{d}x &= \int_0^\infty \int_{-\infty}^\infty e^{-t} \left(\frac{ty}{\pi(x^2+y^2)} \right)^s e^{-2\pi i r x} \mathrm{d}x \frac{\mathrm{d}t}{t} \\ &= \int_0^\infty \int_{-\infty}^\infty e^{-\pi \xi (x^2+y^2)/y} \xi^s e^{-2\pi i r x} \mathrm{d}x \frac{\mathrm{d}\xi}{\xi}. \end{aligned}$$

For $r = 0$ the above expression becomes

$$\begin{aligned} \int_0^\infty \int_{-\infty}^\infty e^{-\pi\xi(x^2+y^2)/y} \xi^s \mathrm{d}x \frac{\mathrm{d}\xi}{\xi} &= \int_0^\infty \sqrt{\frac{y}{\xi}} e^{-\pi \xi y} \xi^s \frac{\mathrm{d}\xi}{\xi} \\ &= \pi^{-s+\frac{1}{2}} y^{1-s} \Gamma(s - \frac{1}{2}). \end{aligned}$$

For $r \neq 0$ we obtain, using the change of variables $\xi \mapsto \frac{\xi}{|r|}$,

$$\begin{aligned} \int_0^\infty \int_{-\infty}^\infty e^{-\pi\xi(x^2+y^2)/y} \xi^s e^{-2\pi i r x} \mathrm{d}x \frac{\mathrm{d}\xi}{\xi} &= \int_0^\infty \sqrt{\frac{y}{\xi}} e^{-y\pi r^2/\xi} e^{-\pi \xi y} \xi^s \frac{\mathrm{d}\xi}{\xi} \\ &= \sqrt{y} |r|^{s-\frac{1}{2}} \int_0^\infty \xi^{s-\frac{1}{2}} e^{-y\pi|r|(1/\xi+\xi)} \frac{\mathrm{d}\xi}{\xi} \\ &= 2|r|^{s-\frac{1}{2}} \sqrt{y} K_{s-\frac{1}{2}}(2\pi|r|y). \end{aligned}$$

Here we used the following integral representation of the modified Bessel function of the second kind $K_\nu(x)$ (see [3] page 182)

$$K_\nu(x) = \frac{1}{2}\int_0^\infty e^{-\frac{x}{2}(t+t^{-1})}t^{-\nu-1}\mathrm{d}t.$$

□

Lemma 14. *For $z = x + iy \in \mathbb{H}$ and $\mathrm{Re}(s) > \frac{1}{2}$ we have*

$$\sum_{n\in\mathbb{Z}} \frac{1}{|z+n|^{2s}} = \frac{\sqrt{\pi}\Gamma(s-\frac{1}{2})}{\Gamma(s)} y^{1-2s} + \frac{2\pi^s}{\Gamma(s)} y^{\frac{1}{2}-s} \sum_{r\neq 0} |r|^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|r|y)\, e^{2\pi i r x}. \tag{15}$$

Proof. Recall the Poisson summation formula

$$\sum_{n\in\mathbb{Z}} \varphi(x+n) = \sum_{n\in\mathbb{Z}} \widehat{\varphi}(n) e^{2\pi i n x},$$

where φ is a continuous function that decays sufficiently rapidly at infinity (for example, $|f(x)| < |x|^{-c}$ with $c > 1$) and where $\widehat{\varphi}(n) = \int_{-\infty}^\infty \varphi(x)e^{-2\pi i n x}\mathrm{d}x$ is the Fourier transform. We apply this formula to $\varphi(x) = |x+iy|^{-2s}$, where $z = x+iy \in \mathbb{H}$ and $\mathrm{Re}(s) > \frac{1}{2}$. The Poisson summation formula gives

$$\sum_{n\in\mathbb{Z}} \frac{1}{|z+n|^{2s}} = \sum_{n\in\mathbb{Z}} \left(\int_{-\infty}^\infty \frac{e^{-2\pi i n x}}{(x^2+y^2)^s}\mathrm{d}x\right) e^{2\pi i n x}.$$

Using Lemma 13 we get the result. □

We would like to adapt the left hand side of Equation 15 to the setting where the sum is twisted by a primitive Dirichlet character χ mod q. For this we need the twisted variant of the Poisson summation formula.

Lemma 15. *Let φ be a function that satisfies the conditions of the Poisson summation formula. Let χ be a primitive Dirichlet character mod q. Then*

$$\sum_{n\in\mathbb{Z}} \chi(n)\varphi(x+n) = \frac{\tau(\chi)}{q} \sum_{n\in\mathbb{Z}} \overline{\chi(n)}\widehat{\varphi}(n/q) e^{2\pi i n x/q}.$$

Proof. From Lemma 2 we know that

$$\chi(n) = \frac{1}{\tau(\overline{\chi})} \sum_{m \bmod q} \overline{\chi(m)} e^{2\pi i n m/q} = \frac{\chi(-1)\tau(\chi)}{q} \sum_{m \bmod q} \overline{\chi(m)} e^{2\pi i n m/q}.$$

Let us now consider $\sum_{n\in\mathbb{Z}} \chi(n)\varphi(n) = \sum_{n\in\mathbb{Z}} \varphi_1(n)$, where

$$\varphi_1(x) = \frac{\chi(-1)\tau(\chi)}{q} \sum_{m \bmod q} \overline{\chi(m)} e^{2\pi i x m/q}\varphi(x).$$

The Fourier transform of $\varphi_1(x)$ is

$$\begin{aligned}\widehat{\varphi_1}(\xi) &= \frac{\chi(-1)\tau(\chi)}{q} \sum_{m \bmod q} \overline{\chi(m)} \int_{-\infty}^\infty \varphi(x) e^{-2\pi i x(\xi - \frac{m}{q})}\mathrm{d}x \\ &= \frac{\chi(-1)\tau(\chi)}{q} \sum_{m \bmod q} \overline{\chi(m)}\widehat{\varphi}\left(\xi - \frac{m}{q}\right).\end{aligned}$$

We apply Poisson summation formula and thus $\sum_{n\in\mathbb{Z}} \varphi_1(n)$ equals

$$\frac{\chi(-1)\tau(\chi)}{q} \sum_{m \bmod q} \sum_{n=-\infty}^{\infty} \overline{\chi(m)}\widehat{\varphi}\left(\frac{nq-m}{q}\right).$$

We can write $\overline{\chi(m)} = \chi(-1)\overline{\chi(nq-m)}$. When n runs through $\mathbb{Z}$ and m through $\mathbb{Z}/q\mathbb{Z}$, the terms $nq-m$ run uniquely through $\mathbb{Z}$. Thus we have shown

$$\sum_{n\in\mathbb{Z}} \chi(n)\varphi(n) = \frac{\tau(\chi)}{q}\sum_{n\in\mathbb{Z}} \overline{\chi(n)}\widehat{\varphi}(n/q).$$

Replacing $\varphi(n)$ by $\varphi(n+x)$ replaces $\widehat{\varphi}(\xi)$ by $\widehat{\varphi}(\xi)e^{2\pi i\xi x}$. This completes the proof. □

We can now determine the Fourier expansion of $E^*(z,s,\chi)$.

Theorem 16. *The function $E^*(z,s,\chi)$ has the Fourier expansion*

$$\begin{aligned} E^*(z,s,\chi) &= L(2s,\chi)y^s \\ &\quad + \tau(\chi)q^{-2s}\sqrt{y}\frac{2\pi^s}{\Gamma(s)}\sum_{k\neq 0}|k|^{\frac{1}{2}-s}\sigma_{2s-1}(|k|,\overline{\chi})K_{s-\frac{1}{2}}(2\pi|k|y)e^{2\pi ikx}, \end{aligned}$$

where $\sigma_s(k,\overline{\chi}) = \sum_{d|k}\overline{\chi(d)}d^s$.

First Proof. We split up the sum (13) into the terms with $m=0$ and those with $m\neq 0$. The assumption that χ is even allows us to combine each positive summand with its negative. We obtain

$$E^*(z,s,\chi) = L(2s,\chi)y^s + \sum_{m=1}^{\infty}\sum_{n\in\mathbb{Z}}\chi(n)\frac{y^s}{|mNq^2z+n|^{2s}}.$$

Substituting into this the twisted variant of the formula of Lemma 14 gives (note that $\chi(0)=0$)

$$\begin{aligned} E^*(z,s,\chi) &= L(2s,\chi)y^s \\ &\quad + \frac{\tau(\chi)}{q}y^s\frac{2\pi^s}{\Gamma(s)}\sum_{m=1}^{\infty}\sum_{n\neq 0}\overline{\chi(n)}\left(\frac{|n|}{q}\right)^{s-\frac{1}{2}}\left(ymNq^2\right)^{\frac{1}{2}-s}K_{s-\frac{1}{2}}(2\pi|n|ymNq)e^{2\pi inxmNq}. \end{aligned}$$

Summing $m\in Nq\mathbb{Z}_{\geq 1}$ and $n\in\mathbb{Z}_{\geq 1}$ is the same as summing their product k over $\mathbb{Z}_{\geq 1}$ and summing for each k over the pairs (n, mNq) such that $Nqmn = k$. Accordingly we can write

$$\begin{aligned} E^*(z,s,\chi) &= L(2s,\chi)y^s \\ &\quad + \tau(\chi)q^{-2s}\sqrt{y}\frac{2\pi^s}{\Gamma(s)}\sum_{m=1}^{\infty}\sum_{n\neq 0}\overline{\chi(n)}|n|^{s-\frac{1}{2}}(mNq)^{\frac{1}{2}-s}K_{s-\frac{1}{2}}(2\pi|n|ymNq)e^{2\pi inxmNq} \\ &= L(2s,\chi)y^s \\ &\quad + \tau(\chi)q^{-2s}\sqrt{y}\frac{2\pi^s}{\Gamma(s)}\sum_{k\neq 0}|k|^{\frac{1}{2}-s}\sigma_{2s-1}(|k|,\overline{\chi})K_{s-\frac{1}{2}}(2\pi|k|y)e^{2\pi ikx}, \end{aligned}$$

where $\sigma_s(k,\chi) = \sum_{d|k}\chi(d)d^s$ is the twisted divisor function. □

Second Proof. Again we start with

$$E^*(z,s,\chi) = L(2s,\chi)y^s + \sum_{m=1}^{\infty}\sum_{n\in\mathbb{Z}}\chi(n)\frac{y^s}{|mNq^2z+n|^{2s}}.$$

Using the substitution $n = tmNq^2 + r$ and Lemma 14 gives

$$
\begin{aligned}
E^*(z,s,\chi) &= L(2s,\chi)y^s \\
&\quad + \frac{y^s}{q^{4s}N^{2s}} \sum_{m=1}^{\infty} |m|^{-2s} \sum_{r=1}^{mNq^2} \chi(r) \sum_{t\in\mathbb{Z}} \left| t + z + \frac{r}{mNq^2} \right|^{-2s} \\
&= L(2s,\chi)y^s + \frac{y^s}{q^{4s}N^{2s}} \sum_{m=1}^{\infty} |m|^{-2s} \sum_{r=1}^{mNq^2} \chi(r) \\
&\quad \times \left[\frac{\sqrt{\pi}\Gamma(s-\frac{1}{2})}{\Gamma(s)} y^{1-2s} + \frac{2\pi^s}{\Gamma(s)} y^{\frac{1}{2}-s} \sum_{t\neq 0} |t|^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|t|y)\, e^{2\pi i t\left(x+\frac{r}{mNq^2}\right)} \right] \\
&= L(2s,\chi)y^s + \frac{2\pi^s\sqrt{y}}{q^{4s}N^{2s}\Gamma(s)} \sum_{t\neq 0} |t|^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|t|y) e^{2\pi i t x} \sum_{m=1}^{\infty} |m|^{-2s} \sum_{r=1}^{mNq^2} \chi(r) e^{\frac{2\pi i t r}{mNq^2}}.
\end{aligned}
$$

In the last equality we used Lemma 1 according to which $\sum_{r=1}^{mNq^2} \chi(r) = \sum_{r=1}^{q} \chi(r) = 0$. By Lemma 4 the last sum equals

$$
\sum_{r=1}^{mNq^2} \chi(r) e^{\frac{2\pi i t r}{mNq^2}} = \begin{cases} mNq\tau(\chi)\overline{\chi}(l) & \text{if } t = mqNl \text{ for some } l \in \mathbb{Z} \\ 0 & \text{else} \end{cases}.
$$

Therefore we get

$$
\begin{aligned}
E^*(z,s,\chi) &= L(2s,\chi)y^s + \frac{2\pi^s\tau(\chi)\sqrt{y}}{q^{4s}N^{2s}\Gamma(s)} \sum_{t\neq 0} |t|^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|t|y) e^{2\pi i t x} \sum_{\substack{m\geq 1 \\ t=mqNl}} |m|^{-2s} mNq\overline{\chi}(l) \\
&= L(2s,\chi)y^s + \frac{2\pi^s\tau(\chi)\sqrt{y}}{q^{4s}N^{2s}\Gamma(s)} \sum_{t\neq 0} |t|^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|t|y) e^{2\pi i t x} \sum_{l|t} \left| \frac{t}{qNl} \right|^{-2s} \frac{t}{l} \overline{\chi}(l) \\
&= L(2s,\chi)y^s + \frac{2\pi^s\tau(\chi)\sqrt{y}}{q^{2s}\Gamma(s)} \sum_{t\neq 0} |t|^{\frac{1}{2}-s} K_{s-\frac{1}{2}}(2\pi|t|y) e^{2\pi i t x} \sum_{l|t} \overline{\chi}(l) l^{2s-1} \\
&= L(2s,\chi)y^s + \frac{2\pi^s\tau(\chi)\sqrt{y}}{q^{2s}\Gamma(s)} \sum_{t\neq 0} |t|^{\frac{1}{2}-s} \sigma_{2s-1}(|t|,\overline{\chi}) K_{s-\frac{1}{2}}(2\pi|t|y) e^{2\pi i t x}.
\end{aligned}
$$

□

A consequence of the above Fourier expansion is the meromorphic continuation in s of the twisted Eisenstein series $E(z,s,\chi)$. We would also like a functional equation for the twisted Eisenstein series. Simply replacing s by $1-s$ in $E(z,s,\chi)$ will not provide us with a functional equation since y^s and y^{1-s} in the constant term will not match. Therefore we also need to appropriately modify the argument z in $E(z,s,\chi)$. For this we consider the Eisenstein series $E(\omega z,s,\chi)$, where $\omega = \begin{pmatrix} 0 & -1 \\ Nq^2 & 0 \end{pmatrix} \in \mathrm{GL}(2,\mathbb{Z})$. Again the automorphic relation $E(\gamma\omega z,s,\chi) = \overline{\chi(\gamma)} E(\omega z,s,\chi)$ holds and consequently $E(\omega z,s,\chi)$ admits a Fourier expansion.

Lemma 17. *The function $E^*(\omega z,s,\chi)$ has the Fourier expansion*

$$
\begin{aligned}
E^*(\omega z,s,\chi) &= \frac{1}{q^{2s}N^s} L(2s-1,\chi) \frac{\sqrt{\pi}\Gamma(s-\frac{1}{2})}{\Gamma(s)} y^{1-s} \\
&\quad + \frac{\sqrt{y}}{q^{2s}N^s} \frac{2\pi^s}{\Gamma(s)} \sum_{r\neq 0} |r|^{s-\frac{1}{2}} \sigma_{1-2s}(r,\chi) K_{s-\frac{1}{2}}(2\pi|r|y) e^{2\pi i r x}.
\end{aligned}
$$

Proof. We will use the fact that the matrix $\omega = \begin{pmatrix} 0 & -1 \\ Nq^2 & 0 \end{pmatrix}$ normalizes the group $\Gamma_0(Nq^2)$. Indeed, we have

$$
\omega \begin{pmatrix} a & b \\ c & d \end{pmatrix} \omega^{-1} = \begin{pmatrix} d & -\frac{c}{Nq^2} \\ -bNq^2 & a \end{pmatrix}.
$$

We now compute

$$
\begin{aligned}
E^*(\omega z, s, \chi) &= \frac{L(2s,\chi)}{2} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(Nq^2)} \chi(\gamma)\operatorname{Im}(\gamma\omega z)^s \\
&= \frac{L(2s,\chi)}{2} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(Nq^2)} \chi(\omega\gamma\omega^{-1})\operatorname{Im}(\omega\gamma z)^s \\
&= \frac{y^s}{q^{2s}N^s}\sum_{a=1}^{\infty}\sum_{b\in\mathbb{Z}} \frac{\chi(a)}{|az+b|^{2s}} \\
&= \frac{y^s}{q^{2s}N^s}\sum_{a=1}^{\infty}\frac{\chi(a)}{a^{2s}}\sum_{d=1}^{a}\sum_{m\in\mathbb{Z}}\left|m+z+\frac{d}{a}\right|^{-2s} \\
&= \frac{1}{q^{2s}N^s} L(2s-1,\chi)\frac{\sqrt{\pi}\Gamma(s-\frac{1}{2})}{\Gamma(s)} y^{1-s} \\
&+ \frac{\sqrt{y}}{q^{2s}N^s}\frac{2\pi^s}{\Gamma(s)}\sum_{r\neq 0}|r|^{s-\frac{1}{2}}K_{s-\frac{1}{2}}(2\pi|r|y)e^{2\pi i r x}\sum_{a=1}^{\infty}\frac{\chi(a)}{a^{2s}}\sum_{d=1}^{a} e^{2\pi i r\frac{d}{a}}.
\end{aligned}
$$

Because

$$
\sum_{d=1}^{a} e^{2\pi i r\frac{d}{a}} = \begin{cases} a & \text{if } a|r \\ 0 & \text{else} \end{cases}
$$

we have

$$
\sum_{a=1}^{\infty}\frac{\chi(a)}{a^{2s}}\sum_{d=1}^{a} e^{2\pi i r\frac{d}{a}} = \sum_{a|r}\frac{\chi(a)}{a^{2s-1}} = \sigma_{1-2s}(r,\chi).
$$

Hence we obtain the claimed Fourier expansion. □

The functional equation for the twisted Eisenstein series takes a simple form when we define

$$
E^{\#}(z,s,\chi) := \pi^{-s}\Gamma(s)E^*(z,s,\chi) = \pi^{-s}\Gamma(s)L(2s,\chi)E(z,s,\chi).
$$

Theorem 18. *The function $E^{\#}(z,s,\chi)$ satisfies the functional equation*

$$
\begin{aligned}
E^{\#}(z,s,\chi) &= \frac{\tau(\chi)}{\sqrt{q}} N^{1-s} q^{\frac{5}{2}-4s} E^{\#}(\omega z, 1-s, \overline{\chi}) \\
&= \frac{\tau(\chi)}{\sqrt{q}} N^{1-s} q^{\frac{5}{2}-4s} E^{\#}\left(\frac{-1}{q^2Nz}, 1-s, \overline{\chi}\right).
\end{aligned}
$$

Proof. This follows from the above Fourier expansions, the functional equation (8) for the Dirichlet L-series, and the symmetry $K_\nu(z) = K_{-\nu}(z)$ of the K-Bessel function.

□

3.3 Hecke operators for $L^2(\Gamma_0(N)\setminus\mathbb{H})$

3.4 Rankin-Selberg convolution for $\Gamma_0(N)$

Let

$$
f(z) = \sum_{n\neq 0} a(n)\sqrt{y}K_{\nu_f}(2\pi|n|y)e^{2\pi i n x}, \tag{16}
$$

$$
g(z) = \sum_{n\neq 0} b(n)\sqrt{y}K_{\nu_g}(2\pi|n|y)e^{2\pi i n x}, \tag{17}
$$

be cuspidal Maass forms for $\Gamma_0(N)$ of the same parity ϵ. Again we fix an even, primitive Dirichlet character χ mod a prime q. We recall that the twist $f_\chi(z)$, defined by Equation (12), is automorphic for $\Gamma_0(q^2N)$ with character χ^2. We define the Rankin-Selberg L-function $L_{f_\chi\times g}(s)$ by

$$L_{f_\chi\times g}(s)=\sum_{n=1}^{\infty}\chi(n)\frac{a(n)b(n)}{n^s}.$$

Because the Fourier coefficients $a(n),b(n)$ satisfy the bound $O(\sqrt{|n|})$ (Lemma 5), the above series converges absolutely for $\mathrm{Re}(s)>2$. By the same methods used to prove Theorem 9, we will shows that the L-function $L_{f_\chi\times g}(s)$ admits a meromorphic continuation to $s\in\mathbb{C}$ and a functional equation. Key is the following lemma, which constructs the L-function $L_{f_\chi\times g}(s)$ as an inner product of $f_\chi(z)E(z,s,\chi^2)$ with $\overline{g(z)}$.

Lemma 19. *For* $\mathrm{Re}(s)$ *sufficiently large we have*

$$\int_{\Gamma_0(Nq^2)\backslash\mathbb{H}}E(z,s,\chi^2)f_\chi(z)g(z)\mathrm{d}\mu(z)=\frac{(-1)^\epsilon}{2^3}\frac{1}{\Gamma(s)\pi^s}\left(\prod\Gamma\left(\frac{s\pm\nu\pm\nu'}{2}\right)\right)L_{f_\chi\times g}(s).$$

Proof. By folding/unfolding it follows that

$$\begin{aligned}\int_{\Gamma_0(Nq^2)\backslash\mathbb{H}}E(z,s,\chi^2)f_\chi(z)g(z)\mathrm{d}\mu(z)&=\frac{1}{2}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(Nq^2)}\int_{\Gamma_0(Nq^2)\backslash\mathbb{H}}\chi(\gamma)^2\,\mathrm{Im}(\gamma z)^s f_\chi(z)g(z)\mathrm{d}\mu(z)\\&=\frac{1}{2}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(Nq^2)}\int_{\Gamma_0(Nq^2)\backslash\mathbb{H}}\mathrm{Im}(\gamma z)^s f_\chi(\gamma z)g(\gamma z)\mathrm{d}\mu(\gamma z)\\&=\frac{1}{2}\int_{\Gamma_\infty\backslash\mathbb{H}}\mathrm{Im}(z)^s f_\chi(z)g(z)\mathrm{d}\mu(z)\\&=\frac{1}{2}\int_0^\infty\int_0^1 y^{s-2}f_\chi(x+iy)g(x+iy)\mathrm{d}x\mathrm{d}y.\end{aligned}$$

For the second equality we used Lemma 12. Upon inserting the Fourier expansions, the integral over the x-coordinates becomes

$$\begin{aligned}\int_0^1 f_\chi(x+iy)g(x+iy)\mathrm{d}x&=y\sum_{n\neq0}\chi(n)a(n)b(-n)K_\nu(2\pi|n|y)K_{\nu'}(2\pi|n|y)\\&=(-1)^\epsilon y\sum_{n\neq0}\chi(n)a(n)b(n)K_\nu(2\pi|n|y)K_{\nu'}(2\pi|n|y).\end{aligned}$$

Then using formula (11) for the Mellin transform of $K_\nu K_{\nu'}$ gives

$$\begin{aligned}\int_{\Gamma_0(Nq^2)\backslash\mathbb{H}}E(z,s,\chi^2)f_\chi(z)g(z)\mathrm{d}\mu(z)&=\frac{(-1)^\epsilon}{2^4}\frac{1}{\Gamma(s)\pi^s}\left(\prod\Gamma\left(\frac{s\pm\nu\pm\nu'}{2}\right)\right)\sum_{n\neq0}\chi(n)\frac{a(n)b(n)}{|n|^s}\\&=\frac{(-1)^\epsilon}{2^3}\frac{1}{\Gamma(s)\pi^s}\left(\prod\Gamma\left(\frac{s\pm\nu\pm\nu'}{2}\right)\right)\sum_{n=1}^{\infty}\chi(n)\frac{a(n)b(n)}{n^s},\end{aligned}$$

where in the last line we used that f and g are of the same parity. ∎

4 References

1. Bateman, H. & Erdélyi, A. *Tables of Integral Transforms* (McGraw Hill, New York, 1954).
2. Macdonald, I. *Symmetric Functions and Hall Polynomials* (Oxford University Press, 1979).
3. Watson, G. N. *A Treatise on the Theory of Bessel Functions* (Cambridge University Press, 1966).