L-functions and the Selberg eigenvalue conjecture

Semester Paper

Matthias Gröbner

Supervised by
Claire Burrin

Department of Mathematics
ETH Zürich
Autumn 2021

Contents

1 Introduction

2 Dirichlet characters and Gauss sums

Let m be a positive integer. A Dirichlet character of modulus m is a function $\chi : \mathbb{Z} \to \mathbb{C}$ such that

- $\chi(ab) = \chi(a)\chi(b)$ for all $a, b \in \mathbb{Z}$,
- $\chi(a) = 0$ if and only if $(a, m) > 1$,
- If $a \equiv b \pmod{m}$, then $\chi(a) = \chi(b)$.

Let χ_1 and χ_2 be Dirichlet characters of modulus m_1 and m_2, respectively, with $m_1|m_2$. If $\chi_2(a) = \chi_1(a)$ for $a \in (\mathbb{Z}/m_2\mathbb{Z})^\times$, then χ_2 is said to be induced by χ_1. A Dirichlet character is called primitive if it is not induced by any Dirichlet character other than itself. A Dirichlet character induced by the identity function is called principal. We denote a principal Dirichlet character by χ_0. By definition a principal Dirichlet character mod m is

$$\chi_0(a) = \begin{cases} 1 & \text{if } (a, m) = 1 \\ 0 & \text{else} \end{cases} . \tag{1}$$

If p is prime, then every nonprincipal Dirichlet character of modulus p is primitive. We call a Dirichlet character even if $\chi(-1) = 1$, odd if $\chi(-1) = -1$.

Lemma 1. *Let χ be a nonprincipal Dirichlet character modulo a prime q. Then we have*

$$\sum_{a=1}^{q} \chi(a) = 0.$$

Proof. Because χ is not principal, there is an integer b such that $\chi(b) \neq \{0, 1\}$. Furthermore, the sum may be restricted to the terms with $(a, q) = 1, 1 \leq a \leq q$. Multiplication by b is a bijection $(\mathbb{Z}/q\mathbb{Z})^\times \to (\mathbb{Z}/q\mathbb{Z})^\times$. Therefore we have

$$\chi(b) \sum_{a=1}^{q} \chi(a) = \sum_{a \in (\mathbb{Z}/q\mathbb{Z})^\times} \chi(ab) = \sum_{c \in (\mathbb{Z}/q\mathbb{Z})^\times} \chi(c),$$

which implies $\sum_{a=1}^{q} \chi(a) = 0$. □

The sum of the n-th roots of unity is zero. We will investigate more closely the twist of this sum by a Dirichlet character mod q, a so-called Gauss sum. More precisely, the Gauss sum $\tau(\chi)$ attached to a primitive Dirichlet character χ mod q is

$$\tau(\chi) = \sum_{a \bmod q} \chi(a) e^{2\pi i a/q}.$$

Lemma 2. *Let χ be a nonprincipal Dirichlet character modulo a prime q. Then for all $n \in \mathbb{Z}$ we have*

$$\tau(\chi)\overline{\chi(n)} = \sum_{a=1}^{q} \chi(a) e^{2\pi i n a/q}.$$

In particular,

$$\tau(\overline{\chi}) = \chi(-1)\overline{\tau(\chi)}.$$

Proof. See the second half of the proof of Lemma 4. □

Lemma 3. *Let χ be a primitive Dirichlet character modulo a prime q. Then*

$$|\tau(\chi)| = \sqrt{q}.$$

Proof. By Lemma 2 we have

$$
\begin{aligned}
|\tau(\chi)|^2 &= \sum_{a=1}^{q} \overline{\chi(a)} e^{-2\pi i a/q} \tau(\chi) \\
&= \sum_{a=1}^{q} e^{-2\pi i a/q} \left(\sum_{b=1}^{q} \chi(b) e^{2\pi i ab/q} \right) \\
&= \sum_{b=1}^{q} \chi(b) \left(\sum_{a=1}^{q} e^{2\pi i a(b-1)/q} \right).
\end{aligned}
$$

If $b = 1$, the inner sum equals q. If $b \neq 1$, the inner sum is zero. Therefore we obtain

$$|\tau(\chi)|^2 = \chi(1) q = q.$$

□

We will also need the following variant of Lemma 2.

Lemma 4. *Let χ be a nonprincipal Dirichlet character modulo a prime q, and let $n, m \in \mathbb{Z}$. Then*

$$\sum_{a=1}^{nq^2} \chi(a) e^{2\pi i m a/nq^2} = \begin{cases} qn\tau(\chi)\overline{\chi}(l) & \text{if } m = lnq \text{ for some } l \in \mathbb{Z} \\ 0 & \text{else} \end{cases}. \tag{2}$$

Proof. Every a in the above sum is of the form $a = a_1 + tq$ with $1 \leq a_1 \leq q$ and $0 \leq t < nq$. Thus we can write

$$\sum_{a=1}^{nq^2} \chi(a) e^{2\pi i m a/nq^2} = \sum_{a_1=1}^{q} \chi(a_1) e^{\frac{2\pi i m a_1}{nq^2}} \sum_{t=0}^{nq-1} e^{\frac{2\pi i m t}{nq}}.$$

The sum over t is zero unless $m = lnq$ for some $l \in \mathbb{Z}$ in which case the sum is nq. Therefore

$$
\begin{aligned}
\sum_{a=1}^{nq^2} \chi(a) e^{2\pi i m a/nq^2} &= nq \sum_{\substack{a \bmod q \\ m = lnq}} \chi(a) e^{\frac{2\pi i m a}{nq^2}} \\
&= nq \sum_{a \bmod q} \chi(a) e^{\frac{2\pi i l a}{q}}
\end{aligned}
$$

If $(l, q) = 1$, then $a \mapsto al$ permutes the residues mod q. In this case we get

$$nq \sum_{a \bmod q} \chi(a) e^{\frac{2\pi i l a}{q}} = nq\tau(\chi)\overline{\chi(l)}.$$

Now suppose that $(l, q) > 1$. Then $\chi(l) = 0$ and we have to show that the left side of Equation 2 vanishes. For this let $l' \in \mathbb{Z}$ be such that $ql' = l$. Then we have

$$
\begin{aligned}
nq \sum_{a \bmod q} \chi(a) e^{\frac{2\pi i l a}{q}} &= nq \sum_{a \bmod q} \chi(a) e^{2\pi i l' a} \\
&= nq \sum_{a \bmod q} \chi(a).
\end{aligned}
$$

The last sum is zero by Lemma 1. This completes the proof of the lemma.

□

We have the following orthogonality relation.

Lemma 5. *Let $a \in \mathbb{Z}/q\mathbb{Z}$ for q prime. The sum of $\chi(a)$ over all non-trivial even primitive Dirichlet characters mod q is*

$$\sum_{\substack{\chi \ (mod\ q) \\ \chi \neq \chi_0,\, \chi(-1)=1}} \chi(a) = \begin{cases} 0 & \text{if } a \equiv 0 \ (mod\ q) \\ \frac{q-1}{2} - 1 & \text{if } a \equiv \pm 1 \ (mod\ q) \\ -1 & \text{else} \end{cases}.$$

Proof. □

We will also make use of the following identity.

Lemma 6. *Let $n \geq 2$, q be prime and m such that $q \nmid m$. Then we have*

$$\sum_{\substack{\chi \ (mod\ q) \\ \chi \neq \chi_0,\, \chi(-1)=1}} \overline{\chi(m)} \tau(\chi)^n = \frac{q-1}{2} \left(K_n(m,q) + K_n(-m,q) \right) - (-1)^n,$$

where we sum over primitive Dirichlet characters and where $K_n(m,q)$ denotes the hyper-Kloosterman sum

$$K_n(m,q) = \sum_{x_1 x_2 \cdots x_n \equiv m \ (mod\ q)} e^{2\pi i \left(\frac{x_1 + \cdots + x_n}{q} \right)}.$$

Proof. ... □

Lemma 7. *Let χ be an even, primitive Dirichelt character of prime modulus q. We have*

$$\frac{1}{q^2} \sum_{\nu} (\tau(\chi\nu)\tau(\overline{\chi}))^2 = \frac{q-1}{q} \left(\frac{\tau(\chi)}{\sqrt{q}} \right)^4 \tau(\overline{\chi}^2),$$

where the sum runs over all primitive Dirichlet characters mod q.

Proof. ... □

3 The case of $\mathrm{SL}(2,\mathbb{Z})$

3.1 Maass forms for $\mathrm{SL}(2,\mathbb{Z})$

Let $\nu \in \mathbb{C}$. A Maass form of type ν for $\mathrm{SL}(2,\mathbb{Z})$ is a non-zero function on $\mathbb{H}$ which satisfies

- $f(\gamma z) = f(z)$ for all $\gamma \in \mathrm{SL}(2,\mathbb{Z})$ and $z \in \mathbb{H}$;
- $\Delta f = \nu(1-\nu)f$;
- $f(x+iy) = O(y^M)$ for some $M > 0$ as $y \to \infty$.

If in addition

$$\int_0^1 f(x+iy)\mathrm{d}x = 0,$$

we call f a cuspidal Maass form or a Maass cusp form. It is not obvious that there should exists non-constant Maass cusp forms. One can prove that there are in fact infinitely many Maass cusp forms, see for example Chapter 4 in [1] or Chapter 4 in [2].

3.1.1 The Fourier expansion of Maass forms

Lemma 8. *Let f be a Maass form of type ν for $\mathrm{SL}(2,\mathbb{Z})$. Then $\nu(1-\nu)$ is real and ≥ 0.*

Proof. ... □

Lemma 9. *Let f be a Maass cusp form of type ν for $\mathrm{SL}(2,\mathbb{Z})$. Then $\nu(1-\nu) \geq 3\pi^2/2$.*

Proof. Let

$$f(z) = \sum_{n\neq 0} a(n)\sqrt{y}K_{\nu-\frac{1}{2}}(2\pi|n|y)e^{2\pi i n x}$$

be a Maass cusp form of type ν for $\mathrm{SL}(2,\mathbb{Z})$. It is an eigenfunction of the Laplace operator Δ with eigenvalue $\lambda = \nu(1-\nu)$. Let $\mathcal{F}$ denote the standard fundamental domain for the action of $\mathrm{SL}(2,\mathbb{Z})$ on $\mathbb{H}$ and let $\mathcal{F}^* = S(\mathcal{F})$, where $S(z) = -1/z$. Note that $\mathcal{F} \cup \mathcal{F}^* \supset \{z \in \mathbb{H} \mid |x| \leq \frac{1}{2}, y \geq \frac{\sqrt{3}}{2}\}$. Using that $\{e^{2\pi i n x}\}_{n\in\mathbb{Z}}$ is a complete orthogonal system for $L^2([-1/2,1/2])$, we can estimate

$$\begin{aligned}
2\lambda\langle f,f\rangle = 2\langle \nabla f, \nabla f\rangle &= \int_{\mathcal{F}\cup\mathcal{F}^*} \left(\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2\right)\mathrm{d}x\mathrm{d}y \\
&\geq \int_{\frac{\sqrt{3}}{2}}^{\infty}\int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\frac{\partial f}{\partial x}\right)^2 \mathrm{d}x\mathrm{d}y \\
&= 4\pi^2 \int_{\frac{\sqrt{3}}{2}}^{\infty} \sum_{n\neq 0} n^2 \big|a(n)\sqrt{y}K_{\nu-\frac{1}{2}}(2\pi|n|y)\big|^2 \mathrm{d}y \\
&\geq 4\pi^2 \frac{3}{4} \int_{\frac{\sqrt{3}}{2}}^{\infty} \sum_{n\neq 0} \big|a(n)\sqrt{y}K_{\nu-\frac{1}{2}}(2\pi|n|y)\big|^2 \frac{\mathrm{d}y}{y^2} \\
&= 3\pi^2 \int_{\frac{\sqrt{3}}{2}}^{\infty}\int_{-\frac{1}{2}}^{\frac{1}{2}} \big|f(z)\big|^2 \frac{\mathrm{d}x\mathrm{d}y}{y^2} \\
&\geq 3\pi^2 \int_{\mathcal{F}} \big|f(z)\big|^2 \mathrm{d}\mu(z) = 3\pi^2\langle f,f\rangle.
\end{aligned}$$

It follows that $\lambda \geq 3\pi^2/2$. □

3.1.2 Even and Odd Maass forms

We define the operator $T_{-1} : L^2(\mathrm{SL}(2,\mathbb{Z})\setminus\mathbb{H}) \to L^2(\mathrm{SL}(2,\mathbb{Z})\setminus\mathbb{H})$ by

$$(T_{-1}f)(x+iy) = f(-x+iy).$$

3.1.3 The non-holomorphic Eisenstein series

In this section we will introduce a (non-cuspidal) Maass form for $\mathrm{SL}(2,\mathbb{Z})$. The group $\mathrm{SL}(2,\mathbb{Z})$ has only a cusp at infinity. The stabilizer of this cups in $\mathrm{SL}(2,\mathbb{Z})$ is

$$\Gamma_\infty = \left\{ \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \mid m \in \mathbb{Z} \right\}.$$

Notice that $\mathrm{Im}(z)$ is Γ_∞-invariant. The Eisenstein series associated to this cusp, defined on $\mathbb{H} \times \mathbb{C}$, is defined by

$$E(z,s) := E_\infty(z,s) = \frac{1}{2} \sum_{\gamma \in \Gamma_\infty \backslash \mathrm{SL}(2,\mathbb{Z})} \mathrm{Im}(\gamma z)^s. \tag{3}$$

For $\mathrm{Re}(s) > 1$, this series converges absolutely and uniformly on compact sets. The Eisenstein series $E(z,s)$ defines an automorphic function with respect to $\mathrm{SL}(2,\mathbb{Z})$, that is, it satisfies $E(\gamma z, s) = E(z,s)$ for all $\gamma \in \mathrm{SL}(2,\mathbb{Z})$. Because $\Delta y^s = s(1-s)y^s$ and because Δ commutes with the $\mathrm{SL}(2,\mathbb{Z})$-action, the Eisenstein series is also an eigenfunction of the Laplacian

$$\Delta E(z,s) = s(1-s)E(z,s).$$

The Eisenstein series $E(z,s)$ is in fact a Maass form. This will come as a consequence of the Fourier expansion of $E(z,s)$.

We first rewrite the Eisenstein series in a form more convenient for explicit computations. The cosets $\Gamma_\infty \setminus \mathrm{SL}(2,\mathbb{Z})$ are determined by the bottom row of a representative

$$\Gamma_\infty \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \left\{ \begin{pmatrix} s & t \\ c & d \end{pmatrix} \,\middle|\, sd - ct = 1 \right\}.$$

A pair of coprime integers (c,d) uniquely determines such a coset. To sum over $(m,n) \in \mathbb{Z}^2 \backslash \{0,0\}$ is the same as to sum over all positive integers M and all pairs (c,d) of coprime integers by taking $(m,n) = (Mc, Md)$. As a consequence, we can write

$$\begin{aligned} E(z,s) &= \frac{1}{2} \sum_{\gamma \in \Gamma_\infty \backslash \mathrm{SL}(2,\mathbb{Z})} \mathrm{Im}(\gamma z)^s \\ &= \frac{1}{2} \sum_{(c,d)=1} \frac{y^s}{|cz+d|^{2s}} \\ &= \frac{1}{2\zeta(2s)} \sum_{\substack{m,n \in \mathbb{Z} \\ (m,n) \neq 0}} \frac{y^s}{|mz+n|^{2s}}. \end{aligned} \tag{4}$$

For the calculation of the Fourier expansion of $E(z,s)$ we need to following two identities.

Lemma 10. *If* $\mathrm{Re}(s) > 1/2$ *and* $r \in \mathbb{R}$, *then*

$$\int_{-\infty}^{\infty} \frac{e^{-2\pi i r x}}{(x^2+y^2)^s} \mathrm{d}x = \frac{\pi^s}{y^s \Gamma(s)} \begin{cases} \pi^{-s+\frac{1}{2}} \Gamma(s-\frac{1}{2}) y^{1-s} & \text{if } r = 0 \\ 2|r|^{s-\frac{1}{2}} \sqrt{y} K_{s-\frac{1}{2}}(2\pi |r| y) & \text{if } r \neq 0 \end{cases}. \tag{5}$$

Proof. Recall the integral representation of the Gamma function

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} \mathrm{d}t.$$

Thus we have

$$\begin{aligned} \frac{y^s}{\pi^s} \Gamma(s) \int_{-\infty}^{\infty} \frac{e^{-2\pi i r x}}{(x^2+y^2)^s} \mathrm{d}x &= \int_0^\infty \int_{-\infty}^{\infty} e^{-t} \left(\frac{ty}{\pi(x^2+y^2)} \right)^s e^{-2\pi i r x} \mathrm{d}x \frac{\mathrm{d}t}{t} \\ &= \int_0^\infty \int_{-\infty}^{\infty} e^{-\pi \xi (x^2+y^2)/y} \xi^s e^{-2\pi i r x} \mathrm{d}x \frac{\mathrm{d}\xi}{\xi}. \end{aligned}$$

For $r = 0$ the above expression becomes

$$\int_0^\infty \int_{-\infty}^\infty e^{-\pi\xi(x^2+y^2)/y} \xi^s \mathrm{d}x \frac{\mathrm{d}\xi}{\xi} = \int_0^\infty \sqrt{\frac{y}{\xi}} e^{-\pi\xi y} \xi^s \frac{\mathrm{d}\xi}{\xi}$$
$$= \pi^{-s+\frac{1}{2}} y^{1-s} \Gamma(s - \frac{1}{2}).$$

For $r \neq 0$ we obtain, using the change of variables $\xi \mapsto \frac{\xi}{|r|}$,

$$\int_0^\infty \int_{-\infty}^\infty e^{-\pi\xi(x^2+y^2)/y} \xi^s e^{-2\pi i r x} \mathrm{d}x \frac{\mathrm{d}\xi}{\xi} = \int_0^\infty \sqrt{\frac{y}{\xi}} e^{-y\pi r^2/\xi} e^{-\pi\xi y} \xi^s \frac{\mathrm{d}\xi}{\xi}$$
$$= \sqrt{y}|r|^{s-\frac{1}{2}} \int_0^\infty \xi^{s-\frac{1}{2}} e^{-y\pi|r|(1/\xi+\xi)} \frac{\mathrm{d}\xi}{\xi}$$
$$= 2|r|^{s-\frac{1}{2}} \sqrt{y} K_{s-\frac{1}{2}}(2\pi|r|y).$$

Here we used the following integral representation of the modified Bessel function of the second kind $K_\nu(x)$ (see [3] page 182)

$$K_\nu(x) = \frac{1}{2} \int_0^\infty e^{-\frac{x}{2}(t+t^{-1})} t^{-\nu-1} \mathrm{d}t.$$

□

Lemma 11. *For $z = x + iy \in \mathbb{H}$ and $\mathrm{Re}(s) > \frac{1}{2}$ we have*

$$\sum_{n\in\mathbb{Z}} \frac{1}{|z+n|^{2s}} = \frac{\sqrt{\pi}\Gamma(s-\frac{1}{2})}{\Gamma(s)} y^{1-2s} + \frac{2\pi^s}{\Gamma(s)} y^{\frac{1}{2}-s} \sum_{r\neq 0} |r|^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|r|y)\, e^{2\pi i r x}. \tag{6}$$

Proof. Recall the Poisson summation formula

$$\sum_{n\in\mathbb{Z}} \varphi(x+n) = \sum_{n\in\mathbb{Z}} \widehat{\varphi}(n) e^{2\pi i n x},$$

where φ is a continuous function that decays sufficiently rapidly at infinity (for example, $|f(x)| < |x|^{-c}$ with $c > 1$) and where $\widehat{\varphi}(n) = \int_{-\infty}^\infty \varphi(x) e^{-2\pi i n x} \mathrm{d}x$ is the Fourier transform. We apply this formula to $\varphi(x) = |x+iy|^{-2s}$, where $z = x + iy \in \mathbb{H}$ and $\mathrm{Re}(s) > \frac{1}{2}$. The Poisson summation formula gives

$$\sum_{n\in\mathbb{Z}} \frac{1}{|z+n|^{2s}} = \sum_{n\in\mathbb{Z}} \left(\int_{-\infty}^\infty \frac{e^{-2\pi i n x}}{(x^2+y^2)^s} \mathrm{d}x \right) e^{2\pi i n x}.$$

Using Lemma 10 we get the result. □

Lemma 12. *The Fourier expansion of $E(z,s)$ is given by*

$$E(z,s) = y^s + \phi(s) y^{1-s} + \frac{2\pi^s}{\Gamma(s)\zeta(2s)} y^{1/2} \sum_{n\neq 0} \sigma_{1-2s}(n) |n|^{s-\frac{1}{2}} K_{s-1/2}(2\pi|n|y) e^{2\pi i n x}, \tag{7}$$

where

$$\phi(s) = \pi^{1/2} \frac{\Gamma(s-1/2)}{\Gamma(s)} \frac{\zeta(2s-1)}{\zeta(2s)},$$
$$\sigma_s(n) = \sum_{\substack{d|n \\ d>0}} d^s.$$

Proof. We split up the sum (4) into the terms with $m = 0$ and those with $m \neq 0$ and combine each positive summand with its negative. We obtain

$$\zeta(2s) E(z,s) = \zeta(2s) y^s + \sum_{m=1}^\infty \sum_{n\in\mathbb{Z}} \frac{y^s}{|mz+n|^{2s}}.$$

Using the substitution $n = tm + r$ and Lemma 11 gives

$$\begin{aligned}
\zeta(2s)E(z,s) &= \zeta(2s)y^s + y^s \sum_{m=1}^{\infty} |m|^{-2s} \sum_{r=1}^{m} \sum_{t\in\mathbb{Z}} \left| t + z + \frac{r}{m} \right|^{-2s} \\
&= \zeta(2s)y^s + y^s \sum_{m=1}^{\infty} |m|^{-2s} \\
&\quad \times \left[\frac{\sqrt{\pi}\Gamma(s-\frac{1}{2})}{\Gamma(s)} m y^{1-2s} + \frac{2\pi^s}{\Gamma(s)} y^{\frac{1}{2}-s} \sum_{t\neq 0} |t|^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|t|y) \sum_{r=1}^{m} e^{2\pi i t\left(x+\frac{r}{m}\right)} \right] \\
&= \zeta(2s)y^s + \frac{\sqrt{\pi}\Gamma(s-\frac{1}{2})\zeta(2s-1)}{\Gamma(s)} y^{1-s} \\
&\quad + \frac{2\pi^s\sqrt{y}}{\Gamma(s)} \sum_{t\neq 0} |t|^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|t|y) e^{2\pi i t x} \sum_{m=1}^{\infty} |m|^{-2s} \sum_{r=1}^{m} e^{\frac{2\pi i t r}{m}}.
\end{aligned}$$

The sum $\sum_{r=1}^{m} e^{\frac{2\pi i t r}{m}}$ is zero unless $t = lm$ for some $l \in \mathbb{Z}$ in which case the sum is m. Therefore we get

$$\begin{aligned}
\zeta(2s)E(z,s) &= \zeta(2s)y^s + \frac{\sqrt{\pi}\Gamma(s-\frac{1}{2})\zeta(2s-1)}{\Gamma(s)} y^{1-s} + \frac{2\pi^s\sqrt{y}}{\Gamma(s)} \sum_{t\neq 0} |t|^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|t|y) e^{2\pi i t x} \sum_{\substack{m\geq 1 \\ t=ml}} |m|^{-2s+1} \\
&= \zeta(2s)y^s + \frac{\sqrt{\pi}\Gamma(s-\frac{1}{2})\zeta(2s-1)}{\Gamma(s)} y^{1-s} + \frac{2\pi^s\sqrt{y}}{\Gamma(s)} \sum_{t\neq 0} |t|^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|t|y) e^{2\pi i t x} \sum_{l|t} \left| \frac{t}{l} \right|^{-2s+1} \\
&= \zeta(2s)y^s + \frac{\sqrt{\pi}\Gamma(s-\frac{1}{2})\zeta(2s-1)}{\Gamma(s)} y^{1-s} + \frac{2\pi^s\sqrt{y}}{\Gamma(s)} \sum_{t\neq 0} \sigma_{1-2s}(t) |t|^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|t|y) e^{2\pi i t x}.
\end{aligned}$$

□

A consequence of the Fourier expansion is the meromorphic continuation of $E(z,s)$ to $s \in \mathbb{C}$. Another consequence of the above Fourier expansion is the functional equation

$$E(z,s) = \frac{1}{\phi(1-s)} E(z, 1-s).$$

3.2 Hecke operators for $L^2(\mathrm{SL}(2,\mathbb{Z}) \setminus \mathbb{H})$

For $n \geq 1$, the Hecke operator $T_n : L^2(\mathrm{SL}(2,\mathbb{Z}) \setminus \mathbb{H}) \to L^2(\mathrm{SL}(2,\mathbb{Z}) \setminus \mathbb{H})$ is defined by

$$(T_n f)(z) = \frac{1}{\sqrt{n}} \sum_{ad=n} \sum_{b=0}^{d-1} f\left(\frac{az+b}{d} \right).$$

The Hecke operators T_n $(n = 1, 2, \dots)$ are self-adjoint in $L^2(\mathrm{SL}(2,\mathbb{Z}) \setminus \mathbb{H})$, that is

$$\langle T_n f, g \rangle = \langle f, T_n g \rangle,$$

for all $f, g \in L^2(\mathrm{SL}(2,\mathbb{Z}) \setminus \mathbb{H})$. Given two positive integers n and m, the Hecke operators T_n and T_m commute. In fact, we have the multilicative rule

$$T_m T_n = \sum_{d|(m,n)} T_{mn/d^2}.$$

Furthermore, the Hecke operators commute with T_{-1} and the Laplacian Δ, and T_{-1} and Δ also commute. It follows that the space $L^2(\mathrm{SL}(2,\mathbb{Z}) \setminus \mathbb{H})$ can be simultaneously diagonalized by the operators $\{\Delta, T_n \,|\, n = -1, 1, 2, \dots\}$. Consequently, we can consider even or odd Maass forms that are simultaneous eigenfunctions for $\{T_n \,|\, n = 1, 2, \dots\}$. For such a Maass form, the following multiplicative relations of the Fourier coefficients hold.

Lemma 13. *Let*

$$f(z) = \sum_{n\neq 0} a(n)\sqrt{y}K_{\nu-\frac{1}{2}}(2\pi|n|y)e^{2\pi inx}$$

be a cuspidal Maass form of type ν for $SL(2,\mathbb{Z})$ which is an eigenfunction of all the Hecke operators. Assume that f is normalized so that $a(1) = 1$. Then

$$T_n f = a(n)f, \qquad \forall n = 1, 2, \ldots$$

Furthermore, we have the multiplicative relations of the Fourier coefficients

$$\begin{aligned} a(m)a(n) &= a(mn), \qquad \text{if } (m,n) = 1, \\ a(m)a(n) &= \sum_{d|(m,n)} a\left(\frac{mn}{d^2}\right), \\ a(p^{k+1}) &= a(p)a(p^k) - a(p^{k-1}), \end{aligned}$$

for all primes p and all $k \geq 1$.

Proof. ... ☐

3.3 L-functions

3.3.1 L-functions associated to Maass forms

Let

$$f(z) = \sum_{n\neq 0} a(n)\sqrt{y}K_{\nu_f}(2\pi|n|y)e^{2\pi inx} \tag{8}$$

be a cuspidal Maass form for $SL(2,\mathbb{Z})$.

Lemma 14. *The coefficients $a(n)$ in the Fourier expansion of $f(z)$ satisfy*

$$a(n) = O(\sqrt{|n|}).$$

Proof. Because f is a cusp form, it is bounded as $\mathrm{Im}(z) \to \infty$. Thus

$$\left|a(n)\sqrt{y}K_{\nu_f}(2\pi|n|y)\right| = \left|\int_0^1 f(x+iy)e^{-2\pi inx}\mathrm{d}x\right| \leq \int_0^1 |f(x+iy)|\mathrm{d}x \leq C,$$

for some constant C (that depends on f). If we choose $y = \frac{1}{|n|}$, the lemma is proved. ☐

The coefficients $a(n)$ behave like a constant on average.

Lemma 15. *For Y sufficiently large we have*

$$\sum_{1\leq n\leq Y} |a(n)|^2 \ll Y.$$

Proof. ... ☐

For $\mathrm{Re}(s) \geq \frac{3}{2}$ we define the L-function $L_f(s)$ associated to $f(z)$ by the absolutely convergent series

$$L_f(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}. \tag{9}$$

Lemma 16. *Let $f(z)$ be a cuspidal Maass form for $SL(2,\mathbb{Z})$ of parity ϵ. Then its L-function $L_f(s)$ can be meromorphically continued to all $\mathbb{C}$ and it satisfies the functional equation*

$$\Lambda_f(s) = \pi^{-s}\Gamma\left(\frac{s+\epsilon+\nu_f}{2}\right)\Gamma\left(\frac{s+\epsilon-\nu_f}{2}\right)L_f(s) = (-1)^{\epsilon}\Lambda_f(1-s).$$

Proof. The idea of the proof is to set $x = 0$ and take the Mellin transform of $f(z)$. We first assume that f is even. For sufficiently large $\mathrm{Re}(s)$ we have

$$\begin{aligned}\int_0^\infty f(iy)y^s \frac{\mathrm{d}y}{y} &= 2\int_0^\infty \sum_{n=1}^\infty a(n)\sqrt{y}K_{\nu_f}(2\pi ny)y^s\frac{\mathrm{d}y}{y} \\ &= \frac{2}{(2\pi)^{s+\frac{1}{2}}}L_f(s+\frac{1}{2})\int_0^\infty K_{\nu_f}(y)y^{s+\frac{1}{2}}\frac{\mathrm{d}y}{y} \\ &= \frac{1}{2\pi^{s+\frac{1}{2}}}L_f(s+\frac{1}{2})\Gamma\left(\frac{s+\frac{1}{2}+\nu_f}{2}\right)\Gamma\left(\frac{s+\frac{1}{2}-\nu_f}{2}\right),\end{aligned}$$

where we used the Mellin transform of the K-Bessel function (see page 127 in [4])

$$\int_0^\infty K_\nu(y)y^s\frac{\mathrm{d}y}{y} = 2^{s-2}\Gamma\left(\frac{s+\nu}{2}\right)\Gamma\left(\frac{s-\nu}{2}\right).$$

The integrals converge absolutely and this gives the meromorphic continuation. Because f is automorphic, we have

$$f(iy) = f(S(iy)) = f(i/y).$$

It follows that

$$\begin{aligned}\int_0^\infty f(iy)y^s\frac{\mathrm{d}y}{y} &= \int_0^1 f(i/y)y^s\frac{\mathrm{d}y}{y} + \int_1^\infty f(iy)y^s\frac{\mathrm{d}y}{y} \\ &= \int_1^\infty f(iy)\left(y^s+y^{-s}\right)\frac{\mathrm{d}y}{y}.\end{aligned}$$

The above integral is invariant under $s \mapsto -s$ and from this the functional equation follows immediately.

If f is odd, the above calculation does not quite work, because $\sum_{n\neq 0} a(n)|n|^{-s} = 0$ and so $\int_0^\infty f(iy)y^s\frac{\mathrm{d}y}{y} = 0$. However, we can deduce the functional equation in the same way as above once we replace in the above calculation $f(iy)$ by

$$\frac{\partial}{\partial x}f(z)\bigg|_{x=0} = 4\pi i\sum_{n=1}^\infty a(n)n\sqrt{y}K_{\nu_f}(2\pi ny).$$

□

Lemma 17. *Let f be a Maass cusp form as in Equation (8) which is an eigenfunction of all the Hecke operators and is normalized so that $a(1) = 1$. Then the L-function $L_f(s) = \sum_{n=1}^\infty \frac{a(n)}{n^s}$ admits the Euler product*

$$L_f(s) = \prod_p \left(1 - a(p)p^{-s} + p^{-2s}\right)^{-1},$$

where the product runs over all primes p.

Proof. It follows from the multiplicativity of the Fourier coefficients (Lemma 13) that

$$\sum_{n=1}^\infty \frac{a(n)}{n^s} = \prod_p\left(\sum_{l=0}^\infty \frac{a(p^l)}{p^{ls}}\right).$$

For a fixed prime p we have

$$\begin{aligned}&\sum_{l=0}^\infty a(p^l)p^{-ls}\left(1-a(p)p^{-s}+p^{-2s}\right) \\ &\qquad = \sum_{l=0}^\infty a(p^l)p^{-ls} - \sum_{l=1}^\infty a(p^{l-1})a(p)p^{-ls} + \sum_{l=2}^\infty a(p^{l-2})p^{-ls} \\ &\qquad = 1 + \sum_{l=2}^\infty \left(a(p^l) - a(p^{l-1})a(p) + a(p^{l-2})\right)p^{-ls} = 1,\end{aligned}$$

where the last equality follows from Lemma 13. This shows that

$$\sum_{l=0}^{\infty} \frac{a(p^l)}{p^{ls}} = \left(1 - a(p)p^{-s} + p^{-2s}\right)^{-1}.$$

□

3.3.2 L-functions associated to Eisenstein series

By formula (7), the non-constant term in the Fourier expansion of the Eisenstein series $E(z,v)$ is given by

$$\frac{2\pi^v}{\Gamma(v)\zeta(2v)} y^{1/2} \sum_{n\neq 0} \sigma_{1-2v}(n)|n|^{v-\frac{1}{2}} K_{v-1/2}(2\pi|n|y)e^{2\pi inx}.$$

We define the L-function $L_{E_v}(s)$ associated to the Eisenstein series $E(z,v)$ to be

$$L_{E_v}(s) = \sum_{n=1}^{\infty} \frac{\sigma_{1-2v}(n)\, n^{v-\frac{1}{2}}}{n^s}.$$

Lemma 18. *The L-function $L_{E_v}(s)$ is simply a product of shifted Riemann zeta functions*

$$L_{E_v}(s) = \zeta(s+v-1/2)\,\zeta(s-v+1/2).$$

Furthermore, we have the functional equation

$$G_{E_v}(s) = G_{E_v}(1-s),$$

where

$$G_{E_v}(s) = \pi^{-s}\Gamma\left(\frac{s+v-\frac{1}{2}}{2}\right)\Gamma\left(\frac{s-v+\frac{1}{2}}{2}\right)\zeta(s+v-1/2)\,\zeta(s-v+1/2).$$

Proof. We calculate

$$\begin{aligned}
\sum_{n=1}^{\infty} \sigma_{1-2v}(n)\, n^{v-\frac{1}{2}-s} &= \sum_{n=1}^{\infty} n^{v-\frac{1}{2}-s} \sum_{d|n} d^{1-2v} \\
&= \sum_{n=1}^{\infty}\sum_{d=1}^{\infty} (dn)^{v-\frac{1}{2}-s} d^{1-2v} \\
&= \sum_{n=1}^{\infty} n^{v-\frac{1}{2}-s} \sum_{d=1}^{\infty} d^{\frac{1}{2}-v-s} \\
&= \zeta(s-v+1/2)\,\zeta(s+v-1/2).
\end{aligned}$$

Now consider

$$\begin{aligned}
G_{E_v}(s) = \pi^{-\frac{s+v-\frac{1}{2}}{2}} &\Gamma\left(\frac{s+v-\frac{1}{2}}{2}\right)\zeta(s+v-1/2) \\
\times\, \pi^{-\frac{s-v+\frac{1}{2}}{2}} &\Gamma\left(\frac{s-v+\frac{1}{2}}{2}\right)\zeta(s-v+1/2).
\end{aligned}$$

The functional equation $G_{E_v}(s) = G_{E_v}(1-s)$ follows directly from the functional equation

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s) \tag{10}$$

of the Riemann zeta function.

□

As expected, the functional equation of $L_{E_v}(s)$ matches the functional equation of a Maass form of even parity (as in Lemma 16). In fact, the functional equation of a Maass form of type ν and of even parity is identical to the functional equation of $L_{E_\nu}(s)$. The reason is that the K-Bessel functions in both Fourier expansions coincide and the proof of the functional equation in Lemma 16 only uses the analytical properties of the K-Bessel function, and does not depend on the arithmetic Fourier coefficients. This gives a method of obtaining functional equations for Maass forms from studying the functional equation of Eisenstein series. We will use this observation several times.

3.3.3 Twisted L-functions associated to Maass forms

Let

$$f(z) = \sum_{n\neq 0} a(n) y^{1/2} K_{\nu-\frac{1}{2}}(2\pi|n|y) e^{2\pi i n x}$$

be a cuspidal Maass form of type ν for $\mathrm{SL}(2,\mathbb{Z})$. Let χ be an even primitive Dirichlet character mod a prime q. The twist of f by χ is defined by

$$f_\chi(z) = \sum_{n\neq 0} a(n)\chi(n) y^{1/2} K_{\nu-\frac{1}{2}}(2\pi|n|y) e^{2\pi i n x}, \tag{11}$$

and the associated L-function by

$$L_{f_\chi}(s) = \sum_{n=1}^{\infty} \frac{\chi(n)a(n)}{n^s}. \tag{12}$$

Following the remark after Lemma 18, the functional equation of the twisted L-function $L_{f_\chi}(s)$ satisfies the functional equation of the twisted L-function $L_{E_v}(s,\chi)$ associated to the Eisenstein series $E(z,v)$. We now determine the functional equation for $L_{E_v}(s,\chi)$.

Including χ in the proof of Lemma 18 shows that the twisted L-function $L_{E_v}(s,\chi)$ is

$$L_{E_v}(s,\chi) = L(s+v-1/2,\chi)\, L(s-v+1/2,\chi),$$

with the Dirichlet L-function $L(s,\chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s}$. The Dirichlet series $L(s,\chi)$ satisfies the functional equation (reference)

$$\Lambda(s,\chi) = \left(\frac{q}{\pi}\right)^{\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) L(s,\chi) = \frac{\tau(\chi)}{\sqrt{q}}\ \Lambda(1-s,\overline{\chi}). \tag{13}$$

From this it follows that the L-function $L_{E_v}(s,\chi)$ satisfies the functional equation

$$\begin{aligned}\Lambda_{E_v}(s,\chi) &= \left(\frac{q}{\pi}\right)^{s} \Gamma\left(\frac{s+v-1/2}{2}\right) \Gamma\left(\frac{s-v+1/2}{2}\right) L_{E_v}(s,\chi) \\ &= \left(\frac{\tau(\chi)}{\sqrt{q}}\right)^{2} \Lambda_{E_v}(1-s,\overline{\chi}).\end{aligned}$$

3.4 Rankin-Selberg convolution for $\mathrm{SL}(2,\mathbb{Z})$

Let

$$f(z) = \sum_{n\neq 0} a(n)\sqrt{y} K_{\nu_f}(2\pi|n|y) e^{2\pi i n x}, \tag{14}$$

$$g(z) = \sum_{n\neq 0} b(n)\sqrt{y} K_{\nu_g}(2\pi|n|y) e^{2\pi i n x}, \tag{15}$$

be cuspidal Maass forms for $SL(2,\mathbb{Z})$. Recall that $f(z)$ (resp. $g(z)$) is an eigenfunction for the Laplacian with eigenvalue $1/4-\nu_f$ (resp. $1/4-\nu_g$). For $\mathrm{Re}(s) > 2$ we define the Rankin-Selberg convolution $L_{f\times g}(s)$ as the absolutely convergent series

$$L_{f\times g}(s) = \zeta(2s) \sum_{n=1}^{\infty} \frac{a(n)\overline{b(n)}}{n^s}.$$

We will prove that $L_{f\times g}$ can be expressed as an inner product of $f\overline{g}$ with an Eisenstein series. This construction gives the meromorphic continuation and functional equation for $L_{f\times g}$.

Theorem 19. *Let $f(z)$ and $g(z)$ be cuspidal Maass forms as in Equation 14 and 15. Then $L_{f\times g}(s)$ can be meromorphically continued to all $s \in \mathbb{C}$ with at most a simple pole at $s = 1$. Furthermore, we have the functional equation*

$$\Lambda_{f\times g}(s) = \pi^{-2s} G_{\nu_f,\nu_g}(s) L_{f\times g}(s) = \Lambda_{f\times g}(1-s),$$

where $G_{\nu_f,\nu_g}(s) = \prod \Gamma\left(\frac{s\pm\nu_f\pm\nu_g}{2}\right)$.

Proof. Let $E(z,s)$ be the non-holomorphic Eisenstein series as defined in Equation 3. For sufficiently large $\mathrm{Re}(s)$, it follows from folding/unfolding that

$$\begin{aligned}
\zeta(2s)\langle f\overline{g}, E(\cdot,\overline{s})\rangle &= \zeta(2s)\int_{SL(2,\mathbb{Z})\backslash\mathbb{H}} f(z)\overline{g(z)E(z,\overline{s})}\mathrm{d}\mu(z) \\
&= \zeta(2s)\sum_{\gamma\in\Gamma_\infty\backslash SL(2,\mathbb{Z}}\int_{SL(2,\mathbb{Z})\backslash\mathbb{H}} f(\gamma z)\overline{g(\gamma z)}\,\mathrm{Im}(\gamma z)^s \mathrm{d}\mu(\gamma z) \\
&= \zeta(2s)\int_{\Gamma_\infty\backslash\mathbb{H}} f(z)\overline{g(z)}\,\mathrm{Im}(z)^s\mathrm{d}\mu(z) \\
&= \zeta(2s)\int_0^\infty\int_0^1 f(z)\overline{g(z)}y^{s-2}\mathrm{d}x\mathrm{d}y \\
&= \zeta(2s)\sum_{n,m\neq 0} a(n)\overline{b(m)}\int_0^\infty K_{\nu_f}(2\pi|n|y)K_{\nu_g}(2\pi|m|y)y^{s-1}\int_0^1 e^{2\pi i(n-m)x}\mathrm{d}x\mathrm{d}y \\
&= \zeta(2s)\sum_{n\neq 0} a(n)\overline{b(n)}\int_0^\infty K_{\nu_f}(2\pi|n|y)K_{\nu_g}(2\pi|n|y)y^s\frac{\mathrm{d}y}{y} \\
&= \frac{\zeta(2s)}{(2\pi)^s}\sum_{n\neq 0}\frac{a(n)\overline{b(n)}}{n^s}\int_0^\infty K_{\nu_f}(y)K_{\nu_g}(y)y^s\frac{\mathrm{d}y}{y} \\
&= (2\pi)^{-s}\,L_{f\times g}(s)\int_0^\infty K_{\nu_f}(y)K_{\nu_g}(y)y^s\frac{\mathrm{d}y}{y}.
\end{aligned}$$

The Mellin transform of $K_\nu(y)K_{\nu'}(y)$ is given by (see page 145 in [4])

$$\int_0^\infty K_\nu(y)K_{\nu'}(y)y^s\frac{\mathrm{d}y}{y} = \frac{2^{s-3}}{\Gamma(s)}\prod\Gamma\left(\frac{s\pm\nu\pm\nu'}{2}\right), \tag{16}$$

which is valid for $\mathrm{Re}(s) > |\mathrm{Re}(\nu)| + |\mathrm{Re}(\nu')|$.

From the calculation it follows that the convolution function $L_{f\times g}(s)$ inherits the analytical properties of the Eisenstein series $E(z,s)$. This means that $L_{f\times g}(s)$ can be meromorphically continued on $\mathbb{C}$. Because the Eisenstein series has a simple pole at $s = 1$ and the Gamma function no zeros, it follows that $L_{f\times g}(s)$ has a simple pole at $s = 1$ if and only if $\langle f, g\rangle \neq 0$. The functional equation follows from the functional equation of the Eisenstein series

$$E^*(z,s) = \pi^{-s}\Gamma(s)\zeta(2s)E(z,s) = E^*(z,1-s).$$

□

Lemma 20. *Let $\alpha_i, \beta_i \in \mathbb{C}$ for $i = 1,2$. Then for $x \in \mathbb{C}$, $|x|$ sufficiently small, we have*

$$\prod_{i=1}^2\prod_{j=1}^2(1-\alpha_i\beta_j x)^{-1} = \sum_{k=0}^\infty S_k(\alpha_1,\alpha_2)S_k(\beta_1,\beta_2)x^k(1-\alpha_1\alpha_2\beta_1\beta_2x^2)^{-1},$$

where $S_k(x_1,x_2)$ is the Schur polynomial

$$S_k(x_1,x_2) = \frac{\det\begin{pmatrix} x_1^{k+1} & x_2^{k+1} \\ 1 & 1\end{pmatrix}}{\det\begin{pmatrix} x_1 & x_2 \\ 1 & 1\end{pmatrix}} = \frac{x_1^{k+1}-x_2^{k+1}}{x_1-x_2}.$$

Proof. This is proved on page 67 in [5] by evaluating the determinant of the matrix

$$\left(\frac{1}{1-\alpha_i\beta_j x}\right)_{1\le i,j\le 2}$$

in two different ways. $\square$

Theorem 21. *Let $f(z)$ and $g(z)$ be cuspidal Maass forms as in Equation* 14 *and* 15*. Assume that $L_f(s)$ and $L_g(s)$ have Euler products*

$$L_f(s) = \prod_p \prod_{i=1}^{2} \left(1 - \frac{\alpha_{i,p}}{p^s}\right)^{-1}, \quad L_g(s) = \prod_p \prod_{j=1}^{2} \left(1 - \frac{\beta_{j,p}}{p^s}\right)^{-1}.$$

Then $L_{f\times g}(s)$ admits the Euler product

$$L_{f\times g}(s) = \prod_p \prod_{i=1}^{2} \prod_{j=1}^{2} \left(1 - \frac{\alpha_{i,p}\,\overline{\beta}_{j,p}}{p^s}\right)^{-1}.$$

Proof. By assumption we have

$$L_{f\times g}(s) = \prod_p \frac{\sum_{k=1}^{\infty} a(p^k)\overline{b(p^k)}p^{-ks}}{(1-p^{-2s})}.$$

In view of Lemma 20, after choosing $x = p^{-s}$, it suffices to show that

$$a(p^k) = S_k(\alpha_{1,p}, \alpha_{2,p}), \quad b(p^k) = S_k(\beta_{1,p}, \beta_{2,p}).$$

The above equalities are obtained inductively from the relations

$$\begin{aligned} 1 &= \alpha_{1,p}\,\alpha_{2,p}, \\ a(p) &= \alpha_{1,p} + \alpha_{2,p}, \\ a(p^{k+1}) &= a(p)a(p^k) - a(p^{k-1}). \end{aligned}$$

$\square$

Following the remark after Lemma 18, the functional equation in Theorem 19 is identical to the functional equation of the Rankin-Selberg L-function of two Eisenstein series E_v and E_w of type v and w, respectively. We now verify this observation. By Lemma 18, the L-functions $L_{E_v}(s)$ and $L_{E_w}(s)$ are given by

$$\begin{aligned} L_{E_v}(s) &= \prod_{i=1}^{2} \zeta(s - \lambda_i(v)) \\ L_{E_w}(s) &= \prod_{i=1}^{2} \zeta(s - \lambda_i(w)), \end{aligned}$$

where $\lambda_1(v) = v - \frac{1}{2}$ and $\lambda_2(v) = -v + \frac{1}{2}$. From the Euler product

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}$$

and from Theorem 21 we deduce that the Rankin-Selberg convolution of E_v and E_w is

$$\begin{aligned} L_{E_v\times E_w}(s) &= \prod_p \prod_{i=1}^{2} \prod_{j=1}^{2} \left(1 - \frac{p^{\lambda_i(v)+\overline{\lambda_j(w)}}}{p^s}\right)^{-1} \\ &= \prod_{i=1}^{2} \prod_{j=1}^{2} \zeta(s - \lambda_i(v) - \overline{\lambda_j(w)}). \end{aligned}$$

It follows from the functional equation (10) for the Riemann Zeta function that the functional equation of $L_{E_v \times E_w}(s)$ is indeed the same as the one stated in Theorem 19. We also note that a corollary of the Rankin-Selberg convolution $L_{E_v \times E_w}(s)$ is the Ramanujan identity

$$\sum_{n=1}^{\infty} \frac{\sigma_a(s)\sigma_b(s)}{n^s} = \frac{\zeta(s)\zeta(s-a)\zeta(s-b)\zeta(s-a-b)}{\zeta(2s-a-b)}.$$

3.5 The Luo-Rudnick-Sarnak Theorem for $\mathrm{SL}(2,\mathbb{Z})$

Let

$$f(z) = \sum_{n\neq 0} a(n)\sqrt{y}K_{\nu-\frac{1}{2}}(2\pi|n|y)e^{2\pi inx}$$

be a Maass cusp form for $\mathrm{SL}(2,\mathbb{Z})$ of type ν so that $a(1) = 1$ and so that $f(z)$ is an eigenfunction of the Hecke operators. We have shown in Lemma 21 that the associated L-function $L_f(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$ has the Euler product

$$\begin{aligned} L_f(s) &= \prod_p \left(1 - a(p)p^{-s} + p^{-2s}\right)^{-1}, \\ &= \prod_p \prod_{i=1}^{2} \left(1 - \frac{\alpha_{i,p}}{p^s}\right)^{-1}, \end{aligned} \tag{17}$$

where for each prime p the $\alpha_{i,p} \in \mathbb{C}$ satisfy

$$\begin{aligned} 1 &= \alpha_{1,p}\,\alpha_{2,p}, \\ a(p) &= \alpha_{1,p} + \alpha_{2,p}. \end{aligned}$$

Here is the Ramanujan conjecture for this setting.

Conjecture 22. *Lef f be a Maass cusp form as above. Then the $\alpha_{i,p}$ in Equation 17 satisfy*

$$|\alpha_{i,p}| = 1,$$

for all primes p and $i = 1, 2$.

Another important conjecture is the Selberg eigenvalue conjecture.

Conjecture 23. *Lef f be a Maass cusp form of type $\nu \in \mathbb{C}$ as above. Then*

$$\nu(1-\nu) \geq \frac{1}{4},$$

or equivalently,

$$\mathrm{Re}(\nu) = \frac{1}{2},$$

or equivalently,

$$\mathrm{Re}(\lambda_i(\nu)) = 0,$$

for $i = 1, 2$ and where $\lambda_1(\nu) = \nu - \frac{1}{2}$ and $\lambda_2(\nu) = -\nu + \frac{1}{2}$.

These two conjectures can in fact be treated on an equal footing. The Selberg eigenvalue conjecture is not hard to prove for $\mathrm{SL}(2,\mathbb{Z})$. In fact, in Lemma 9 we proved the much better lower bound $\nu(1-\nu) \geq 3\pi^2/2$. The above conjectures can be generalized to $\mathrm{SL}(n,\mathbb{Z})$ for $n \geq 2$ and to congruence subgroups and in this setting the conjectures are not proved. The current best bound for both the Ramanujan conjecture and the Selberg eigenvalue conjecture for Maass forms for $\mathrm{SL}(n,\mathbb{Z})$ was obtained by Luo, Rudnick and Sarnak ([6], [7]). In this section we will illustrate their method specialized to Maass forms for $\mathrm{SL}(2,\mathbb{Z})$. In this setting, the following is their theorem.

Theorem 24. *Lef f be a Maass cusp form of type ν for $\mathrm{SL}(2,\mathbb{Z})$ as above. Then, using the above notation, we have*

$$|\alpha_{i,p}| \leq p^{\frac{1}{2}-\frac{1}{5}}, \tag{18}$$

$$\mathrm{Re}(\lambda_i(\nu)) \leq \frac{1}{2} - \frac{1}{5}, \tag{19}$$

for all primes p and for $i = 1, 2$.

We will give a proof of the bound (19) towards the Selberg eigenvalue conjecture. The proof of the bound (18) for the Ramanujan conjecture is based on the same argument, but requires more care, and we will not go into details.

The proof of Theorem 24 is based on the Rankin-Selberg convolution L-function. Let us recall the material that we will need. Let f be a Maass form as in Theorem 24. It has the Euler product

$$L_f(s) = \prod_p \prod_{i=1}^{2} \left(1 - \frac{\alpha_{i,p}}{p^s}\right)^{-1}.$$

Let χ be a primitive character mod a prime q. The twisted L-function $L_{f_\chi}(s) = \sum_{n=1}^{\infty} \frac{\chi(n)a(n)}{n^s}$ has the Euler product

$$L_{f_\chi}(s) = \prod_p \prod_{i=1}^{2} \left(1 - \frac{\alpha_{i,p}\,\chi(p)}{p^s}\right)^{-1}.$$

We now take the Rankin-Selberg convolution $L_{f_\chi \times f}(s)$ of f_χ and f. According to Theorem 21, it is given by

$$L_{f_\chi \times f}(s) = \prod_p \prod_{i=1}^{2} \prod_{j=1}^{2} \left(1 - \frac{\alpha_{i,p}\,\overline{\alpha}_{j,p}\,\chi(p)}{p^s}\right)^{-1}. \tag{20}$$

According to Section 3.3.2, the functional equation of $L_{f_\chi \times f}(s)$ can be deduced from the functional equation for the Eisenstein L-functions $L_{E_v}(s,\chi)$ and $L_{E_v}(s)$. We recall from Section 3.3.2 and 3.3.3 that these L-functions take the simple form

$$\begin{aligned} L_{E_v}(s) &= \prod_{i=1}^{2} \zeta(s - \lambda_i(\nu)), \\ L_{E_v}(s,\chi) &= \prod_{j=1}^{2} L(s - \lambda_j(\nu), \chi). \end{aligned}$$

From the Euler products

$$\begin{aligned} \zeta(s) &= \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}, \\ L(s,\chi) &= \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}, \end{aligned}$$

and from Theorem 21 we deduce that the Rankin-Selberg convolution of E_ν and $E_{\nu,\chi}$ is

$$\begin{aligned} L_{E_{\nu,\chi} \times E_\nu}(s) &= \prod_p \prod_{i=1}^{2} \prod_{j=1}^{2} \left(1 - \frac{\chi(p) p^{\lambda_i(\nu) + \overline{\lambda_j(\nu)}}}{p^s}\right)^{-1} \\ &= \prod_{i=1}^{2} \prod_{j=1}^{2} L(s - \lambda_i(\nu) - \overline{\lambda_j(\nu)}, \chi). \end{aligned}$$

The functional equation (13) of the Dirichlet L-series gives us now the functional equation

$$\begin{aligned} \Lambda_{E_{\nu,\chi} \times E_\nu}(s) &= \prod_{i=1}^{2} \prod_{j=1}^{2} \left(\frac{q}{\pi}\right)^{\frac{s - \lambda_i(\nu) - \overline{\lambda_j(\nu)}}{2}} \Gamma\left(\frac{s - \lambda_i(\nu) - \overline{\lambda_j(\nu)}}{2}\right) L_{E_{\nu,\chi} \times E_\nu}(s) \\ &= \left(\frac{\tau(\chi)}{\sqrt{q}}\right)^4 \Lambda_{E_v}(1 - s, \overline{\chi}), \end{aligned} \tag{21}$$

which, as we have just mentioned, is also the functional equation satisfied by $L_{f_\chi \times f}(s)$. We may, as in the proof of Theorem 19, construct $L_{f_\chi \times f}(s)$ as the inner product

$$\langle f, f_\chi\, E(\cdot, s)\rangle,$$

proving that $L_{f_\chi \times f}(s)$ has a meromorphic continuation to $s \in \mathbb{C}$. If χ is non-trivial, which we will assume from now on, then $\Lambda_{f_\chi \times f}(s)$ is in fact entire.

The idea of the proof for the bound (19) in Theorem 24 is to balance poles of the Gamma factors in $\Lambda_{f_\chi \times f}(s)$ with zeros of $L_{f_\chi \times f}(s)$. We are thus led to examine zeros of $L_{f_\chi \times f}(s)$. The following theorem gives us zero-free regions for a collection of L-functions $L_{f_\chi \times f}(s)$.

Theorem 25. *Let f be a Maass form as above. Then for any real number $\beta > 1 - \frac{2}{5}$, there are even primitive Dirichlet characters χ such that*

$$L_{f_\chi \times f}(\beta) \neq 0.$$

Before proving the theorem, let us show that it implies the bound (19) in Theorem 24.

Proof of (19) in Theorem 24. We argue by contradiction. Let f be a Maass cusp form as in Theorem 24 and assume that for some $1 \leq i \leq 2$ we have $\mathrm{Re}(\lambda_i(\nu)) = \beta$ for some $\beta > \frac{1}{2} - \frac{1}{5}$. Then by Theorem 25 there is an even primitive Dirichlet character χ such that

$$L_{f_\chi \times f}(2\beta) \neq 0.$$

On the other hand, the Gamma factor

$$\prod_{i=1}^{2} \Gamma\left(\frac{s - 2\,\mathrm{Re}(\lambda_i(\nu))}{2}\right)$$

in $\Lambda_{f_\chi \times f}(s)$ has a pole at $s = 2\beta$. Because $\Lambda_{f_\chi \times f}(s)$ is holomorphic, we obtain a contradiction. □

Proof of Theorem 25. According to Lemma 14, we have the bound $|\alpha_{i,p}| \ll \sqrt{p}$ for $i = 1, 2$. Therefore, for $s \geq 1$, the Euler factor $\left(1 - \frac{\alpha_{i,p}\,\overline{\alpha}_{j,p}\,\chi(p)}{p^s}\right)^{-1}$ in Equation (20) is non-zero for every p, thus also $L_{f_\chi \times f}(\beta) \neq 0$ for $\beta \geq 1$. We can thus assume $1 - \frac{2}{5} < \beta < 1$, and the theorem is a consequence of the following lemma. □

Lemma 26. *Let $\beta \in (1 - \frac{2}{5}, 1)$. Then for Q sufficiently large we have*

$$\sum_{q \sim Q} \sum_{\chi} L_{f_\chi \times f}(\beta) \approx Q^2 / \log(Q) \neq 0, \tag{22}$$

where q is prime, where the notations $q \sim Q$ means $Q \leq q \leq 2Q$ and where the inner sum is taken over non-trivial even primitive Dirichlet characters χ mod q.

Proof. We write the Dirichlet series of $L_{f_\chi \times f}(s)$ as

$$L_{f_\chi \times f}(s) = \sum_{n=1}^{\infty} \chi(n) \frac{b(n)}{n^s}.$$

By Lemma 15, this Dirichlet series converges absolutely for $\mathrm{Re}(s) \geq 1$. Therefore $\beta \in (1 - \frac{2}{5}, 1)$ does not lie in the region of absolute convergence of the Dirichlet series of $L_{f_\chi \times f}(s)$. The method of approximate functional equations allows us to compute values of L-functions inside regions where the Dirichlet series of the L-function fails to converge. The first step of the proof is thus the derivation of an approximate function equation for $L_{f_\chi \times f}(\beta)$.

Step 1. *Set $\beta_0 = 2\max_{1 \leq i \leq 2} \mathrm{Re}(\lambda_i(\nu))$. There are two real-valued functions h_1 and h_2 that satisfy*

$$\begin{aligned} &h_1(y), h_2(y) = O_A(y^{-A}) \quad \text{as } y \to \infty, \\ &h_1(y) = 1 + O_A(y^{A}) \quad \text{as } y \to 0, \\ &h_2(y) \ll 1 + y^{1-\beta_0-\beta-\epsilon} \quad \text{as } y \to 0, \end{aligned}$$

for all $A \geq 1$ and all $\epsilon > 0$ and such that for any $Y > 1$ we have the approximate functional equation

$$\begin{aligned} L_{f_\chi \times f}(\beta) = &\sum_{n=1}^{\infty} \frac{b(n)}{n^\beta} \chi(n)\, h_1\left(\frac{n}{Y}\right) \\ &- \frac{\tau(\chi)^4 q^{-4\beta}}{\pi^{2-4\beta}} \sum_{n=1}^{\infty} \frac{b(n)}{n^{1-\beta}} \overline{\chi(n)}\, h_2\left(\frac{nY\pi^4}{q^4}\right). \end{aligned} \tag{23}$$

Let $h : (0, \infty) \to \mathbb{R}$ be a non-negative smooth, compactly supported function such that $\int_0^\infty h(y) \frac{\mathrm{d}y}{y} = 1$. Let

$$\tilde{h}(s) = \int_0^\infty h(y) y^s \frac{\mathrm{d}y}{y}$$

be the Mellin transform of h, which is holomorphic, bounded on vertical strips and satisfies $\tilde{h}(0) = 1$. We define

$$h_1(y) = \frac{1}{2\pi i} \int_{\mathrm{Re}(s)=2} \tilde{h}(s) y^{-s} \frac{\mathrm{d}s}{s}.$$

By a shift of the line of integration to the right (left), we get the behaviour as $y \to \infty$ ($y \to 0$). Next, for $y > 0$, we define

$$h_2(y) = \frac{1}{2\pi i} \int_{\mathrm{Re}(s)=1} \tilde{h}(-s) G(-s+\beta) y^{-s} \frac{\mathrm{d}s}{s}, \tag{24}$$

where

$$G(s) = \frac{\prod_{i=1}^2 \prod_{j=1}^2 \Gamma\left(\frac{1-s-\lambda_i(\nu)-\overline{\lambda_j(\nu)}}{2}\right)}{\prod_{i=1}^2 \prod_{j=1}^2 \Gamma\left(\frac{s-\lambda_i(\nu)-\overline{\lambda_j(\nu)}}{2}\right)}.$$

By Stirling's formula, $G(s)$ has at most polynomial growth in vertical strips. Therefore, by shifting the contour in Equation (24) to the right, we see that $h_2(y)$ decays rapidly as $y \to \infty$. For the behaviour as $y \to 0$, we shift the integration line to the left. (...) This gives the bound $h_2(y) \ll 1 + y^{1-\beta_0-\beta-\epsilon}$ as $y \to 0$.

Now we come to the derivation of an approximate functional equation. To this end, for $Y > 1$, consider the integral

$$\begin{aligned} \frac{1}{2\pi i} \int_{\mathrm{Re}(s)=2} \tilde{h}(s) L_{f_\chi \times f}(s+\beta) Y^s \frac{\mathrm{d}s}{s} &= \sum_{n=1}^{\infty} \frac{b(n)\chi(n)}{n^\beta} \frac{1}{2\pi i} \int_{\mathrm{Re}(s)=2} \tilde{h}(s) \left(\frac{Y}{n}\right)^s \frac{\mathrm{d}s}{s} \\ &= \sum_{n=1}^{\infty} \frac{b(n)\chi(n)}{n^\beta}\, h_1\left(\frac{n}{Y}\right). \end{aligned} \tag{25}$$

Since $L_{f_\chi \times f}(s)$ is bounded in vertical strips, we may shift the integration line in the above integral to $\mathrm{Re}(s) = -1$, thereby picking up a pole at $s = 0$, and we obtain

$$\begin{aligned} &\frac{1}{2\pi i} \int_{\mathrm{Re}(s)=2} \tilde{h}(s) L_{f_\chi \times f}(s+\beta) Y^s \frac{\mathrm{d}s}{s} \\ &\quad = L_{f_\chi \times f}(\beta) + \frac{1}{2\pi i} \int_{\mathrm{Re}(s)=-1} \tilde{h}(s) L_{f_\chi \times f}(s+\beta) Y^s \frac{\mathrm{d}s}{s} \\ &\quad = L_{f_\chi \times f}(\beta) + \frac{1}{2\pi i} \left(\frac{\tau(\chi)}{\sqrt{q}}\right)^4 \left(\frac{q}{\pi}\right)^{2-4\beta} \int_{\mathrm{Re}(s)=-1} \tilde{h}(s) G(s+\beta) L_{f_{\overline{\chi}} \times f}(1-s-\beta) \left(\frac{\pi^4 Y}{q^4}\right)^s \frac{\mathrm{d}s}{s}, \end{aligned}$$

where in the last step we applied the functional equation (21). On changing variables $s \mapsto -s$ the last expression equals

$$L_{f_\chi \times f}(\beta) + \frac{1}{2\pi i} \left(\frac{\tau(\chi)}{\sqrt{q}}\right)^4 \left(\frac{q}{\pi}\right)^{2-4\beta} \int_{\mathrm{Re}(s)=1} \tilde{h}(-s) G(-s+\beta) L_{f_{\overline{\chi}} \times f}(1+s-\beta) \left(\frac{\pi^4 Y}{q^4}\right)^{-s} \frac{\mathrm{d}s}{s},$$

which equals

$$L_{f_\chi \times f}(\beta) + \left(\frac{\tau(\chi)}{\sqrt{q}}\right)^4 \left(\frac{q}{\pi}\right)^{2-4\beta} \sum_{n=1}^{\infty} \overline{\chi}(n) \frac{b(n)}{n^{1-\beta}} h_2 \left(\frac{nY\pi^4}{q^4}\right). \tag{26}$$

Combining (25) and (26) gives the approximate functional equation (23).

Step 2. *We average the approximate functional equation over even primitive Dirichlet characters to deduce (22).*

According to the approximate functional equation (23), we can decompose $\sum_{q\sim Q}\sum_\chi L_{f_\chi\times f}(\beta)$ into $T_1 - T_2$, where

$$T_1 = \sum_{q\sim Q}\sum_{\chi}\sum_{n=1}^{\infty} \frac{b(n)}{n^\beta}\chi(n)\, h_1\left(\frac{n}{Y}\right),$$

$$T_2 = \sum_{q\sim Q}\sum_{\chi} \frac{\tau(\chi)^4 q^{-4\beta}}{\pi^{2-4\beta}} \sum_{n=1}^{\infty} \frac{b(n)}{n^{1-\beta}} \overline{\chi(n)}\, h_2\left(\frac{nY\pi^4}{q^4}\right).$$

By Lemma 5, we can rewrite T_1 as

$$T_1 = \sum_{q\sim Q} \frac{q-1}{2} \sum_{n\equiv \pm 1 \pmod q} \frac{b(n)}{n^\beta} h_1\left(\frac{n}{Y}\right) - \sum_{q\sim Q}\sum_{(n,q)=1} \frac{b(n)}{n^\beta} h_1\left(\frac{n}{Y}\right).$$

In the end we will choose Y to be very large, and so we use that $h_1\left(\frac{1}{Y}\right) \to 1$ as $Y\to\infty$. The contribution to T_1 from the term $n=1$ is thus

$$\sum_{q\sim Q} \frac{q-1}{2} h_1\left(\frac{1}{Y}\right) = \sum_{q\sim Q}\frac{q-1}{2}.$$

We write the sum over $n \equiv 1 \pmod q, n\neq 1$ as

$$\sum_{q\sim Q}\frac{q-1}{2}\sum_{d=1}^{\infty} \frac{b(1+dq)}{(1+dq)^\beta} h_1\left(\frac{1+dq}{Y}\right).$$

The number of divisors of a positive integer n, as n gets large, is bounded by $O_\epsilon(n^\epsilon)$ for any $\epsilon>0$, see *e.g* Theorem 315 in [8]. Thus the number of representations of n of the form $n = 1+dq$ with $d,q\geq 1$ is bounded by $O_\epsilon(n^\epsilon)$ for any $\epsilon>0$. Accordingly we have

$$\sum_{q\sim Q}\frac{q-1}{2}\sum_{d=1}^{\infty} \frac{b(1+dq)}{(1+dq)^\beta} h_1\left(\frac{1+dq}{Y}\right) \ll Q\sum_{n=1}^{\infty} n^\epsilon \frac{b(n)}{n^\beta}\left|h_1\left(\frac{n}{Y}\right)\right|.$$

For large enough Y we deduce from the properties of h_1 and from Lemma 15 that

$$\sum_{n=1}^{\infty} n^{-\beta+\epsilon} b(n) \left|h_1\left(\frac{n}{Y}\right)\right| = Y^{-\beta+\epsilon}\sum_{1\leq n\leq Y} b(n) \ll Y^{1-\beta+\epsilon}.$$

The remaining terms in T_1 are bounded similarly and we find that

$$T_1 = \sum_{q\sim Q}\frac{q-1}{2} + O(Q\,Y^{1-\beta+\epsilon}). \tag{27}$$

We now consider the term T_2. It contains the sum

$$\sum_{\substack{\chi \pmod q,\\ \chi\neq\chi_0,\, \chi(-1)=1}} \overline{\chi(n)}\tau(\chi)^4$$

which is zero if $q \mid n$ and otherwise equals by Lemma 6

$$\frac{q-1}{2}\left(K_4(n,q)+K_4(-n,q)\right)-1.$$

It is a consequence of Deligne's proof of the Weil conjectures that the hyper-Kloosterman sum $K_d(n,q)$ is bounded by

$$K_d(n,q) \ll q^{\frac{d-1}{2}}.$$

Hence if $q \nmid n$ we get

$$\sum_{\substack{\chi \pmod q,\\ \chi\neq\chi_0,\,\chi(-1)=1}} \overline{\chi(n)}\tau(\chi)^4 \ll q^{\frac{5}{2}}.$$

We can now bound the contribution of T_2. We have

$$\begin{aligned}
T_2 &= \sum_{q\sim Q}\sum_{\chi}\frac{\tau(\chi)^4 q^{-4\beta}}{\pi^{2-4\beta}}\sum_{n=1}^{\infty}\frac{b(n)}{n^{1-\beta}}\overline{\chi(n)}\,h_2\left(\frac{nY\pi^4}{q^4}\right)\\
&\ll \sum_{q\sim Q}\frac{q^{\frac{5}{2}}q^{-4\beta}}{\pi^{2-4\beta}}\sum_{(n,q)=1}\frac{b(n)}{n^{1-\beta}}\left|h_2\left(\frac{nY\pi^4}{q^4}\right)\right|\\
&\ll \sum_{q\sim Q}\frac{q^{\frac{5}{2}}q^{-4\beta}}{\pi^{2-4\beta}}\int_1^{\infty}\left|h_2\left(\frac{rY\pi^4}{q^4}\right)\right|r^{\beta}\frac{\mathrm{d}r}{r}\\
&= \sum_{q\sim Q}\frac{q^{\frac{5}{2}}}{\pi^2 Y^{\beta}}\int_{\frac{Y\pi^4}{q^4}}^{\infty}|h_2(r)|r^{\beta}\frac{\mathrm{d}r}{r}\\
&\ll \sum_{q\sim Q}\frac{q^{\frac{5}{2}}}{Y^{\beta}}\\
&\ll Q^{1+\frac{5}{2}}Y^{-\beta}.
\end{aligned} \tag{28}$$

From the upper bounds (27) and (28) it follows that

$$\begin{aligned}
\sum_{q\sim Q}\sum_{\chi}L_{f_\chi\times f}(\beta) &= \sum_{q\sim Q}\frac{q-1}{2}+O(QY^{1-\beta+\epsilon}+Q^{1+\frac{5}{2}}Y^{-\beta})\\
&= \frac{1}{2}\sum_{q\sim Q}q+O(QY^{1-\beta+\epsilon}+Q^{1+\frac{5}{2}}Y^{-\beta})+O(Q).
\end{aligned}$$

Choosing $Y=Q^{\frac{5}{2}}$ results in

$$\sum_{q\sim Q}\sum_{\chi}L_{f_\chi\times f}(\beta)=\frac{1}{2}\sum_{q\sim Q}q+O(Q^{1+\frac{5}{2}(1-\beta+\epsilon)})+O(Q).$$

The prime number theorem yields

$$\sum_{q\sim Q}q \approx \frac{(2Q)^2}{2\log(2Q)}-\frac{Q^2}{2\log(Q)}\approx\frac{3}{2}\frac{Q^2}{\log(Q)}.$$

For $\beta\in(1-\frac{2}{5},1)$ and Q sufficiently large, the term $\sum_{q\sim Q}q\approx Q^2/\log(Q)$ dominates the O-terms. In particular, the average $\sum_{q\sim Q}\sum_{\chi}L_{f_\chi\times f}(\beta)$ is non-zero and therefore $L_{f_\chi\times f}(\beta)$ is non-zero for some χ. This completes the proof of the lemma.

□

4 The case of $\Gamma_0(N)$

Throughout this section we fix a positive integer N, which we assume to be square-free. Let

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) \,|\, c \equiv 0 \pmod{N} \right\}.$$

Denote the space of Maass cusp forms of level N by $\mathcal{C}(\Gamma_0(N)\setminus\mathbb{H})$.

4.1 Hecke operators and Atkin-Lehner theory for $\Gamma_0(N)$

The Hecke operator T_n on $\mathcal{C}(\Gamma_0(N)\setminus\mathbb{H})$ is defined by the formula

$$(T_n f)(z) = \frac{1}{\sqrt{n}} \sum_{\substack{ad=n \\ (a,N)=1}} \sum_{b=0}^{d-1} f\left(\frac{az+b}{d}\right).$$

There is an orthonormal basis of $\mathcal{C}(\Gamma_0(N)\setminus\mathbb{H})$ which consists of eigenfunctions of all the Hecke operators T_n, for $(n,N)=1$. Such an eigencuspform f thus satisfies

$$T_n f = \lambda(n) f, \quad \text{if}(n,N) = 1. \tag{29}$$

We wish to have a basis of eigenfunctions of all the Hecke operators without exceptions. This becomes possibly after putting aside certain Maass cusp forms that come from lower levels. This theory of newforms was developed originally for modular forms, but the results also apply in the context of Maass forms, see *e.g.* [9] for a review.

The space of newforms $\mathcal{C}(\Gamma_0(N)\setminus\mathbb{H})^{\mathrm{new}}$ is the orthogonal complement of the (old) subspace generated by all forms $f(dz)$, where $d|N$ and where $f \in \mathcal{C}(\Gamma_0(N')\setminus\mathbb{H})$ for some $N' < N$ such that $N'd|N$. For f such a newform, Equation (29) holds for all n. Furthermore, the first coefficient in the Fourier expansion of f does not vanish, so f may be normalized by setting $a(1) = 1$. In that case, for all n, we have $a(n) = \lambda(n)$.

An important role plays the Atkin-Lehner operator W_p on $\mathcal{C}(\Gamma_0(N)\setminus\mathbb{H})$. Let p be a prime divisor of N satisfying $(p, N/p) = 1$. Let a,b,c,t be any integers such that

$$W_p = \begin{pmatrix} pa & b \\ Nc & pt \end{pmatrix} \tag{30}$$

has determinant p. The matrix W_p normalizes the group $\Gamma_0(N)$, and induces a linear operator on the space of cusp forms $\mathcal{C}(\Gamma_0(N)\setminus\mathbb{H})$ through $W_p f(z) := f(W_p z)$.

If f is a Maass cusp eigenform, then it is also an eigenfunction of the operator W_p with eigenvalue ξ_p a complex number of modulus one, that is,

$$W_p f = \xi_p f, \quad \text{and } |\xi_p| = 1.$$

The eigenvalue ξ_p will appear in the functional equation for L-functions that are associated to f.

4.2 Twisted Maass Forms for $\Gamma_0(N)$

Let

$$f(z) = \sum_{n\neq 0} a(n) y^{1/2} K_{\nu-\frac{1}{2}}(2\pi|n|y) e^{2\pi i n x}$$

be a cuspidal Maass form for $\Gamma_0(N)$ of type ν. Let χ be a primitive Dirichlet character mod a prime q. The twist of f by χ is defined by

$$f_\chi(z) = \sum_{n\neq 0} a(n)\chi(n) y^{1/2} K_{\nu-\frac{1}{2}}(2\pi|n|y) e^{2\pi i n x}. \tag{31}$$

Lemma 27. *The twist of a cuspidal Maass form for $\Gamma_0(N)$ is an automorphic form for $\Gamma_0(q^2N)$ with nebentypus χ^2. That is, for all $\gamma \in \Gamma_0(q^2N)$, we have*

$$f_\chi(\gamma z) = \chi(\gamma)^2 f_\chi(z).$$

Proof. By Lemma 3, the Gauss sum $\tau(\overline{\chi})$ is not zero. Then Lemma 2 allows us to write

$$f_\chi(z) = \frac{1}{\tau(\overline{\chi})} \sum_{l=1}^{q} \overline{\chi(l)} f(z + \frac{l}{q}).$$

Now let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(q^2N)$. Then

$$\begin{pmatrix} 1 & l/q \\ 0 & 1 \end{pmatrix} \gamma \in \Gamma_0(N) \begin{pmatrix} 1 & d^2l/q \\ 0 & 1 \end{pmatrix}.$$

Thus for all $z \in \mathbb{H}$, the point $\gamma z + \frac{l}{q}$ lies in the $\Gamma_0(N)$-orbit of $z + \frac{d^2 l}{q}$. It follows that

$$\begin{aligned}
f_\chi(\gamma z) &= \frac{1}{\tau(\overline{\chi})} \sum_{l=1}^{q} \overline{\chi(l)} f(\gamma z + \frac{l}{q}) \\
&= \frac{1}{\tau(\overline{\chi})} \sum_{l=1}^{q} \overline{\chi(l)} f(z + \frac{d^2 l}{q}) \\
&= \frac{1}{\tau(\overline{\chi})} \sum_{l=1}^{q} \overline{\chi(ld^2)} \chi(d)^2 f(z + \frac{d^2 l}{q}) \\
&= \chi(d)^2 f_\chi(z).
\end{aligned}$$

□

Let W_p be the Atkin-Lehner operator on $\mathcal{C}(\Gamma_0(Nq^2) \setminus \mathbb{H})$. If f is a Maass newform, a similar computation as above gives the following result.

Lemma 28. *The twist f_χ of a newform $f \in \mathcal{C}(\Gamma_0(N) \setminus \mathbb{H})^{\text{new}}$ is an eigenfunction of W_p with eigenvalue $\chi(p)\xi_p$, that is,*

$$W_p f_\chi = \chi(p)\xi_p f_\chi.$$

Proof. As in the previous proof we write

$$f_\chi(z) = \frac{1}{\tau(\overline{\chi})} \sum_{l=1}^{q} \overline{\chi(l)} f(z + \frac{l}{q}).$$

Moreover we have

$$\begin{pmatrix} 1 & l/q \\ 0 & 1 \end{pmatrix} W_p \in \Gamma_0(N) W_p \begin{pmatrix} 1 & lp/q \\ 0 & 1 \end{pmatrix}.$$

Thus for all $z \in \mathbb{H}$, the point $W_p(z) + \frac{l}{q}$ lies in the $\Gamma_0(N)$-orbit of $W_p \begin{pmatrix} 1 & lp/q \\ 0 & 1 \end{pmatrix} z = W_p(z + lp/q)$. It then follows that

$$\begin{aligned}
W_p f_\chi(z) &= \frac{1}{\tau(\overline{\chi})} \sum_{l=1}^{q} \overline{\chi(l)}\, f(W_p(z) + \frac{l}{q}) \\
&= \frac{\chi(p)}{\tau(\overline{\chi})} \sum_{l=1}^{q} \overline{\chi(lp)}\, W_p f(z + \frac{lp}{q}) \\
&= \chi(p)\xi_p \frac{1}{\tau(\overline{\chi})} \sum_{l=1}^{q} \overline{\chi(lp)}\, f(z + \frac{lp}{q}) \\
&= \chi(p)\xi_p f_\chi(z).
\end{aligned}$$

□

4.3 Eisenstein series of level Nq^2 and nebentypus χ

Let q be a prime. In this section we assume that χ is an even, non-principal (and thus primitive) Dirichlet character mod q. For $\mathrm{Re}(s) > 1$ we define the Eisenstein series $E(z,s,\chi)$ of level Nq^2 and with nebentypus χ by the absolutely convergent series

$$E(z,s,\chi) = \frac{1}{2}\sum_{\gamma} \chi(d)\,\mathrm{Im}(\gamma z)^s,$$

where the sum goes over a set of coset representatives $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\infty \setminus \Gamma_0(Nq^2)$. We will sometimes write $\chi(\gamma)$ to denote $\chi(d)$.

The Eisenstein series $E(z,s,\chi)$ satisfies indeed the automorphic relation $E(\gamma z,s,\chi) = \overline{\chi(\gamma)}E(z,s,\chi)$ for all $\gamma \in \Gamma_0(Nq^2)$. It follows in particular that the function $E(z,s,\chi)$ is invariant under $z \mapsto z+1$. Hence it has a Fourier expansion (our assumption that χ is even simplifies the calculation of the Fourier expansion). We determine the Fourier expansion of $E(z,s,\chi)$ by adapting the calculation we did in Section 3.1.3 for the non-holomorphic Eisenstein series $E(z,s)$.

The cosets $\Gamma_\infty \setminus \Gamma_0(Nq^2)$ are determined by the bottom row of a representative

$$\Gamma_\infty \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \left\{ \begin{pmatrix} s & t \\ c & d \end{pmatrix} \,\middle|\, sd - ct = 1, Nq^2|c \right\}.$$

A pair of coprime integers (c,d), subject to $Nq^2|c$, uniquely determines such a coset. To sum over $(m,n) \in \mathbb{Z}^2 \setminus \{0,0\}$ is the same as to sum over all positive integers M and all pairs (c,d) of coprime integers by taking $(m,n) = (Mc, Md)$. As a consequence, we can write

$$\begin{aligned} E^*(z,s,\chi) &:= L(2s,\chi)E(z,s,\chi) \\ &= \frac{L(2s,\chi)}{2} \sum_{\gamma \in \Gamma_\infty \setminus \Gamma_0(Nq^2)} \chi(d)\,\mathrm{Im}(\gamma z)^s \\ &= \frac{L(2s,\chi)}{2} \sum_{\substack{(c,d)=1 \\ Nq^2|c}} \chi(d) \frac{y^s}{|cz+d|^{2s}} \\ &= \frac{1}{2} \sum_{\substack{m,n \in \mathbb{Z} \\ (m,n)\neq 0}} \chi(n) \frac{y^s}{|mNq^2 z + n|^{2s}}. \end{aligned} \tag{32}$$

We would like to adapt the left hand side of Equation 6 to the setting where the sum is twisted by a primitive Dirichlet character χ mod q. For this we need the twisted variant of the Poisson summation formula.

Lemma 29. *Let φ be a function that satisfies the conditions of the Poisson summation formula. Let χ be a primitive Dirichlet character mod q. Then*

$$\sum_{n\in\mathbb{Z}} \chi(n)\varphi(x+n) = \frac{\tau(\chi)}{q} \sum_{n\in\mathbb{Z}} \overline{\chi(n)}\widehat{\varphi}(n/q)e^{2\pi i n x/q}.$$

Proof. From Lemma 2 we know that

$$\chi(n) = \frac{1}{\tau(\overline{\chi})} \sum_{m \bmod q} \overline{\chi(m)} e^{2\pi i n m/q} = \frac{\chi(-1)\tau(\chi)}{q} \sum_{m \bmod q} \overline{\chi(m)} e^{2\pi i n m/q}.$$

Let us now consider $\sum_{n\in\mathbb{Z}} \chi(n)\varphi(n) = \sum_{n\in\mathbb{Z}} \varphi_1(n)$, where

$$\varphi_1(x) = \frac{\chi(-1)\tau(\chi)}{q} \sum_{m \bmod q} \overline{\chi(m)} e^{2\pi i x m/q} \varphi(x).$$

The Fourier transform of $\varphi_1(x)$ is

$$\begin{aligned}\widehat{\varphi_1}(\xi) &= \frac{\chi(-1)\tau(\chi)}{q} \sum_{m \bmod q} \overline{\chi(m)} \int_{-\infty}^{\infty} \varphi(x) e^{-2\pi i x\left(\xi - \frac{m}{q}\right)} \mathrm{d}x \\ &= \frac{\chi(-1)\tau(\chi)}{q} \sum_{m \bmod q} \overline{\chi(m)} \widehat{\varphi}\left(\xi - \frac{m}{q}\right).\end{aligned}$$

We apply Poisson summation formula and thus $\sum_{n\in\mathbb{Z}} \varphi_1(n)$ equals

$$\frac{\chi(-1)\tau(\chi)}{q} \sum_{m \bmod q} \sum_{n=-\infty}^{\infty} \overline{\chi(m)} \widehat{\varphi}\left(\frac{nq-m}{q}\right).$$

We can write $\overline{\chi(m)} = \chi(-1)\overline{\chi(nq-m)}$. When n runs through $\mathbb{Z}$ and m through $\mathbb{Z}/q\mathbb{Z}$, the terms $nq-m$ run uniquely through $\mathbb{Z}$. Thus we have shown

$$\sum_{n\in\mathbb{Z}} \chi(n)\varphi(n) = \frac{\tau(\chi)}{q} \sum_{n\in\mathbb{Z}} \overline{\chi(n)} \widehat{\varphi}(n/q).$$

Replacing $\varphi(n)$ by $\varphi(n+x)$ replaces $\widehat{\varphi}(\xi)$ by $\widehat{\varphi}(\xi)e^{2\pi i \xi x}$. This completes the proof. $\square$

We can now determine the Fourier expansion of $E^*(z,s,\chi)$. We present two calculations, one making directly use of the twisted Poisson summation formula and the other not.

Theorem 30. *The function $E^*(z,s,\chi)$ has the Fourier expansion*

$$E^*(z,s,\chi) = L(2s,\chi)y^s + \frac{2\pi^s \tau(\chi)\sqrt{y}}{q^{4s-1}N^{2s-1}\Gamma(s)} \sum_{t\neq 0} |t|^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|t|y) e^{2\pi i t x} \sum_{qNl|t} \left|\frac{t}{qNl}\right|^{1-2s} \overline{\chi}(l).$$

First Proof. We split up the sum (32) into the terms with $m=0$ and those with $m\neq 0$. The assumption that χ is even allows us to combine each positive summand with its negative. We obtain

$$E^*(z,s,\chi) = L(2s,\chi)y^s + \sum_{m=1}^{\infty} \sum_{n\in\mathbb{Z}} \chi(n) \frac{y^s}{|mNq^2 z + n|^{2s}}.$$

Using the twisted Poisson summation formula (29) in the proof of Lemma 11 allows us to deduce

$$\begin{aligned}E^*(z,s,\chi) &= L(2s,\chi)y^s \\ &\quad + \frac{\tau(\chi)}{q} y^s \frac{2\pi^s}{\Gamma(s)} \sum_{m=1}^{\infty} \sum_{n\neq 0} \overline{\chi(n)} \left(\frac{|n|}{q}\right)^{s-\frac{1}{2}} \left(ymNq^2\right)^{\frac{1}{2}-s} K_{s-\frac{1}{2}}(2\pi|n|ymNq) e^{2\pi i n x m N q}.\end{aligned}$$

Summing $m \in Nq\mathbb{Z}_{\geq 1}$ and $n \in \mathbb{Z}_{\geq 1}$ is the same as summing their product k over $\mathbb{Z}_{\geq 1}$ and summing for each k over the pairs (n, mNq) such that $Nqmn = k$. Accordingly we can write

$$\begin{aligned}E^*(z,s,\chi) &= L(2s,\chi)y^s \\ &\quad + \tau(\chi) q^{-2s} \sqrt{y} \frac{2\pi^s}{\Gamma(s)} \sum_{m=1}^{\infty} \sum_{n\neq 0} \overline{\chi(n)} |n|^{s-\frac{1}{2}} (mNq)^{\frac{1}{2}-s} K_{s-\frac{1}{2}}(2\pi|n|ymNq) e^{2\pi i n x m N q} \\ &= L(2s,\chi)y^s \\ &\quad + \frac{\tau(\chi)}{q^{4s-1}N^{2s-1}} \sqrt{y} \frac{2\pi^s}{\Gamma(s)} \sum_{k\neq 0} |k|^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|k|y) e^{2\pi i k x} \sum_{\substack{m\geq 1 \\ k=mqNn}} |m|^{1-2s} \overline{\chi(n)} \\ &= L(2s,\chi)y^s \\ &\quad + \frac{\tau(\chi)}{q^{4s-1}N^{2s-1}} \sqrt{y} \frac{2\pi^s}{\Gamma(s)} \sum_{k\neq 0} |k|^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|k|y) e^{2\pi i k x} \sum_{Nqn|k} \left|\frac{k}{Nqn}\right|^{1-2s} \overline{\chi(n)}.\end{aligned}$$

$\square$

Second Proof. Again we start with

$$E^*(z,s,\chi) = L(2s,\chi)y^s + \sum_{m=1}^{\infty}\sum_{n\in\mathbb{Z}} \chi(n)\frac{y^s}{|mNq^2z+n|^{2s}}.$$

Using the substitution $n = tmNq^2 + r$ and Lemma 11 gives

$$\begin{aligned}
E^*(z,s,\chi) &= L(2s,\chi)y^s \\
&\quad + \frac{y^s}{q^{4s}N^{2s}} \sum_{m=1}^{\infty} |m|^{-2s} \sum_{r=1}^{mNq^2} \chi(r) \sum_{t\in\mathbb{Z}} \left| t + z + \frac{r}{mNq^2}\right|^{-2s} \\
&= L(2s,\chi)y^s + \frac{y^s}{q^{4s}N^{2s}} \sum_{m=1}^{\infty} |m|^{-2s} \sum_{r=1}^{mNq^2} \chi(r) \\
&\quad \times \left[\frac{\sqrt{\pi}\Gamma(s-\frac{1}{2})}{\Gamma(s)} y^{1-2s} + \frac{2\pi^s}{\Gamma(s)} y^{\frac{1}{2}-s} \sum_{t\neq 0} |t|^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|t|y)\, e^{2\pi i t\left(x + \frac{r}{mNq^2}\right)} \right] \\
&= L(2s,\chi)y^s + \frac{2\pi^s\sqrt{y}}{q^{4s}N^{2s}\Gamma(s)} \sum_{t\neq 0} |t|^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|t|y) e^{2\pi i t x} \sum_{m=1}^{\infty} |m|^{-2s} \sum_{r=1}^{mNq^2} \chi(r) e^{\frac{2\pi i t r}{mNq^2}}.
\end{aligned}$$

In the last equality we used Lemma 1 according to which $\sum_{r=1}^{mNq^2}\chi(r) = \sum_{r=1}^{q}\chi(r) = 0$. By Lemma 4 the last sum equals

$$\sum_{r=1}^{mNq^2} \chi(r) e^{\frac{2\pi i t r}{mNq^2}} = \begin{cases} mNq\tau(\chi)\overline{\chi}(l) & \text{if } t = mqNl \text{ for some } l \in \mathbb{Z} \\ 0 & \text{else} \end{cases}.$$

Therefore we get

$$\begin{aligned}
E^*(z,s,\chi) &= L(2s,\chi)y^s + \frac{2\pi^s\tau(\chi)\sqrt{y}}{q^{4s}N^{2s}\Gamma(s)} \sum_{t\neq 0} |t|^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|t|y) e^{2\pi i t x} \sum_{\substack{m\geq 1 \\ t = mqNl}} |m|^{-2s} mNq\overline{\chi}(l) \\
&= L(2s,\chi)y^s + \frac{2\pi^s\tau(\chi)\sqrt{y}}{q^{4s-1}N^{2s-1}\Gamma(s)} \sum_{t\neq 0} |t|^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|t|y) e^{2\pi i t x} \sum_{qNl|t} \left|\frac{t}{qNl}\right|^{1-2s} \overline{\chi}(l).
\end{aligned}$$

□

Because all terms in the above Fourier expansion are entire, the Fourier expansion gives the analytic continuation to all of $s \in \mathbb{C}$ of the Eisenstein series $E(z,s,\chi)$. We would also like to obtain a functional equation for the twisted Eisenstein series $E(z,s,\chi)$. To this end, we express the completed Eisenstein series

$$E^{\#}(z,s,\chi) := \pi^{-s}\Gamma(s)E^*(z,s,\chi) = \pi^{-s}\Gamma(s)L(2s,\chi)E(z,s,\chi)$$

as the Mellin transform of a theta series. The zeros of $L(2s,\chi)$ cancel with the poles of $\Gamma(s)$, hence $E^{\#}(z,s,\chi)$ is also an entire function.

Theorem 31. *The completed Eisenstein series $E^{\#}(z,s,\chi)$ satisfies the functional equation*

$$E^{\#}(z,s,\chi) = \frac{\tau(\chi)}{\sqrt{q}} N^{1-2s} q^{\frac{5}{2}-5s} E^{\#}\left(\frac{-1}{q^3N^2z}, 1-s, \overline{\chi}\right). \tag{33}$$

Proof. Twisted theta series and Mellin transform. □

4.4 Rankin-Selberg convolution for $\Gamma_0(N)$

Let

$$f(z) = \sum_{n\neq 0} a(n)\sqrt{y}K_{\nu_f}(2\pi|n|y)e^{2\pi i n x}, \tag{34}$$

$$g(z) = \sum_{n\neq 0} b(n)\sqrt{y}K_{\nu_g}(2\pi|n|y)e^{2\pi i n x}, \tag{35}$$

be cuspidal Maass newforms for $\Gamma_0(N)$ of the same parity ϵ. We fix an even, primitive Dirichlet character χ of prime modulus q, and such that $(q, N) = 1$. The character χ induces an imprimitive character mod Nq^2 which we denote by χ'. It is given by

$$\chi'(a) = \begin{cases} \chi(a) & \text{if } (a, N) = 1 \\ 0 & \text{else} \end{cases}.$$

We recall that the twist $f_\chi(z)$, defined by Equation (31), is automorphic for $\Gamma_0(q^2N)$ with nebentypus χ^2. We define the Rankin-Selberg L-function $L_{f_\chi\times g}(s)$ by

$$L_{f_\chi\times g}(s) = L(2s, \chi'^2)\sum_{n=1}^{\infty}\chi(n)\frac{a(n)b(n)}{n^s}.$$

Because the Fourier coefficients $a(n), b(n)$ satisfy the bound $O(\sqrt{|n|})$ (Lemma 14), the above series converges absolutely for $\mathrm{Re}(s) > 2$. We will show this function admits a meromorphic continuation to $s \in \mathbb{C}$ and a functional equation. Key is the following lemma, which constructs the series

$$\sum_{n=1}^{\infty}\chi(n)\frac{a(n)b(n)}{n^s}$$

as an inner product of $f_\chi(z)E(z, s, \chi^2)$ with $\overline{g(z)}$.

Lemma 32. *For* $\mathrm{Re}(s)$ *sufficiently large we have*

$$\int_{\Gamma_0(Nq^2)\backslash\mathbb{H}} E(z, s, \chi^2)f_\chi(z)g(z)\mathrm{d}\mu(z) = \frac{(-1)^\epsilon}{2^3}\frac{1}{\Gamma(s)\pi^s}\left(\prod\Gamma\left(\frac{s\pm\nu\pm\nu'}{2}\right)\right)\sum_{n=1}^{\infty}\chi(n)\frac{a(n)b(n)}{n^s}.$$

Proof. By folding/unfolding it follows that

$$\begin{aligned}\int_{\Gamma_0(Nq^2)\backslash\mathbb{H}} E(z, s, \chi^2)f_\chi(z)g(z)\mathrm{d}\mu(z) &= \frac{1}{2}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(Nq^2)}\int_{\Gamma_0(Nq^2)\backslash\mathbb{H}}\chi(\gamma)^2\,\mathrm{Im}(\gamma z)^s f_\chi(z)g(z)\mathrm{d}\mu(z)\\ &= \frac{1}{2}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(Nq^2)}\int_{\Gamma_0(Nq^2)\backslash\mathbb{H}}\mathrm{Im}(\gamma z)^s f_\chi(\gamma z)g(\gamma z)\mathrm{d}\mu(\gamma z)\\ &= \frac{1}{2}\int_{\Gamma_\infty\backslash\mathbb{H}}\mathrm{Im}(z)^s f_\chi(z)g(z)\mathrm{d}\mu(z)\\ &= \frac{1}{2}\int_0^\infty\int_0^1 y^{s-2}f_\chi(x+iy)g(x+iy)\mathrm{d}x\mathrm{d}y.\end{aligned}$$

For the second equality we used Lemma 27. After inserting the Fourier expansions, the integral over the x-coordinate becomes

$$\begin{aligned}\int_0^1 f_\chi(x+iy)g(x+iy)\mathrm{d}x &= y\sum_{n\neq 0}\chi(n)a(n)b(-n)K_\nu(2\pi|n|y)K_{\nu'}(2\pi|n|y)\\ &= (-1)^\epsilon y\sum_{n\neq 0}\chi(n)a(n)b(n)K_\nu(2\pi|n|y)K_{\nu'}(2\pi|n|y).\end{aligned}$$

Then formula (16) for the Mellin transform of $K_\nu K_{\nu'}$ gives

$$\begin{aligned}\int_{\Gamma_0(Nq^2)\backslash\mathbb{H}} E(z,s,\chi^2)f_\chi(z)g(z)\mathrm{d}\mu(z) &= \frac{(-1)^\epsilon}{2^4}\frac{1}{\Gamma(s)\pi^s}\left(\prod\Gamma\left(\frac{s\pm\nu\pm\nu'}{2}\right)\right)\sum_{n\neq0}\chi(n)\frac{a(n)b(n)}{|n|^s}\\ &= \frac{(-1)^\epsilon}{2^3}\frac{1}{\Gamma(s)\pi^s}\left(\prod\Gamma\left(\frac{s\pm\nu\pm\nu'}{2}\right)\right)\sum_{n=1}^{\infty}\chi(n)\frac{a(n)b(n)}{n^s},\end{aligned}$$

where in the last line we used that f and g are of the same parity. □

To express the L-function $L_{f_\chi\times g}(s)$ as a Rankin-Selberg convolution, we still need to deal with the Dirichlet L-series $L(2s,\chi'^2)$. To this end, we note that

$$\begin{aligned}\pi^{-s}\Gamma(s)L(2s,\chi')E(z,s,\chi) &= \pi^{-s}\Gamma(s)\frac{1}{2}\sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq0}}\chi'(n)\frac{y^s}{|mNq^2z+n|^{2s}}\\ &= \pi^{-s}\Gamma(s)\frac{1}{2}\sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq0\\(n,N)=1}}\chi(n)\frac{y^s}{|mNq^2z+n|^{2s}}\\ &= \pi^{-s}\Gamma(s)\frac{1}{2}\sum_{d|N}\mu(d)\chi(d)\sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq0}}\chi(n)\frac{y^s}{|mNq^2z+dn|^{2s}}\\ &= \pi^{-s}\Gamma(s)\frac{1}{2}\sum_{d|N}\mu(d)\chi(d)d^{-s}\sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq0}}\chi(n)\frac{(y/d)^s}{|mNq^2z/d+n|^{2s}}\\ &= \sum_{d|N}\mu(d)\chi(d)d^{-s}E^{\#}\left(z/d,s,\chi\right),\end{aligned}$$

where $\mu(n)$ is the Möbius function. We can now write the L-function $L_{f_\chi\times g}(s)$ as the Rankin-Selberg convolution

$$\frac{1}{4\pi^{2s}}\prod\Gamma\left(\frac{s\pm\nu\pm\nu'}{2}\right)L_{f_\chi\times g}(s) = (-1)^\epsilon\sum_{d|N}\mu(d)\chi^2(d)d^{-s}\int_{\Gamma_0(Nq^2)\backslash\mathbb{H}}E^{\#}(z/d,s,\chi^2)f_\chi(z)g(z)\mathrm{d}\mu(z). \tag{36}$$

Lemma 33. *Let d be a divisor of N and p a prime dividing N but not d. Then*

$$\int_{\Gamma_0(Nq^2)\backslash\mathbb{H}}E^{\#}(z/dp,s,\chi^2)f_\chi(z)g(z)\mathrm{d}\mu(z) = \overline{\chi(p)}\xi_p^{-1}\eta_p^{-1}\int_{\Gamma_0(Nq^2)\backslash\mathbb{H}}E^{\#}(z/d,s,\chi^2)f_\chi(z)g(z)\mathrm{d}\mu(z).$$

Proof. ... □

Using Lemma 33 we can write for a divisor d of N

$$\int_{\Gamma_0(Nq^2)\backslash\mathbb{H}}E^{\#}(z/d,s,\chi^2)f_\chi(z)g(z)\mathrm{d}\mu(z) = \chi(N/d)\prod_{p|(N/d)}\xi_p\eta_p\int_{\Gamma_0(Nq^2)\backslash\mathbb{H}}E^{\#}(z/N,s,\chi^2)f_\chi(z)g(z)\mathrm{d}\mu(z) \tag{37}$$

On the other hand, we can write

$$\begin{aligned}\sum_{d|N}\mu(d)\chi(d)d^{-s}\prod_{p|d}\xi_p^{-1}\eta_p^{-1} &= \sum_{d|N}\mu(d)\prod_{p|d}\chi(p)\xi_p^{-1}\eta_p^{-1}p^{-s}\\ &= \prod_{p|N}(1-\chi(p)\xi_p^{-1}\eta_p^{-1}p^{-s}).\end{aligned} \tag{38}$$

We define the completed Rankin-Selberg L-function to be

$$\Lambda_{f_\chi \times g}(s) = \left(\frac{q}{\pi}\right)^{2s} \left(\prod \Gamma\left(\frac{s \pm \nu \pm \nu'}{2}\right)\right) \left(\prod_{p|N} (1 - \chi(p)\xi_p^{-1}\eta_p^{-1}p^{-s})^{-1}\right) L_{f_\chi \times g}(s). \tag{39}$$

Combining Equation (37)-(39) allows us to rewrite $\Lambda_{f_\chi \times g}(s)$ in the form

$$\Lambda_{f_\chi \times g}(s) = (-1)^\epsilon 4q^{2s}\chi(N) \left(\prod_{p|N} \xi_p \eta_p\right) \int_{\Gamma_0(Nq^2)\backslash\mathbb{H}} E^{\#}(z/N, s, \chi^2) f_\chi(z) g(z) \mathrm{d}\mu(z). \tag{40}$$

It now follows from Section 4.3 that the function $s \mapsto \Lambda_{f_\chi \times g}(s)$ is entire. It remains to establish a functional equation for the completed Rankin-Selberg L-function $\Lambda_{f_\chi \times g}(s)$.

Theorem 34. *The function $\Lambda_{f_\chi \times g}(s)$ satisfies the functional equation*

$$\Lambda_{f_\chi \times g}(s) = w(f, g)\chi^2(N) \left(\frac{\tau(\chi)}{\sqrt{q}}\right)^4 N^{1-2s} \Lambda_{\tilde{f}_{\overline{\chi}} \times \tilde{g}}(1 - s), \tag{41}$$

where $w(f, g)$ is a complex number of modulus 1.

Let I denote the integral on the RHS of (40). To prove Theorem 34, we apply the functional equation (33) of the Eisenstein series $E^{\#}$ to I. This results in

$$\begin{aligned} I &= \tau(\chi^2)N^{1-2s}q^{2-5s} \int_{\Gamma_0(Nq^2)\backslash\mathbb{H}} E^{\#}(-1/Nq^3 z, 1 - s, \overline{\chi}^2) f_\chi(z) g(z) \mathrm{d}\mu(z) \\ &= \tau(\chi^2)N^{1-2s}q^{2-5s} J, \end{aligned}$$

where

$$J = \int_{\Gamma_0(Nq^2)\backslash\mathbb{H}} E^{\#}(z/q, 1 - s, \overline{\chi}^2) f_\chi(-1/Nq^2 z) g(-1/Nq^2 z) \mathrm{d}\mu(z).$$

Here we made the change of variables $z \mapsto \begin{pmatrix} 0 & -1/(q\sqrt{N}) \\ q\sqrt{N} & 0 \end{pmatrix} z$.

The rest of the proof of the functional equation of $\Lambda_{f_\chi \times g}(s)$ consists in deriving a managable expression for J. The next lemma allows us to get rid of the dominator q in the Eisenstein series inside of J.

Lemma 35. *We have*

$$E^{\#}(z/q, s, \overline{\chi}^2) = \sum_{u \bmod q} E^{\#}(z, s, \overline{\chi}^2) \circ \begin{pmatrix} 1 & 0 \\ Nq^2 & 1 \end{pmatrix}^u.$$

Proof. ... ☐

Lemma 36. *We have*

$$J = q^{s-1}\chi(N) \left(\frac{\tau(\chi)}{\sqrt{q}}\right)^4 \tau(\overline{\chi}^2) \int_{\Gamma_0(Nq^2)\backslash\mathbb{H}} \tilde{f}_{\overline{\chi}}(z)\tilde{g}(z) E^{\#}(z, 1 - s, \overline{\chi}^2) \mathrm{d}\mu(z). \tag{42}$$

Proof. Many steps. ☐

Proof of Theorem 34. Applying Lemma 33 inductively gives

$$\begin{aligned} &\int_{\Gamma_0(Nq^2)\backslash\mathbb{H}} \tilde{f}_{\overline{\chi}}(z)\tilde{g}(z) E^{\#}(z, 1 - s, \overline{\chi}^2) \mathrm{d}\mu(z) \\ &\qquad = \overline{\chi(N)} \left(\prod_{p|N} \overline{\xi}_p \overline{\eta}_p\right) \int_{\Gamma_0(Nq^2)\backslash\mathbb{H}} \tilde{f}_{\overline{\chi}}(z)\tilde{g}(z) E^{\#}(z/N, 1 - s, \overline{\chi}^2) \mathrm{d}\mu(z). \end{aligned}$$

This last identity together with (40) and (42) imply that

$$\begin{aligned}
\Lambda_{f_\chi \times g}(s) &= (-1)^\epsilon 4 q^{2s} \chi(N) \left(\prod_{p|N} \xi_p \eta_p \right) \tau(\chi^2) N^{1-2s} q^{2-5s} J \\
&= \frac{1}{q} \chi^2(N) \tau(\chi^2) \tau(\overline{\chi}^2) N^{1-2s} \left(\frac{\tau(\chi)}{\sqrt{q}} \right)^4 \left(\prod_{p|N} \overline{\xi}_p \overline{\eta}_p \right) \Lambda_{\tilde{f}_{\overline{\chi}} \times \tilde{g}}(1-s) \\
&= w(f,g) \chi^2(N) N^{1-2s} \left(\frac{\tau(\chi)}{\sqrt{q}} \right)^4 \Lambda_{\tilde{f}_{\overline{\chi}} \times \tilde{g}}(1-s).
\end{aligned}$$

Here, using Lemma 3,

$$w(f,g) = \frac{1}{q} \tau(\chi^2) \tau(\overline{\chi}^2) \left(\prod_{p|N} \overline{\xi}_p \overline{\eta}_p \right) = \prod_{p|N} \overline{\xi}_p \overline{\eta}_p$$

is a complex number of modulus one. This completes the proof of the theorem. □

4.5 The Luo-Rudnick-Sarnak Theorem for $\Gamma_0(N)$

Let

$$f(z) = \sum_{n \neq 0} a(n)\sqrt{y}K_\nu(2\pi|n|y)e^{2\pi inx} \tag{43}$$

be a Maass cusp form for $\Gamma_0(N)$. The Selberg eigenvalue conjecture asserts that $\frac{1}{4} - \nu^2 \geq \frac{1}{4}$, or equivalently that $\mathrm{Re}(\nu) = 0$. In this section we will prove the Luo-Rudnick-Sarnak Theorem for $\Gamma_0(N)$, providing the best current lower bound towards the Selberg eigenvalue conjecture.

Theorem 37. *Lef f be a Maass cusp form for $\Gamma_0(N)$ as above. Then we have*

$$\mathrm{Re}(\nu) \leq \frac{1}{2} - \frac{1}{5}. \tag{44}$$

The proof of Theorem 37 consists in translating the proof of Theorem 24 into the $\Gamma_0(N)$ setting. Without loss of generality we may assume f to be a Maass cusp newform. We fix an even, primitive Dirichlet character χ mod a prime q, with $(q, N) = 1$. In Section 4.4 we have shown that the completed Rankin-Selberg L-function

$$\begin{aligned}\Lambda_{f_\chi \times \overline{f}}(s) = \left(\frac{q}{\pi}\right)^{2s} \Gamma\left(\frac{s + 2\,\mathrm{Re}(\nu)}{2}\right)\Gamma\left(\frac{s - 2\,\mathrm{Re}(\nu)}{2}\right)\Gamma\left(\frac{s + 2\,\mathrm{Im}(\nu)}{2}\right)\\ \times \Gamma\left(\frac{s + 2\,\mathrm{Im}(\nu)}{2}\right)\left(\prod_{p|N}(1 - \chi(p)|\xi_p|^{-2}p^{-s})^{-1}\right) L_{f_\chi \times \overline{f}}(s)\end{aligned} \tag{45}$$

is entire. Key to prove Theorem 24 was the existence of Rankin-Selberg L-functions $L_{f_\chi \times \overline{f}}(s)$ that have no zeros in predescribed regions. This result also holds in the context of $\Gamma_0(N)$-Maass newforms.

Lemma 38. *Let f be a Maass form as above. Then for any real number $\beta > 1 - \frac{2}{5}$, there are even primitive Dirichlet characters χ such that*

$$L_{f_\chi \times f}(\beta) \neq 0.$$

As in the $\mathrm{SL}(2, \mathbb{Z})$ setting, one shows that Theorem 38 implies Theorem 37. The proof of Theorem 38 also goes along the same lines as the proof of Theorem 25. In that proof, one simply has to replace the functional equation (21), valid for the $\mathrm{SL}(2, \mathbb{Z})$ setting, with the corresponding functional equation (41) that holds in the $\Gamma_0(N)$ setting.

5 References

1. Bergeron, N. *The Spectrum of Hyperbolic Surfaces* (Springer, 2016).
2. Goldfeld, D. *Automorphic Forms and L-Functions for the Group GL(n,R)* (Cambridge University Press, 2006).
3. Watson, G. N. *A Treatise on the Theory of Bessel Functions* (Cambridge University Press, 1966).
4. Bateman, H. & Erdélyi, A. *Tables of Integral Transforms (Volume II)* (McGraw Hill, New York, 1954).
5. Macdonald, I. *Symmetric Functions and Hall Polynomials* (Oxford University Press, 1979).
6. Luo, W., Rudnick, Z. & Sarnak, P. On Selberg's eigenvalue conjecture. *Geom. Funct. Anal.* **5,** 387–401. (1995).
7. Luo, W., Rudnick, Z. & Sarnak, P. On the generalized Ramanujan conjecture for GL(n). *Proceedings of Symposia in Pure Mathematics* **66.2** (1999).
8. Hardy, G. & Wright, E. *An introduction to the theory of numbers (Sixth ed.)* (eds Heath-Brown, D. R & Silverman, J. H.) (Oxford University Press, 2008).
9. Kowalski, E., Michel, P. & VanderKam, J. Rankin-Selberg L-functions in the level aspect. *Duke Mathematical Journal* **114,** 123 –191. `https://doi.org/10.1215/S0012-7094-02-11416-1` (2002).