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1 Dirichlet characters

Let m be a positive integer. A Dirichlet character of modulus m is a function $\chi : \mathbb{Z} \to \mathbb{C}$ such that

- $\chi(ab) = \chi(a)\chi(b)$ for all $a, b \in \mathbb{Z}$,
- $\chi(a) = 0$ if and only if $(a, m) > 1$,
- If $a \equiv b \pmod{m}$, then $\chi(a) = \chi(b)$.

Let χ_1 and χ_2 be Dirichlet characters of modulus m_1 and m_2, respectively, with $m_1|m_2$. If $\chi_2(a) = \chi_1(a)$ for $a \in (\mathbb{Z}/m_2\mathbb{Z})^\times$, then χ_2 is said to be induced by χ_1. A Dirichlet character is called primitive if it is not induced by any Dirichlet character other than itself. A Dirichlet character induced by the identity function is called principal. We denote a principal Dirichlet character by χ_0. By definition a principal Dirichlet character mod m is

$$\chi_0(a) = \begin{cases} 1 & \text{if } (a, m) = 1 \\ 0 & \text{else} \end{cases} . \tag{1}$$

If p is prime, then every nonprincipal Dirichlet character of modulus p is primitive. We call a Dirichlet character even if $\chi(-1) = 1$, odd if $\chi(-1) = -1$.

Lemma 1. *Let χ be a nonprincipal Dirichlet character modulo a prime q. Then we have*

$$\sum_{a=1}^{q} \chi(a) = 0.$$

Proof. Because χ is not principal, there is an integer b such that $\chi(b) \neq \{0, 1\}$. Furthermore, the sum may be restricted to the terms with $(a, q) = 1, 1 \leq a \leq q$. Multiplication by b is a bijection $(\mathbb{Z}/q\mathbb{Z})^\times \to (\mathbb{Z}/q\mathbb{Z})^\times$. Therefore we have

$$\chi(b) \sum_{a=1}^{q} \chi(a) = \sum_{a \in (\mathbb{Z}/q\mathbb{Z})^\times} \chi(ab) = \sum_{c \in (\mathbb{Z}/q\mathbb{Z})^\times} \chi(c),$$

which implies $\sum_{a=1}^{q} \chi(a) = 0$. □

The sum of the n-th roots of unity is zero. We will investigate more closely the twist of this sum by a Dirichlet character mod q, a so-called Gauss sum. More precisely, the Gauss sum $\tau(\chi)$ attached to a primitive Dirichlet character χ mod q is

$$\tau(\chi) = \sum_{a \bmod q} \chi(a) e^{2\pi i a/q}.$$

Lemma 2. *Let χ be a nonprincipal Dirichlet character modulo a prime q. Then for all $n \in \mathbb{Z}$ we have*

$$\tau(\chi)\overline{\chi(n)} = \sum_{a=1}^{q} \chi(a) e^{2\pi i n a/q}.$$

Proof. See the second half of the proof of Lemma 4. □

Lemma 3. *Let χ be a primitive Dirichlet character modulo a prime q. Then*

$$|\tau(\chi)| = \sqrt{q}.$$

Proof. By Lemma 2 we have

$$\begin{aligned}
|\tau(\chi)|^2 &= \sum_{a=1}^{q} \overline{\chi(a)} e^{-2\pi i a/q} \tau(\chi) \\
&= \sum_{a=1}^{q} e^{-2\pi i a/q} \left(\sum_{b=1}^{q} \chi(b) e^{2\pi i a b/q} \right) \\
&= \sum_{b=1}^{q} \chi(b) \left(\sum_{a=1}^{q} e^{2\pi i a(b-1)/q} \right).
\end{aligned}$$

If $b = 1$, the inner sum equals q. If $b \neq 1$, the inner sum is zero. Therefore we obtain

$$|\tau(\chi)|^2 = \chi(1) q = q.$$

□

We will also need the following variant of Lemma 2.

Lemma 4. *Let χ be a nonprincipal Dirichlet character modulo a prime q, and let $n, m \in \mathbb{Z}$. Then*

$$\sum_{a=1}^{nq^2} \chi(a) e^{2\pi i m a/nq^2} = \begin{cases} qn\tau(\chi)\overline{\chi}(l) & \text{if } m = lnq \text{ for some } l \in \mathbb{Z} \\ 0 & \text{else} \end{cases}. \tag{2}$$

Proof. Every a in the above sum is of the form $a = a_1 + tq$ with $1 \leq a_1 \leq q$ and $0 \leq t < nq$. Thus we can write

$$\sum_{a=1}^{nq^2} \chi(a) e^{2\pi i m a/nq^2} = \sum_{a_1=1}^{q} \chi(a_1) e^{\frac{2\pi i m a_1}{nq^2}} \sum_{t=0}^{nq-1} e^{\frac{2\pi i m t}{nq}}.$$

The sum over t is zero unless $m = lnq$ for some $l \in \mathbb{Z}$ in which case the sum is nq. Therefore

$$\begin{aligned}
\sum_{a=1}^{nq^2} \chi(a) e^{2\pi i m a/nq^2} &= nq \sum_{\substack{a \bmod q \\ m = lnq}} \chi(a) e^{\frac{2\pi i m a}{nq^2}} \\
&= nq \sum_{a \bmod q} \chi(a) e^{\frac{2\pi i l a}{q}}
\end{aligned}$$

If $(l, q) = 1$, then $a \mapsto al$ permutes the residues mod q. In this case we get

$$nq \sum_{a \bmod q} \chi(a) e^{\frac{2\pi i l a}{q}} = nq\tau(\chi)\overline{\chi(l)}.$$

Now suppose that $(l, q) > 1$. Then $\chi(l) = 0$ and we have to show that the left side of Equation 2 vanishes. For this let $l' \in \mathbb{Z}$ be such that $ql' = l$. Then we have

$$\begin{aligned}
nq \sum_{a \bmod q} \chi(a) e^{\frac{2\pi i l a}{q}} &= nq \sum_{a \bmod q} \chi(a) e^{2\pi i l' a} \\
&= nq \sum_{a \bmod q} \chi(a).
\end{aligned}$$

The last sum is zero by Lemma 1. This completes the proof of the lemma.

□

We have the following orthogonality relation.

Lemma 5. *Let $a \in \mathbb{Z}/q\mathbb{Z}$ for q prime. The sum of $\chi(a)$ over all non-trivial even primitive Dirichlet characters mod q is*

$$\sum_{\substack{\chi \ (mod\ q) \\ \chi \neq \chi_0,\, \chi(-1)=1}} \chi(a) = \begin{cases} 0 & \text{if } a \equiv 0 \ (mod\ q) \\ \frac{q-1}{2} - 1 & \text{if } a \equiv \pm 1 \ (mod\ q) \\ -1 & \text{else} \end{cases}.$$

Proof. □

We will also make use of the following identity.

Lemma 6. *Let $n \geq 2$, q be prime and m such that $q \nmid m$. Then we have*

$$\sum_{\substack{\chi \ (mod\ q) \\ \chi \neq \chi_0,\, \chi(-1)=1}} \overline{\chi(m)} \tau(\chi)^n = \frac{q-1}{2} \left(K_n(m,q) + K_n(-m,q) \right) - (-1)^n,$$

where we sum over primitive Dirichlet characters and where $K_n(m,q)$ denotes the hyper-Kloosterman sum

$$K_n(m,q) = \sum_{x_1 x_2 \cdots x_n \equiv m \ (mod\ q)} e^{2\pi i \left(\frac{x_1 + \cdots + x_n}{q} \right)}.$$

Proof. ... □

2 $SL(2,\mathbb{Z})$

2.1 Maass forms for $\mathrm{SL}(2,\mathbb{Z})$

Let $\nu \in \mathbb{C}$. A Maass form of type ν for $\mathrm{SL}(2,\mathbb{Z})$ is a non-zero function on $\mathbb{H}$ which satisfies

- $f(\gamma z) = f(z)$ for all $\gamma \in \mathrm{SL}(2,\mathbb{Z})$ and $z \in \mathbb{H}$;
- $\Delta f = \nu(1-\nu) f$;
- $f(x+iy) = O(y^M)$ for some $M > 0$ as $y \to \infty$.

If in addition

$$\int_0^1 f(x+iy)\mathrm{d}x = 0,$$

we call f a cuspidal Maass form or a Maass cusp form. It is not obvious that there should exists non-constant Maass cusp forms. One can prove that there are in fact infinitely many Maass cusp forms, see for example Chapter 4 in [1] or Chapter 4 in [2].

2.1.1 The Fourier expansion of Maass forms

Lemma 7. *Let f be a Maass form of type ν for* $\mathrm{SL}(2,\mathbb{Z})$*. Then $\nu(1-\nu)$ is real and* ≥ 0.

Proof. ... □

Lemma 8. *Let f be a Maass form of type ν for* $\mathrm{SL}(2,\mathbb{Z})$*. Then* $\nu(1-\nu) \geq 3\pi^2/2$.

Proof. Let

$$f(z) = \sum_{n\neq 0} a(n)\sqrt{y}K_{\nu-\frac{1}{2}}(2\pi|n|y)e^{2\pi i n x}$$

be a Maass cusp form of type ν for $\mathrm{SL}(2,\mathbb{Z})$. It is an eigenfunction of the Laplace operator Δ with eigenvalue $\lambda = \nu(1-\nu)$. Let $\mathcal{F}$ denote the standard fundamental domain for the action of $\mathrm{SL}(2,\mathbb{Z})$ on $\mathbb{H}$ and let $\mathcal{F}^* = S(\mathcal{F})$, where $S(z) = -1/z$. Note that $\mathcal{F} \cup \mathcal{F}^* \supset \{z \in \mathbb{H} \mid |x| \leq \frac{1}{2}, y \geq \frac{\sqrt{3}}{2}\}$. Using that $\{e^{2\pi i n x}\}_{n\in\mathbb{Z}}$ is a complete orthogonal system for $L^2([-1/2,1/2])$, we can estimate

$$\begin{aligned}
2\lambda\langle f, f\rangle = 2\langle \nabla f, \nabla f\rangle &= \int_{\mathcal{F}\cup\mathcal{F}^*} \left(\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2\right)\mathrm{d}x\mathrm{d}y \\
&\geq \int_{\frac{\sqrt{3}}{2}}^{\infty}\int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\frac{\partial f}{\partial x}\right)^2 \mathrm{d}x\mathrm{d}y \\
&= 4\pi^2 \int_{\frac{\sqrt{3}}{2}}^{\infty} \sum_{n\neq 0} n^2 \left|a(n)\sqrt{y}K_{\nu-\frac{1}{2}}(2\pi|n|y)\right|^2 \mathrm{d}y \\
&\geq 4\pi^2 \frac{3}{4} \int_{\frac{\sqrt{3}}{2}}^{\infty} \sum_{n\neq 0} \left|a(n)\sqrt{y}K_{\nu-\frac{1}{2}}(2\pi|n|y)\right|^2 \frac{\mathrm{d}y}{y^2} \\
&= 3\pi^2 \int_{\frac{\sqrt{3}}{2}}^{\infty}\int_{-\frac{1}{2}}^{\frac{1}{2}} \left|f(z)\right|^2 \frac{\mathrm{d}x\mathrm{d}y}{y^2} \\
&\geq 3\pi^2 \int_{\mathcal{F}} \left|f(z)\right|^2 \mathrm{d}\mu(z) = 3\pi^2\langle f, f\rangle.
\end{aligned}$$

It follows that $\lambda \geq 3\pi^2/2$. □

2.1.2 Even and Odd Maass forms

We define the operator $T_{-1} : L^2(\mathrm{SL}(2,\mathbb{Z}) \setminus \mathbb{H}) \to L^2(\mathrm{SL}(2,\mathbb{Z}) \setminus \mathbb{H})$ by

$$(T_{-1}f)(x+iy) = f(-x+iy).$$

2.1.3 The non-holomorphic Eisenstein series

In this section we will introduce a (non-cuspidal) Maass form for $\mathrm{SL}(2,\mathbb{Z})$. The group $\Gamma = \mathrm{SL}(2,\mathbb{Z})$ has only a cusp at infinity. The stabilizer of this cups in Γ is

$$\Gamma_\infty = \left\{ \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \,|\, m \in \mathbb{Z} \right\}.$$

Notice that $\mathrm{Im}(z)$ is Γ_∞-invariant. The Eisenstein series associated to this cusp, defined on $\mathbb{H} \times \mathbb{C}$, is defined by

$$E(z,s) := E_\infty(z,s) = \frac{1}{2} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \mathrm{Im}(\gamma z)^s. \tag{3}$$

For $\mathrm{Re}(s) > 1$, this series converges absolutely and uniformly on compact sets. The Eisenstein series $E(z,s)$ defines an automorphic function with respect to Γ, that is, it satisfies $E(\gamma z, s) = E(z,s)$ for all $\gamma \in \Gamma$. Because $\Delta y^s = s(1-s)y^s$ and because Δ commutes with the Γ-action, the Eisenstein series is also an eigenfunction of the Laplacian

$$\Delta E(z,s) = s(1-s)E(z,s).$$

The Eisenstein series $E(z,s)$ is in fact a Maass form. This will come as a consequence of the Fourier expansion of $E(z,s)$.

Lemma 9. *The Fourier expansion of $E(z,s)$ is given by*

$$E(z,s) = y^s + \phi(s)y^{1-s} + \frac{2\pi^s}{\Gamma(s)\zeta(2s)} y^{1/2} \sum_{n \neq 0} \sigma_{1-2s}(n)|n|^{s-\frac{1}{2}} K_{s-1/2}(2\pi|n|y)e^{2\pi i n x}, \tag{4}$$

where

$$\phi(s) = \pi^{1/2} \frac{\Gamma(s-1/2)}{\Gamma(s)} \frac{\zeta(2s-1)}{\zeta(2s)}.$$

Proof. ... □

A consequence of the Fourier expansion is the meromorphic continuation of $E(z,s)$ to $s \in \mathbb{C}$. Another consequence of the above Fourier expansion is the functional equation

$$E(z,s) = \frac{1}{\phi(1-s)} E(z, 1-s).$$

2.2 Hecke operators for $L^2(\mathrm{SL}(2,\mathbb{Z}) \setminus \mathbb{H})$

For $n \geq 1$, the Hecke operator $T_n : L^2(\mathrm{SL}(2,\mathbb{Z}) \setminus \mathbb{H}) \to L^2(\mathrm{SL}(2,\mathbb{Z}) \setminus \mathbb{H})$ is defined by

$$(T_n f)(z) = \frac{1}{\sqrt{n}} \sum_{ad=n} \sum_{b=0}^{d-1} f\left(\frac{az+b}{d} \right).$$

The Hecke operators T_n $(n = 1, 2, \dots)$ are self-adjoint in $L^2(\mathrm{SL}(2,\mathbb{Z}) \setminus \mathbb{H})$, that is

$$\langle T_n f, g \rangle = \langle f, T_n g \rangle,$$

for all $f, g \in L^2(\mathrm{SL}(2,\mathbb{Z}) \setminus \mathbb{H})$. Given two positive integers n and m, the Hecke operators T_n and T_m commute. In fact, we have the multilicative rule

$$T_m T_n = \sum_{d|(m,n)} T_{mn/d^2}.$$

Furthermore, the Hecke operators commute with T_{-1} and the Laplacian Δ, and T_{-1} and Δ also commute. It follows that the space $L^2(\mathrm{SL}(2,\mathbb{Z}) \setminus \mathbb{H})$ can be simultaneously diagonalized by the operators $\{\Delta, T_n |\, n = -1, 1, 2, \dots\}$. Consequently, we can consider even or odd Maass forms that are simultaneous eigenfunctions for $\{T_n |\, n = 1, 2, \dots\}$. For such a Maass form, the following multiplicative relations of the Fourier coefficients hold.

Lemma 10. *Let*

$$f(z) = \sum_{n \neq 0} a(n)\sqrt{y}K_{\nu-\frac{1}{2}}(2\pi|n|y)e^{2\pi inx}$$

be a cuspidal Maass form of type ν for $SL(2,\mathbb{Z})$ which is an eigenfunction of all the Hecke operators. Assume that f is normalized so that $a(1) = 1$. Then

$$T_n f = a(n)f, \qquad \forall n = 1, 2, \dots$$

Furthermore, we have the multiplicative relations of the Fourier coefficients

$$\begin{aligned} a(m)a(n) &= a(mn), \qquad \text{if } (m,n) = 1, \\ a(m)a(n) &= \sum_{d|(m,n)} a\left(\frac{mn}{d^2}\right), \\ a(p^{k+1}) &= a(p)a(p^k) - a(p^{k-1}), \end{aligned}$$

for all primes p and all $k \geq 1$.

Proof. ... $\square$

2.3 L-functions

2.3.1 L-functions associated to Maass forms

Let

$$f(z) = \sum_{n \neq 0} a(n)\sqrt{y}K_{\nu_f}(2\pi|n|y)e^{2\pi inx} \tag{5}$$

be a cuspidal Maass form for $SL(2,\mathbb{Z})$.

Lemma 11. *The coefficients $a(n)$ in the Fourier expansion of $f(z)$ satisfy*

$$a(n) = O(\sqrt{|n|}).$$

Proof. Because f is a cusp form, it is bounded as $\mathrm{Im}(z) \to \infty$. Thus

$$\left| a(n)\sqrt{y}K_{\nu_f}(2\pi|n|y) \right| = \left| \int_0^1 f(x+iy)e^{-2\pi inx}\mathrm{d}x \right| \leq \int_0^1 |f(x+iy)|\mathrm{d}x \leq C,$$

for some constant C (that depends on f). If we choose $y = \frac{1}{|n|}$, the lemma is proved. $\square$

The coefficients $a(n)$ behave like a constant on average.

Lemma 12. *For Y sufficiently large we have*

$$\sum_{1 \leq n \leq Y} |a(n)|^2 \ll Y.$$

Proof. ... $\square$

For $\mathrm{Re}(s) \geq \frac{3}{2}$ we define the L-function $L_f(s)$ associated to $f(z)$ by the absolutely convergent series

$$L_f(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}. \tag{6}$$

Lemma 13. *Let $f(z)$ be a cuspidal Maass form for $SL(2,\mathbb{Z})$ of parity ϵ. Then its L-function $L_f(s)$ can be meromorphically continued to all $\mathbb{C}$ and it satisfies the functional equation*

$$\Lambda_f(s) = \pi^{-s}\Gamma\left(\frac{s+\epsilon+\nu_f}{2}\right)\Gamma\left(\frac{s+\epsilon-\nu_f}{2}\right)L_f(s) = (-1)^{\epsilon}\Lambda_f(1-s).$$

Proof. The idea of the proof is to set $x = 0$ and take the Mellin transform of $f(z)$. We first assume that f is even. For sufficiently large $\mathrm{Re}(s)$ we have

$$\begin{aligned}\int_0^\infty f(iy)y^s \frac{\mathrm{d}y}{y} &= 2\int_0^\infty \sum_{n=1}^\infty a(n)\sqrt{y}K_{\nu_f}(2\pi ny)y^s\frac{\mathrm{d}y}{y} \\ &= \frac{2}{(2\pi)^{s+\frac{1}{2}}}L_f(s+\frac{1}{2})\int_0^\infty K_{\nu_f}(y)y^{s+\frac{1}{2}}\frac{\mathrm{d}y}{y} \\ &= \frac{1}{2\pi^{s+\frac{1}{2}}}L_f(s+\frac{1}{2})\Gamma\left(\frac{s+\frac{1}{2}+\nu_f}{2}\right)\Gamma\left(\frac{s+\frac{1}{2}-\nu_f}{2}\right),\end{aligned}$$

where we used the Mellin transform of the K-Bessel function (see page 127 in [3])

$$\int_0^\infty K_\nu(y)y^s\frac{\mathrm{d}y}{y} = 2^{s-2}\Gamma\left(\frac{s+\nu}{2}\right)\Gamma\left(\frac{s-\nu}{2}\right).$$

The integrals converge absolutely and this gives the meromorphic continuation. Because f is automorphic, we have

$$f(iy) = f(S(iy)) = f(i/y).$$

It follows that

$$\begin{aligned}\int_0^\infty f(iy)y^s\frac{\mathrm{d}y}{y} &= \int_0^1 f(i/y)y^s\frac{\mathrm{d}y}{y} + \int_1^\infty f(iy)y^s\frac{\mathrm{d}y}{y} \\ &= \int_1^\infty f(iy)\left(y^s + y^{-s}\right)\frac{\mathrm{d}y}{y}.\end{aligned}$$

The above integral is invariant under $s \mapsto -s$ and from this the functional equation follows immediately.

If f is odd, the above calculation does not quite work, because $\sum_{n\neq 0} a(n)|n|^{-s} = 0$ and so $\int_0^\infty f(iy)y^s\frac{\mathrm{d}y}{y} = 0$. However, we can deduce the functional equation in the same way as above once we replace in the above calculation $f(iy)$ by

$$\frac{\partial}{\partial x}f(z)\bigg|_{x=0} = 4\pi i\sum_{n=1}^\infty a(n)n\sqrt{y}K_{\nu_f}(2\pi ny).$$

□

Lemma 14. *Let f be a Maass cusp form as in Equation (5) which is an eigenfunction of all the Hecke operators and is normalized so that $a(1) = 1$. Then the L-function $L_f(s) = \sum_{n=1}^\infty \frac{a(n)}{n^s}$ admits the Euler product*

$$L_f(s) = \prod_p \left(1 - a(p)p^{-s} + p^{-2s}\right)^{-1},$$

where the product runs over all primes p.

Proof. It follows from the multiplicativity of the Fourier coefficients (Lemma 10) that

$$\sum_{n=1}^\infty \frac{a(n)}{n^s} = \prod_p\left(\sum_{l=0}^\infty \frac{a(p^l)}{p^{ls}}\right).$$

For a fixed prime p we have

$$\begin{aligned}&\sum_{l=0}^\infty a(p^l)p^{-ls}\left(1 - a(p)p^{-s} + p^{-2s}\right) \\ &\qquad = \sum_{l=0}^\infty a(p^l)p^{-ls} - \sum_{l=1}^\infty a(p^{l-1})a(p)p^{-ls} + \sum_{l=2}^\infty a(p^{l-2})p^{-ls} \\ &\qquad = 1 + \sum_{l=2}^\infty\left(a(p^l) - a(p^{l-1})a(p) + a(p^{l-2})\right)p^{-ls} = 1,\end{aligned}$$

where the last equality follows from Lemma 10. This shows that

$$\sum_{l=0}^{\infty} \frac{a(p^l)}{p^{ls}} = \left(1 - a(p)p^{-s} + p^{-2s}\right)^{-1}.$$

□

2.3.2 L-functions associated to Eisenstein series

By formula (4), the non-constant term in the Fourier expansion of the Eisenstein series $E(z,v)$ is given by

$$\frac{2\pi^v}{\Gamma(v)\zeta(2v)} y^{1/2} \sum_{n\neq 0} \sigma_{1-2v}(n)|n|^{v-\frac{1}{2}} K_{v-1/2}(2\pi|n|y)e^{2\pi inx}.$$

We define the L-function $L_{E_v}(s)$ associated to the Eisenstein series $E(z,v)$ to be

$$L_{E_v}(s) = \sum_{n=1}^{\infty} \frac{\sigma_{1-2v}(n)\, n^{v-\frac{1}{2}}}{n^s}.$$

Lemma 15. *The L-function $L_{E_v}(s)$ is simply a product of shifted Riemann zeta functions*

$$L_{E_v}(s) = \zeta(s+v-1/2)\,\zeta(s-v+1/2).$$

Furthermore, we have the functional equation

$$G_{E_v}(s) = G_{E_v}(1-s),$$

where

$$G_{E_v}(s) = \pi^{-s}\Gamma\left(\frac{s+v-\frac{1}{2}}{2}\right)\Gamma\left(\frac{s-v+\frac{1}{2}}{2}\right)\zeta(s+v-1/2)\,\zeta(s-v+1/2).$$

Proof. We calculate

$$\begin{aligned}
\sum_{n=1}^{\infty} \sigma_{1-2v}(n)\, n^{v-\frac{1}{2}-s} &= \sum_{n=1}^{\infty} n^{v-\frac{1}{2}-s} \sum_{d|n} d^{1-2v} \\
&= \sum_{n=1}^{\infty}\sum_{d=1}^{\infty} (dn)^{v-\frac{1}{2}-s} d^{1-2v} \\
&= \sum_{n=1}^{\infty} n^{v-\frac{1}{2}-s} \sum_{d=1}^{\infty} d^{\frac{1}{2}-v-s} \\
&= \zeta(s-v+1/2)\,\zeta(s+v-1/2).
\end{aligned}$$

Now consider

$$\begin{aligned}
G_{E_v}(s) = {} & \pi^{-\frac{s+v-\frac{1}{2}}{2}}\Gamma\left(\frac{s+v-\frac{1}{2}}{2}\right)\zeta(s+v-1/2) \\
& \times \pi^{-\frac{s-v+\frac{1}{2}}{2}}\Gamma\left(\frac{s-v+\frac{1}{2}}{2}\right)\zeta(s-v+1/2).
\end{aligned}$$

The functional equation $G_{E_v}(s) = G_{E_v}(1-s)$ follows directly from the functional equation

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s) \tag{7}$$

of the Riemann zeta function.

□

As expected, the functional equation of $L_{E_v}(s)$ matches the functional equation of a Maass form of even parity (as in Lemma 13). In fact, the functional equation of a Maass form of type ν and of even parity is identical to the functional equation of $L_{E_\nu}(s)$. The reason is that the K-Bessel functions in both Fourier expansions coincide and the proof of the functional equation in Lemma 13 only uses the analytical properties of the K-Bessel function, and does not depend on the arithmetic Fourier coefficients. This gives a method of obtaining functional equations for Maass forms from studying the functional equation of Eisenstein series. We will use this observation several times.

2.3.3 Twisted L-functions associated to Maass forms

Let

$$f(z) = \sum_{n\neq 0} a(n) y^{1/2} K_{\nu-\frac{1}{2}}(2\pi|n|y) e^{2\pi i n x}$$

be a cuspidal Maass form of type ν for $\mathrm{SL}(2,\mathbb{Z})$. Let χ be an even primitive Dirichlet character mod a prime q. The twist of f by χ is defined by

$$f_\chi(z) = \sum_{n\neq 0} a(n)\chi(n) y^{1/2} K_{\nu-\frac{1}{2}}(2\pi|n|y) e^{2\pi i n x}, \tag{8}$$

and the associated L-function by

$$L_{f_\chi}(s) = \sum_{n=1}^{\infty} \frac{\chi(n)a(n)}{n^s}. \tag{9}$$

Following the remark after Lemma 15, the functional equation of the twisted L-function $L_{f_\chi}(s)$ satisfies the functional equation of the twisted L-function $L_{E_v}(s,\chi)$ associated to the Eisenstein series $E(z,v)$. We now determine the functional equation for $L_{E_v}(s,\chi)$.

Including χ in the proof of Lemma 15 shows that the twisted L-function $L_{E_v}(s,\chi)$ is

$$L_{E_v}(s,\chi) = L(s+v-1/2,\chi)\, L(s-v+1/2,\chi),$$

with the Dirichlet L-function $L(s,\chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s}$. The Dirichlet series $L(s,\chi)$ satisfies the functional equation (reference)

$$\Lambda(s,\chi) = \left(\frac{q}{\pi}\right)^{\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) L(s,\chi) = \frac{\tau(\chi)}{\sqrt{q}}\, \Lambda(1-s,\overline{\chi}). \tag{10}$$

From this it follows that the L-function $L_{E_v}(s,\chi)$ satisfies the functional equation

$$\begin{aligned}\Lambda_{E_v}(s,\chi) &= \left(\frac{q}{\pi}\right)^{s} \Gamma\left(\frac{s+v-1/2}{2}\right) \Gamma\left(\frac{s-v+1/2}{2}\right) L_{E_v}(s,\chi) \\ &= \left(\frac{\tau(\chi)}{\sqrt{q}}\right)^2 \Lambda_{E_v}(1-s,\overline{\chi}).\end{aligned}$$

2.4 Rankin-Selberg convolution for $\mathrm{SL}(2,\mathbb{Z})$

Let

$$f(z) = \sum_{n\neq 0} a(n)\sqrt{y} K_{\nu_f}(2\pi|n|y) e^{2\pi i n x}, \tag{11}$$

$$g(z) = \sum_{n\neq 0} b(n)\sqrt{y} K_{\nu_g}(2\pi|n|y) e^{2\pi i n x}, \tag{12}$$

be cuspidal Maass forms for $SL(2,\mathbb{Z})$. Recall that $f(z)$ (resp. $g(z)$) is an eigenfunction for the Laplacian with eigenvalue $1/4-\nu_f$ (resp. $1/4-\nu_g$). For $\mathrm{Re}(s) > 2$ we define the Rankin-Selberg convolution $L_{f\times g}(s)$ as the absolutely convergent series

$$L_{f\times g}(s) = \zeta(2s) \sum_{n=1}^{\infty} \frac{a(n)\overline{b(n)}}{n^s}.$$

We will prove that $L_{f\times g}$ can be expressed as an inner product of $f\overline{g}$ with an Eisenstein series. This construction gives the meromorphic continuation and functional equation for $L_{f\times g}$.

Theorem 16. *Let $f(z)$ and $g(z)$ be cuspidal Maass forms as in Equation 11 and 12. Then $L_{f\times g}(s)$ can be meromorphically continued to all $s\in\mathbb{C}$ with at most a simple pole at $s=1$. Furthermore, we have the functional equation*

$$\Lambda_{f\times g}(s)=\pi^{-2s}G_{\nu_f,\nu_g}(s)L_{f\times g}(s)=\Lambda_{f\times g}(1-s),$$

where $G_{\nu_f,\nu_g}(s)=\prod\Gamma\left(\frac{s\pm\nu_f\pm\nu_g}{2}\right)$.

Proof. Let $E(z,s)$ be the non-holomorphic Eisenstein series as defined in Equation 3. For sufficiently large $\mathrm{Re}(s)$, we have

$$\begin{aligned}
\zeta(2s)\langle f\overline{g},E(\cdot,\overline{s})\rangle &= \zeta(2s)\int_{SL(2,\mathbb{Z})\backslash\mathbb{H}} f(z)\overline{g(z)\overline{E(z,\overline{s})}}\mathrm{d}\mu(z)\\
&= \zeta(2s)\sum_{\gamma\in\Gamma_\infty\backslash SL(2,\mathbb{Z}}\int_{SL(2,\mathbb{Z})\backslash\mathbb{H}} f(\gamma z)\overline{g(\gamma z)}\,\mathrm{Im}(\gamma z)^s\mathrm{d}\mu(\gamma z)\\
&= \zeta(2s)\int_{\Gamma_\infty\backslash\mathbb{H}} f(z)\overline{g(z)}\,\mathrm{Im}(z)^s\mathrm{d}\mu(z)\\
&= \zeta(2s)\int_0^\infty\int_0^1 f(z)\overline{g(z)}y^{s-2}\mathrm{d}x\mathrm{d}y\\
&= \zeta(2s)\sum_{n,m\neq 0}a(n)\overline{b(m)}\int_0^\infty K_{\nu_f}(2\pi|n|y)K_{\nu_g}(2\pi|m|y)y^{s-1}\int_0^1 e^{2\pi i(n-m)x}\mathrm{d}x\mathrm{d}y\\
&= \zeta(2s)\sum_{n\neq 0}a(n)\overline{b(n)}\int_0^\infty K_{\nu_f}(2\pi|n|y)K_{\nu_g}(2\pi|n|y)y^s\frac{\mathrm{d}y}{y}\\
&= \frac{\zeta(2s)}{(2\pi)^s}\sum_{n\neq 0}\frac{a(n)\overline{b(n)}}{n^s}\int_0^\infty K_{\nu_f}(y)K_{\nu_g}(y)y^s\frac{\mathrm{d}y}{y}\\
&= (2\pi)^{-s}\,L_{f\times g}(s)\int_0^\infty K_{\nu_f}(y)K_{\nu_g}(y)y^s\frac{\mathrm{d}y}{y}.
\end{aligned}$$

The Mellin transform of $K_\nu(y)K_{\nu'}(y)$ is given by (see page 145 in [3])

$$\int_0^\infty K_\nu(y)K_{\nu'}(y)y^s\frac{\mathrm{d}y}{y}=\frac{2^{s-3}}{\Gamma(s)}\prod\Gamma\left(\frac{s\pm\nu\pm\nu'}{2}\right), \tag{13}$$

which is valid for $\mathrm{Re}(s)>|\mathrm{Re}(\nu)|+|\mathrm{Re}(\nu')|$.

From the calculation it follows that the convolution function $L_{f\times g}(s)$ inherits the analytical properties of the Eisenstein series $E(z,s)$. This means that $L_{f\times g}(s)$ can be meromorphically continued on $\mathbb{C}$. Because the Eisenstein series has a simple pole at $s=1$ and the Gamma function no zeros, it follows that $L_{f\times g}(s)$ has a simple pole at $s=1$ if and only if $\langle f,g\rangle\neq 0$. The functional equation follows from the functional equation of the Eisenstein series

$$E^*(z,s)=\pi^{-s}\Gamma(s)\zeta(2s)E(z,s)=E^*(z,1-s).$$

□

Lemma 17. *Let $\alpha_i,\beta_i\in\mathbb{C}$ for $i=1,2$. Then for $x\in\mathbb{C}$, $|x|$ sufficiently small, we have*

$$\prod_{i=1}^2\prod_{j=1}^2(1-\alpha_i\beta_j x)^{-1}=\sum_{k=0}^\infty S_k(\alpha_1,\alpha_2)S_k(\beta_1,\beta_2)x^k(1-\alpha_1\alpha_2\beta_1\beta_2x^2)^{-1},$$

where $S_k(x_1,x_2)$ is the Schur polynomial

$$S_k(x_1,x_2)=\frac{\det\begin{pmatrix}x_1^{k+1} & x_2^{k+1}\\ 1 & 1\end{pmatrix}}{\det\begin{pmatrix}x_1 & x_2\\ 1 & 1\end{pmatrix}}=\frac{x_1^{k+1}-x_2^{k+1}}{x_1-x_2}.$$

Proof. This is proved on page 67 in [4] by evaluating the determinant of the matrix

$$\left(\frac{1}{1-\alpha_i\beta_j x}\right)_{1\le i,j\le 2}$$

in two different ways. □

Theorem 18. *Let $f(z)$ and $g(z)$ be cuspidal Maass forms as in Equation 11 and 12. Assume that $L_f(s)$ and $L_g(s)$ have Euler products*

$$L_f(s)=\prod_p\prod_{i=1}^{2}\left(1-\frac{\alpha_{i,p}}{p^s}\right)^{-1},\quad L_g(s)=\prod_p\prod_{j=1}^{2}\left(1-\frac{\beta_{j,p}}{p^s}\right)^{-1}.$$

Then $L_{f\times g}(s)$ admits the Euler product

$$L_{f\times g}(s)=\prod_p\prod_{i=1}^{2}\prod_{j=1}^{2}\left(1-\frac{\alpha_{i,p}\,\overline{\beta}_{j,p}}{p^s}\right)^{-1}.$$

Proof. By assumption we have

$$L_{f\times g}(s)=\prod_p\frac{\sum_{k=1}^{\infty}a(p^k)\overline{b(p^k)}p^{-ks}}{(1-p^{-2s})}.$$

In view of Lemma 17, after choosing $x=p^{-s}$, it suffices to show that

$$a(p^k)=S_k(\alpha_{1,p},\alpha_{2,p}),\quad b(p^k)=S_k(\beta_{1,p},\beta_{2,p}).$$

The above equalities are obtained inductively from the relations

$$\begin{aligned}1&=\alpha_{1,p}\,\alpha_{2,p},\\ a(p)&=\alpha_{1,p}+\alpha_{2,p},\\ a(p^{k+1})&=a(p)a(p^k)-a(p^{k-1}).\end{aligned}$$

□

Following the remark after Lemma 15, the functional equation in Theorem 16 is identical to the functional equation of the Rankin-Selberg L-function of two Eisenstein series E_v and E_w of type v and w, respectively. We now verify this claim. By Lemma 15, the L-functions $L_{E_v}(s)$ and $L_{E_w}(s)$ are given by

$$\begin{aligned}L_{E_v}(s)&=\prod_{i=1}^{2}\zeta(s-\lambda_i(v))\\ L_{E_w}(s)&=\prod_{i=1}^{2}\zeta(s-\lambda_i(w)),\end{aligned}$$

where $\lambda_1(v)=v-\frac{1}{2}$ and $\lambda_2(v)=-v+\frac{1}{2}$. From the Euler product

$$\zeta(s)=\prod_p\left(1-\frac{1}{p^s}\right)^{-1}$$

and from Theorem 18 we deduce that the Rankin-Selberg convolution of E_v and E_w is

$$\begin{aligned}L_{E_v\times E_w}(s)&=\prod_p\prod_{i=1}^{2}\prod_{j=1}^{2}\left(1-\frac{p^{\lambda_i(v)+\overline{\lambda_j(w)}}}{p^s}\right)^{-1}\\ &=\prod_{i=1}^{2}\prod_{j=1}^{2}\zeta(s-\lambda_i(v)-\overline{\lambda_j(w)}).\end{aligned}$$

It follows from the functional equation (7) for the Riemann Zeta function that the functional equation of $L_{E_v\times E_w}(s)$ is indeed the same as the one stated in Theorem 16. We also note that a corollary of the Rankin-Selberg convolution $L_{E_v\times E_w}(s)$ is the Ramanujan identity

$$\sum_{n=1}^{\infty}\frac{\sigma_a(s)\sigma_b(s)}{n^s}=\frac{\zeta(s)\zeta(s-a)\zeta(s-b)\zeta(s-a-b)}{\zeta(2s-a-b)}.$$

3 $\Gamma_0(N)$

Let $\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) \,|\, c \equiv 0 \pmod N \right\}$.

3.1 Twisted Maass Forms

Let

$$f(z) = \sum_{n \neq 0} a(n) y^{1/2} K_\nu(2\pi|n|y) e^{2\pi i n x}$$

be a cuspidal Maass form for $\mathrm{SL}(2,\mathbb{Z})$. Let χ be a primitive Dirichlet character mod a prime q. The twist of f by χ is defined by

$$f_\chi(z) = \sum_{n \neq 0} a(n)\chi(n) y^{1/2} K_\nu(2\pi|n|y) e^{2\pi i n x}. \tag{14}$$

Lemma 19. *The twist of a cuspidal Maass form for $\Gamma_0(N)$ is an automorphic form for $\Gamma_0(q^2N)$ with character χ^2. That is, for all $\gamma \in \Gamma_0(q^2N)$, we have*

$$f_\chi(\gamma z) = \chi(\gamma)^2 f_\chi(z).$$

Proof. By Lemma 3, the Gauss sum $\tau(\overline{\chi})$ is not zero. Then Lemma 2 allows us to write

$$f_\chi(z) = \frac{1}{\tau(\overline{\chi})} \sum_{a=1}^{q} \overline{\chi(a)} f(z + \frac{a}{q}).$$

Now let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(q^2N)$. Then

$$\begin{pmatrix} 1 & a/q \\ 0 & 1 \end{pmatrix} \gamma \in \Gamma_0(N) \begin{pmatrix} 1 & d^2a/q \\ 0 & 1 \end{pmatrix}.$$

Thus for all $z \in \mathbb{H}$, the point $\gamma z + \frac{a}{q}$ lies in the $\Gamma_0(N)$-orbit of $z + \frac{d^2a}{q}$. It follows that

$$\begin{aligned} f_\chi(\gamma z) &= \frac{1}{\tau(\overline{\chi})} \sum_{a=1}^{q} \overline{\chi(a)} f(\gamma z + \frac{a}{q}) \\ &= \frac{1}{\tau(\overline{\chi})} \sum_{a=1}^{q} \overline{\chi(a)} f(z + \frac{d^2a}{q}) \\ &= \frac{1}{\tau(\overline{\chi})} \sum_{a=1}^{q} \overline{\chi(ad^2)} \chi(d)^2 f(z + \frac{d^2a}{q}) \\ &= \chi(d)^2 f_\chi(z). \end{aligned}$$

□

3.2 Twisted Eisenstein series

Let q be a prime. In this section we assume that χ is an even, non-principal (and thus primitive) Dirichlet character mod q, and that $(q,N) = 1$ (at the moment the last condition is not used.). For $\mathrm{Re}(s) > 1$ we define the twisted Eisenstein series by the absolutely convergent series

$$E(z,s,\chi) = \frac{1}{2} \sum_{\gamma} \chi(d)\, \mathrm{Im}(\gamma z)^s,$$

where the sum goes over a set of coset representatives $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\infty \setminus \Gamma_0(Nq^2)$. We will sometimes write $\chi(\gamma)$ to denote $\chi(d)$.

The Eisenstein series $E(z,s,\chi)$ satisfies the automorphic relation $E(\gamma z,s,\chi)=\overline{\chi(\gamma)}E(z,s,\chi)$ for all $\gamma\in\Gamma_0(Nq^2)$. In particular, the function $E(z,s,\chi)$ is invariant under $z\mapsto z+1$. Hence it has a Fourier expansion (our assumption that χ is even simplifies the calculation of the Fourier expansion). In order to determine its Fourier expansion we first establish a few identities.

The cosets $\Gamma_\infty\setminus\Gamma_0(Nq^2)$ are determined by the bottom row of a representative

$$\Gamma_\infty\begin{pmatrix}a & b\\ c & d\end{pmatrix}=\left\{\begin{pmatrix}s & t\\ c & d\end{pmatrix}\,\middle|\, sd-ct=1, Nq^2|c\right\}.$$

A pair of coprime integers (c,d), subject to $Nq^2|c$, uniquely determines such a coset. To sum over $(m,n)\in\mathbb{Z}\setminus\{0,0\}$ is the same as to sum over all positive integers M and all pairs (c,d) of coprime integers by taking $(m,n)=(Mc,Md)$. As a consequence, we can write

$$\begin{aligned}E^*(z,s,\chi)&:=L(2s,\chi)E(z,s,\chi)\\&=\frac{L(2s,\chi)}{2}\sum_{\gamma\in\Gamma_\infty\setminus\Gamma_0(Nq^2)}\chi(d)\operatorname{Im}(\gamma z)^s\\&=\frac{L(2s,\chi)}{2}\sum_{\substack{(c,d)=1\\ Nq^2|c}}\chi(d)\frac{y^s}{|cz+d|^{2s}}\\&=\frac{1}{2}\sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq 0}}\chi(n)\frac{y^s}{|mNq^2z+n|^{2s}}.\end{aligned}\tag{15}$$

Each summand $\chi(n)\frac{y^s}{|mNq^2z+n|^{2s}}$ in Equation 15 is hit by a summand $\chi(\tilde n)\frac{y^s}{|\tilde m q^2z+\tilde n|^{2s}}$ in Equation 15 when specialized to $N=1$ (choose $\tilde m=Nm$ and $\tilde n=n$). But not all terms in the latter sum appear in the former sum. However, the proof of Theorem 23 shows that these terms do not contribute to the sum (without the twisting by χ this would not be true). As a consequence, the Eisenstein series $E^*(z,s,\chi)$ does in fact not depend on N.

Lemma 20. *If* $\operatorname{Re}(s)>1/2$ *and* $r\in\mathbb{R}$, *then*

$$\int_{-\infty}^{\infty}\frac{e^{-2\pi i r x}}{(x^2+y^2)^s}\mathrm{d}x=\frac{\pi^s}{y^s\Gamma(s)}\begin{cases}\pi^{-s+\frac12}\Gamma(s-\frac12)y^{1-s} & \text{if } r=0\\ 2|r|^{s-\frac12}\sqrt{y}K_{s-\frac12}(2\pi|r|y) & \text{if } r\neq 0\end{cases}.\tag{16}$$

Proof. Recall the integral representation of the Gamma function

$$\Gamma(s)=\int_0^\infty t^{s-1}e^{-t}\mathrm{d}t.$$

Thus we have

$$\begin{aligned}\frac{y^s}{\pi^s}\Gamma(s)\int_{-\infty}^{\infty}\frac{e^{-2\pi i r x}}{(x^2+y^2)^s}\mathrm{d}x&=\int_0^\infty\int_{-\infty}^{\infty}e^{-t}\left(\frac{ty}{\pi(x^2+y^2)}\right)^s e^{-2\pi i r x}\mathrm{d}x\frac{\mathrm{d}t}{t}\\&=\int_0^\infty\int_{-\infty}^{\infty}e^{-\pi\xi(x^2+y^2)/y}\xi^s e^{-2\pi i r x}\mathrm{d}x\frac{\mathrm{d}\xi}{\xi}.\end{aligned}$$

For $r=0$ the above expression becomes

$$\begin{aligned}\int_0^\infty\int_{-\infty}^{\infty}e^{-\pi\xi(x^2+y^2)/y}\xi^s\mathrm{d}x\frac{\mathrm{d}\xi}{\xi}&=\int_0^\infty\sqrt{\frac{y}{\xi}}e^{-\pi\xi y}\xi^s\frac{\mathrm{d}\xi}{\xi}\\&=\pi^{-s+\frac12}y^{1-s}\Gamma(s-\frac12).\end{aligned}$$

For $r\neq 0$ we obtain, using the change of variables $\xi\mapsto\frac{\xi}{|r|}$,

$$\begin{aligned}\int_0^\infty\int_{-\infty}^{\infty}e^{-\pi\xi(x^2+y^2)/y}\xi^s e^{-2\pi i r x}\mathrm{d}x\frac{\mathrm{d}\xi}{\xi}&=\int_0^\infty\sqrt{\frac{y}{\xi}}e^{-y\pi r^2/\xi}e^{-\pi\xi y}\xi^s\frac{\mathrm{d}\xi}{\xi}\\&=\sqrt{y}|r|^{s-\frac12}\int_0^\infty\xi^{s-\frac12}e^{-y\pi|r|(1/\xi+\xi)}\frac{\mathrm{d}\xi}{\xi}\\&=2|r|^{s-\frac12}\sqrt{y}K_{s-\frac12}(2\pi|r|y).\end{aligned}$$

Here we used the following integral representation of the modified Bessel function of the second kind $K_\nu(x)$ (see [5] page 182)

$$K_\nu(x) = \frac{1}{2}\int_0^\infty e^{-\frac{x}{2}(t+t^{-1})}t^{-\nu-1}\mathrm{d}t.$$

$\square$

Lemma 21. *For $z = x + iy \in \mathbb{H}$ and $\mathrm{Re}(s) > \frac{1}{2}$ we have*

$$\sum_{n\in\mathbb{Z}} \frac{1}{|z+n|^{2s}} = \frac{\sqrt{\pi}\Gamma(s-\frac{1}{2})}{\Gamma(s)}y^{1-2s} + \frac{2\pi^s}{\Gamma(s)}y^{\frac{1}{2}-s}\sum_{r\neq 0}|r|^{s-\frac{1}{2}}K_{s-\frac{1}{2}}(2\pi|r|y)\,e^{2\pi irx}. \tag{17}$$

Proof. Recall the Poisson summation formula

$$\sum_{n\in\mathbb{Z}}\varphi(x+n) = \sum_{n\in\mathbb{Z}}\widehat{\varphi}(n)e^{2\pi inx},$$

where φ is a continuous function that decays sufficiently rapidly at infinity (for example, $|f(x)| < |x|^{-c}$ with $c > 1$) and where $\widehat{\varphi}(n) = \int_{-\infty}^{\infty}\varphi(x)e^{-2\pi inx}\mathrm{d}x$ is the Fourier transform. We apply this formula to $\varphi(x) = |x+iy|^{-2s}$, where $z = x+iy \in \mathbb{H}$ and $\mathrm{Re}(s) > \frac{1}{2}$. The Poisson summation formula gives

$$\sum_{n\in\mathbb{Z}}\frac{1}{|z+n|^{2s}} = \sum_{n\in\mathbb{Z}}\left(\int_{-\infty}^{\infty}\frac{e^{-2\pi inx}}{(x^2+y^2)^s}\mathrm{d}x\right)e^{2\pi inx}.$$

Using Lemma 20 we get the result. $\square$

We would like to adapt the left hand side of Equation 17 to the setting where the sum is twisted by a primitive Dirichlet character χ mod q. For this we need the twisted variant of the Poisson summation formula.

Lemma 22. *Let φ be a function that satisfies the conditions of the Poisson summation formula. Let χ be a primitive Dirichlet character mod q. Then*

$$\sum_{n\in\mathbb{Z}}\chi(n)\varphi(x+n) = \frac{\tau(\chi)}{q}\sum_{n\in\mathbb{Z}}\overline{\chi(n)}\widehat{\varphi}(n/q)e^{2\pi inx/q}.$$

Proof. From Lemma 2 we know that

$$\chi(n) = \frac{1}{\tau(\overline{\chi})}\sum_{m \bmod q}\overline{\chi(m)}e^{2\pi inm/q} = \frac{\chi(-1)\tau(\chi)}{q}\sum_{m \bmod q}\overline{\chi(m)}e^{2\pi inm/q}.$$

Let us now consider $\sum_{n\in\mathbb{Z}}\chi(n)\varphi(n) = \sum_{n\in\mathbb{Z}}\varphi_1(n)$, where

$$\varphi_1(x) = \frac{\chi(-1)\tau(\chi)}{q}\sum_{m \bmod q}\overline{\chi(m)}e^{2\pi ixm/q}\varphi(x).$$

The Fourier transform of $\varphi_1(x)$ is

$$\begin{aligned}\widehat{\varphi_1}(\xi) &= \frac{\chi(-1)\tau(\chi)}{q}\sum_{m \bmod q}\overline{\chi(m)}\int_{-\infty}^{\infty}\varphi(x)e^{-2\pi ix(\xi-\frac{m}{q})}\mathrm{d}x\\ &= \frac{\chi(-1)\tau(\chi)}{q}\sum_{m \bmod q}\overline{\chi(m)}\widehat{\varphi}\left(\xi-\frac{m}{q}\right).\end{aligned}$$

We apply Poisson summation formula and thus $\sum_{n\in\mathbb{Z}}\varphi_1(n)$ equals

$$\frac{\chi(-1)\tau(\chi)}{q}\sum_{m \bmod q}\sum_{n=-\infty}^{\infty}\overline{\chi(m)}\widehat{\varphi}\left(\frac{nq-m}{q}\right).$$

We can write $\overline{\chi(m)} = \chi(-1)\overline{\chi(nq-m)}$. When n runs through $\mathbb{Z}$ and m through $\mathbb{Z}/q\mathbb{Z}$, the terms $nq-m$ run uniquely through $\mathbb{Z}$. Thus we have shown

$$\sum_{n\in\mathbb{Z}} \chi(n)\varphi(n) = \frac{\tau(\chi)}{q}\sum_{n\in\mathbb{Z}} \overline{\chi(n)}\widehat{\varphi}(n/q).$$

Replacing $\varphi(n)$ by $\varphi(n+x)$ replaces $\widehat{\varphi}(\xi)$ by $\widehat{\varphi}(\xi)e^{2\pi i\xi x}$. This completes the proof. □

We can now determine the Fourier expansion of $E^*(z,s,\chi)$.

Theorem 23. *The function $E^*(z,s,\chi)$ has the Fourier expansion*

$$\begin{aligned} E^*(z,s,\chi) &= L(2s,\chi)y^s \\ &\quad + \tau(\chi)q^{-2s}\sqrt{y}\frac{2\pi^s}{\Gamma(s)}\sum_{k\neq 0}|k|^{\frac{1}{2}-s}\sigma_{2s-1}(|k|,\overline{\chi})K_{s-\frac{1}{2}}(2\pi|k|y)e^{2\pi ikx}, \end{aligned}$$

where $\sigma_s(k,\overline{\chi}) = \sum_{d|k}\overline{\chi(d)}d^s$.

First Proof. We split up the sum (15) into the terms with $m=0$ and those with $m\neq 0$. The assumption that χ is even allows us to combine each positive summand with its negative. We obtain

$$E^*(z,s,\chi) = L(2s,\chi)y^s + \sum_{m=1}^{\infty}\sum_{n\in\mathbb{Z}}\chi(n)\frac{y^s}{|mNq^2z+n|^{2s}}.$$

Substituting into this the twisted variant of the formula of Lemma 21 gives (note that $\chi(0)=0$)

$$\begin{aligned} E^*(z,s,\chi) &= L(2s,\chi)y^s \\ &\quad + \frac{\tau(\chi)}{q}y^s\frac{2\pi^s}{\Gamma(s)}\sum_{m=1}^{\infty}\sum_{n\neq 0}\overline{\chi(n)}\left(\frac{|n|}{q}\right)^{s-\frac{1}{2}}\left(ymNq^2\right)^{\frac{1}{2}-s}K_{s-\frac{1}{2}}(2\pi|n|ymNq)e^{2\pi inxmNq}. \end{aligned}$$

Summing $m\in Nq\mathbb{Z}_{\geq 1}$ and $n\in\mathbb{Z}_{\geq 1}$ is the same as summing their product k over $\mathbb{Z}_{\geq 1}$ and summing for each k over the pairs (n,mNq) such that $Nqmn=k$. Accordingly we can write

$$\begin{aligned} E^*(z,s,\chi) &= L(2s,\chi)y^s \\ &\quad + \tau(\chi)q^{-2s}\sqrt{y}\frac{2\pi^s}{\Gamma(s)}\sum_{m=1}^{\infty}\sum_{n\neq 0}\overline{\chi(n)}|n|^{s-\frac{1}{2}}(mNq)^{\frac{1}{2}-s}K_{s-\frac{1}{2}}(2\pi|n|ymNq)e^{2\pi inxmNq} \\ &= L(2s,\chi)y^s \\ &\quad + \tau(\chi)q^{-2s}\sqrt{y}\frac{2\pi^s}{\Gamma(s)}\sum_{k\neq 0}|k|^{\frac{1}{2}-s}\sigma_{2s-1}(|k|,\overline{\chi})K_{s-\frac{1}{2}}(2\pi|k|y)e^{2\pi ikx}, \end{aligned}$$

where $\sigma_s(k,\chi) = \sum_{d|k}\chi(d)d^s$ is the twisted divisor function. □

Second Proof. Again we start with

$$E^*(z,s,\chi) = L(2s,\chi)y^s + \sum_{m=1}^{\infty}\sum_{n\in\mathbb{Z}}\chi(n)\frac{y^s}{|mNq^2z+n|^{2s}}.$$

Using the substitution $n = tmNq^2 + r$ and Lemma 21 gives

$$
\begin{aligned}
E^*(z,s,\chi) &= L(2s,\chi)y^s \\
&\quad + \frac{y^s}{q^{4s}N^{2s}} \sum_{m=1}^{\infty} |m|^{-2s} \sum_{r=1}^{mNq^2} \chi(r) \sum_{t\in\mathbb{Z}} \left| t + z + \frac{r}{mNq^2} \right|^{-2s} \\
&= L(2s,\chi)y^s + \frac{y^s}{q^{4s}N^{2s}} \sum_{m=1}^{\infty} |m|^{-2s} \sum_{r=1}^{mNq^2} \chi(r) \\
&\quad \times \left[\frac{\sqrt{\pi}\Gamma(s-\frac{1}{2})}{\Gamma(s)} y^{1-2s} + \frac{2\pi^s}{\Gamma(s)} y^{\frac{1}{2}-s} \sum_{t\neq 0} |t|^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|t|y)\, e^{2\pi i t\left(x+\frac{r}{mNq^2}\right)} \right] \\
&= L(2s,\chi)y^s + \frac{2\pi^s\sqrt{y}}{q^{4s}N^{2s}\Gamma(s)} \sum_{t\neq 0} |t|^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|t|y) e^{2\pi i t x} \sum_{m=1}^{\infty} |m|^{-2s} \sum_{r=1}^{mNq^2} \chi(r) e^{\frac{2\pi i t r}{mNq^2}}.
\end{aligned}
$$

In the last equality we used Lemma 1 according to which $\sum_{r=1}^{mNq^2} \chi(r) = \sum_{r=1}^{q} \chi(r) = 0$. By Lemma 4 the last sum equals

$$
\sum_{r=1}^{mNq^2} \chi(r) e^{\frac{2\pi i t r}{mNq^2}} = \begin{cases} mNq\tau(\chi)\overline{\chi}(l) & \text{if } t = mqNl \text{ for some } l \in \mathbb{Z} \\ 0 & \text{else} \end{cases} .
$$

Therefore we get

$$
\begin{aligned}
E^*(z,s,\chi) &= L(2s,\chi)y^s + \frac{2\pi^s\tau(\chi)\sqrt{y}}{q^{4s}N^{2s}\Gamma(s)} \sum_{t\neq 0} |t|^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|t|y) e^{2\pi i t x} \sum_{\substack{m\geq 1 \\ t=mqNl}} |m|^{-2s} mNq\overline{\chi}(l) \\
&= L(2s,\chi)y^s + \frac{2\pi^s\tau(\chi)\sqrt{y}}{q^{4s}N^{2s}\Gamma(s)} \sum_{t\neq 0} |t|^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|t|y) e^{2\pi i t x} \sum_{l|t} \left| \frac{t}{qNl} \right|^{-2s} \frac{t}{l} \overline{\chi}(l) \\
&= L(2s,\chi)y^s + \frac{2\pi^s\tau(\chi)\sqrt{y}}{q^{2s}\Gamma(s)} \sum_{t\neq 0} |t|^{\frac{1}{2}-s} K_{s-\frac{1}{2}}(2\pi|t|y) e^{2\pi i t x} \sum_{l|t} \overline{\chi}(l) l^{2s-1} \\
&= L(2s,\chi)y^s + \frac{2\pi^s\tau(\chi)\sqrt{y}}{q^{2s}\Gamma(s)} \sum_{t\neq 0} |t|^{\frac{1}{2}-s} \sigma_{2s-1}(|t|,\overline{\chi}) K_{s-\frac{1}{2}}(2\pi|t|y) e^{2\pi i t x}.
\end{aligned}
$$

□

A consequence of the above Fourier expansion is the meromorphic continuation in s of the twisted Eisenstein series $E(z,s,\chi)$. We would also like a functional equation for the twisted Eisenstein series. Simply replacing s by $1-s$ in $E(z,s,\chi)$ will not provide us with a functional equation since y^s and y^{1-s} in the constant term will not match. Therefore we also need to appropriately modify the argument z in $E(z,s,\chi)$. For this we consider the Eisenstein series $E(\omega z,s,\chi)$, where $\omega = \begin{pmatrix} 0 & -1 \\ Nq^2 & 0 \end{pmatrix} \in \mathrm{GL}(2,\mathbb{Z})$. Again the automorphic relation $E(\gamma\omega z,s,\chi) = \overline{\chi(\gamma)} E(\omega z,s,\chi)$ holds and consequently $E(\omega z,s,\chi)$ admits a Fourier expansion.

Lemma 24. *The function $E^*(\omega z,s,\chi)$ has the Fourier expansion*

$$
\begin{aligned}
E^*(\omega z,s,\chi) &= \frac{1}{q^{2s}N^s} L(2s-1,\chi) \frac{\sqrt{\pi}\Gamma(s-\frac{1}{2})}{\Gamma(s)} y^{1-s} \\
&\quad + \frac{\sqrt{y}}{q^{2s}N^s} \frac{2\pi^s}{\Gamma(s)} \sum_{r\neq 0} |r|^{s-\frac{1}{2}} \sigma_{1-2s}(r,\chi) K_{s-\frac{1}{2}}(2\pi|r|y) e^{2\pi i r x}.
\end{aligned}
$$

Proof. We will use the fact that the matrix $\omega = \begin{pmatrix} 0 & -1 \\ Nq^2 & 0 \end{pmatrix}$ normalizes the group $\Gamma_0(Nq^2)$. Indeed, we have

$$
\omega \begin{pmatrix} a & b \\ c & d \end{pmatrix} \omega^{-1} = \begin{pmatrix} d & -\frac{c}{Nq^2} \\ -bNq^2 & a \end{pmatrix}.
$$

We now compute

$$\begin{aligned}
E^*(\omega z, s, \chi) &= \frac{L(2s,\chi)}{2} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(Nq^2)} \chi(\gamma)\,\mathrm{Im}(\gamma\omega z)^s \\
&= \frac{L(2s,\chi)}{2} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(Nq^2)} \chi(\omega\gamma\omega^{-1})\,\mathrm{Im}(\omega\gamma z)^s \\
&= \frac{y^s}{q^{2s}N^s} \sum_{a=1}^{\infty}\sum_{b\in\mathbb{Z}} \frac{\chi(a)}{|az+b|^{2s}} \\
&= \frac{y^s}{q^{2s}N^s} \sum_{a=1}^{\infty} \frac{\chi(a)}{a^{2s}} \sum_{d=1}^{a}\sum_{m\in\mathbb{Z}} \left| m + z + \frac{d}{a}\right|^{-2s} \\
&= \frac{1}{q^{2s}N^s} L(2s-1,\chi) \frac{\sqrt{\pi}\Gamma(s-\frac{1}{2})}{\Gamma(s)} y^{1-s} \\
&+ \frac{\sqrt{y}}{q^{2s}N^s}\frac{2\pi^s}{\Gamma(s)} \sum_{r\neq 0} |r|^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|r|y) e^{2\pi i r x} \sum_{a=1}^{\infty} \frac{\chi(a)}{a^{2s}} \sum_{d=1}^{a} e^{2\pi i r \frac{d}{a}}.
\end{aligned}$$

Because

$$\sum_{d=1}^{a} e^{2\pi i r\frac{d}{a}} = \begin{cases} a & \text{if } a|r \\ 0 & \text{else}\end{cases}$$

we have

$$\sum_{a=1}^{\infty} \frac{\chi(a)}{a^{2s}} \sum_{d=1}^{a} e^{2\pi i r\frac{d}{a}} = \sum_{a|r} \frac{\chi(a)}{a^{2s-1}} = \sigma_{1-2s}(r,\chi).$$

Hence we obtain the claimed Fourier expansion. □

The functional equation for the twisted Eisenstein series takes a simple form when we define

$$E^{\#}(z,s,\chi) := \pi^{-s}\Gamma(s)E^*(z,s,\chi) = \pi^{-s}\Gamma(s)L(2s,\chi)E(z,s,\chi).$$

Theorem 25. *The function $E^{\#}(z,s,\chi)$ satisfies the functional equation*

$$\begin{aligned}
E^{\#}(z,s,\chi) &= \frac{\tau(\chi)}{\sqrt{q}} N^{1-s} q^{\frac{5}{2}-4s} E^{\#}(\omega z, 1-s, \overline{\chi}) \\
&= \frac{\tau(\chi)}{\sqrt{q}} N^{1-s} q^{\frac{5}{2}-4s} E^{\#}\left(\frac{-1}{q^2Nz}, 1-s, \overline{\chi}\right).
\end{aligned}$$

Proof. This follows from the above Fourier expansions, the functional equation (10) for the Dirichlet L-series, and the symmetry $K_\nu(z) = K_{-\nu}(z)$ of the K-Bessel function.

□

3.3 Rankin-Selberg convolution for $\Gamma_0(N)$

Let

$$f(z) = \sum_{n\neq 0} a(n)\sqrt{y}K_{\nu_f}(2\pi|n|y)e^{2\pi i n x}, \tag{18}$$

$$g(z) = \sum_{n\neq 0} b(n)\sqrt{y}K_{\nu_g}(2\pi|n|y)e^{2\pi i n x}, \tag{19}$$

be cuspidal Maass forms for $\Gamma_0(N)$ of the same parity ϵ. Again we fix an even, primitive Dirichlet character χ mod a prime q. We recall that the twist $f_\chi(z)$, defined by Equation (14), is automorphic for $\Gamma_0(q^2N)$ with character χ^2. We define the Rankin-Selberg L-function $L_{f_\chi\times g}(s)$ by

$$L_{f_\chi\times g}(s)=\sum_{n=1}^{\infty}\chi(n)\frac{a(n)b(n)}{n^s}.$$

Because the Fourier coefficients $a(n),b(n)$ satisfy the bound $O(\sqrt{|n|})$ (Lemma 11), the above series converges absolutely for $\mathrm{Re}(s)>2$. By the same methods used to prove Theorem 16, we will show that the L-function $L_{f_\chi\times g}(s)$ admits a meromorphic continuation to $s\in\mathbb{C}$ and a functional equation. Key is the following lemma, which constructs the L-function $L_{f_\chi\times g}(s)$ as an inner product of $f_\chi(z)E(z,s,\chi^2)$ with $\overline{g(z)}$.

Lemma 26. *For* $\mathrm{Re}(s)$ *sufficiently large we have*

$$\int_{\Gamma_0(Nq^2)\backslash\mathbb{H}}E(z,s,\chi^2)f_\chi(z)g(z)\mathrm{d}\mu(z)=\frac{(-1)^\epsilon}{2^3}\frac{1}{\Gamma(s)\pi^s}\left(\prod\Gamma\left(\frac{s\pm\nu\pm\nu'}{2}\right)\right)L_{f_\chi\times g}(s).$$

Proof. By folding/unfolding it follows that

$$\begin{aligned}\int_{\Gamma_0(Nq^2)\backslash\mathbb{H}}E(z,s,\chi^2)f_\chi(z)g(z)\mathrm{d}\mu(z)&=\frac{1}{2}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(Nq^2)}\int_{\Gamma_0(Nq^2)\backslash\mathbb{H}}\chi(\gamma)^2\,\mathrm{Im}(\gamma z)^s f_\chi(z)g(z)\mathrm{d}\mu(z)\\&=\frac{1}{2}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(Nq^2)}\int_{\Gamma_0(Nq^2)\backslash\mathbb{H}}\mathrm{Im}(\gamma z)^s f_\chi(\gamma z)g(\gamma z)\mathrm{d}\mu(\gamma z)\\&=\frac{1}{2}\int_{\Gamma_\infty\backslash\mathbb{H}}\mathrm{Im}(z)^s f_\chi(z)g(z)\mathrm{d}\mu(z)\\&=\frac{1}{2}\int_0^\infty\int_0^1 y^{s-2}f_\chi(x+iy)g(x+iy)\mathrm{d}x\mathrm{d}y.\end{aligned}$$

For the second equality we used Lemma 19. Upon inserting the Fourier expansions, the integral over the x-coordinates becomes

$$\begin{aligned}\int_0^1 f_\chi(x+iy)g(x+iy)\mathrm{d}x&=y\sum_{n\neq 0}\chi(n)a(n)b(-n)K_\nu(2\pi|n|y)K_{\nu'}(2\pi|n|y)\\&=(-1)^\epsilon y\sum_{n\neq 0}\chi(n)a(n)b(n)K_\nu(2\pi|n|y)K_{\nu'}(2\pi|n|y).\end{aligned}$$

Then formula (13) for the Mellin transform of $K_\nu K_{\nu'}$ gives

$$\begin{aligned}\int_{\Gamma_0(Nq^2)\backslash\mathbb{H}}E(z,s,\chi^2)f_\chi(z)g(z)\mathrm{d}\mu(z)&=\frac{(-1)^\epsilon}{2^4}\frac{1}{\Gamma(s)\pi^s}\left(\prod\Gamma\left(\frac{s\pm\nu\pm\nu'}{2}\right)\right)\sum_{n\neq 0}\chi(n)\frac{a(n)b(n)}{|n|^s}\\&=\frac{(-1)^\epsilon}{2^3}\frac{1}{\Gamma(s)\pi^s}\left(\prod\Gamma\left(\frac{s\pm\nu\pm\nu'}{2}\right)\right)\sum_{n=1}^{\infty}\chi(n)\frac{a(n)b(n)}{n^s},\end{aligned}$$

where in the last line we used that f and g are of the same parity. $\square$

4 The Luo-Rudnick-Sarnak Theorem

4.1 $\mathrm{SL}(2,\mathbb{Z})$

Let

$$f(z) = \sum_{n\neq 0} a(n)\sqrt{y} K_{\nu-\frac{1}{2}}(2\pi|n|y)e^{2\pi i n x}$$

be a Maass cusp form for $\mathrm{SL}(2,\mathbb{Z})$ of type ν so that $a(1) = 1$ and so that $f(z)$ is an eigenfunction of the Hecke operators. We have shown in Lemma 18 that the associated L-function $L_f(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$ has the Euler product

$$\begin{aligned} L_f(s) &= \prod_p \left(1 - a(p)p^{-s} + p^{-2s}\right)^{-1}, \\ &= \prod_p \prod_{i=1}^{2} \left(1 - \frac{\alpha_{i,p}}{p^s}\right)^{-1}, \end{aligned} \tag{20}$$

where for each prime p the $\alpha_{i,p} \in \mathbb{C}$ satisfy

$$\begin{aligned} 1 &= \alpha_{1,p}\,\alpha_{2,p}, \\ a(p) &= \alpha_{1,p} + \alpha_{2,p}. \end{aligned}$$

Here is the Ramanujan conjecture for this setting.

Conjecture 27. *Lef f be a Maass cusp form as above. Then the $\alpha_{i,p}$ in Equation 20 satisfy*

$$|\alpha_{i,p}| = 1,$$

for all primes p and $i = 1, 2$.

Another important conjecture is the Selberg eigenvalue conjecture.

Conjecture 28. *Lef f be a Maass cusp form of type $\nu \in \mathbb{C}$ as above. Then*

$$\nu(1-\nu) \geq \frac{1}{4},$$

or equivalently,

$$\mathrm{Re}(\nu) = \frac{1}{2},$$

or equivalently,

$$\mathrm{Re}(\lambda_i(\nu)) = 0,$$

for $i = 1, 2$ and where $\lambda_1(\nu) = \nu - \frac{1}{2}$ and $\lambda_2(\nu) = -\nu + \frac{1}{2}$.

These two conjectures can in fact be treated in an unified manner. The Selberg eigenvalue conjecture is not hard to prove for $\mathrm{SL}(2,\mathbb{Z})$. In fact, in Lemma 8 we proved the much better lower bound $\nu(1-\nu) \geq 3\pi^2/2$. The above conjectures can be generalized to $\mathrm{SL}(n,\mathbb{Z})$ for $n \geq 2$ and to congruence subgroups and in this setting the conjectures are not proved. The current best bound for both the Ramanujan conjecture and the Selberg eigenvalue conjecture for Maass forms for $\mathrm{SL}(n,\mathbb{Z})$ was obtained by Luo, Rudnick and Sarnak. In this section we will illustrate their method specialized to Maass forms for $\mathrm{SL}(2,\mathbb{Z})$. Here is their theorem.

Theorem 29. *Lef f be a Maass cusp form of type ν for $\mathrm{SL}(2,\mathbb{Z})$ as above. Then, using the above notation, we have*

$$|\alpha_{i,p}| \leq p^{\frac{1}{2}-\frac{1}{5}}, \tag{21}$$

$$\mathrm{Re}(\lambda_i(\nu)) \leq \frac{1}{2} - \frac{1}{5}, \tag{22}$$

for all primes p and for $i = 1, 2$.

We will give a proof of the bound (22) towards the Selberg eigenvalue conjecture. The proof of the bound (21) for the Ramanujan conjecture is based on the same argument, but requires more care, and we will not go into details.

The proof of Theorem 29 is based on the Rankin-Selberg convolution L-function. Let us recall the material that we will need. Let f be a Maass form as in Theorem 29. It has the Euler product

$$L_f(s) = \prod_p \prod_{i=1}^{2} \left(1 - \frac{\alpha_{i,p}}{p^s}\right)^{-1}.$$

Let χ be a primitive character mod a prime q. The twisted L-function $L_{f_\chi}(s) = \sum_{n=1}^{\infty} \frac{\chi(n)a(n)}{n^s}$ has the Euler product

$$L_{f_\chi}(s) = \prod_p \prod_{i=1}^{2} \left(1 - \frac{\alpha_{i,p}\,\chi(p)}{p^s}\right)^{-1}.$$

We now take the Rankin-Selberg convolution $L_{f_\chi \times f}(s)$ of f_χ and f. According to Theorem 18, it is given by

$$L_{f_\chi \times f}(s) = \prod_p \prod_{i=1}^{2} \prod_{j=1}^{2} \left(1 - \frac{\alpha_{i,p}\,\overline{\alpha}_{j,p}\,\chi(p)}{p^s}\right)^{-1}. \tag{23}$$

According to Section 2.3.2, the functional equation of $L_{f_\chi \times f}(s)$ can be deduced from the functional equation for the Eisenstein L-functions $L_{E_v}(s,\chi)$ and $L_{E_v}(s)$. We recall from Section 2.3.2 and 2.3.3 that these L-functions take the simple form

$$\begin{aligned} L_{E_v}(s) &= \prod_{i=1}^{2} \zeta(s - \lambda_i(\nu)), \\ L_{E_v}(s,\chi) &= \prod_{j=1}^{2} L(s - \lambda_j(\nu), \chi). \end{aligned}$$

From the Euler products

$$\begin{aligned} \zeta(s) &= \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}, \\ L(s,\chi) &= \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}, \end{aligned}$$

and from Theorem 18 we deduce that the Rankin-Selberg convolution of E_ν and $E_{\nu,\chi}$ is

$$\begin{aligned} L_{E_{\nu,\chi} \times E_\nu}(s) &= \prod_p \prod_{i=1}^{2} \prod_{j=1}^{2} \left(1 - \frac{\chi(p) p^{\lambda_i(\nu) + \overline{\lambda_j(\nu)}}}{p^s}\right)^{-1} \\ &= \prod_{i=1}^{2} \prod_{j=1}^{2} L(s - \lambda_i(\nu) - \overline{\lambda_j(\nu)}, \chi). \end{aligned}$$

The functional equation (10) of the Dirichlet L-series gives us now the functional equation

$$\begin{aligned} \Lambda_{E_{\nu,\chi} \times E_\nu}(s) &= \prod_{i=1}^{2} \prod_{j=1}^{2} \left(\frac{q}{\pi}\right)^{\frac{s - \lambda_i(\nu) - \overline{\lambda_j(\nu)}}{2}} \Gamma\left(\frac{s - \lambda_i(\nu) - \overline{\lambda_j(\nu)}}{2}\right) L_{E_{\nu,\chi} \times E_\nu}(s) \\ &= \left(\frac{\tau(\chi)}{\sqrt{q}}\right)^4 \Lambda_{E_v}(1 - s, \overline{\chi}), \end{aligned} \tag{24}$$

which, as we have just mentioned, is also the functional equation satisfied by $L_{f_\chi \times f}(s)$. We may, as in the proof of Theorem 16, construct $L_{f_\chi \times f}(s)$ as the inner product

$$\langle f, f_\chi\, E(\cdot, s) \rangle,$$

proving that $L_{f_\chi \times f}(s)$ has a meromorphic continuation to $s \in \mathbb{C}$. If χ is non-trivial, which we will assume from now on, then $\Lambda_{f_\chi \times f}(s)$ is in fact holomorphic.

The idea of the proof for the bound (22) in Theorem 29 is to balance poles of the Gamma factors in $\Lambda_{f_\chi \times f}(s)$ with zeros of $L_{f_\chi \times f}(s)$. We are thus led to examine zeros of $L_{f_\chi \times f}(s)$. The following lemma gives us zero-free regions for an infinite family of L-functions $L_{f_\chi \times f}(s)$.

Lemma 30. *Let f be a Maass form as above. Then for any real number $\beta > 1 - \frac{2}{5}$, there are even primitive Dirichlet characters χ such that*

$$L_{f_\chi \times f}(\beta) \neq 0.$$

Before proving the lemma, let us show that it implies the bound (22) in Theorem 29.

Proof of (22) in Theorem 29. We argue by contradiction. Let f be a Maass cusp form as in Theorem 29 and assume that for some $1 \leq i \leq 2$ we have $\mathrm{Re}(\lambda_i(\nu)) = \beta$ for some $\beta > \frac{1}{2} - \frac{1}{5}$. Then by Lemma 30 there is an even primitive Dirichlet character χ such that

$$L_{f_\chi \times f}(2\beta) \neq 0.$$

On the other hand, the Gamma factor

$$\prod_{i=1}^{2} \Gamma\left(\frac{s - 2\,\mathrm{Re}(\lambda_i(\nu))}{2}\right)$$

in $\Lambda_{f_\chi \times f}(s)$ has a pole at $s = 2\beta$. Because $\Lambda_{f_\chi \times f}(s)$ is holomorphic, we obtain a contradiction. □

Proof of Lemma 30. According to Lemma 11, we have the bound $|\alpha_{i,p}| \ll \sqrt{p}$ for $i = 1, 2$. Therefore, for $s \geq 1$, the Euler factor $\left(1 - \frac{\alpha_{i,p}\,\overline{\alpha}_{j,p}\,\chi(p)}{p^s}\right)^{-1}$ in Equation (23) is non-zero for every p, thus also $L_{f_\chi \times f}(\beta) \neq 0$ for $\beta \geq 1$. We can thus assume $1 - \frac{2}{5} < \beta < 1$, and the theorem is a consequence of the following lemma. □

Lemma 31. *Let $\beta \in (1 - \frac{2}{5}, 1)$. Then for Q sufficiently large we have*

$$\sum_{q \sim Q} \sum_{\chi} L_{f_\chi \times f}(\beta) \approx Q^2 / \log(Q) \neq 0, \tag{25}$$

where q is prime, where the notations $q \sim Q$ means $Q \leq q \leq 2Q$ and where the inner sum is taken over non-trivial even primitive Dirichlet characters χ mod q.

Proof. We write the Dirichlet series of $L_{f_\chi \times f}(s)$ as

$$L_{f_\chi \times f}(s) = \sum_{n=1}^{\infty} \chi(n) \frac{b(n)}{n^s}.$$

By Lemma 12, this Dirichlet series converges absolutely for $\mathrm{Re}(s) \geq 1$. Therefore $\beta \in (1 - \frac{2}{5}, 1)$ does not lie in the region of absolute convergence of the Dirichlet series of $L_{f_\chi \times f}(s)$. The method of approximate functional equations allows us to compute values of L-functions inside regions where the Dirichlet series of the L-function fails to converge. The first step of the proof is thus the derivation of an approximate function equation for $L_{f_\chi \times f}(\beta)$.

Step 1. *Set $\beta_0 = 2 \max_{1 \leq i \leq 2} \mathrm{Re}(\lambda_i(\nu))$. There are two real-valued functions h_1 and h_2 that satisfy*

$$\begin{aligned}
&h_1(y), h_2(y) = O_A(y^{-A}) \quad \text{as } y \to \infty, \\
&h_1(y) = 1 + O_A(y^{A}) \quad \text{as } y \to 0, \\
&h_2(y) \ll 1 + y^{1-\beta_0-\beta-\epsilon} \quad \text{as } y \to 0,
\end{aligned}$$

for all $A \geq 1$ and all $\epsilon > 0$ and such that for any $Y > 1$ we have the approximate functional equation

$$\begin{aligned} L_{f_\chi \times f}(\beta) = &\sum_{n=1}^{\infty} \frac{b(n)}{n^\beta} \chi(n)\, h_1\left(\frac{n}{Y}\right) \\ &- \frac{\tau(\chi)^4 q^{-4\beta}}{\pi^{2-4\beta}} \sum_{n=1}^{\infty} \frac{b(n)}{n^{1-\beta}} \overline{\chi(n)}\, h_2\left(\frac{nY\pi^4}{q^4}\right). \end{aligned} \tag{26}$$

Let $h : (0,\infty) \to \mathbb{R}$ be a non-negative smooth, compactly supported function such that $\int_0^\infty h(y) \frac{\mathrm{d}y}{y} = 1$. Let

$$\tilde{h}(s) = \int_0^\infty h(y) y^s \frac{\mathrm{d}y}{y}$$

be the Mellin transform of h, which is holomorphic, bounded on vertical strips and satisfies $\tilde{h}(0) = 1$. We define

$$h_1(y) = \frac{1}{2\pi i} \int_{\mathrm{Re}(s)=2} \tilde{h}(s) y^{-s} \frac{\mathrm{d}s}{s}.$$

By a shift of the line of integration to the right (left), we get the behaviour as $y \to \infty$ ($y \to 0$). Next, for $y > 0$, we define

$$h_2(y) = \frac{1}{2\pi i} \int_{\mathrm{Re}(s)=1} \tilde{h}(-s) G(-s+\beta) y^{-s} \frac{\mathrm{d}s}{s}, \tag{27}$$

where

$$G(s) = \frac{\prod_{i=1}^2 \prod_{j=1}^2 \Gamma\left(\frac{1-s-\lambda_i(\nu)-\overline{\lambda_j(\nu)}}{2}\right)}{\prod_{i=1}^2 \prod_{j=1}^2 \Gamma\left(\frac{s-\lambda_i(\nu)-\overline{\lambda_j(\nu)}}{2}\right)}.$$

By Stirling's formula, $G(s)$ has at most polynomial growth in vertical strips. Therefore, by shifting the contour in Equation (27) to the right, we see that $h_2(y)$ decays rapidly as $y \to \infty$. For the behaviour as $y \to 0$, we shift the integration line to the left. (...) This gives the bound $h_2(y) \ll 1 + y^{1-\beta_0-\beta-\epsilon}$ as $y \to 0$.

Now we come to the derivation of an approximate functional equation. To this end, for $Y > 1$, consider the integral

$$\begin{aligned} \frac{1}{2\pi i} \int_{\mathrm{Re}(s)=2} \tilde{h}(s) L_{f_\chi \times f}(s+\beta) Y^s \frac{\mathrm{d}s}{s} &= \sum_{n=1}^{\infty} \frac{b(n)\chi(n)}{n^\beta} \frac{1}{2\pi i} \int_{\mathrm{Re}(s)=2} \tilde{h}(s) \left(\frac{Y}{n}\right)^s \frac{\mathrm{d}s}{s} \\ &= \sum_{n=1}^{\infty} \frac{b(n)\chi(n)}{n^\beta}\, h_1\left(\frac{n}{Y}\right). \end{aligned} \tag{28}$$

Since $L_{f_\chi \times f}(s)$ is bounded in vertical strips, we may shift the integration line in the above integral to $\mathrm{Re}(s) = -1$, thereby picking up a pole at $s = 0$, and we obtain

$$\begin{aligned} &\frac{1}{2\pi i} \int_{\mathrm{Re}(s)=2} \tilde{h}(s) L_{f_\chi \times f}(s+\beta) Y^s \frac{\mathrm{d}s}{s} \\ &\quad = L_{f_\chi \times f}(\beta) + \frac{1}{2\pi i} \int_{\mathrm{Re}(s)=-1} \tilde{h}(s) L_{f_\chi \times f}(s+\beta) Y^s \frac{\mathrm{d}s}{s} \\ &\quad = L_{f_\chi \times f}(\beta) + \frac{1}{2\pi i} \left(\frac{\tau(\chi)}{\sqrt{q}}\right)^4 \left(\frac{q}{\pi}\right)^{2-4\beta} \int_{\mathrm{Re}(s)=-1} \tilde{h}(s) G(s+\beta) L_{f_{\overline{\chi}} \times f}(1-s-\beta) \left(\frac{\pi^4 Y}{q^4}\right)^s \frac{\mathrm{d}s}{s}, \end{aligned}$$

where in the last step we applied the functional equation (24). On changing variables $s \mapsto -s$ the last expression equals

$$L_{f_\chi \times f}(\beta) + \frac{1}{2\pi i} \left(\frac{\tau(\chi)}{\sqrt{q}}\right)^4 \left(\frac{q}{\pi}\right)^{2-4\beta} \int_{\mathrm{Re}(s)=1} \tilde{h}(-s) G(-s+\beta) L_{f_{\overline{\chi}} \times f}(1+s-\beta) \left(\frac{\pi^4 Y}{q^4}\right)^{-s} \frac{\mathrm{d}s}{s},$$

which equals

$$L_{f_\chi \times f}(\beta) + \left(\frac{\tau(\chi)}{\sqrt{q}}\right)^4 \left(\frac{q}{\pi}\right)^{2-4\beta} \sum_{n=1}^{\infty} \overline{\chi}(n) \frac{b(n)}{n^{1-\beta}} h_2 \left(\frac{nY\pi^4}{q^4}\right). \tag{29}$$

Combining (28) and (29) gives the approximate functional equation (26).

Step 2. *We average the approximate functional equation over even primitive Dirichlet characters to deduce (25).*

According to the approximate functional equation (26), we can decompose $\sum_{q\sim Q}\sum_{\chi} L_{f_\chi\times f}(\beta)$ into $T_1 - T_2$, where

$$\begin{aligned} T_1 &= \sum_{q\sim Q}\sum_{\chi}\sum_{n=1}^{\infty} \frac{b(n)}{n^{\beta}} \chi(n)\, h_1\left(\frac{n}{Y}\right), \\ T_2 &= \sum_{q\sim Q}\sum_{\chi} \frac{\tau(\chi)^4 q^{-4\beta}}{\pi^{2-4\beta}} \sum_{n=1}^{\infty} \frac{b(n)}{n^{1-\beta}} \overline{\chi(n)}\, h_2\left(\frac{nY\pi^4}{q^4}\right). \end{aligned}$$

By Lemma 5, we can rewrite T_1 as

$$T_1 = \sum_{q\sim Q} \frac{q-1}{2} \sum_{n\equiv \pm 1 \pmod q} \frac{b(n)}{n^{\beta}} h_1\left(\frac{n}{Y}\right) - \sum_{q\sim Q}\sum_{(n,q)=1} \frac{b(n)}{n^{\beta}} h_1\left(\frac{n}{Y}\right).$$

In the end we will choose Y to be very large, and so we use that $h_1\left(\frac{1}{Y}\right) \to 1$ as $Y\to\infty$. The contribution to T_1 from the term $n=1$ is thus

$$\sum_{q\sim Q}\frac{q-1}{2} h_1\left(\frac{1}{Y}\right) = \sum_{q\sim Q}\frac{q-1}{2}.$$

We write the sum over $n \equiv 1 \pmod q, n \neq 1$ as

$$\sum_{q\sim Q}\frac{q-1}{2}\sum_{d=1}^{\infty}\frac{b(1+dq)}{(1+dq)^{\beta}} h_1\left(\frac{1+dq}{Y}\right).$$

The number of representations of n of the form $n = 1+dq$ with $d, q \geq 1$ is $O_\epsilon(n^\epsilon)$ for any $\epsilon > 0$ (?). Accordingly we have

$$\sum_{q\sim Q}\frac{q-1}{2}\sum_{d=1}^{\infty}\frac{b(1+dq)}{(1+dq)^{\beta}} h_1\left(\frac{1+dq}{Y}\right) \ll^{?} Q\sum_{n=1}^{\infty} n^{\epsilon}\frac{b(n)}{n^{\beta}}\left|h_1\left(\frac{n}{Y}\right)\right|.$$

For large enough Y we deduce from the properties of h_1 and from Lemma 12 that

$$\sum_{n=1}^{\infty} n^{-\beta+\epsilon} b(n)\left|h_1\left(\frac{n}{Y}\right)\right| = Y^{-\beta+\epsilon}\sum_{1\leq n\leq Y} b(n) \ll Y^{1-\beta+\epsilon}.$$

The remaining terms in T_1 are bounded similarly and we find that

$$T_1 = \sum_{q\sim Q}\frac{q-1}{2} + O(Q\,Y^{1-\beta+\epsilon}). \tag{30}$$

We now consider the term T_2. It contains the sum

$$\sum_{\substack{\chi \pmod q,\\ \chi\neq\chi_0,\, \chi(-1)=1}} \overline{\chi(n)}\tau(\chi)^4$$

which is zero if $q \mid n$ and otherwise equals by Lemma 6

$$\frac{q-1}{2}\left(K_4(n,q)+K_4(-n,q)\right)-1.$$

It is a consequence of Deligne's proof of the Weil conjectures that the hyper-Kloosterman sum $K_d(n,q)$ is bounded by

$$K_d(n,q) \ll q^{\frac{d-1}{2}}.$$

Hence if $q \nmid n$ we get

$$\sum_{\substack{\chi \pmod q,\\ \chi\neq\chi_0,\, \chi(-1)=1}} \overline{\chi(n)}\tau(\chi)^4 \ll q^{\frac{5}{2}}.$$

The term T_2 is accordingly

$$\begin{aligned}
T_2 &= \sum_{q\sim Q}\sum_{\chi} \frac{\tau(\chi)^4 q^{-4\beta}}{\pi^{2-4\beta}} \sum_{n=1}^{\infty} \frac{b(n)}{n^{1-\beta}}\overline{\chi(n)}\, h_2\left(\frac{nY\pi^4}{q^4}\right)\\
&\ll \sum_{q\sim Q} \frac{q^{\frac{5}{2}}q^{-4\beta}}{\pi^{2-4\beta}} \sum_{(n,q)=1} \frac{b(n)}{n^{1-\beta}}\left|h_2\left(\frac{nY\pi^4}{q^4}\right)\right|\\
&\ll \sum_{q\sim Q} \frac{q^{\frac{5}{2}}q^{-4\beta}}{\pi^{2-4\beta}} \int_1^{\infty} \left|h_2\left(\frac{rY\pi^4}{q^4}\right)\right| r^{\beta}\frac{\mathrm{d}r}{r}\\
&= \sum_{q\sim Q} \frac{q^{\frac{5}{2}}}{\pi^2 Y^{\beta}} \int_{\frac{Y\pi^4}{q^4}}^{\infty} |h_2(r)| r^{\beta}\frac{\mathrm{d}r}{r}\\
&\ll ???\\
&\ll \sum_{q\sim Q} \frac{q^{\frac{5}{2}}}{Y^{\beta}}\\
&\ll \sum_{q\sim Q} q^{1+\frac{5}{2}}Y^{-\beta}\\
&\ll ???\\
&\ll Q^{1+\frac{5}{2}}Y^{-\beta}.
\end{aligned} \tag{31}$$

From the upper bounds (30) and (31) it follows that

$$\sum_{q\sim Q}\sum_{\chi} L_{f_\chi\times f}(\beta) = \sum_{q\sim Q}\frac{q-1}{2} + O(QY^{1-\beta+\epsilon} + Q^{1+\frac{5}{2}}Y^{-\beta}.$$

Then choosing $Y = Q^{\frac{5}{2}}$ gives

$$\sum_{q\sim Q}\sum_{\chi} L_{f_\chi\times f}(\beta) = \sum_{q\sim Q}\frac{q-1}{2} + O(Q^{1+\frac{5}{2}(1-\beta+\epsilon)}.$$

For $\beta \in (1-\frac{2}{5}, 1)$ and Q sufficiently large, the term $\sum_{q\sim Q}\frac{q-1}{2} \approx Q^2/\log(Q)$ dominates the O-term. In particular, the average $\sum_{q\sim Q}\sum_{\chi} L_{f_\chi\times f}(\beta)$ is non-zero and therefore $L_{f_\chi\times f}(\beta)$ is non-zero for some χ. This completes the proof of the lemma.

□

4.2 Congruence subgroups

5 References

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