L-functions and the Selberg eigenvalue conjecture

Contents

1 Dirichlet characters

Let m be a positive integer. A Dirichlet character of modulus m is a function $\chi : \mathbb{Z} \to \mathbb{C}$ such that

- $\chi(ab) = \chi(a)\chi(b)$ for all $a, b \in \mathbb{Z}$,
- $\chi(a) = 0$ if and only if $(a, m) > 1$,
- If $a \equiv b \pmod{m}$, then $\chi(a) = \chi(b)$.

Let χ_1 and χ_2 be Dirichlet characters of modulus m_1 and m_2, respectively, with $m_1|m_2$. If $\chi_2(a) = \chi_1(a)$ for $a \in (\mathbb{Z}/m_2\mathbb{Z})^\times$, then χ_2 is said to be induced by χ_1. A Dirichlet character is called primitive if it is not induced by any Dirichlet character other than itself. A Dirichlet character induced by the identity function is called principal. We denote a principal Dirichlet character by χ_0. By definition a principal Dirichlet character mod m is

$$\chi_0(a) = \begin{cases} 1 & \text{if } (a, m) = 1 \\ 0 & \text{else} \end{cases} . \tag{1}$$

If p is prime, then every nonprincipal Dirichlet character of modulus p is primitive. We call a Dirichlet character even if $\chi(-1) = 1$, odd if $\chi(-1) = -1$.

Lemma 1. *Let χ be a nonprincipal Dirichlet character modulo a prime q. Then we have*

$$\sum_{a=1}^{q} \chi(a) = 0.$$

Proof. Because χ is not principal, there is an integer b such that $\chi(b) \neq \{0, 1\}$. Furthermore, the sum may be restricted to the terms with $(a, q) = 1, 1 \leq a \leq q$. Multiplication by b is a bijection $(\mathbb{Z}/q\mathbb{Z})^\times \to (\mathbb{Z}/q\mathbb{Z})^\times$. Therefore we have

$$\chi(b) \sum_{a=1}^{q} \chi(a) = \sum_{a \in (\mathbb{Z}/q\mathbb{Z})^\times} \chi(ab) = \sum_{c \in (\mathbb{Z}/q\mathbb{Z})^\times} \chi(c),$$

which implies $\sum_{a=1}^{q} \chi(a) = 0$. □

The Gauss sum $\tau(\chi)$ attached to a primitive Dirichlet character χ mod q is

$$\tau(\chi) = \sum_{a \bmod q} \chi(a) e^{2\pi i a/q}.$$

Lemma 2. *Let χ be a nonprincipal Dirichlet character modulo a prime q. Then for all $n \in \mathbb{Z}$ we have*

$$\tau(\chi)\overline{\chi(n)} = \sum_{a=1}^{q} \chi(a) e^{2\pi i n a/q}.$$

Proof. See the second half of the proof of Lemma 3. □

We will also need the following variant of Lemma 2.

Lemma 3. *Let χ be a nonprincipal Dirichlet character modulo a prime q, and let $n, m \in \mathbb{Z}$. Then*

$$\sum_{a=1}^{nq^2} \chi(a) e^{2\pi i m a/nq^2} = \begin{cases} qn\tau(\chi)\overline{\chi}(l) & \text{if } m = lnq \text{ for some } l \in \mathbb{Z} \\ 0 & \text{else} \end{cases} . \tag{2}$$

Proof. Every a in the above sum is of the form $a = a_1 + tq$ with $1 \leq a_1 \leq q$ and $0 \leq t < nq$. Thus we can write

$$\sum_{a=1}^{nq^2} \chi(a) e^{2\pi i m a / nq^2} = \sum_{a_1=1}^{q} \chi(a_1) e^{\frac{2\pi i m a_1}{nq^2}} \sum_{t=0}^{nq-1} e^{\frac{2\pi i m t}{nq}}.$$

The sum over t is zero unless $m = lnq$ for some $l \in \mathbb{Z}$ in which case the sum is nq. Therefore

$$\begin{aligned}\sum_{a=1}^{nq^2} \chi(a) e^{2\pi i m a / nq^2} &= nq \sum_{\substack{a \bmod q \\ m = lnq}} \chi(a) e^{\frac{2\pi i m a}{nq^2}} \\ &= nq \sum_{a \bmod q} \chi(a) e^{\frac{2\pi i l a}{q}}\end{aligned}$$

If $(l, q) = 1$, then $a \mapsto al$ permutes the residues mod q. In this case we get

$$nq \sum_{a \bmod q} \chi(a) e^{\frac{2\pi i l a}{q}} = nq \tau(\chi) \overline{\chi(l)}.$$

Now suppose that $(l, q) > 1$. Then $\chi(l) = 0$ and we have to show that the left side of Equation 2 vanishes. For this let $l' \in \mathbb{Z}$ be such that $ql' = l$. Then we have

$$\begin{aligned}nq \sum_{a \bmod q} \chi(a) e^{\frac{2\pi i l a}{q}} &= nq \sum_{a \bmod q} \chi(a) e^{2\pi i l' a} \\ &= nq \sum_{a \bmod q} \chi(a).\end{aligned}$$

The last sum is zero by Lemma 1. This completes the proof of the lemma.

$\square$

2 Integrals

Lemma 4. *If* $\mathrm{Re}(s) > 1/2$ *and* $r \in \mathbb{R}$*, then*

$$\int_{-\infty}^{\infty} \frac{e^{-2\pi irx}}{(x^2+y^2)^s}\mathrm{d}x = \frac{\pi^s}{y^s\Gamma(s)} \begin{cases} \pi^{-s+\frac{1}{2}}\Gamma(s-\frac{1}{2})y^{1-s} & \text{if } r=0 \\ 2|r|^{s-\frac{1}{2}}\sqrt{y}K_{s-\frac{1}{2}}(2\pi|r|y) & \text{if } r\neq 0 \end{cases}. \tag{3}$$

Proof. Recall the integral representation of the Gamma function

$$\Gamma(s) = \int_0^\infty t^{s-1}e^{-t}\mathrm{d}t.$$

Thus we have

$$\begin{aligned} \frac{y^s}{\pi^s}\Gamma(s)\int_{-\infty}^{\infty}\frac{e^{-2\pi irx}}{(x^2+y^2)^s}\mathrm{d}x &= \int_0^\infty\int_{-\infty}^{\infty} e^{-t}\left(\frac{ty}{\pi(x^2+y^2)}\right)^s e^{-2\pi irx}\mathrm{d}x\frac{\mathrm{d}t}{t} \\ &= \int_0^\infty\int_{-\infty}^{\infty} e^{-\pi\xi(x^2+y^2)/y}\xi^s e^{-2\pi irx}\mathrm{d}x\frac{\mathrm{d}\xi}{\xi}. \end{aligned}$$

For $r=0$ the above expression becomes

$$\begin{aligned} \int_0^\infty\int_{-\infty}^{\infty} e^{-\pi\xi(x^2+y^2)/y}\xi^s\mathrm{d}x\frac{\mathrm{d}\xi}{\xi} &= \int_0^\infty\sqrt{\frac{y}{\xi}}e^{-\pi\xi y}\xi^s\frac{\mathrm{d}\xi}{\xi} \\ &= \pi^{-s+\frac{1}{2}}y^{1-s}\Gamma(s-\frac{1}{2}). \end{aligned}$$

For $r\neq 0$ we obtain, using the change of variables $\xi \mapsto \frac{\xi}{|r|}$,

$$\begin{aligned} \int_0^\infty\int_{-\infty}^{\infty} e^{-\pi\xi(x^2+y^2)/y}\xi^s e^{-2\pi irx}\mathrm{d}x\frac{\mathrm{d}\xi}{\xi} &= \int_0^\infty\sqrt{\frac{y}{\xi}}e^{-y\pi r^2/\xi}e^{-\pi\xi y}\xi^s\frac{\mathrm{d}\xi}{\xi} \\ &= \sqrt{y}|r|^{s-\frac{1}{2}}\int_0^\infty \xi^{s-\frac{1}{2}}e^{-y\pi|r|(1/\xi+\xi)}\frac{\mathrm{d}\xi}{\xi} \\ &= 2|r|^{s-\frac{1}{2}}\sqrt{y}K_{s-\frac{1}{2}}(2\pi|r|y). \end{aligned}$$

Here we used the following integral representation of the modified Bessel function of the second kind $K_\nu(x)$ (see [1] page 182)

$$K_\nu(x) = \frac{1}{2}\int_0^\infty e^{-\frac{x}{2}(t+t^{-1})}t^{-\nu-1}\mathrm{d}t.$$

□

Lemma 5. *For* $z = x+iy \in \mathbb{H}$ *and* $\mathrm{Re}(s) > \frac{1}{2}$ *we have*

$$\sum_{n\in\mathbb{Z}}\frac{1}{|z+n|^{2s}} = \frac{\sqrt{\pi}\Gamma(s-\frac{1}{2})}{\Gamma(s)}y^{1-2s} + \frac{2\pi^s}{\Gamma(s)}y^{\frac{1}{2}-s}\sum_{r\neq 0}|r|^{s-\frac{1}{2}}K_{s-\frac{1}{2}}(2\pi|r|y)\,e^{2\pi irx}. \tag{4}$$

Proof. Recall the Poisson summation formula

$$\sum_{n\in\mathbb{Z}}\varphi(x+n) = \sum_{n\in\mathbb{Z}}\widehat{\varphi}(n)e^{2\pi inx},$$

where φ is a continuous function that decays sufficiently rapidly at infinity (for example, $|f(x)| < |x|^{-c}$ with $c>1$) and where $\widehat{\varphi}(n) = \int_{-\infty}^{\infty}\varphi(x)e^{-2\pi inx}\mathrm{d}x$ is the Fourier transform. We apply this formula to $\varphi(x) = |x+iy|^{-2s}$, where $z = x+iy \in \mathbb{H}$ and $\mathrm{Re}(s) > \frac{1}{2}$. The Poisson summation formula gives

$$\sum_{n\in\mathbb{Z}}\frac{1}{|z+n|^{2s}} = \sum_{n\in\mathbb{Z}}\left(\int_{-\infty}^{\infty}\frac{e^{-2\pi inx}}{(x^2+y^2)^s}\mathrm{d}x\right)e^{2\pi inx}.$$

Using Lemma 4 we get the result. □

3 Level 1

Fourier expansion of

$$\zeta(2s)E(z,s) = \frac{1}{2} \sum_{\substack{m,n\in\mathbb{Z} \\ (m,n)\neq 0}} \frac{y^s}{|mz+n|^{2s}}. \tag{5}$$

We split up the sum (5) into the terms with $m = 0$ and those with $m \neq 0$ and combine each positive summand with its negative. We obtain

$$\zeta(2s)E(z,s) = \zeta(2s)y^s + \sum_{m=1}^{\infty} \sum_{n\in\mathbb{Z}} \frac{y^s}{|mz+n|^{2s}}.$$

Using the substitution $n = tm + r$ and Lemma 5 gives

$$\begin{aligned}
\zeta(2s)E(z,s) &= \zeta(2s)y^s + y^s \sum_{m=1}^{\infty} |m|^{-2s} \sum_{r=1}^{m} \sum_{t\in\mathbb{Z}} \left| t + z + \frac{r}{m} \right|^{-2s} \\
&= \zeta(2s)y^s + y^s \sum_{m=1}^{\infty} |m|^{-2s} \\
&\quad \times \left[\frac{\sqrt{\pi}\Gamma(s-\frac{1}{2})}{\Gamma(s)} m y^{1-2s} + \frac{2\pi^s}{\Gamma(s)} y^{\frac{1}{2}-s} \sum_{t\neq 0} |t|^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|t|y) \sum_{r=1}^{m} e^{2\pi i t\left(x+\frac{r}{m}\right)} \right] \\
&= \zeta(2s)y^s + \frac{\sqrt{\pi}\Gamma(s-\frac{1}{2})\zeta(2s-1)}{\Gamma(s)} y^{1-s} \\
&\quad + \frac{2\pi^s\sqrt{y}}{\Gamma(s)} \sum_{t\neq 0} |t|^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|t|y) e^{2\pi i t x} \sum_{m=1}^{\infty} |m|^{-2s} \sum_{r=1}^{m} e^{\frac{2\pi i t r}{m}}.
\end{aligned}$$

The sum $\sum_{r=1}^{m} e^{\frac{2\pi i t r}{m}}$ is zero unless $t = lm$ for some $l \in \mathbb{Z}$ in which case the sum is m. Therefore we get

$$\begin{aligned}
\zeta(2s)E(z,s) &= \zeta(2s)y^s + \frac{\sqrt{\pi}\Gamma(s-\frac{1}{2})\zeta(2s-1)}{\Gamma(s)} y^{1-s} + \frac{2\pi^s\sqrt{y}}{\Gamma(s)} \sum_{t\neq 0} |t|^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|t|y) e^{2\pi i t x} \sum_{\substack{m\geq 1 \\ t=ml}} |m|^{-2s+1} \\
&= \zeta(2s)y^s + \frac{\sqrt{\pi}\Gamma(s-\frac{1}{2})\zeta(2s-1)}{\Gamma(s)} y^{1-s} + \frac{2\pi^s\sqrt{y}}{\Gamma(s)} \sum_{t\neq 0} |t|^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|t|y) e^{2\pi i t x} \sum_{l|t} \left| \frac{t}{l} \right|^{-2s+1} \\
&= \zeta(2s)y^s + \frac{\sqrt{\pi}\Gamma(s-\frac{1}{2})\zeta(2s-1)}{\Gamma(s)} y^{1-s} + \frac{2\pi^s\sqrt{y}}{\Gamma(s)} \sum_{t\neq 0} \sigma_{1-2s}(t) |t|^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|t|y) e^{2\pi i t x}.
\end{aligned}$$

4 Level N

Fourier expansion of

$$E(z,s)=\frac{1}{2}\sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq 0}}\frac{y^s}{|mNz+n|^{2s}}. \tag{6}$$

We split up the sum (6) into the terms with $m=0$ and those with $m\neq 0$ and combine each positive summand with its negative. We obtain

$$E(z,s)=\zeta(2s)y^s+\sum_{m=1}^{\infty}\sum_{n\in\mathbb{Z}}\frac{y^s}{|mNz+n|^{2s}}.$$

Using the substitution $n=tmN+r$ and Lemma 5 gives

$$\begin{aligned}
E(z,s)&=\zeta(2s)y^s+\frac{y^s}{N^{2s}}\sum_{m=1}^{\infty}|m|^{-2s}\sum_{r=1}^{mN}\sum_{t\in\mathbb{Z}}\left|t+z+\frac{r}{mN}\right|^{-2s}\\
&=\zeta(2s)y^s+\frac{y^s}{N^{2s}}\sum_{m=1}^{\infty}|m|^{-2s}\\
&\quad\times\left[\frac{\sqrt{\pi}\Gamma(s-\frac{1}{2})}{\Gamma(s)}mNy^{1-2s}+\frac{2\pi^s}{\Gamma(s)}y^{\frac{1}{2}-s}\sum_{t\neq 0}|t|^{s-\frac{1}{2}}K_{s-\frac{1}{2}}(2\pi|t|y)\sum_{r=1}^{mN}e^{2\pi it\left(x+\frac{r}{mN}\right)}\right]\\
&=\zeta(2s)y^s+\frac{\sqrt{\pi}\Gamma(s-\frac{1}{2})\zeta(2s-1)}{\Gamma(s)N^{2s-1}}y^{1-s}\\
&\quad+\frac{2\pi^s\sqrt{y}}{\Gamma(s)N^{2s}}\sum_{t\neq 0}|t|^{s-\frac{1}{2}}K_{s-\frac{1}{2}}(2\pi|t|y)e^{2\pi itx}\sum_{m=1}^{\infty}|m|^{-2s}\sum_{r=1}^{mN}e^{\frac{2\pi itr}{mN}}.
\end{aligned}$$

The sum $\sum_{r=1}^{mN}e^{\frac{2\pi itr}{mN}}$ is zero unless $t=lmN$ for some $l\in\mathbb{Z}$ in which case the sum is mN. Therefore we get

$$\begin{aligned}
E(z,s)&=\zeta(2s)y^s+\frac{\sqrt{\pi}\Gamma(s-\frac{1}{2})\zeta(2s-1)}{\Gamma(s)N^{2s-1}}y^{1-s}\\
&\quad+\frac{2\pi^s\sqrt{y}}{\Gamma(s)N^{2s}}\sum_{t\neq 0}|t|^{s-\frac{1}{2}}K_{s-\frac{1}{2}}(2\pi|t|y)e^{2\pi itx}\sum_{\substack{m\geq 1\\t=mNl}}|m|^{-2s+1}N\\
&=\zeta(2s)y^s+\frac{\sqrt{\pi}\Gamma(s-\frac{1}{2})\zeta(2s-1)}{\Gamma(s)N^{2s-1}}y^{1-s}\\
&\quad+\frac{2\pi^s\sqrt{y}}{\Gamma(s)N^{2s}}\sum_{t\neq 0}|t|^{s-\frac{1}{2}}K_{s-\frac{1}{2}}(2\pi|t|y)e^{2\pi itx}\sum_{Nl|t}\left|\frac{t}{Nl}\right|^{-2s+1}N\\
&=\zeta(2s)y^s+\frac{\sqrt{\pi}\Gamma(s-\frac{1}{2})\zeta(2s-1)}{\Gamma(s)N^{2s-1}}y^{1-s}\\
&\quad+\frac{2\pi^s\sqrt{y}}{\Gamma(s)N^{2s-1}}\sum_{t\neq 0}\sigma_{1-2s}(t)|t|^{s-\frac{1}{2}}K_{s-\frac{1}{2}}(2\pi|t|y)e^{2\pi itx}.
\end{aligned}$$

5 Level N and χ mod q

Let χ be an even primitive Dirichlet character mod q. We determine the Fourier expansion of

$$E^*(z,s,\chi) := \frac{1}{2} \sum_{\substack{m,n\in\mathbb{Z} \\ (m,n)\neq 0}} \chi(n) \frac{y^s}{|mNz+n|^{2s}}. \tag{7}$$

We start with

$$E^*(z,s,\chi) = L(2s,\chi)y^s + \sum_{m=1}^{\infty} \sum_{n\in\mathbb{Z}} \chi(n) \frac{y^s}{|mNz+n|^{2s}}.$$

Using the substitution $n = tmN + r$ and Lemma 5 gives

$$\begin{aligned} E^*(z,s,\chi) &= L(2s,\chi)y^s \\ &\quad + \frac{y^s}{N^{2s}} \sum_{m=1}^{\infty} |m|^{-2s} \sum_{r=1}^{mN} \chi(tmN+r) \sum_{t\in\mathbb{Z}} \left| t + z + \frac{r}{mN} \right|^{-2s} \\ &= L(2s,\chi)y^s + \frac{y^s}{N^{2s}} \sum_{m=1}^{\infty} |m|^{-2s} \sum_{r=1}^{mN} \chi(tmN+r) \\ &\quad \times \left[\frac{\sqrt{\pi}\Gamma(s-\frac{1}{2})}{\Gamma(s)} y^{1-2s} + \frac{2\pi^s}{\Gamma(s)} y^{\frac{1}{2}-s} \sum_{t\neq 0} |t|^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|t|y)\, e^{2\pi i t\left(x+\frac{r}{mN}\right)} \right]. \end{aligned}$$

6 Level q^2 and χ mod q

Let χ be an even primitive Dirichlet character mod q. We determine the Fourier expansion of

$$E^*(z,s,\chi) := \frac{1}{2} \sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq 0}} \chi(n) \frac{y^s}{|mq^2 z+n|^{2s}}. \tag{8}$$

We start with

$$E^*(z,s,\chi) = L(2s,\chi)y^s + \sum_{m=1}^{\infty}\sum_{n\in\mathbb{Z}} \chi(n)\frac{y^s}{|mq^2z+n|^{2s}}.$$

Using the substitution $n = tmq^2 + r$ and Lemma 5 gives

$$\begin{aligned}
E^*(z,s,\chi) &= L(2s,\chi)y^s \\
&\quad + \frac{y^s}{q^{4s}} \sum_{m=1}^{\infty} |m|^{-2s} \sum_{r=1}^{mq^2} \chi(r) \sum_{t\in\mathbb{Z}} \left| t+z+\frac{r}{mq^2}\right|^{-2s} \\
&= L(2s,\chi)y^s + \frac{y^s}{q^{4s}} \sum_{m=1}^{\infty} |m|^{-2s} \sum_{r=1}^{mq^2} \chi(r) \\
&\quad \times \left[\frac{\sqrt{\pi}\Gamma(s-\frac{1}{2})}{\Gamma(s)} y^{1-2s} + \frac{2\pi^s}{\Gamma(s)} y^{\frac{1}{2}-s} \sum_{t\neq 0} |t|^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|t|y)\, e^{2\pi i t\left(x+\frac{r}{mq^2}\right)} \right] \\
&= L(2s,\chi)y^s + \frac{2\pi^s\sqrt{y}}{q^{4s}\Gamma(s)} \sum_{t\neq 0} |t|^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|t|y) e^{2\pi i t x} \sum_{m=1}^{\infty} |m|^{-2s} \sum_{r=1}^{mq^2} \chi(r) e^{\frac{2\pi i t r}{mq^2}}.
\end{aligned}$$

In the last equality we used Lemma 1 according to which $\sum_{r=1}^{mq^2} \chi(r) = \sum_{r=1}^{q} \chi(r) = 0$. By Lemma 3 the last sum equals

$$\sum_{r=1}^{mq^2} \chi(r) e^{\frac{2\pi i t r}{mq^2}} = \begin{cases} mq\tau(\chi)\overline{\chi}(l) & \text{if } t = mql \text{ for some } l\in\mathbb{Z} \\ 0 & \text{else} \end{cases}.$$

Therefore we get

$$\begin{aligned}
E^*(z,s,\chi) &= L(2s,\chi)y^s + \frac{2\pi^s \tau(\chi)\sqrt{y}}{q^{4s}\Gamma(s)} \sum_{t\neq 0} |t|^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|t|y) e^{2\pi i t x} \sum_{\substack{m\geq 1\\ t=mql}} |m|^{-2s} mq\overline{\chi}(l) \\
&= L(2s,\chi)y^s + \frac{2\pi^s \tau(\chi)\sqrt{y}}{q^{4s-1}\Gamma(s)} \sum_{t\neq 0} |t|^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|t|y) e^{2\pi i t x} \sum_{ql|t} \left|\frac{t}{ql}\right|^{1-2s} \overline{\chi}(l).
\end{aligned}$$

7 Level Nq^2 and χ mod q

Let χ be an even primitive Dirichlet character mod q. We determine the Fourier expansion of

$$E^*(z,s,\chi) := \frac{1}{2} \sum_{\substack{m,n\in\mathbb{Z} \\ (m,n)\neq 0}} \chi(n) \frac{y^s}{|mNq^2 z + n|^{2s}}. \tag{9}$$

We start with

$$E^*(z,s,\chi) = L(2s,\chi) y^s + \sum_{m=1}^{\infty} \sum_{n\in\mathbb{Z}} \chi(n) \frac{y^s}{|mNq^2 z + n|^{2s}}.$$

Using the substitution $n = tmNq^2 + r$ and Lemma 5 gives

$$\begin{aligned}
E^*(z,s,\chi) &= L(2s,\chi)y^s \\
&\quad + \frac{y^s}{q^{4s}N^{2s}} \sum_{m=1}^{\infty} |m|^{-2s} \sum_{r=1}^{mNq^2} \chi(r) \sum_{t\in\mathbb{Z}} \left| t + z + \frac{r}{mNq^2} \right|^{-2s} \\
&= L(2s,\chi)y^s + \frac{y^s}{q^{4s}N^{2s}} \sum_{m=1}^{\infty} |m|^{-2s} \sum_{r=1}^{mNq^2} \chi(r) \\
&\quad \times \left[\frac{\sqrt{\pi}\Gamma(s-\frac{1}{2})}{\Gamma(s)} y^{1-2s} + \frac{2\pi^s}{\Gamma(s)} y^{\frac{1}{2}-s} \sum_{t\neq 0} |t|^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|t|y)\, e^{2\pi i t\left(x + \frac{r}{mNq^2}\right)} \right] \\
&= L(2s,\chi)y^s + \frac{2\pi^s\sqrt{y}}{q^{4s}N^{2s}\Gamma(s)} \sum_{t\neq 0} |t|^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|t|y) e^{2\pi i t x} \sum_{m=1}^{\infty} |m|^{-2s} \sum_{r=1}^{mNq^2} \chi(r) e^{\frac{2\pi i t r}{mNq^2}}.
\end{aligned}$$

In the last equality we used Lemma 1 according to which $\sum_{r=1}^{mNq^2} \chi(r) = \sum_{r=1}^{q} \chi(r) = 0$. By Lemma 3 the last sum equals

$$\sum_{r=1}^{mNq^2} \chi(r) e^{\frac{2\pi i t r}{mNq^2}} = \begin{cases} mNq\tau(\chi)\overline{\chi}(l) & \text{if } t = mqNl \text{ for some } l \in \mathbb{Z} \\ 0 & \text{else} \end{cases}.$$

Therefore we get

$$\begin{aligned}
E^*(z,s,\chi) &= L(2s,\chi)y^s + \frac{2\pi^s\tau(\chi)\sqrt{y}}{q^{4s}N^{2s}\Gamma(s)} \sum_{t\neq 0} |t|^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|t|y) e^{2\pi i t x} \sum_{\substack{m\geq 1 \\ t = mqNl}} |m|^{-2s} mNq\overline{\chi}(l) \\
&= L(2s,\chi)y^s + \frac{2\pi^s\tau(\chi)\sqrt{y}}{q^{4s-1}N^{2s-1}\Gamma(s)} \sum_{t\neq 0} |t|^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|t|y) e^{2\pi i t x} \sum_{qNl|t} \left| \frac{t}{qNl} \right|^{1-2s} \overline{\chi}(l).
\end{aligned}$$