Communications in Mathematical Physics manuscript No.
(will be inserted by the editor)

Something Hyperbolic

MG[1,2], **MH**[2], **WI**[2,3], **PJ**[2], **ST**[2]

[1] Institute of Physics, ETH Zürich, Zürich, Switzerland.
[2] Physik-Institut, Universität Zürich, Zürich, Switzerland
[3] Institute of Astronomy, Madingley Road, Cambridge CB3 0HA

Received: date / Accepted: date

Abstract: Insert your abstract here.

1. Introduction

Your text comes here. Separate text sections with

1.1. Subsection title. as required. Don't forget to give each section and subsection a unique label (see Sect. **??**).

2. Aspects of gravitational wave emission

In this section we briefly describe the multipole decomposition formalism in linearised general relativity as given by Thorne [1]. Then this formalism is applied to determine in the mass quadrupole approximation the energy loss, angular momentum loss and energy spectrum for binary systems on elliptic, parabolic and hyperbolic Keplerian orbits.

2.1. Multipole expansion formalism. In the wave zone, gravitational waves can be treated as linearised perturbations propagating on a flat background metric. This background metric can be characterized by a Minkowskiian coordinate system whose origin coincides with the source of mass content responsible for the gravitational waves. In this coordinate frame the transverse and traceless part of the metric perturbation has the form

$$h_{ij}^{TT} = \frac{1}{r} A_{ij}(t - r, \theta, \phi), \tag{1}$$

where A_{ij} is a traceless and transverse tensor. The angular dependence of the radiation field A_{ij} can be decomposed into spherical harmonics. The most suitable set of harmonics are the pure-spin harmonics with well-defined transversality and helicity properties (for more information see [1]). Because A_{ij} is traceless and transverse, it contains only the transverse and traceless harmonics $\mathbf{T}^{\mathrm{E}2,lm}$ and $\mathbf{T}^{\mathrm{B}2,lm}$. The purely longitudinal, purely transversal or mixed longitudinal and transversal components are not encountered in standard general relativity, but need to be included in other metric theories of gravity. For $l = 0$ and $l = 1$ there are no transverse traceless harmonics. Therefore the general form of radiation field has the form

$$h_{ij}^{TT} = \frac{1}{r} \sum_{l=2}^{\infty} \sum_{m=-l}^{l} \left[{}^{(l)}I^{lm}(t-r)\, T_{ij}^{\mathrm{E}2,lm} + {}^{(l)}S^{lm}(t-r)\, T_{ij}^{\mathrm{B}2,lm} \right]. \tag{2}$$

The expansion coefficients ${}^{(l)}I^{lm}$ and ${}^{(l)}S^{lm}$ are the l-th time derivative of I^{lm} and S^{lm}. The components I^{lm} and S^{lm} are said to be responsible for mass multipole radiation and current multipole radiation respectively. For that reason, the leading order terms I^{2m} are responsible for the mass quadrupole radiation, terms associated with S^{2m} are called current quadrupole contributions, terms arising from I^{3m} are called mass octupole contributions, etc. The explicit expression of I^{lm} is given by

$$I^{lm} = \frac{16\pi}{(2l+1)!!} \left(\frac{(l+1)(l+2)}{2l(l-1)} \right)^{1/2} \mathcal{G}_{A_l} \mathcal{Y}_{A_l}^{lm*}, \tag{3}$$

where $\mathcal{Y}^{lm}$ are symmetric trace free tensors (for their properties see again [1]) and where $\mathcal{G}_{A_l}$ stands for the mass multipole moments. The general form of the mass multipole moments $\mathcal{G}_{A_l}$ is given by a sum of several integrals involving the energy-stress tensor and spherical Bessel functions, see equation (5.9a) in [1]. Our calculations will only involve the second mass moments, for which $\mathcal{G}_{A_2}$ in the Newtonian limit becomes equation (8). We also don't need expressions for the current multipole contributions S^{lm} in this work.
In the next sections we will frequently make use of the following STF tensors

$$\mathcal{Y}^{22} = \left(\frac{15}{32\pi} \right)^{1/2} \begin{pmatrix} 1 & i & 0 \\ i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{4}$$

$$\mathcal{Y}^{21} = -\left(\frac{15}{32\pi} \right)^{1/2} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & i \\ 1 & i & 0 \end{pmatrix}, \tag{5}$$

$$\mathcal{Y}^{20} = \left(\frac{5}{16\pi} \right)^{1/2} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \tag{6}$$

together with $\mathcal{Y}^{2-m} = (-1)^m \left(\mathcal{Y}^{2m}\right)^*$.

2.1.1. Energy loss. In the multipole formalism the total radiated power is given by

$$P = \frac{dE}{dt} = \frac{1}{32\pi} \sum_{l=2}^{\infty} \sum_{m=-l}^{l} \left\langle \, |^{(l+1)}I^{lm}|^2 + |^{(l+1)}S^{lm}|^2 \right\rangle . \tag{7}$$

In case of a periodic motion, the brackets denote orbital averaging of the expression. In case of unbound motion, the brackets are to be understood as the integral of the expression over the entire orbit. In what follows, we only consider mass quadrupole radiation, that is, we consider only contributions arising from I^{2m}. If the source is non-relativistic and has a negligible self-gravity, it is possible to express the second mass moment $\mathcal{G}_{a_1 a_2}$ in terms of the Newtonian mass density ρ

$$\mathcal{G}_{a_1 a_2} = \int d^3x \, \rho \, x_{a_1} x_{a_2} . \tag{8}$$

Then

$$I^{2m} = \frac{16\pi}{5\sqrt{3}} \int d^3x \, \rho \, \mathcal{Y}^{2m*}_{a_1 a_2} \, x_{a_1} x_{a_2} \tag{9}$$

$$\equiv \frac{16\pi}{5\sqrt{3}} \int d^3x \, \rho \, r^2 Y^{2m*} . \tag{10}$$

2.1.2. Angular momentum loss. In the multipole expansion formalism, the change in angular momentum of a system due to gravitational wave emission is given by

$$\frac{dL_j}{dt} = \sum_{l=2}^{\infty} \frac{(l+1)(l+2)}{(l-1)l!(2l+1)!!} \left\langle \epsilon_{jpq} \, {}^{(l)}\mathcal{G}_{pA_{l-1}} \, {}^{(l+1)}\mathcal{G}_{qA_{l-1}} \right\rangle \tag{11}$$

$$+ \sum_{l=2}^{\infty} \frac{4l^2(l+2)}{(l-1)(l+1)!(2l+1)!!} \left\langle \epsilon_{jpq} \, {}^{(l)}\mathcal{S}_{pA_{l-1}} \, {}^{(l+1)}\mathcal{S}_{qA_{l-1}} \right\rangle . \tag{12}$$

In the mass quadrupole approximation this reduces to

$$\frac{dL_j}{dt} = \frac{2}{5} \left\langle \epsilon_{jpq} \, {}^{(2)}\mathcal{G}_{pa} \, {}^{(3)}\mathcal{G}_{qa} \right\rangle . \tag{13}$$

2.2. Application to binaries on Keplerian orbits. In this section we calculate, for binaries on a Keplerian orbit and for all $e \geqslant 0$, the energy flux, angular momentum flux and energy spectrum resulting from gravitational wave emission in the mass quadrupole approximation.

The energy emitted by gravitational waves from an elliptic Keplerian binary system was first calculated by Peters & Mathews [2]. The first calculation of the angular momentum loss for elliptic orbits is due to Peters [3]. The analogous expressions for binary systems on hyperbolic orbits were first derived by Hansen [4], both contain minor numerical mistakes. The error in Hansen's expression for the energy flux was rectified by Turner [5] and the first correct expression for the angular momentum flux is found in Junker & Schäfer [6]. The first determination of the elliptic frequency spectrum is given in Peters and Mathews's paper. The parabolic limit of the elliptic energy spectrum was taken in Berry and Gair [7]. A first complete calculation of the hyperbolic frequency spectrum is found in DeVittori *et al.* [8], their computation was subsequently corrected by García-Bellido & Nesseris [9].

In what follows we will rederive all the above results in an unified way.

2.2.1. Energy loss. We choose a coordinate system whose origin coincides with the center of mass of the system and we let the binary lie in the $\theta = \pi/2$-plane. The two masses can then be specified by $(r_1 \cos\phi, r_1 \sin\phi)$ and $(-r_2\cos\phi, -r_2 \sin\phi)$ with $r_i = \frac{\mu r}{m_i}$. We obtain using equation (10)

$$I^{2m} = \frac{16\pi}{5\sqrt{3}} \left[m_1 \frac{\mu^2 r^2}{m_1^2} Y^{2m*}\left(\frac{\pi}{2}, \phi\right) + m_2 \frac{\mu^2 r^2}{m_2^2} Y^{2m*}\left(\frac{\pi}{2}, \phi + \pi\right)\right]. \tag{14}$$

Because the $Y^{2\pm1*}\left(\frac{\pi}{2}, \phi\right)$ terms vanish, it is possible to set in the above expression $Y^{2m*}\left(\frac{\pi}{2}, \phi\right)$ equal to $Y^{2m*}\left(\frac{\pi}{2}, \phi+\pi\right)$. Hence we can write

$$I^{2m} = \frac{16\pi}{5\sqrt{3}} \mu r^2 Y^{2m*}\left(\frac{\pi}{2}, \phi\right). \tag{15}$$

In the quadrupole approximation the power radiated by the system reduces from expression (7) to

$$P = \frac{1}{32\pi} \sum_{m=-2}^{2} |{}^{(3)}I^{2m}|^2. \tag{16}$$

The orbit equations

$$\begin{aligned} r &= \frac{r_p(1+e)}{1+e\cos\phi}, \\ r^2 \frac{d\phi}{dt} &= [M r_p (1+e)]^{1/2}, \end{aligned} \tag{17}$$

allow the calculation of the time derivatives of I^{2m}. Employing these identities and using the explicit expression of the spherical harmonics Y^{2m*}, the third time derivatives of I^{2m} read

$$\begin{aligned} {}^{(3)}I^{20} &= \left(\frac{64\pi}{15}\right)^{1/2} \frac{\mu M^{3/2}}{[r_p(1+e)]^{5/2}} (1+e\cos\phi)^2 e \sin\phi, \\ {}^{(3)}I^{2-2} &= \left(\frac{32\pi}{5}\right)^{1/2} \frac{\mu M^{3/2}}{[r_p(1+e)]^{5/2}} (1+e\cos\phi)^2 [-e\sin\phi - 4i(1+e\cos\phi)] e^{2i\phi}, \\ {}^{(3)}I^{22} &= \left({}^{(3)}I^{2-2}\right)^*. \end{aligned}$$

The radiated power as a function of ϕ along the orbit is then

$$P(\phi) = \frac{8}{15} \frac{\mu^2 M^3}{[r_p(1+e)]^5} (1+e\cos\phi)^4 \left[e^2 \sin^2\phi + 12(1+e\cos\phi)^2\right]. \tag{18}$$

Note that this expression is valid for all values of $e \geq 0$.

Hyperbolic orbit and parabolic orbit $e \geq 1$:. The asymptotes of an hyperbolic orbit are given by $\cos\psi = -1/e$. Thus the total energy released by a binary on an hyperbolic orbit is

$$
\begin{aligned}
E &= \int_{-\psi}^{\psi} P(\phi)\frac{dt}{d\phi}\,d\phi && (19)\\
&= \frac{8}{15}\frac{\eta^2 M^{9/2}}{(r(1+e))^{7/2}}\int_{-\psi}^{\psi} d\phi\,(1+e\cos\phi)^2\left[e^2\sin^2\phi + 12(1+e\cos\phi)^2\right] && (20)\\
&= \frac{\eta^2 M^{9/2}}{r_p^{7/2}(1+e)^{7/2}}\varrho(e), && (21)
\end{aligned}
$$

where

$$
\begin{aligned}
\varrho(e) &= \frac{1}{180}\,[(12(96+292e^2+37e^4)\phi + 48e(96+73e^2)\sin\phi \\
&\quad + 24e^2(71+12e^2)\sin 2\phi + 368e^3\sin 3\phi + 33e^4\sin 4\phi]_{-\psi}^{\psi} \\
&= \frac{64}{5}\left[\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right)\cos^{-1}\left(-\frac{1}{e}\right) + \frac{301}{144}(e^2-1)^{1/2}\left(1+\frac{673}{602}e^2\right)\right].
\end{aligned}
$$

For a parabolic orbit, $e = 1$, the radiated energy becomes

$$E = \frac{85\pi}{12\sqrt{2}}\frac{\eta^2 M^{9/2}}{r_p^{7/2}}. \qquad (22)$$

Elliptic orbit $e < 1$:. In case of elliptic orbits, the average of $P(\phi)$ over one orbital period gives

$$
\begin{aligned}
\frac{dE}{dt} = P &= \frac{\Omega}{2\pi}\int_0^{2\pi} P(\phi)\frac{dt}{d\phi}\,d\phi \\
&= \frac{8}{15}\frac{1}{(1-e^2)^{7/2}}\frac{\mu^2 M^3}{a^5}\int_0^{2\pi}\frac{d\phi}{2\pi}(1+e\cos\phi)^2\left[e^2\sin^2\phi + 12(1+e\cos\phi)^2\right].
\end{aligned}
$$

After performing the integral we arrive at

$$\frac{dE}{dt} = \frac{32}{5}\frac{\mu^2 M^3}{a^5}\frac{1}{(1-e^2)^{7/2}}\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right). \qquad (23)$$

There is a simple way to deduce the energy loss of a parabolic orbit from the energy loss of an elliptic orbit. The energy radiated per orbit is $E = \frac{2\pi}{\Omega}P$. For a parabolic orbit, the energy radiated per orbit is the total energy radiated. By analytically continuing the orbital parameters to $e = 1$, we then obtain indeed equation (22). An extension of the results from $e < 1$ beyond $e = 1$ is, when resorting to the parametrization used here, not possible.

2.2.2. Angular momentum loss. We now proceed to determine the GW-induced angular momentum loss of a binary on a Keplerian orbit. To this end we choose a coordinate system such that the orbits lies in the (x, y)-plane; then we can set $L = L_z$. Therefore we have

$$\frac{dL}{dt} = \frac{2}{5}\left({}^{(2)}\mathcal{G}_{1a}{}^{(3)}\mathcal{G}_{2a} - {}^{(2)}\mathcal{G}_{2a}{}^{(3)}\mathcal{G}_{1a}\right). \tag{24}$$

The second mass moments in our chosen coordinate frame are given by

$$\mathcal{G} = \mu r^2 \begin{bmatrix} \cos^2\phi & \cos\phi\sin\phi & 0 \\ \cos\phi\sin\phi & \sin^2\phi & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{25}$$

By making use of the orbit equations (17), we can calculate the required derivatives and obtain

$$\begin{aligned}
{}^{(2)}\mathcal{G}_{12} &= -\frac{\mu M}{r_p(1+e)}\sin\phi\,[4\cos\phi + e(3+\cos(2\phi))], \\
{}^{(3)}\mathcal{G}_{12} &= -\left(\frac{\mu^2 M^3}{4[r_p(1+e)]^5}\right)^{1/2}(1+e\cos\phi)^2(5e\cos\phi + 8\cos(2\phi) + 3e\cos(3\phi)), \\
{}^{(2)}\mathcal{G}_{11} &= -\frac{\mu M}{2r_p(1+e)}(3e\cos\phi + 4\cos(2\phi) + e\cos(3\phi)), \\
{}^{(3)}\mathcal{G}_{11} &= \left(\frac{\mu^2 M^3}{[r_p(1+e)]^5}\right)^{1/2}(1+e\cos\phi)^2(4+3e\cos\phi)\sin(2\phi), \\
{}^{(2)}\mathcal{G}_{22} &= \frac{\mu M}{2r_p(1+e)}[7e\cos\phi + 4\cos(2\phi) + e(4e+\cos(3\phi))], \\
{}^{(3)}\mathcal{G}_{22} &= -\left(\frac{\mu^2 M^3}{[r_p(1+e)]^5}\right)^{1/2}(1+e\cos\phi)^2[8\cos\phi + e(5+3\cos(2\phi))]\sin\phi.
\end{aligned}$$

The angular momentum emitted as a function of ϕ along the orbit is then

$$\frac{dL}{dt} = \frac{4}{5}\frac{\mu^2 M^{5/2}}{[r_p(1+e)]^{7/2}}(1+e\cos\phi)^3(8+e^2+12e\cos\phi+3e^2\cos(2\phi)). \tag{26}$$

Hyperbolic orbit and parabolic orbit $e \geq 1$:. The angular momentum loss integrated over the entire hyperbolic orbit is

$$L = \int_{-\psi}^{\psi}\frac{dL}{dt}\frac{dt}{d\phi}\,d\phi \tag{27}$$

$$= \frac{4}{5}\frac{\mu^2 M^2}{[r_p(1+e)]^2}\int_{-\psi}^{\psi}d\phi\,(1+e\cos\phi)(8+e^2+12e\cos\phi+3e^2\cos(2\phi)) \tag{28}$$

$$= \frac{4}{5}\frac{\mu^2 M^2}{[r_p(1+e)]^2}\,\varpi|_{-\psi}^{\psi}, \tag{29}$$

where

$$\varpi = 8\phi + 7e^2\phi + 20e\sin\phi + \frac{5}{2}e^3\sin\phi + \frac{9}{2}e^2\sin(2\phi) + \frac{1}{2}e^3\sin(3\phi).$$

The evaluation of ϖ between $-\psi = -\cos^{-1}(-1/e)$ and $\psi = \cos^{-1}(-1/e)$ gives

$$\varpi|_{-\psi}^{\psi} = 2\left(8+7e^2\right)\cos^{-1}\left(-\frac{1}{e}\right) + 2(e^2-1)^{1/2}\left(13+2e^2\right). \tag{30}$$

For a parabolic orbit, $e=1$, the emitted angular momentum becomes

$$L = 6\pi\frac{\mu^2 M^2}{r_p^2}. \tag{31}$$

Elliptic orbit $e<1$*:*. In case of elliptic orbits, the angular momentum change averaged over one orbital period is

$$\begin{aligned}\frac{dL}{dt} &= \frac{\Omega}{2\pi}\int_0^{2\pi}\frac{dL}{dt}\frac{dt}{d\phi}\,d\phi \\ &= \frac{4}{10\pi}\frac{\mu^2 M^{5/2}}{r_p^{7/2}}\frac{(1-e)^{3/2}}{(1+e)^2}\,\varpi|_0^{2\pi}.\end{aligned}$$

The expression $\varpi|_0^{2\pi}$ evaluates to $2\pi(8+7e^2)$ and we have

$$\frac{dL}{dt} = \frac{32}{5}\frac{\mu^2 M^{5/2}}{a^{7/2}}\frac{1}{(1-e^2)^2}\left(1+\frac{7}{8}e^2\right). \tag{32}$$

Analogous to previous subsection, we can perform an analytical continuation of the above result to $e=1$, thereby again obtaining the parabolic limit (31) of the hyperbolic case.

2.2.3. Frequency spectrum. In this section we compute for all $e \geq 0$ the frequency spectrum of the radiated power in the mass quadrupole approximation. The key ingredient for this is to perform a Fourier decomposition of the binary's second mass moment.

Elliptic orbit $e < 1$:. We parametrize an elliptic Keplerian orbit in the following form

$$x(\beta) = a(\cos u - e), \tag{33}$$
$$y(\beta) = b \sin u. \tag{34}$$

Here a is the semi-major axis and b is the semi-minor axis of the orbit. The eccentric anomaly u is defined by the Kepler equation

$$u - e \sin u = \Omega t \equiv \beta. \tag{35}$$

The second mass moment has then the form

$$\mathcal{G} = \mu \begin{bmatrix} a^2(\cos u - e)^2 & ab(\cos u - e)\sin u & 0 \\ ab(\cos u - e)\sin u & b^2 \sin^2 u & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{36}$$

$$= \mu \begin{bmatrix} a^2(-2e\cos u + \frac{1}{2}\cos(2u) + e^2 + \frac{1}{2}) & ab(-e\sin u + \frac{1}{2}\sin(2u)) & 0 \\ ab(-e\sin u + \frac{1}{2}\sin(2u)) & \frac{b^2}{2}(1 - \cos(2u)) & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{37}$$

The emitted power in the mass quadrupole approximation is obtained from the formulae (3) and (16)

$$P = \frac{8\pi}{75} \sum_{m=-2}^{2} \left\langle \left| {}^{(3)}\mathcal{G}_{a_1 a_2} \left(\mathcal{Y}^{2m}_{a_1 a_2}\right)^* \right|^2 \right\rangle \tag{38}$$

$$= \frac{8\pi}{75} \sum_{m=-2}^{2} \frac{1}{2T} \int_{-T}^{T} dt \left| {}^{(3)}\mathcal{G}_{a_1 a_2} \left(\mathcal{Y}^{2m}_{a_1 a_2}\right)^* \right|^2, \tag{39}$$

with the orbital period T. To find the frequency decomposition of the emitted power we can make use of Parseval's identity. This identity asserts equality between the integral of the norm of a periodic function f and the sum of the norm of the function's Fourier coefficients $\mathcal{F}(f)$

$$\frac{1}{2T} \int_{-T}^{T} dt\, |f(t)|^2 = \sum_{n=-\infty}^{\infty} |\mathcal{F}(f)|^2. \tag{40}$$

The effect of the third derivative is to multiply the power spectrum by $(n\Omega)^6$ while at the same time replacing ${}^{(3)}\mathcal{G}_{a_1 a_2}$ by $\mathcal{G}_{a_1 a_2}$. Hence we can write for the radiated power in the n-th harmonic of the orbital angular frequency

$$P_n = 2 \times \frac{8\pi}{75} (n\Omega)^6 \sum_{m=-2}^{2} \left| \mathcal{F}\left(\mathcal{G}_{a_1 a_2} \left(\mathcal{Y}^{2m}_{a_1 a_2}\right)^* \right) \right|^2 \tag{41}$$

$$= \frac{1}{15} (n\Omega)^6 \, \mu^2 \, \Psi, \tag{42}$$

where

$$
\begin{aligned}
\Psi = &\left|2a^2 e\,\mathcal{F}(\cos(u)) - \frac{1}{2}\mathcal{F}(\cos(2u))\left(a^2-b^2\right)\right|^2 \\
&+ 3\left|-2a^2 e\,\mathcal{F}(\cos(u)) + \frac{1}{2}\mathcal{F}(\cos(2u))\left(a^2+b^2\right)\right|^2 \\
&+ 12\left|-abe\,\mathcal{F}(\sin(u)) + \frac{ab}{2}\mathcal{F}(\sin(2u))\right|^2 .
\end{aligned} \tag{43}
$$

The factor 2 appearing in equation (41) accounts for the counting of the negative terms in the sum (40). We now proceed with the calculation of the Fourier coefficients of $\sin(mu)$

$$
\begin{aligned}
\mathcal{F}(\sin(mu)) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} d\beta\ \sin(mu)\, e^{-in\beta} \\
&= -\frac{1}{2in\pi}\int_{-\pi}^{\pi} du\,\frac{d}{du}\left(e^{-in\beta}\right)\ \sin(mu) \\
&= \frac{m}{2in\pi}\int_{-\pi}^{\pi} du\, e^{-in\beta}\ \cos(mu) \\
&= \frac{m}{2in\pi}\int_{-\pi}^{\pi} du\ \cos(n\beta)\ \cos(mu) \\
&= \frac{m}{in\pi}\int_{0}^{\pi} du\ \cos(mu)\ \cos(nu - ne\sin u) \\
&= \frac{m}{2in}\int_{0}^{\pi} \frac{du}{\pi}\ \left[\cos(u(n+m) - ne\sin u) + \cos(u(n-m) - ne\sin u)\right].
\end{aligned}
$$

By using the following integral representation of the Bessel functions of the first kind

$$
J_n(z) = \int_0^{\pi} \frac{du}{\pi}\cos(nu - z\sin u), \tag{44}
$$

we arrive at

$$
\mathcal{F}(\sin(mu)) = \frac{m}{2in}\left[J_{n+m}(ne) + J_{n-m}(ne)\right]. \tag{45}
$$

A similar calculation gives

$$
\mathcal{F}(\cos(mu)) = \frac{m}{2n}\left[-J_{n+m}(ne) + J_{n-m}(ne)\right]. \tag{46}
$$

By the use of the recurrence relations

$$
J_{n-1}(z) + J_{n+1}(z) = \frac{2n}{z}J_n(z), \tag{47}
$$

$$
J_{n-1}(z) - J_{n+1}(z) = 2J_n'(z), \tag{48}
$$

we can express the Fourier coefficients entirely in terms of J_n and its derivative. We obtain

$$\begin{aligned}
\mathcal{F}(\sin(u)) &= \frac{1}{ine}J_n(ne),\\
\mathcal{F}(\cos(u)) &= \frac{1}{n}J_n'(ne),\\
\mathcal{F}(\sin(2u)) &= \frac{1}{in}\left[\left(\frac{4}{e^2}-2\right)J_n(ne)-\frac{4}{ne}J_n'(ne)\right],\\
\mathcal{F}(\cos(2u)) &= \frac{1}{n}\left[\frac{4}{e}J_n'(ne)-\frac{4}{ne^2}J_n(ne)\right].
\end{aligned} \tag{49}$$

We can insert the above Fourier coefficients into expression (43) and we obtain after rearranging

$$\begin{aligned}
P_n = {} & \frac{16}{15}\frac{n^2\Omega^6}{e^4}\mu^2\\
& \times\left\{J_n^2(ne)\left[a^4+b^4+a^2b^2\left(1+3n^2-6n^2e^2+3n^2e^4\right)\right]+\right.\\
& \quad J_n'^2(ne)\left[a^4\left(e^6n^2+n^2e^2-2e^4n^2\right)+b^4n^2e^2+a^2b^2\left(n^2e^2+3e^2-e^4n^2\right)\right]+\\
& \quad \left.J_n(ne)J_n'(ne)\left[2a^4\left(ne^3-ne\right)-2b^4ne+a^2b^2\left(7ne^3-8ne\right)\right]\right\}.
\end{aligned} \tag{50}$$

Parseval's identity allowed us at one stroke to implement the orbital averaging and to decompose the emitted radiation into its harmonics. In appendix (**??**) there is a slightly less straightforward derivation of the frequency spectrum P_n that goes around Parseval's identity.

Hyperbolic orbit $e > 1$*: .* The calculation of the frequency spectrum of an hyperbolic orbit follows the same lines as the calculation of the elliptic one. In fact, by an analytic continuation argument the structure of the hyperbolic frequency spectrum is the same as the elliptic one. However, we will first close our eyes to this fact and present a detailed outline of the calculation. In the end we argue how the result is inferred from the elliptic case.

We parametrize the coordinates on an hyperbolic orbit as

$$x = a(\cosh u - e), \tag{51}$$

$$y = b\sinh u = -a(e^2-1)^{\frac{1}{2}}\sinh u, \tag{52}$$

where the hyperbolic anomaly u satisfies the hyperbolic Kepler equation

$$e\sinh(u) - u = \Omega t \equiv \frac{\omega}{\nu}t. \tag{53}$$

Recall that a is strictly negative and that $\Omega = \sqrt{\frac{M}{-a^3}} = \left(\frac{M(e-1)^3}{r_p^3}\right)^{1/2}$. The second mass moment (8) has accordingly the form

$$\mathcal{G} = \mu\begin{bmatrix} a^2(e^2-2e\cosh u+\frac{1}{2}\cosh(2u)+\frac{1}{2}) & -ab(e\sinh u-\frac{1}{2}\sinh(2u)) & 0\\ -ab(e\sinh u-\frac{1}{2}\sinh(2u)) & \frac{b^2}{2}\cosh(2u)-\frac{b^2}{2} & 0\\ 0 & 0 & 0\end{bmatrix}. \tag{54}$$

We can decompose the total radiated energy in the frequency domain as follows

$$E = \frac{8\pi}{75}\int_{-\infty}^{\infty} dt \sum_{m=-2}^{2} \left| {}^{(3)}\mathcal{G}_{a_1a_2} \left(\mathcal{Y}^{2m}_{a_1a_2}\right)^* \right|^2 \tag{55}$$

$$= \frac{1}{2\pi}\frac{8\pi}{75}\int_{-\infty}^{\infty} d\omega \sum_{m=-2}^{2} \left| \widehat{\left[{}^{(3)}\mathcal{G}_{a_1a_2} \left(\mathcal{Y}^{2m}_{a_1a_2}\right)^*\right]} \right|^2 \tag{56}$$

$$\equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \, P(\omega) = \frac{1}{\pi}\int_{0}^{\infty} d\omega \, P(\omega), \tag{57}$$

where the second equality follows from Plancherel theorem and the hat represents the Fourier transform. We can take the derivative

$$P(\omega) = \frac{8\pi}{75}\,\omega^6 \sum_{m=-2}^{2} \left| \left(\mathcal{Y}^{2m}_{a_1a_2}\right)^* \widehat{\mathcal{G}_{a_1a_2}} \right|^2, \tag{58}$$

and it remains to calculate the Fourier transform of $\mathcal{G}_{a_1a_2}$, that is

$$\begin{aligned} \widehat{\mathcal{G}_{a_1a_2}} &= \int_{-\infty}^{\infty} du \, \frac{dt(u)}{du} \, e^{-i\omega t(u)} \, \mathcal{G}_{a_1a_2} \\ &= -\frac{1}{i\omega}\int_{-\infty}^{\infty} du \, \frac{d}{du}\left(e^{-i\omega t(u)}\right) \mathcal{G}_{a_1a_2}. \end{aligned}$$

We will outline the remaining calculation for the $\sinh(nu)$ terms

$$\begin{aligned} \widehat{\sinh(nu)} &= -\frac{1}{i\omega}\int_{-\infty}^{\infty} du \, \frac{d}{du}\left(e^{-i\omega t(u)}\right) \sinh(nu) \\ &= \frac{n}{i\omega}\int_{-\infty}^{\infty} du \, e^{-i\omega t(u)} \, \cosh(nu) \\ &= \frac{n}{2i\omega}\int_{-\infty}^{\infty} du \, e^{-i\nu(e\sinh u - u)} \, \left(e^{nu} + e^{-nu}\right) \\ &= \frac{n}{2i\omega}\int_{-\infty}^{\infty} du \, \left(e^{-i\nu e\sinh u + (i\nu+n)u} + e^{-i\nu e\sinh u + (i\nu-n)u}\right). \end{aligned}$$

This result can be expressed in terms of modified Bessel functions of the second kind $K_\alpha(x)$. One possible integral representation of $K_\alpha(x)$ has the form (see appendix (2.3) or *e.g.* page 182 in the Bessel function treatise [10])

$$K_\alpha(x) = \frac{1}{2} e^{\frac{1}{2}\alpha\pi i}\int_{-\infty}^{\infty} dt \, e^{-ix\sinh t + \alpha t}. \tag{59}$$

This formula is valid for positive x. One could equivalently express the Fourier integrals in terms of Hankel functions of the first kind $H^{(1)}_\alpha(ix)$ with purely imaginary argument. Appendix (2.3) provides a brief discussion of this matter.

Using the above integral representation of $K_\alpha(x)$ we can write for the Fourier transforms[1]

$$\widehat{\sinh(nu)} = \frac{n}{i\omega} e^{\frac{1}{2}\nu\pi} e^{-\frac{1}{2}n\pi i} \left[K_{i\nu+n}(\nu e) + e^{i\pi n} K_{i\nu-n}(\nu e)\right]. \tag{60}$$

For the relevant terms for $\widehat{\mathcal{G}_{a_1 a_2}}$ we get

$$\begin{aligned}\widehat{\sinh(u)} &= -\frac{1}{\omega} e^{\frac{1}{2}\nu\pi} \left[K_{i\nu+1}(\nu e) - K_{i\nu-1}(\nu e)\right], \\ \widehat{\sinh(2u)} &= -\frac{2}{i\omega} e^{\frac{1}{2}\nu\pi} \left[K_{i\nu+2}(\nu e) + K_{i\nu-2}(\nu e)\right].\end{aligned} \tag{61}$$

The calculation of $\widehat{\cosh(nu)}$ proceeds the same way and one finds

$$\widehat{\cosh(nu)} = \frac{n}{i\omega} e^{\frac{1}{2}\nu\pi} e^{-\frac{1}{2}n\pi i} \left[K_{i\nu+n}(\nu e) - e^{i\pi n} K_{i\nu-n}(\nu e)\right], \tag{62}$$

so that we have

$$\begin{aligned}\widehat{\cosh(u)} &= -\frac{1}{\omega} e^{\frac{1}{2}\nu\pi} \left[K_{i\nu+1}(\nu e) + K_{i\nu-1}(\nu e)\right], \\ \widehat{\cosh(2u)} &= -\frac{2}{i\omega} e^{\frac{1}{2}\nu\pi} \left[K_{i\nu+2}(\nu e) - K_{i\nu-2}(\nu e)\right].\end{aligned} \tag{63}$$

We can now use the recurrence relations

$$\begin{aligned}K_\alpha(x) &= -\frac{x}{2\alpha}[K_{\alpha-1}(x) - K_{\alpha+1}(x)], \\ K'_\alpha(x) &= -\frac{1}{2}[K_{\alpha-1}(x) + K_{\alpha+1}(x)],\end{aligned} \tag{64}$$

to derive the following identities

$$\begin{aligned}K_{\alpha+2}(x) + K_{\alpha-2}(x) &= \left(\frac{4\alpha^2}{x^2} + 2\right) K_\alpha(x) - \frac{4}{x} K'_\alpha(x), \\ K_{\alpha+2}(x) - K_{\alpha-2}(x) &= -\frac{4\alpha}{x} K'_\alpha(x) + \frac{4\alpha}{x^2} K_\alpha(x).\end{aligned} \tag{65}$$

Combining the above we arrive at

$$\begin{aligned}\widehat{\sinh(u)} &= -\frac{2i}{\omega e}\, e^{\frac{1}{2}\nu\pi}\, K_{i\nu}(\nu e), \\ \widehat{\sinh(2u)} &= -\frac{2}{i\omega} e^{\frac{1}{2}\nu\pi} \left[\left(\frac{-4}{e^2} + 2\right) K_{i\nu}(\nu e) - \frac{4}{\nu e} K'_{i\nu}(\nu e)\right], \\ \widehat{\cosh(u)} &= \frac{2}{\omega}\, e^{\frac{1}{2}\nu\pi}\, K'_{i\nu}(\nu e), \\ \widehat{\cosh(2u)} &= \frac{8}{e\omega}\, e^{\frac{1}{2}\nu\pi} \left[K'_{i\nu}(\nu e) - \frac{1}{\nu e} K_{i\nu}(\nu e)\right].\end{aligned} \tag{66}$$

[1] Strictly speaking, when $\mathrm{Re}(\alpha) \in \mathbb{Z}$, the function $K_\alpha(x)$ has to be interpreted as the limiting value $\lim_{\beta\to\alpha} K_\beta(x)$ for $\mathrm{Re}(\beta) \in \mathbb{R} \setminus \mathbb{Z}$.

From equation (58) for the radiated power in the frequency domain we find

$$P(\omega) = \frac{1}{30}\,\omega^6\,\mu^2\,\Pi, \tag{67}$$

where

$$\begin{aligned}\Pi = &\left|2a^2 e\,\widehat{\cosh(u)} - \frac{1}{2}\widehat{\cosh(2u)}\left(a^2+b^2\right)\right|^2 \\ &+ 3\left|-2a^2 e\,\widehat{\cosh(u)} + \frac{1}{2}\widehat{\cosh(2u)}\left(a^2-b^2\right)\right|^2 \\ &+ 12\left|abe\,\widehat{\sinh(u)} - \frac{ab}{2}\widehat{\sinh(2u)}\right|^2 .\end{aligned} \tag{68}$$

By plugging in the explicit expressions of the Fourier transforms (66), we can bring $P(\omega)$ into a similar form as (50). We obtain after factoring out terms

$$\begin{aligned}P(\omega) = &\frac{64}{30}\,\frac{\omega^2}{e^4}\,\mu^2\,\exp(\nu\pi) \\ &\times\Big\{K_{i\nu}^2(\nu e)\left[(a^4+b^4)\Omega^2 + a^2b^2\left(3\omega^2 - 6\omega^2e^2 + 3\omega^2e^4 - \Omega^2\right)\right] + \\ &\quad K_{i\nu}'^2(\nu e)\left[a^4\left(e^2\omega^2 + \omega^2e^6 - 2\omega^2e^4\right) + b^4\omega^2e^2 + a^2b^2\left(-\omega^2e^2 + \omega^2e^4 + 3\Omega^2e^2\right)\right] + \\ &\quad K_{i\nu}(\nu e)\,K_{i\nu}'(\nu e)\left[2a^4\left(e^3-e\right)\omega\Omega - 2b^4\omega\Omega e + a^2b^2\left(-7e^3+8e\right)\omega\Omega\right]\Big\} .\end{aligned} \tag{69}$$

The similarity of the calculation and final expressions of the elliptic and hyperbolic case is not a surprise. In fact, for a deduction of the hyperbolic frequency spectrum from the elliptic one, it suffices to modify the calculation of the elliptic case as follows:

1. The Kepler parameters for elliptic orbits are replaced by the respective hyperbolic ones.
2. The hyperbolic Kepler equation (53) is obtained from the elliptic one (equation (35)) by setting $u \mapsto iu$. Hence $\sin(nu) \mapsto i\sinh(nu)$ and $\cos(nu) \mapsto \cosh(nu)$.
3. The Fourier coefficients become Fourier transforms and $n\Omega \mapsto \omega$.
4. The energy radiated per orbit into the n-th harmonic, *i.e.* $E(n) = \frac{2\pi}{\Omega}P(n)$, becomes the energy spectrum multiplied by Ω.

The power emitted at zero frequency is non-zero for $e > 1$; this effect is called gravitational wave memory. To see this, we expand $K_0(\nu e)$ and $K_0'(\nu e)$ to first order in ν around $\nu = 0$. The result is (see *e.g.* formula (9.6.13) in Abramowitz and Stegun [11])

$$K_0(\nu e) = -\gamma - \log\left(\frac{e}{2}\right) - \log(\nu) + \mathcal{O}(\nu^2), \tag{70}$$

$$K_0'(\nu e) = -\frac{1}{e\nu} + \frac{e\nu}{4}\left(1 - 2\gamma - 2\log\left(\frac{e}{2}\right) - 2\log(\nu)\right) + \mathcal{O}(\nu^2), \tag{71}$$

where γ denotes Euler's constant. Inserting the above expressions into the equation (68) and then taking the limit of equation (67) towards $\omega = 0$ gives

$$\lim_{\omega \to 0} P(\omega) = \frac{32}{5} \frac{\mu^2 M^2 (e^2 - 1)}{a^2 e^4}. \tag{72}$$

The gravitational memory effect that results from hyperbolic encounters has been calculated up to 1.5 post-Newtonian accuracy in De Vittori *et al.* [12]. They find that only the cross polarization state of the radiation field contributes to the memory effect and that in the Newtonian limit this state behaves as $|h_\times|^2 \propto \frac{\mu^2 M^2}{a^2} \frac{e^2-1}{e^4}$, which agrees with expression (72).

In figure (**??**) we plot the frequency spectrum (69) for eccentricities ranging from $e = 1.5$ to $e = 3$. Both the value of the frequency $\bar{\omega}$, where $P(\bar{\omega})$ is maximal, and the value $P(\bar{\omega})$ decrease with increasing hyperbolicity. As expected, the power emitted at zero frequency is non-zero for $e > 1$.

Parabolic orbit $e = 1$*:* . In this section we determine the parabolic energy spectrum by taking the appropriate limit of the corresponding elliptic (or hyperbolic) result.

The total energy emitted into the n-harmonic $E(n) = \frac{2\pi}{\Omega} P(n)$ during one elliptic orbit is equivalent the energy spectrum $\frac{dE(\omega)}{d\omega}$ multiplied by Ω,

$$\left.\frac{dE(\omega)}{d\omega}\right|_{\text{elliptic}} = \frac{2\pi}{\Omega^2} P(n), \tag{73}$$

and given that the involved expressions are subject to the substitution

$$n \mapsto \frac{\omega}{\Omega} = \frac{r_p^{3/2}}{M^{1/2}} \frac{\omega}{(1-e)^{3/2}} \equiv \tilde{\nu} \frac{1}{(1-e)^{3/2}}. \tag{74}$$

We recast the elliptic energy spectrum using expressions (42), (43) and (49) to the form

$$\left.\frac{dE(\omega)}{d\omega}\right|_{\text{elliptic}} = \frac{8\pi}{15} \mu^2 \omega^4 \frac{r_p^4}{(1-e)^4} \Psi, \tag{75}$$

where

$$\begin{aligned} \Psi = & \left| \frac{(1-e)^{3/2}}{\tilde{\nu}} J_n(ne) \right|^2 \\ & + 3 \left| 2J_n'(ne) \frac{1-e^2}{e} - \frac{(1-e)^{3/2}}{\tilde{\nu}} \frac{2-e^2}{e^2} J_n(ne) \right|^2 \\ & + 12 \left| \frac{(1-e^2)^{\frac{3}{2}}}{e^2} J_n(ne) - \frac{(1-e^2)^{\frac{1}{2}} (1-e)^{3/2}}{e\tilde{\nu}} J_n'(ne) \right|^2 . \end{aligned} \tag{76}$$

In the limit of $e \to 1$, $n \to \infty$. Therefore we need $J_n(nz)$ and $J_n'(nz)$ as $n \to \infty$. We have the following asymptotic expansion for $J_n(nz)$ (this Bessel function expansion and the ones to follow are derived in Olver [13])

$$J_n(nz) \sim \left(\frac{4\zeta}{1-z^2} \right)^{1/4} \left\{ \frac{\mathrm{Ai}(n^{2/3}\zeta)}{n^{1/3}} \sum_{s=0}^{\infty} \frac{A_s(\zeta)}{n^{2s}} + \frac{\mathrm{Ai}'(n^{2/3}\zeta)}{n^{5/3}} \sum_{s=0}^{\infty} \frac{B_s(\zeta)}{n^{2s}} \right\}, \tag{77}$$

as $n \to \infty$ and provided that $|\arg(z)| < \pi$. In the above formula Ai denotes the Airy function of the first kind and ζ is given by

$$\frac{2}{3}(-\zeta)^{3/2} = (z^2-1)^{1/2} - \arccos\left(\frac{1}{z}\right). \tag{78}$$

The coefficients A_s and B_s are given by

$$A_s(\zeta) = \sum_{m=0}^{2s} b_m \zeta^{-\frac{3}{2}m} U_{2s-m}, \quad \zeta^{\frac{1}{2}} B_s(\zeta) = -\sum_{m=0}^{2s+1} a_m \zeta^{-\frac{3}{2}m} U_{2s-m+1}, \tag{79}$$

in which $U_0 = 1$ and with $u = (1 - z^2)^{-\frac{1}{2}}$

$$U_{s+1} = \frac{1}{2}u^2(1-u^2)\frac{dU_s}{du} + \frac{1}{8}\int_0^u du\, U_s(1-5u^2).$$

The remaining coefficients are recursively defined by $a_0 = b_0 = 1$ and

$$a_s = \frac{(2s+1)(2s+3)\cdots(6s-1)}{s!\,(144)^s}, \quad b_s = -\frac{6s+1}{6s-1}a_s. \tag{80}$$

The Bessel function derivative $J_n'(ne)$ has the asymptotic expansion

$$J_n'(nz) \sim -\frac{2}{z}\left(\frac{1-z^2}{4\zeta}\right)^{1/4}\left\{\frac{\mathrm{Ai}(n^{2/3}\zeta)}{n^{4/3}}\sum_{s=0}^{\infty}\frac{C_s(\zeta)}{n^{2s}} + \frac{\mathrm{Ai}'(n^{2/3}\zeta)}{n^{2/3}}\sum_{s=0}^{\infty}\frac{D_s(\zeta)}{n^{2s}}\right\}, \tag{81}$$

where

$$C_s(\zeta) = \chi(\zeta)A_s(\zeta) + A_s'(\zeta) + \zeta B_s(\zeta),$$
$$D_s(\zeta) = A_s(\zeta) + \chi(\zeta)B_{s-1}(\zeta) + B_{s-1}'(\zeta),$$

and

$$\chi(\zeta) = \frac{4 - z^2\left(\frac{4\zeta}{1-z^2}\right)^{3/2}}{16\zeta}.$$

Let us also aim to specialize the hyperbolic energy spectrum to the parabolic case. From expressions (57), (66), (67) and (68) we deduce that the hyperbolic energy spectrum has the form

$$\left.\frac{dE(\omega)}{d\omega}\right|_{\text{hyperbolic}} = \frac{8}{15\pi}\,\omega^4\,\mu^2\left(\frac{r_p}{1-e}\right)^4 e^{\nu\pi}\,\Pi, \tag{82}$$

where

$$\begin{aligned}\Pi = &\left|\frac{(e-1)^{3/2}}{\tilde{\nu}}K_{i\nu}(\nu e)\right|^2 \\ &+ 3\left|2K_{i\nu}'\frac{1-e^2}{e} - \frac{(e-1)^{3/2}}{\tilde{\nu}}\frac{2-e^2}{e^2}K_{i\nu}(\nu e)\right|^2 \\ &+ 12\left|\frac{(e^2-1)^{\frac{3}{2}}}{e^2}K_{i\nu}(\nu e) - \frac{(e^2-1)^{\frac{1}{2}}(e-1)^{3/2}}{e\tilde{\nu}}K_{i\nu}'(\nu e)\right|^2.\end{aligned} \tag{83}$$

To obtain the parabolic energy spectrum, we need to accommodate to the limit as $\nu = \frac{\tilde{\nu}}{(e-1)^{3/2}} \to \infty$.

The modified Bessel functions of the second kind are related to the purely imaginary Hankel functions

$$K_\alpha(z) = \frac{1}{2}\pi i e^{\frac{\alpha\pi i}{2}} H_\alpha^{(1)}(iz); \tag{84}$$

in our case

$$K_{i\nu}(\nu e) = \frac{1}{2}\pi i e^{-\frac{\nu\pi}{2}} H_{i\nu}^{(1)}(i\nu e). \tag{85}$$

The respective asymptotic expansions for the Hankel function of the first kind are

$$\begin{aligned} H_n^{(1)}(nz) \sim 2e^{-\frac{\pi i}{3}} \left(\frac{4\zeta}{1-z^2}\right)^{1/4} &\left\{ \frac{\mathrm{Ai}(e^{\frac{2\pi i}{3}} n^{2/3}\zeta)}{n^{1/3}} \sum_{s=0}^{\infty} \frac{A_s(\zeta)}{n^{2s}} \right. \\ &\left. + \frac{e^{\frac{2\pi i}{3}}\mathrm{Ai}'(e^{\frac{2\pi i}{3}} n^{2/3}\zeta)}{n^{5/3}} \sum_{s=0}^{\infty} \frac{B_s(\zeta)}{n^{2s}} \right\}, \end{aligned} \tag{86}$$

$$\begin{aligned} H_n^{(1)\prime}(nz) \sim \frac{4e^{\frac{2\pi i}{3}}}{z} \left(\frac{1-z^2}{4\zeta}\right)^{1/4} &\left\{ \frac{\mathrm{Ai}(e^{\frac{2\pi i}{3}} n^{2/3}\zeta)}{n^{4/3}} \sum_{s=0}^{\infty} \frac{C_s(\zeta)}{n^{2s}} \right. \\ &\left. + \frac{e^{\frac{2\pi i}{3}}\mathrm{Ai}'(e^{\frac{2\pi i}{3}} n^{2/3}\zeta)}{n^{2/3}} \sum_{s=0}^{\infty} \frac{D_s(\zeta)}{n^{2s}} \right\}. \end{aligned} \tag{87}$$

The expansions (77), (81), (86) and (87) hold with n replaced by $\xi = ne^{i\theta}$, provided that $\theta < |\frac{\pi}{2}|$ (as we need purely complex orders, we will resort to the limiting value). Therefore we obtain the desired expansions

$$\begin{aligned} K_{i\nu}(\nu e) \sim \pi e^{-\frac{\nu\pi}{2}} \left(\frac{4\zeta}{1-e^2}\right)^{1/4} &\left\{ \frac{\mathrm{Ai}(-\nu^{2/3}\zeta)}{\nu^{1/3}} \sum_{s=0}^{\infty} (-1)^s \frac{A_s(\zeta)}{\nu^{2s}} \right. \\ &\left. + \frac{\mathrm{Ai}'(-\nu^{2/3}\zeta)}{\nu^{5/3}} \sum_{s=0}^{\infty} (-1)^s \frac{B_s(\zeta)}{\nu^{2s}} \right\}, \end{aligned} \tag{88}$$

$$\begin{aligned} K'_{i\nu}(\nu e) \sim -\pi e^{-\frac{\nu\pi}{2}} \frac{2}{e} \left(\frac{1-e^2}{4\zeta}\right)^{1/4} &\left\{ \frac{\mathrm{Ai}(-\nu^{2/3}\zeta)}{\nu^{4/3}} \sum_{s=0}^{\infty} (-1)^s \frac{C_s(\zeta)}{\nu^{2s}} \right. \\ &\left. - \frac{\mathrm{Ai}'(-\nu^{2/3}\zeta)}{\nu^{2/3}} \sum_{s=0}^{\infty} (-1)^s \frac{D_s(\zeta)}{\nu^{2s}} \right\}. \end{aligned} \tag{89}$$

We will now specialize the above expressions to the limit of $e \to 1$. A power counting argument shows that, in this limit, only the leading term in each Bessel function expansion evaluates to a non-zero value in the spectra (75) and (82). Expanding equation (78) around $z = 1$ yields the following value for ζ

$$\zeta = 2^{\frac{1}{3}}(1-z). \tag{90}$$

Applying the above limit of ζ and restoring $n = \tilde{\nu}\frac{1}{(1-e)^{3/2}}$ and $\nu = \frac{\tilde{\nu}}{(e-1)^{3/2}}$ in the respective variables we find the following expressions for the leading terms of the Bessel function expansions (77), (81), (88) and (89)

$$J_n(ne) \sim \left(\frac{2}{\tilde{\nu}}\right)^{1/3} (1-e)^{\frac{1}{2}}\,\mathrm{Ai}(2^{\frac{1}{3}}\tilde{\nu}^{\frac{2}{3}}), \tag{91}$$

$$J_n'(ne) \sim -\left(\frac{2}{\tilde{\nu}}\right)^{2/3} \frac{1-e}{e}\,\mathrm{Ai}'(2^{\frac{1}{3}}\tilde{\nu}^{\frac{2}{3}}), \tag{92}$$

$$K_{i\nu}(\nu e) \sim \pi e^{-\frac{\nu\pi}{2}} \left(\frac{2}{\tilde{\nu}}\right)^{1/3} (e-1)^{\frac{1}{2}}\,\mathrm{Ai}(2^{\frac{1}{3}}\tilde{\nu}^{\frac{2}{3}}), \tag{93}$$

$$K_{i\nu}'(\nu e) \sim \pi e^{-\frac{\nu\pi}{2}} \left(\frac{2}{\tilde{\nu}}\right)^{2/3} \frac{e-1}{e}\,\mathrm{Ai}'(2^{\frac{1}{3}}\tilde{\nu}^{\frac{2}{3}}). \tag{94}$$

An inspection of the structure of the energy spectra (75) and (82) and of the above expansions makes it evident that the parabolic limit of the elliptic energy spectrum and the parabolic limit of the hyperbolic energy spectrum coincide. Indeed, both limits give the parabolic energy spectrum as

$$\begin{aligned}\left.\frac{dE(\omega)}{d\omega}\right|_{\text{parabolic}} &= \frac{128\pi}{5}\,\omega^4\,\mu^2\,r_p^4 \\ &\times \Bigg\{ \mathrm{Ai}^2(2^{\frac{1}{3}}\tilde{\nu}^{\frac{2}{3}}) \left(\frac{2}{\tilde{\nu}}\right)^{2/3} \left(2 + \frac{1}{12\tilde{\nu}^2}\right) \\ &\quad + \mathrm{Ai}'^2(2^{\frac{1}{3}}\tilde{\nu}^{\frac{2}{3}}) \left(\frac{2}{\tilde{\nu}}\right)^{4/3} \\ &\quad + \mathrm{Ai}(2^{\frac{1}{3}}\tilde{\nu}^{\frac{2}{3}})\,\mathrm{Ai}'(2^{\frac{1}{3}}\tilde{\nu}^{\frac{2}{3}}) \frac{1}{\tilde{\nu}^2} \Bigg\}. \end{aligned} \tag{95}$$

2.3. A small note on Bessel function contours. The material of this section originates from [10]. The modified Bessel functions of the second kind are usually defined through the modified Bessel functions of the first kind $I_\nu(z)$ by the equation

$$K_\nu(z) = \frac{\pi}{2} \frac{I_{-\nu}(z) - I_\nu(z)}{\sin(\nu\pi)}. \tag{96}$$

We see that $K_\nu(z) = K_{-\nu}(z)$.

Let's consider the contour $\mathcal{C}_1$ that emerges at $0 \exp(i\omega)$, that tends towards $\infty \exp(-i\omega)$ and that from there goes straight back to the origin (we exclude the origin by including a quarter circle, whose radius we let tend to 0). When $z > 0$, we may take $\omega \to \pi/2$ and the integral

$$\frac{1}{2} \int_{\mathcal{C}_1} du\, u^{-\nu-1} \exp\left\{-\frac{1}{2} z \left(u + \frac{1}{u}\right)\right\} \tag{97}$$

is zero by Cauchy's theorem. The integral of the integrand along the segments which are in the purely complex plane is exactly the modified Bessel function of the second kind with positive argument $K_\nu(x)$. Hence we have

$$K_\nu(x) = \frac{1}{2} \int_0^{-i\infty} du\, u^{-\nu-1} \exp\left\{-\frac{1}{2} x \left(u + \frac{1}{u}\right)\right\} \tag{98}$$

$$= \frac{1}{2} e^{-i\pi\nu/2} \int_0^{\infty} du\, u^{-\nu-1} \exp\left\{-\frac{1}{2} ix \left(u - \frac{1}{u}\right)\right\}. \tag{99}$$

The substitution $u = e^t$ yields

$$K_\nu(x) = \frac{1}{2} e^{-\frac{i\pi\nu}{2}} \int_{-\infty}^{\infty} dt \exp\left\{-ix \sinh(t) - \nu t\right\}, \tag{100}$$

and from which we obtain equation (59) after changing the sign of ν

$$K_\nu(x) = \frac{1}{2} e^{\frac{i\pi\nu}{2}} \int_{-\infty}^{\infty} dt \exp\left\{-ix \sinh(t) + \nu t\right\}. \tag{101}$$

Next consider the contour $\mathcal{C}_2$ that emerges at $0 \exp(i\omega)$, that tends towards $\infty \exp(i(\pi - \omega))$ and that from there goes straight back to the origin (we again exclude the origin by including a quarter circle, whose radius we let tend to 0). When $z = ix$, with $x > 0$ we may take $\omega \to 0$ and the integral

$$\frac{1}{\pi i} \int_{\mathcal{C}_2} du\, u^{-\nu-1} \exp\left\{\frac{1}{2} z \left(u - \frac{1}{u}\right)\right\} \tag{102}$$

is again zero by Cauchy's theorem. The integral of the integrand along the segments which are in the purely complex plane is the Hankel function of the first kind with purely complex argument $H_\nu^{(1)}(ix)$. So it holds that

$$H_\nu^{(1)}(ix) = \frac{1}{\pi i} \int_0^{-\infty} du\, u^{-\nu-1} \exp\left\{\frac{1}{2} ix \left(u - \frac{1}{u}\right)\right\} \tag{103}$$

$$= \frac{1}{\pi i} e^{-i\pi\nu} \int_0^{\infty} du\, u^{-\nu-1} \exp\left\{-\frac{1}{2} ix \left(u - \frac{1}{u}\right)\right\}. \tag{104}$$

Therefore we have established, for positive x, that

$$K_\nu(x) = \frac{\pi i}{2} e^{\frac{i\pi\nu}{2}} H_\nu^{(1)}(ix). \tag{105}$$

The segments of the contour $\mathcal{C}_2$ that represent the Hankel function of the first kind may be deformed to yield other integral representations. For example, we can substitute $\log(u) = t$ in the integrand (102) and obtain

$$H_\nu^{(1)}(z) = \frac{1}{\pi i} \int_{\mathcal{C}_3} dt\, e^{z \sinh(t) - \nu t}, \tag{106}$$

which is valid for $|\arg(z)| < \frac{\pi}{2}$. The contour $\mathcal{C}_3$ starts at $-\infty$ and tends in the upper half plane towards $\infty + i\pi$. We may deform this contour to consist of the three straight segments $(-\infty, 0] \cup [0, i\pi] \cup [i\pi, i\pi + \infty]$.

References

1. K.S. Thorne, Rev. Mod. Phys. **52**, 299 (1980). DOI 10.1103/RevModPhys.52.299. URL https://link.aps.org/doi/10.1103/RevModPhys.52.299
2. P.C. Peters, J. Mathews, Phys. Rev. **131**, 435 (1963). DOI 10.1103/PhysRev.131.435. URL https://link.aps.org/doi/10.1103/PhysRev.131.435
3. P.C. Peters, Phys. Rev. **136**, B1224 (1964). DOI 10.1103/PhysRev.136.B1224. URL https://link.aps.org/doi/10.1103/PhysRev.136.B1224
4. R.O. Hansen, Phys. Rev. D **5**, 1021 (1972). DOI 10.1103/PhysRevD.5.1021. URL https://link.aps.org/doi/10.1103/PhysRevD.5.1021
5. M. Turner, The Astrophysical Journal **216**, 610 (1977). DOI 10.1086/155501
6. W. Junker, G. Schaefer, Monthly Notices of the Royal Astronomical Society **254**, 146 (1992). DOI 10.1093/mnras/254.1.146
7. C.P.L. Berry, J.R. Gair, Phys. Rev. D **82**, 107501 (2010). DOI 10.1103/PhysRevD.82.107501. URL https://link.aps.org/doi/10.1103/PhysRevD.82.107501
8. L. De Vittori, P. Jetzer, A. Klein, Phys. Rev. D **86**, 044017 (2012). DOI 10.1103/PhysRevD.86.044017. URL https://link.aps.org/doi/10.1103/PhysRevD.86.044017
9. J. García-Bellido, S. Nesseris, Physics of the Dark Universe **21**, 61 (2018). DOI https://doi.org/10.1016/j.dark.2018.06.001. URL http://www.sciencedirect.com/science/article/pii/S2212686418300578
10. G.N. Watson, *A Treatise on the Theory of Bessel Functions* (Cambridge University Press, 1966)
11. M. Abramowitz, I. Stegun, *Handbook of Mathematical Functions* (DOVER PUBN INC, 1965)
12. L. De Vittori, A. Gopakumar, A. Gupta, P. Jetzer, Phys. Rev. D **90**, 124066 (2014). DOI 10.1103/PhysRevD.90.124066. URL https://link.aps.org/doi/10.1103/PhysRevD.90.124066
13. F.W.J. Olver, Philosophical Transactions of the Royal Society of London. Series A, Mathematical and Physical Sciences **247**(930), 328 (1954). URL http://www.jstor.org/stable/91484

Communicated by name