Rotating Spacetimes

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Abstract

In this thesis I give an overview over some instructive examples of rotating spacetimes. The *van Stockum dust*, the *Gödel spacetime* and its stationary and expanding generalizations are covered as well as the *Tipler cylinder*.

A selection of measurable and observable effects is given along with several estimates of a possible global angular velocity of the universe based on different arguments and sets of data. The estimates do not agree, and recent data suggests that there is indeed no global rotation of the universe.

Contents

1 Introduction **2**

1.1 Notation and Conventions 3

2 Rotation of Spacetimes **4**

2.1 Nonrotating Frames of Reference 4

2.2 Expansion, Shear and Vorticity 5

3 Examples of Rotating Spacetimes **7**

3.1 Van Stockum dust 7

3.2 Spacetimes of Gödel type 9

3.2.1 The original Gödel metric 9

3.2.2 Stationary cosmological spacetimes of Gödel type . . . 11

3.2.3 The expanding Gödel type cosmological model 14

3.3 The Tipler cylinder 15

4 Observable Effects **18**

4.1 Frame dragging 18

4.2 Cosmological observations 20

4.2.1 The Cosmic Microwave Background 20

4.2.2 Parallax 21

4.2.3 Apparent magnitude m versus red shift z 22

4.2.4 Number of Sources N versus red shift z 23

4.2.5 Polarization rotation effect 24

4.2.6 The Formation of Galaxies 25

5 Conclusion **28**

Chapter 1

Introduction

In his *Principia Mathematica* Newton (1846) postulated a concept of absolute space and time (Newton, 1846, p. 77) relative to which all movement and rotation would take place. The *fixed stars* were taken as resting, and the bucket argument (Newton, 1846, p. 81), presuming that water at rest with regard to "absolute space" would retain its flat surface, even if the bucket holding it were spinning, seemed to show that rotation was indeed absolute.

This view remained unchallenged until Mach (1901) asked whether this might only be the effect of the water's non-rotation with regard to the Earth and other celestial bodies (Mach, 1901, pp. 242–243). Mach's considerations apparently played a major role in Einstein's development of the theory of General Relativity (Thirring, 1918, p. 33) and in his and Thirring's investigation of frame dragging by distant masses (see section 4.1).

The notion that the distant stars or galaxies were "fixed" was challenged by the discovery of cosmological solutions of Einsteins field equations in which the matter in the universe is globally rotating. Some examples of such spacetimes will be covered in chapter 3.

Since spacetimes lack an "outside" relative to which their matter content rotates intrinsic criteria for rotation and vorticity need to be found. These topics will be illuminated shortly in chapter 2.

An overview over a selection of—locally and cosmologically—observable effects in rotating spacetimes will be given in chapter 4.

1.1 Notation and Conventions

Throughout this thesis I will use the $(+ - - -)$ metric sign convention. Units are chosen such that the speed of light $c = 1$ and the constant of gravitation $G = 1$.

Partial derivation will be denoted equivalently as $\frac{\partial}{\partial x^\mu} = \partial_{x^\mu} = \partial_\mu$.

4-vectors and tensors will be denoted by bold roman letters, their components by greek indices, so that e.g. $\mathbf{u} = u^\mu \partial_\mu$ and $\mathbf{B} = B_{\mu\nu} dx^\mu dx^\nu$. 3-vectors will be denoted by bold italic letters, so that e.g. $\boldsymbol{v} = (v^1, v^2, v^3)$.

Chapter 2

Rotation of Spacetimes

2.1 Nonrotating Frames of Reference

The first difficulty encountered in treating the rotation of spacetimes is defining non-rotation. Since there is—unless the spacetime is asymptotically flat—no global background of "fixed stars" a local formulation needs to be found. Such a local, non-rotating frame of reference for an arbitrary—possibly accelerated—observer with 4-velocity $\mathbf{u}$ ist given through the Fermi-Walker-transport of a tetrad along his world line (Thorne et al., 1973, pp. 170–172).

At each instant, the basis vectors $\mathbf{e}_0$, $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ are required to be orthonormal ($\mathbf{e}_\mu \cdot \mathbf{e}_\nu = \eta_{\mu\nu}$), form a restframe ($\mathbf{e}_0 = \mathbf{u}$) and be "nonrotating", i.e. Fermi-Walker transported along the world line:

$$\frac{dv^\mu}{d\tau} = (a^\mu u^\nu - u^\mu a^\nu)v_\nu, \text{ where } \mathbf{a} = \frac{d\mathbf{u}}{d\tau} \tag{2.1}$$

The infinitesimal Lorentz transformation $\Omega^{\mu\nu} = (a^\mu u^\nu - u^\mu a^\nu)$ indeed yields

$$\frac{du^\mu}{d\tau} = a^\mu(u^\nu u_\nu) - u^\mu(a^\nu u_\nu) = a^\mu \text{ since } \mathbf{u}\cdot\mathbf{u} = 1 \text{ and } \mathbf{u}\cdot\mathbf{a} = 0$$

while a spacelike vector $\mathbf{w}$ orthogonal to $\mathbf{u}$ and $\mathbf{a}$ ($\mathbf{w}\cdot\mathbf{u} = \mathbf{w}\cdot\mathbf{a} = 0$) remains unchanged:

$$(a^\mu u^\nu - u^\mu a^\nu)w_\nu = a^\mu(u^\nu w_\nu) - u^\mu(a^\nu w_\nu) = 0$$

Gyroscopes accelerated along with the observer by forces applied to their center of mass, i.e. without exerting a torque, will remain stationary with regard to this tetrad (Thorne et al., 1973, p. 165).

Any event in a neighbourhood of the world line will then be determined uniquely by the proper distance s along a spacelike geodesic emanating from the observer's position at the time τ with the original direction $\mathbf{n} = n^j \mathbf{e}_j$, where $\mathbf{n}$ is a unit spacelike 4-vector, $\mathbf{n} \cdot \mathbf{n} = -1$. The four numbers $(x^0, x^1, x^2, x^3) \equiv (\tau, sn^1, sn^2, sn^3)$ are then the coordinates of the event in the observer's *proper reference frame* (Thorne et al., 1973, p. 330).

2.2 Expansion, Shear and Vorticity

Consider a congruence of timelike geodesics with the normalized tangent vector field $\mathbf{u}$, $\mathbf{u} \cdot \mathbf{u} = 1$. Then the (purely spatial) tensor field

$$B_{\mu\nu} = \nabla_\nu u_\mu \tag{2.2}$$

measures the failure of the infinitesimal spatial displacement $\boldsymbol{\eta}$ between "neighbouring" geodesics to be parallel transported (Wald, 1984, p. 217).

Defining the *spatial metric* h_{ab} on the spatial hypersurfaces perpendicular to $\mathbf{u}$,

$$h_{\mu\nu} := g_{\mu\nu} - u_\mu u_\nu \tag{2.3}$$

$\mathbf{B}$ can be decomposed into the *expansion scalar* θ, *shear tensor* $\sigma_{\mu\nu}$ and *vorticity tensor*[1] $\omega_{\mu\nu}$ such that

$$B_{\mu\nu} = \frac{1}{3}\theta h_{\mu\nu} + \sigma_{\mu\nu} + \omega_{\mu\nu} \tag{2.4}$$

with

$$\theta := B^\mu{}_\mu = h^{\mu\nu} B_{\mu\nu} \tag{2.5}$$

is the *trace* of $\mathbf{B}$,

$$\sigma_{\mu\nu} := B_{(\mu\nu)} - \frac{1}{3}\theta h_{\mu\nu} \tag{2.6}$$

the *traceless symmetric* part of $\mathbf{B}$ and

$$\omega_{\mu\nu} := B_{[\mu\nu]} \tag{2.7}$$

[1] also known as *twist* (Wald, 1984) or *rotation* (Thorne et al., 1973)

the *antisymmetric* part of $\mathbf{B}$.

The expansion also corresponds to the fractional change of volume of a small ball of test particles flowing along the geodesics, i.e. $\theta = \frac{1}{V}\frac{dV}{d\tau}$ with the proper time τ. Similarly shear and vorticity describe the deformation and rotation of the particle cloud. The congruence is locally hypersurface orthogonal if and only if $\boldsymbol{\omega} = 0$ (Wald, 1984, p. 217).

θ, $\boldsymbol{\sigma}$ and $\boldsymbol{\omega}$ are connected through *Raychaudhuri's equation* (Wald, 1984, p. 218)

$$\frac{d\theta}{d\tau} = -\frac{1}{3}\theta^2 - 2\left(\sigma^2 - \omega^2\right) - R_{\mu\nu}u^\mu u^\nu \tag{2.8}$$

with

$$\sigma^2 := \frac{1}{2}\sigma_{\mu\nu}\sigma^{\mu\nu}, \quad \omega^2 := \frac{1}{2}\omega_{\mu\nu}\omega^{\mu\nu} \tag{2.9}$$

It collects the various influences on the geodesics' tendency to come together or fly apart.

Nonzero vorticity promotes the latter, while nonzero shear and expansion (or contraction) cause the ball to recollapse. Using the Einstein field equations the last term of eq. (2.8) can be rewritten as

$$R_{\mu\nu}u^\mu u^\nu = 8\pi\left[T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}\right]u^\mu u^\nu = 8\pi\left[T_{\mu\nu}u^\mu u^\nu - \frac{1}{2}T\right] \tag{2.10}$$

This means that the presence of any matter observing the *strong energy condition*

$$T_{\mu\nu}u^\mu u^\nu \geq \frac{1}{2}T \quad \text{for all timelike } \mathbf{u} \tag{2.11}$$

will also cause a tendency to recollapse.

Chapter 3

Examples of Rotating Spacetimes

3.1 Van Stockum dust

Discovered independently by Lanczos (1924) and van Stockum (1937), this exact solution of the Einstein field equations first introduced the idea of a rotating universe. Lanczos (1924) conceived it as an alternative to the stationary, spherically symmetric "Einstein cylinder" solution supported by a carefully chosen cosmological constant, where the latter could be abolished at the cost of also losing spherical symmetry. Lanczos derives his solution as a particular example of such a spacetime, while employing a very strict notion of "stationary" spacetimes by demanding that a free particle should remain at rest in the coordinate system, i.e. that the timelike coordinate lines be geodesics.

While Lanczos (1924) does not explicitly mention rotation, van Stockum (1937) rederived the same solution by demanding the spacetime to be cylindrically symmetric and filled with a fluid in rigid rotation. The worldlines of the matter should therefore have nonzero vorticity, but zero shear and expansion. This is satisfied by a frame of the form

$$\mathbf{e}_0 = \partial_t, \quad \mathbf{e}_1 = f(r)\partial_z, \quad \mathbf{e}_2 = f(r)\partial_r, \quad \mathbf{e}_3 = \frac{1}{r}\partial_\phi - h(r)\partial_t \tag{3.1}$$

with the corresponding dual coframe

$$\sigma^0 = -dt + h(r)rd\phi, \quad \sigma^1 = \frac{1}{f(r)}dz, \quad \sigma^2 = \frac{1}{f(r)}dr, \quad \sigma^3 = rd\phi \tag{3.2}$$

with indeterminate functions $f(r)$ and $h(r)$. The metric tensor is then

$$\begin{aligned} g &= \sigma^0 \otimes \sigma^0 - \sigma^1 \otimes \sigma^1 - \sigma^2 \otimes \sigma^2 - \sigma^3 \otimes \sigma^3, && (3.3a) \\ ds^2 &= dt^2 - 2h(r)rd\phi dt - \left(1 - h(r)^2\right) r^2 d\phi^2 - \frac{dr^2 + dz^2}{f(r)^2} && (3.3b) \end{aligned}$$

Computing the Einstein tensor and demanding it to match the stress-energy-tensor of a perfect fluid

$$G^{\mu\nu} = 8\pi\rho \,\mathrm{diag}\,(1,0,0,0) + 8\pi p \,\mathrm{diag}\,(0,1,1,1) \tag{3.4}$$

yields the differential equations

$$f'' = \frac{(f')^2}{f} + \frac{f'}{r}, \quad \left(h'\right)^2 + \frac{2h'h}{r} + \frac{h^2}{r^2} = \frac{4f'}{rf} \tag{3.5}$$

which solve to

$$\begin{aligned} f(r) &= e^{\frac{\omega^2 r^2}{2}}, \quad h(r) = \omega r && (3.6a) \\ ds^2 &= dt^2 - 2\omega r^2 d\phi dt - \left(1 - \omega^2 r^2\right) r^2 d\phi^2 && (3.6b) \\ &\quad - e^{-\omega^2 r^2} \left(dr^2 + dz^2\right) && (3.6c) \end{aligned}$$

The pressure p vanishes (so we are actually dealing with dust), but the mass density $\rho = \frac{\omega^2}{2\pi} e^{\omega^2 r^2}$ increases very quickly with distance r from the axis of rotation. The world lines do not expand or shear, but they rotate around the central axis with an angular velocity of ω (van Stockum, 1937, p. 145).

The frame is comoving but spinning about $\mathbf{e}_1$. A nonrotating inertial frame can be obtained by spinning $\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ in the $\mathbf{e}_2$-$\mathbf{e}_3$ plane by $\theta = \omega t e^{\frac{a^2 r^2}{2}}$ which can be found by demanding that the covariant derivatives $\nabla_{\mathbf{e}_0}\mathbf{e}_1$, $\nabla_{\mathbf{e}_0}\mathbf{e}_2$ and $\nabla_{\mathbf{e}_0}\mathbf{e}_3$ vanish.

The most interesting feature of the van Stockum dust universe is that it contains closed timelike curves (CTCs). Consider a null vector $\mathbf{u}$ in the t-ϕ-plane

$$\begin{aligned} 0 &= u_\mu u^\mu = g_{\mu\nu} u^\mu u^\nu && (3.7a) \\ &= u^0 u^0 - 2\omega r^3 u^0 u^\phi - \left(1 - (\omega r)^2\right) u^\phi u^\phi && (3.7b) \\ \implies \quad \frac{u^0}{u^\phi} &= \omega r^3 \pm \sqrt{\left(\omega r^3\right)^2 - \omega^2 r^2 + 1} && (3.7c) \end{aligned}$$

When $\omega r = 1$ the lower branch of u^0/u^ϕ is zero, for $\omega r > 1$ even negative. The future lightcone of any event where $r \geq \omega^{-1}$ therefore contains a timelike curve $r, t, z = \text{const.}$, $\phi \in [0, 2\pi)$ that returns to the same event, a closed timelike curve. Lanczos (1924) also showed that null geodesics, i.e. light rays, emitted from the axis of rotation in the $\mathbf{e}_2$-$\mathbf{e}_3$ plane only reach the $r = \omega^{-1}$ circle and then return to the central axis at a later time. For rays deviating from this plane the limiting circle is even smaller.

3.2 Spacetimes of Gödel type

3.2.1 The original Gödel metric

In Gödel (1949) a stationary homogeneous rotating spacetime with zero expansion and shear of the form

$$ds^2 = a^2 \left(dt^2 - 2e^x dt dy + \frac{1}{2} e^{2x} dy^2 - dx^2 - dz^2 \right) \tag{3.8}$$

which can also be written as (Hawking and Ellis, 1975, p. 168)

$$ds^2 = dt^2 + \frac{1}{2} e^{2\sqrt{2}\omega x} dy^2 + 2e^{\sqrt{2}\omega x} dy dt - dx^2 - dz^2 \tag{3.9}$$

was proposed, where a pressureless perfect fluid (dust) of energy density ρ is supported by a negative cosmological constant of carefully chosen magnitude

$$-\Lambda = \omega^2 = \frac{1}{2a^2} = 4\pi\rho \tag{3.10}$$

This choice makes the model somewhat artificial. Since the nontrivial covariant field equations reduce to (Ellis, 1996, p. 36)

$$2\omega^2 = 8\pi(\rho + p) \tag{3.11a}$$

$$\Lambda + 2\omega^2 = 4\pi(\rho + 3p) \tag{3.11b}$$

therefore

$$\Lambda = 4\pi(-\rho + p) \tag{3.11c}$$

Choosing $p = 0$ gives Gödels original view. Alternatively

$$\Lambda = 0 \quad \Longrightarrow \quad p = \rho = \frac{\omega^2}{8\pi} \tag{3.12}$$

allows the matter to equivalently be interpreted as a perfect fluid without a cosmological constant and the equation of state $p = \rho$.

Transforming into axisymmetric comoving coordinates centered on any particular world line

$$e^{\sqrt{2}\omega x} = \cosh 2r + \cos\phi \sinh 2r \qquad (3.13a)$$

$$\omega y e^{\sqrt{2}\omega x} = \sin\phi \sinh 2r \qquad (3.13b)$$

$$\tan \frac{1}{2}\left(\phi + \omega t - \sqrt{2}t'\right) = e^{-2r} \tan \frac{1}{2}\phi \qquad (3.13c)$$

gives the line element

$$ds^2 = \frac{2}{\omega^2}\Bigg(dt'^2 - dz^2 - dr^2 - \left(\sinh^2 r - \sinh^4 r\right) d\phi^2 \qquad (3.14a)$$

$$- 2\sqrt{2}\sinh^2 r d\phi dt'\Bigg) \qquad (3.14b)$$

where $0 \leq \phi \leq 2\pi$ and $\phi = 0$ is identified with $\phi = 2\pi$. The flow vector is then $\mathbf{u} = \frac{\omega}{\sqrt{2}}\partial_{t'}$.

An argument along the lines of eq. (3.7) again shows that, albeit being locally well-behaved everywhere, simply connected and geodesically complete (Hawking and Ellis, 1975), the Gödel spacetime contains CTCs when $r > \ln\left(1 + \sqrt{2}\right)$. Since the choice of the $r = 0$ world line is arbitrary, there are infinitely many such CTCs at any event. There are however no closed timelike geodesics (Ellis, 1996)—an observer trying to reach his own past is required to accelerate by non-gravitational means.

Kundt (1956) first investigated the geodesic movement of test particles and light rays in Gödel's metric. Hawking and Ellis (1975) depicted the null geodesics originating at the $r = 0$ world line as spiraling outward until they reach the limiting caustic $r = \ln\left(1 + \sqrt{2}\right)$ and then refocusing back onto a later event on the same worldline. Observers riding along with the dust can thereby see themselves in their own past. Massive particles will similarly reach some smaller limiting circle and refocus on the original worldline (Hawking and Ellis, 1975, p. 170).

3.2.2 Stationary cosmological spacetimes of Gödel type[1]

The Gödel metric (3.8) is the special shear- and expansion free case of the more general class of cosmological models of the form

$$ds^2 = dt^2 - 2\sqrt{\sigma}e^{mx}dtdy - \left(dx^2 + ke^{2mx}dy^2 + dz^2\right) \tag{3.15}$$

where m, k and σ are constant parameters. The spacetime rotates in the x-y-plane with magnitude

$$\omega = \sqrt{\frac{1}{2}\omega_{\mu\nu}\omega^{\mu\nu}} = \frac{m}{2}\sqrt{\frac{\sigma}{\sigma + k}} \tag{3.16}$$

the metric determinant is $\det g = -(k+\sigma)e^{2mx}$, and for the metric to have Lorentzian signature it must be assumed that $k + \sigma > 0$.

A convenient choice for an orthonormal tetrad h^a_μ such that $g_{\mu\nu} = h^a_\mu h^b_\nu \eta_{ab}$ ($\eta_{ab} = \mathrm{diag}(+1, -1, -1, -1)$ the Minkowski metric) is

$$h^{\hat{0}}_0 = h^{\hat{1}}_1 = h^{\hat{3}}_3 = 1, \quad h^{\hat{0}}_2 = -\sqrt{\sigma}e^{mx}, \quad h^{\hat{2}}_2 = e^{mx}\sqrt{k+\sigma} \tag{3.17}$$

with the local Lorentz coframe form

$$\vartheta^a = h^a_\mu dx^\mu \tag{3.18}$$

The curvature form for (3.15) is then

$$R^{ab} = \omega^2 \vartheta^a \wedge \vartheta^b + \frac{km^2}{k+\sigma}\left(\delta^a_{\hat{1}}\delta^b_{\hat{2}} - \delta^b_{\hat{1}}\delta^a_{\hat{2}}\right)\vartheta^{\hat{1}} \wedge \vartheta^{\hat{2}} \tag{3.19}$$

where $a, b = 0, 1, 2$. The range of indices reflects the fact that (3.15) is the direct product of a $(2+1)$-dimensional manifold with the curvature of (3.19) and 1-dimensional flat space.

The components of the Einstein tensor are

$$G_{\hat{0}\hat{0}} = -\omega^2\left(1 + 4\frac{k}{\sigma}\right), \quad G_{\hat{1}\hat{1}} = G_{\hat{2}\hat{2}} = \omega^2, \quad G_{\hat{3}\hat{3}} = \omega^2\left(3 + 4\frac{k}{\sigma}\right) \tag{3.20}$$

Considering dust ($T_{ab} = \rho u_a u_b$) and substituting the four-velocity of comoving matter $u^a = \delta^a_{\hat{0}}$ allows finding the parameters (3.10) of the original Gödel solution (compare equation 3.11) and $\frac{k}{\sigma} = -\frac{1}{2}$.

[1]This chapter mostly follows Obukhov (2000)

Three isometries evident from (3.15) are generated by the Killing vector fields

$$\boldsymbol{\xi}_{(0)} = \partial_t, \quad \boldsymbol{\xi}_{(1)} = \partial_y, \quad \boldsymbol{\xi}_{(2)} = \partial_z \tag{3.21}$$

The coordinate transformation

$$\begin{aligned}
(t, x, y) &\mapsto (\bar{\tau}, \bar{r}, \bar{\varphi}) && (3.22a)\\
e^{mx} &= e^{\bar{\varphi}} \cosh(m\bar{r}) && (3.22b)\\
ye^{mx} &= \frac{\sinh(m\bar{r}}{m\sqrt{k+\sigma}} && (3.22c)\\
\tan\left[m\sqrt{\frac{k+\sigma}{\sigma}}(t-\bar{\tau})\right] &= \sinh(m\bar{r}) && (3.22d)
\end{aligned}$$

yields a fourth Killing vector field

$$\boldsymbol{\xi}_{(3)} = \partial_{\bar{\varphi}} = \frac{1}{m}\partial_x - y\partial_y \tag{3.23}$$

and yet another transformation

$$\begin{aligned}
(t, x, y) &\mapsto (\tau, r, \varphi) && (3.24a)\\
e^{mx} &= \cosh(mr) + \cos\varphi \sinh(mr) && (3.24b)\\
ye^{mx} &= \frac{\sin\varphi \sinh(mr)}{m\sqrt{k+\sigma}} && (3.24c)\\
\tan\left[m\sqrt{\frac{k+\sigma}{\sigma}}(t-\bar{\tau})\right] &= \frac{\sin\varphi}{\cos\varphi + \coth\left(\frac{mr}{2}\right)} && (3.24d)
\end{aligned}$$

results in the another isometry generated by

$$\partial_\varphi = \frac{1}{m}\sqrt{\frac{\sigma}{k+\sigma}}\boldsymbol{\xi}_{(0)} + \frac{1}{2m\sqrt{k+\sigma}}\boldsymbol{\xi}_{(1)} - \sqrt{k+\sigma}\boldsymbol{\xi}_{(4)} \tag{3.25}$$

with the fifth Killing vector field

$$\boldsymbol{\xi}_{(4)} = \frac{\sqrt{\sigma}e^{-mx}}{m(k+\sigma)}\partial_t + y\partial_x + \frac{1}{2}\left[\frac{e^{-2mx}}{m(k+\sigma)} - my^2\right]\partial_y \tag{3.26}$$

In examining the causal structure it is instructive to compute the squares of the Killing vectors:

$$\boldsymbol{\xi}_{(0)} \cdot \boldsymbol{\xi}_{(0)} = 1 \tag{3.27a}$$

$$\boldsymbol{\xi}_{(1)} \cdot \boldsymbol{\xi}_{(1)} = -ke^{2mx} \tag{3.27b}$$

$$\boldsymbol{\xi}_{(2)} \cdot \boldsymbol{\xi}_{(2)} = -1 \tag{3.27c}$$

$$\boldsymbol{\xi}_{(3)} \cdot \boldsymbol{\xi}_{(3)} = -\left(ky^2 e^{2mx} + \frac{1}{m^2}\right) \tag{3.27d}$$

$$\boldsymbol{\xi}_{(4)} \cdot \boldsymbol{\xi}_{(4)} = -\frac{k}{4}\left(my^2 e^{mx} + \frac{e^{-mx}}{m(k+\sigma)}\right)^2 \tag{3.27e}$$

$$\tag{3.27f}$$

$\boldsymbol{\xi}_{(0)}$ is obviously strictly timelike, its integral curves coinciding with the coordinate time lines.

For $k > 0$ the other four Killing fields $\boldsymbol{\xi}_{(1)}, \ldots, \boldsymbol{\xi}_{(4)}$ are strictly spacelike and the spacetime is completely causal, containing no CTCs. The first three Killing vector fields represent the homogeneity of the $t = \text{const.}$ slices.

For $k < 0$ the Killing fields $\boldsymbol{\xi}_{(1)}$ and $\boldsymbol{\xi}_{(4)}$ are everywhere timelike, while $\boldsymbol{\xi}_{(3)}$ can be space- or timelike depending on the value of x and y. There exist then closed timelike curves, for example the closed curve $\{t(\varphi), x(\varphi), y(\varphi)\}$ as defined in (3.24) for $\tau, r, z = \text{const.}$ with $\cosh^2\left(\frac{mr}{2}\right) > \left|\frac{\sigma}{k}\right|$. As with the Gödel metric the CTCs are not geodesics. It should be noted that the coordinate $\bar{\varphi}$ curve is timelike for $\cosh^2(m\bar{r}) > \left|\frac{\sigma}{k}\right|$ but not closed.

For $k = 0$ the curvature (3.19) reduces to $R^{ab} = \omega^2 \vartheta^a \wedge \vartheta^b$. $a, b = 0, 1, 2$, describing a $2+1$-dimensional manifold of constant curvature, i.e. de Sitter or anti de Sitter spacetime. The symmetries of this spacetime result in the two additional Killing vector fields

$$\boldsymbol{\xi}_{(5)} = \sin(mt)\partial_t - \cos(mt)\partial_x + \frac{e^{-mx}}{\sqrt{\sigma}}\sin(mt)\partial_y \tag{3.28a}$$

$$\boldsymbol{\xi}_{(6)} = \cos(mt)\partial_t + \sin(mt)\partial_x + \frac{e^{-mx}}{\sqrt{\sigma}}\cos(mt)\partial_y \tag{3.28b}$$

$$\tag{3.28c}$$

Both are strictly spacelike

$$\boldsymbol{\xi}_{(5)} \cdot \boldsymbol{\xi}_{(5)} = \boldsymbol{\xi}_{(6)} \cdot \boldsymbol{\xi}_{(6)} = -1 \tag{3.29}$$

3.2.3 The expanding Gödel type cosmological model

Introducing a time dependent scale factor $a(t)$ into the Gödel type metric (3.15) results in the line element

$$ds^2 = dt^2 - 2\sqrt{\sigma}a(t)e^{mx}dtdy - a^2(t)\left(dx^2 + ke^{2mx}dy^2 + dz^2\right) \tag{3.30}$$

with constants $m, \sigma, k > 0$. Note that the choice $k > 0$ ensures the causality of the spacetime. This model is usually called the *Gödel type model with rotation and expansion.* Global rotation is about the z axis with magnitude

$$\omega = \sqrt{\frac{1}{2}\omega_{\mu\nu}\omega^{\mu\nu}} = \frac{m}{2a(t)}\sqrt{\frac{\sigma}{\sigma + k}} \tag{3.31}$$

decreasing with the expansion of the universe.

There are three Killing vector fields

$$\boldsymbol{\xi}_{(1)} = \frac{1}{m}\partial_z - y\partial_y, \quad \boldsymbol{\xi}_{(2)} = \partial_y, \quad \boldsymbol{\xi}_{(3)} = \partial_z \tag{3.32}$$

and a conformal Killing vector field

$$\boldsymbol{\xi}_{(0)} = a(t)\partial_t \tag{3.33}$$

A suitable choice of a tetrad h^a_μ, $g_{\mu\nu} = h^a_\mu h^b_\nu \eta_{ab}$ is, generalizing (3.17),

$$\begin{aligned} h^{\hat{0}}_0 &= 1 && \text{(3.34a)} \\ h^{\hat{0}}_2 &= -a(t)\sqrt{\sigma}e^{mx} && \text{(3.34b)} \\ h^{\hat{1}}_1 = h^{\hat{3}}_3 &= a(t) && \text{(3.34c)} \\ h^{\hat{2}}_2 &= a(t)e^{mx}\sqrt{k+\sigma} && \text{(3.34d)} \end{aligned}$$

Korotky and Obukhov (1996) give a solution for null geodesics. Because of the homogeneity of the spacetime (3.30) it is sufficient to consider geodesics originating at an event $P = (t = t_0, x, y, z = 0)$ such that in the tetrad (3.34) the tangent vector $\mathbf{k}$ to the geodesic is

$$k^a|_P = \left(h^a_\mu k^\mu\right)\big|_P = (1, \sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)|_P \tag{3.35}$$

where (θ, ϕ) are spherical coordinates on the celestial sphere of the observer at P.

For $\sin\theta\sin\phi + \sqrt{\frac{\sigma}{\sigma+k}} \neq 0$ the geodesics are then described by

$$e^{-mx} = \frac{\sqrt{\sigma} + \sqrt{k+\sigma}\sin\theta\sin\Phi(t)}{\sqrt{\sigma} + \sqrt{k+\sigma}\sin\theta\sin\phi} \tag{3.36a}$$

$$y = \frac{\sin\theta\left(\cos\Phi(t) - \cos\phi\right)}{m\left(\sqrt{\sigma} + \sqrt{k+\sigma}\sin\theta\sin\phi\right)} \tag{3.36b}$$

$$z = \frac{k+\sigma}{k}\cos\theta\left[\int_{t_0}^{t}\frac{dt'}{a(t')} + \sqrt{\frac{\sigma}{\sigma+k}}\frac{\Phi(t)-\phi}{m}\right] \tag{3.36c}$$

$$\tag{3.36d}$$

where $\Phi(t)$ satisfies

$$\frac{d\Phi(t)}{dt} = -\frac{m}{a(t)}\frac{\sqrt{\frac{\sigma}{k+\sigma}} + \sin\theta\sin\phi}{1 + \sqrt{\frac{\sigma}{k+\sigma}}\sin\theta\sin\phi} \tag{3.37}$$

3.3 The Tipler cylinder

While the spacetimes of Lanczos-van Stockum and Gödel type are cosmological model describing universes filled with matter, Tipler (1974) showed that causality violation also occurs as the result of a "localized" matter configuration.

Tiplers spacetime consists of a cylindrical slice of van Stockum dust (see section 3.1) cut off at a boundary $R < \frac{1}{\omega}$. Outside the cylinder there is vacuum.

Expressing the metric as

$$ds^2 = Fdt^2 - 2Md\varphi dt - Ld\varphi^2 - H\left(dr^2 + dz^2\right) \tag{3.38}$$

the interior solution of the cylinder is given by (compare eq. (3.6))

$$F = 1, \quad M = \omega r^2, \quad L = \left(1 - \omega^2 r^2\right) r^2, \quad H = e^{-\omega^2 r^2} \tag{3.39}$$

for $r < R < \frac{1}{\omega}$.

There are three distinct exterior solutions depending on ωR, the "velocity" of the matter at the edge of the cylinder. All can be expressed in the form of eq. (3.38).

For $0 < \omega R < \frac{1}{2}$ the parameters of the exterior solution are (Tipler, 1974)

$$
\begin{aligned}
F &= \frac{r \sinh(\varepsilon - \theta)}{R \sinh \varepsilon} && (3.40a)\\
M &= \frac{r \sinh(\varepsilon + \theta}{\sinh 2\varepsilon} && (3.40b)\\
L &= \frac{Rr \sinh(3\varepsilon + \theta}{2 \sinh 2\varepsilon \cosh \varepsilon} && (3.40c)\\
H &= e^{-\omega^2 r^2} \left(\frac{r}{R}\right)^{-2\omega^2 r^2} && (3.40d)
\end{aligned}
$$

with

$$
\begin{aligned}
\theta &= \sqrt{1 - 4\omega^2 R^2} \ln\left(\frac{r}{R}\right) && (3.41a)\\
\varepsilon &= \tanh^{-1} \sqrt{1 - 4\omega^{2}{}^{\text{‘}} R^2} && (3.41b)
\end{aligned}
$$

For $\omega R = \frac{1}{2}$

$$
\begin{aligned}
F &= \frac{r}{R}\left[1 - \ln\left(\frac{r}{R}\right)\right] && (3.42a)\\
M &= \frac{r}{2}\left[1 + \ln\left(\frac{r}{R}\right)\right] && (3.42b)\\
L &= \frac{Rr}{4}\left[3 + \ln\left(\frac{r}{R}\right)\right] && (3.42c)\\
H &= e^{-\frac{1}{4}} \left(\frac{r}{R}\right)^{-\frac{1}{2}} && (3.42d)
\end{aligned}
$$

For $\omega R > \frac{1}{2}$

$$
\begin{aligned}
F &= \frac{r \sin(\beta - \gamma)}{R \sin \beta} && (3.43a)\\
M &= \frac{r \sin(\beta + \gamma)}{\sin 2\beta} && (3.43b)\\
L &= \frac{Rr \sin(3\beta + \gamma)}{2 \sin 2\beta cos\beta} && (3.43c)\\
H &= e^{-\omega^2 R^2} \left(\frac{r}{R}\right)^{-2\omega^2 R^2} && (3.43d)
\end{aligned}
$$

with

$$
\begin{aligned}
\gamma &= \sqrt{4\omega^2 R^2 - 1} \ln\left(\frac{r}{R}\right) && (3.44a)\\
\beta &= \tan^{-1} \sqrt{4\omega^2 R^2 - 1} && (3.44b)
\end{aligned}
$$

In the first two cases, $\omega R \leq \frac{1}{2}$, there are no closed timelike curves, and the spacetime is causal everywhere. For $\omega R > \frac{1}{2}$ there are however CTCs through every event outside the cylinder (Tipler, 1974, p. 2204).

Tipler presumed that this might as well be the case for a sufficiently fast spinning cylinder of finite length, at least close to the surface $r = R$. Hawking (1992) on the other hand proved that it is impossible to create stable closed timelike curves within a compact region of a spacetime without violation of the weak energy condition, i.e. without negative energy densities. Hawking extends this to a general *chronology protection conjecture*, reckoning that *the laws of physics do not allow the appearance of closed timelike curves.*

Chapter 4

Observable Effects

4.1 Frame dragging

While the Newtonian theory of mechanics employs a concept of an absolute space relative to which any rotation could be defined, there is no such space in General Relativity. Thirring (1918)[1] calculated—strongly influenced by Albert Einstein, who had performed similar calculations in earlier versions of his theory of gravitation, see Pfister (2007)—the effect a rotating, distant shell of material on an observer close to its center. This shell is taken as a model of the background of distant stars. According to Machs principle, as explained by Einstein (quoted in Thirring (1918)), the centrifugal and Coriolis forces experienced by an observer rotating relative to this distant background should be the same if the distant stars are rotating relative to the observer, who is now taken to be non-rotating.

Thirring (1918) finds the acceleration due to the rotation of the shell experienced by a test particle close to its center to be (in the form given by (Pfister, 2007, p. 1744))

$$\boldsymbol{a} = -\frac{8M}{3R}\left(\boldsymbol{\omega} \times \boldsymbol{v}\right) - \frac{4M}{15R}\left[\boldsymbol{\omega} \times \left(\boldsymbol{\omega} \times \boldsymbol{r}\right) + 2\left(\boldsymbol{\omega} \cdot \boldsymbol{r}\right)\boldsymbol{\omega}\right] \tag{4.1}$$

with M the mass of the shell, R its radius, $\boldsymbol{\omega}$ its angular 3-velocity and $\boldsymbol{r}$, $\boldsymbol{v}$ the location and 3-velocity of the observer, where $|\boldsymbol{\omega}|\, R \ll 1$ and $|\boldsymbol{v}| \ll 1$ are assumed.

[1] note also the erratum (Thirring, 1921)

The first term of the right hand side of eq. (4.1) is valid for all points $|r| < R$, while the second term arises from the distribution of mass on the shell. It is not physically correct since, among other effects, Thirring (1918) disregards stresses in the shell (Pfister, 2007, p. 1744). It is possible to construct the shell such that the interior is flat and therefore quasi-Newtonian conditions with the "correct" centrifugal and Coriolis forces as expected from the relativity of rotation arise, but this requires a prolate shape of the shell and differential rotation (Pfister, 2007, p. 1746).

Despite its deficiencies the general idea of (Thirring, 1918) does hold: in the interior of a rotating massive shell the spacetime is "dragged along", producing Coriolis and centrifugal effects. These vanish only in a coordinate system rotating against the background of the (asymptotically flat) spacetime outside the shell. This frame dragging effect is known as the *Lense-Thirring effect*.

Later in the same year, Lense and Thirring (1918) applied the methods of (Thirring, 1918) to the exterior far field of a rotating, massive spherical body. They find a Coriolis acceleration (again in the form given by (Pfister, 2007, p. 1743))

$$\boldsymbol{b} = \frac{4MR^2}{5r^2}\boldsymbol{v} \times \left[\boldsymbol{\omega} - 3\frac{(\boldsymbol{\omega}\cdot\boldsymbol{r})\,\boldsymbol{r}}{r^2}\right] \tag{4.2}$$

where again M and R are the mass and radius of the rotating body, $\boldsymbol{\omega}$ its angular 3-velocity, $\boldsymbol{r}$ the 3-location of the test body and $r = |\boldsymbol{r}|$ its distance from the center of the body. This result is valid for all $r > R$ (Pfister, 2007, p.1743). Lense and Thirring (1918) also calculate the effect of this acceleration in terms of the orbital elements used in astronomy and apply it to the motions of the planets and moons in the solar system (apparently (Pfister, 2007, p. 1742) this was the contribution of Lense to the paper). The effects they find are too small to be measured astronomically—especially considering the many other disturbances in gravitational many-body-systems.

The Lense-Thirring effect has however been experimentally confirmed in the movement of the orbital nodes of the LAGEOS 1 and 2 satellites (Ciufolini and Pavlis, 2004) orbiting Earth. Data analysis of the Gravity Probe B experiment[2], launched in 2004 specifically for measuring the precession

[2] `http://einstein.stanford.edu`

of gyroscopes in the Earth's gravitational field, is expected to be completed in 2010. Preliminary results (NASA, 2008) confirm the expected Lense-Thirring effect, although—due to technical difficulties—not with the accuracy the experiment was designed for.

4.2 Cosmological observations

A possible global rotation of the universe would be very hard to measure locally. The frame dragging of local bodies (Earth, the Sun, very likely even of the Milky Way) would drown any locally measurable effect.

On the other hand a rotating universe is necessarily anisotropic, and this should show in astronomical observations statistically varying dependent on the portion of sky observed. Such anisotropies have repeatedly been found, most notably by Birch (1982) who first proposed global rotation as a possible explanation. The significance of such results has always been a matter of dispute—in the case of Birch (1982) first by Phinney and Webster (1983) (and also see the response Birch (1983)); Kendall and Young (1984) confirm Birch's findings, while Bietenholz and Kronberg (1984) do not.

4.2.1 The Cosmic Microwave Background

One of the strongest indications for an overall isotropy of the universe is the isotropy of the Cosmic Microwave Background (CMB). The observed temperature T_P of thermal radiation is related to the tempterature T_S at the source through the red shift z by

$$T_P = \frac{T_S}{1+z} = T_S \frac{(k^\mu u_\mu)_P}{(k^\mu u_\mu)_S} \tag{4.3}$$

with the cosmological redshift (Ellis, 2009, p. 624)

$$z = \frac{(k^\mu u_\mu)_S}{(k^\mu u_\mu)_P} - 1 \tag{4.4}$$

where $\mathbf{k}$ is the wave vector and $\mathbf{u}$ the matter velocity. The factor $\frac{(k^\mu u_\mu)_P}{(k^\mu u_\mu)_S}$ is in general dependent on the direction of pbservation.

Note however that in the case of the expanding Gödel type model (section 3.2.3), where

$$\xi^\mu_{(0)} = a(t)\partial_t = a(t)u^\mu \tag{4.5}$$

the CMB temperature only depends on the scale factor $a(t)$: Since the scalar product of the wave vector with any conformal Killing vector field

$$k_\mu \xi^\mu = \text{const.} \quad \Longrightarrow \quad a(t_P)(k_\mu u^\mu)_P = a(t_S)(k_\mu u^\mu)_S \tag{4.6a}$$

and therefore

$$T_P = T_S \frac{a(t_S)}{a(t_P)} \tag{4.7}$$

so that the rotation of spacetimes of type (3.30) would not indtroduce any anisotropy into the observed CMB.

4.2.2 Parallax

The generalized concept of *parallax* as relative motion of distant galaxies on the celestial sphere has been introduced and formalized by Hasse and Perlick (1988). A a spacetime with a normalized timelike vector field $\mathbf{u}$ describing the average motion of matter in a universe is said to be *parallax free* if for any two sources A_0, A_1 and an observer P, all moving along with $\mathbf{u}$, the angle between A_0 and A_1 on the celestial sphere of P is constant.

There are five equivalent necessary and sufficient conditions for a spacetime to be parallax free:

- The spacetime is *parallax free in the weak sense*, i.e. if A_0 and A_1 appear in the same direction as seen from P at any time, they will also appear so at any later time
- $\mathbf{u}$ is proportional to some conformal Killing vector field
- there exists a scalar function f such that

  $$\frac{1}{2}\mathcal{L}_\mathbf{u} g_{\mu\nu} = \left(u^\lambda \partial_\lambda f\right) g_{\mu\nu} - u_{(\mu}\partial_{\nu)} f \tag{4.8}$$

  where $\mathcal{L}_\mathbf{u}$ denotes the Lie derivative with respect to $\mathbf{u}$.
- $\mathbf{u}$ is shear free and the one-form

  $$\rho = \left(u^\nu \nabla_\nu u_\mu - \frac{\vartheta}{3} u_\mu\right) dx^\mu \tag{4.9}$$

  is closed ($d\rho = 0$)

- there is a *red shift potential*, i.e. a scalar function f such that for a light signal emitted at the event S and received at P the red shift $z = \frac{\Delta\lambda}{\lambda}$ is given by

$$\ln(z+1) = f(P) - f(S) \tag{4.10}$$

Note that the expanding Gödel type spacetime (section 3.2.3)—and as special cases the stationary Gödel type spacetime (section 3.2.2) and the classicel Gödel model (section 3.2.1)—fulfill these conditions (e.g. the second one by eq. (4.5)) and are therefore parallax free. This is also true for the van Stockum dust (section 3.1).

4.2.3 Apparent magnitude m versus red shift z

While the red shift of a distant galaxy is given by eq. (4.4), the measured energy flux at L_P at the observer P (emitted at S) and thus the apparent magnitude $m = -\frac{5}{2}\log_{10} L_P$ depend on the *area distance* r defined by the relation (Ellis, 2009, p. 631)

$$dA_S = r^2 d\Omega_P \tag{4.11}$$

where the solid angle $d\Omega_P$ at P is subtended by the intrinsic perpendicular area dA_S at S. A reverse area distance can be defined similarly:

$$dA_P = r_S^2 d\Omega_S \tag{4.12}$$

They are related through the *reciprocity theorem* (Ellis, 2009, p. 631) by

$$r_S^2 = r^2 (1+z)^2 \tag{4.13}$$

If the total output L of the source is known (or can be statistically estimated) then the corresponding energy flux at the source is (provided the emission is isotropic, so caution is necessary, especially in the case of Active Galactic Nuclei!) $L_S = \frac{L}{4\pi}$ and the absolute luminosity $M = -\frac{5}{2}\log_{10} L_S$. Then (Ellis, 2009, p. 634)

$$L_P = \frac{L_S}{r_S^2(1+z)^2} = \frac{L_S}{r^2(1+z)^4} \tag{4.14a}$$

$$m = M + 5\log_{10} r(1+z)^2 \tag{4.14b}$$

For the Gödel type universe of section 3.2.2 the relation between r and z depends on the direction (θ, ϕ) at which the light source appears on the observer's celestial sphere. Equation (4.14) is then, up to first order in z (Obukhov, 2000)

$$
\begin{aligned}
m &= M - 5\log_{10} H_0 + 5\log_{10} z + \frac{5}{2}(\log_{10} e)(1-q_0)z && (4.15a)\\
&\quad - 5\log_{10}\left(1 + \sqrt{\frac{\sigma}{\sigma+k}}\sin\theta\sin\phi\right) && (4.15b)\\
&\quad - \frac{5}{2}(\log_{10} e)\frac{\omega_0}{H_0} z \sin\theta\cos\phi \left[\frac{\sqrt{\frac{\sigma}{\sigma+k}} + \sin\theta\sin\phi}{\left(1+\sqrt{\frac{\sigma}{\sigma+k}}\sin\theta\sin\phi\right)^2}\right] && (4.15c)\\
&\quad + O(z^2) && (4.15d)
\end{aligned}
$$

where σ, k are the parameters of eq. (3.15), $H_0 = \frac{\dot{a}}{a}\big|_P$ the Hubble constant, $\omega_0 = \omega(t = t_0)$ the rotation value and $q_0 = -\frac{\ddot{a}a^2}{\dot{a}^2}\Big|_P$ the deceleration parameter, all taken at the time of observation.

4.2.4 Number of Sources N versus red shift z

An observer P looking at a small patch $d\Omega$ of his night sky sees a projection of the images of all the sources along the bundle of null geodesics approaching from that direction. The distance dl between two infinitesimally close cross-sections dA of that bundle is given by

$$dl = (k^\mu u_\mu)ds \tag{4.16}$$

where s is a parameter designating the position of an observer on the geodesic, while $dA = r^2 d\Omega$ as in eq. (4.11). The cross-sections then enclose a volume $dV = dAdl$ containing $dN = -n_S(k^\mu u_\mu)_S r^2 d\Omega ds$ sources where n_S is the density of sources in dV.

By integration over s the apparent number density on the celestial sphere of the observer at P can be obtained, and by further integration the global difference of the number of sources visible in two hemispheres N^+, N^- can be estimated. For the Gödel type universe (section 3.2.2) it is (Obukhov, 2000)

$$\frac{N^+ - N^-}{N^+ + N^-} = \frac{1}{2}\sqrt{\frac{\sigma}{\sigma+k}}\left(3 - \frac{\sigma}{\sigma+k}\right) + O(z^2) \tag{4.17}$$

where sources up to red shift z are considered. Note that there is no correction linear in z.

This treatment however neglects any evolution of the sources. This is a severe restriction, as the evolution of physical properties of the sources often dominates over these geometrical effects.

4.2.5 Polarization rotation effect

In the geometric optics approximation (Ellis, 2009, pp. 622–624) the polarization of light is not affected by the spacetime in the sense that, along the null geodesic on which the light propagates, any directions of polarization are parallel transported, while numerical parameters remain constant (Ellis, 2009, p. 627).

On the other hand shape, size and orientation of the source's image are naturally affected by the gravitational field. If there is, for a certain class of emitters, a known relationship between the polarization of the light and a distinguished geometrical axis, e.g. the major axis of an elliptical image, the difference η between the observed angles will depend on the distance and the gravitational field along the path of the light rays.

For a given spacetime, this effect can be calculated from the knowledge of the null geodesics, as e.g. Korotky and Obukhov (1996) have done for the Gödel type metric of section 3.2.2. They find

$$\eta = \omega_0 r cos\theta + O(z^2) \tag{4.18}$$

with ω_0, r and θ as before.

Birch (1982) has found an anisotropy in the angle difference between the position angles of elongation and polarization in a sample of 94 3CR radio sources. The data is consistent with a direction and magnitude of cosmic rotation (Obukhov, 2000)

$$l^\circ = 295^\circ \pm 25^\circ, \quad b^\circ = 24^\circ \pm 20^\circ, \quad \omega_0 = (1.8 \pm 0.8) H_0 \simeq (1.4 \pm 0.6) \cdot 10^{-10}\,\mathrm{yr}^{-1} \tag{4.19}$$

Nodland and Ralston (1997) discovered an anisotropy of the form (4.18) in radio galaxy sources of $z \geq 0.3$. While they propose a modification of electrodynamics, introducing an interaction with a "cosmic spin", Obukhov (2000) reinterprets the effect as arising from cosmic rotation with

$$l^\circ = 50^\circ \pm 20^\circ, \quad b^\circ = -30^\circ \pm 25^\circ, \quad \omega_0 = (6.5 \pm 0.5) H_0 \simeq (5.0 \pm 0.4) \cdot 10^{-10}\,\mathrm{yr}^{-1} \tag{4.20}$$

4.2.6 The Formation of Galaxies

Li (1998) investigated the effects of a global rotation of the matter on the formation of a galaxy from the collapse of a cloud of gas. For a perfect fluid with the stress-energy-tensor $T_{\mu\nu} = (\rho + p) u_\mu u_\nu + p g_{\mu\nu}$ the Raychaudhuri equation (2.8) becomes

$$\dot{\theta} + \frac{1}{3}\theta^2 + 2(\sigma^2 - \omega^2) = -4\pi(\rho + 3p) \tag{4.21}$$

The conservation of energy and angular momentum gives

$$\dot{\rho} = -\left(\rho + p\right)\theta, \quad \omega \rho a^5 = \mathrm{const.} \tag{4.22}$$

with the scale factor $a = a(t)$. For dust it follows that $\rho_d \propto a^{-3} \propto (1 + z)^3$ and $\omega_d \propto a^{-2} \propto (1 + z)^2$, while for radiation $\rho_r \propto a^{-4} \propto (1 + z)^4$ and $\omega_r \propto a^{-1} \propto 1 + z$. The shear falls of as $\sigma \propto a^{-3}$ and can be neglected for sufficiently late times (Li, 1998, p. 499).

Integration of eq. (4.21) for dust, neglecting the shear term, gives

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}\rho - \frac{2}{3}\omega_d^2 - \frac{\kappa}{a^2} \tag{4.23}$$

with the integral constant κ. Note that $\rho = \rho_r + \rho_d$ is the total mass energy density of dust and radiation, but because the universe is dominated by dust for redshifts $z = \frac{a_0}{a} - 1 \lesssim 10^4$ we consider the angular velocity of dust.

In a homogeneous universe galaxies form from small fluctuations in density. For simplicity the fluctuations can be assumed to be spherical. Relative to gyroscopic frames, the matter with total mass M within the (also spherical) region destined to collapse initially has the angular momentum $J_i = \frac{2}{5} M r_i^2 \omega_i$ relative to *gyroscopic frames*, with r_i the radius of the region and ω_i the angular velocity of the universe (and thus the matter in the region) at that epoch. On the other hand the angular momentum relative to *galactic frames*, centered at the fluctuation and co-rotating with the universe, is naturally zero.

At any later time (denoted by f) the angular momentum relative to gyroscopic frames is $J_f = J + \beta M r_f^2 \omega_f$, where β is a parameter dependent on the distribution of matter within the region and J is the angular momentum relative to the galactic frames. Using $M = \frac{4}{3}\pi \rho_{di} r_i^3$, $\omega \propto (1+z)^2$ and $\rho_d \propto (1+z)^3$ the conservation of angular momentum, $J_i = J_f$, leads to

$$J = \frac{2}{5}\left(\frac{3}{4\pi\rho_{d0}}\right)^{\frac{2}{3}} \omega_0 M^{\frac{5}{3}} - \beta r_f^2 (1+z_f)^2 \omega_0 M \tag{4.24}$$

The index 0 denotes the time of observation, i.e. $z = 0$. If z is not too large the second term of eq. (4.24) is sufficiently small (Li, 1998, p. 501), so

$$J \simeq k M^{\frac{5}{3}}, \quad k = \frac{2}{5}\left(\frac{3}{4\pi\rho_{d0}}\right)^{\frac{2}{3}} \omega_0 \tag{4.25}$$

This corresponds well to the empirical relation $J \propto M^{\frac{5}{3}}$ (Li, 1998), which is conventionally explained through the virial theorem with the additional assumption that galaxies have a constant density. Taking $\rho_{d0} = 1.0575 \cdot 10^{-28}\,\mathrm{kg\,m^{-3}}$ and $k \approx 0.004\,\mathrm{m^2\,s^{-1}\,kg^{-\frac{2}{3}}}$ (Li, 1998) the current angular velocity of the universe is approximately

$$\omega_0 \simeq 6 \cdot 10^{-21}\,\frac{\mathrm{rad}}{\mathrm{s}} \simeq 2 \cdot 10^{-13}\,\frac{\mathrm{rad}}{\mathrm{yr}} \tag{4.26}$$

The shape resulting from the collapse depends on the value of the parameter (Li, 1998, pp. 502–504)

$$\vartheta = \frac{H_0^2}{\omega_0^2}\frac{3\delta_i}{1+z_i} \tag{4.27}$$

where $\delta_i = \frac{\delta\rho_i}{\rho_i} > 0$ is the relative fluctuation of density at $z = z_i$. For $\vartheta \gtrsim 1$ the collaps time in the equatorial and polar plane (relative to the cosmic rotation) are similar, so that the dust cloud collapses into dynamic equilibrium in an ellipsoidal shape. For $0 < \vartheta \ll 1$ the collaps in the equatorial plane is delayed by the centrifugal force of the rotation. The cloud collapses into a disklike shape with matter rotating around the core in the equatorial plane. If $-1 < \vartheta \leq 0$ the matter is not bound in the equatorial direction, but does collapse in the polar direction. The result is a thinning, ever-expanding disk of material.

The first two cases appear to correspond well to elliptical and spiral galaxies respectively, while the expanding disk of the last case may be a seed for wall structures in the universe. While this seems to imply a strong alignment of the axis of disk galaxies, this effect is weakened by deviations from the spherical shape of the fluctuation assumed earlier (Li, 1998, pp. 504–505). Then the angular momentum of the proto-galaxy will in general not have the same direction as the angular velocity (which at that epoch coincides with that of the universe). When the evolution of the galaxy decouples from interaction with its surroundings the components of the angular velocity orthogonal to the angular momentum will dissipate, so the final axis of revolution of the galaxy will no longer coincide with that of the universe. The distributions of spins of galaxies can therefore be expected to be somewhat random, although a strong dipole anisotropy should still show.

Land et al. (2008) find no dipole anisotropy in the direction of the spiral arms (S- or Z-like) in a sample of $\sim 37\,000$ spiral galaxies from the Sloan Digital Sky Survey.

Chapter 5

Conclusion

Although the interest in spacetimes and cosmologies with globally rotating matter has been somewhat sparse compared with more conventional approaches, some very interesting phenomena have been found in this area since the conception of General Relativity.

Most notably the occurence of closed timelike curves in apparently simple spacetime models such as the Gödel (section 3.2.1) and van Stockum (section 3.1) spacetimes shows that such—somewhat pathological—behaviour is a possibility inherent in General Relativity.

Whether our universe is actually rotating has been an open question since the discovery of an apparent anisotropy in radioastronomical observations by Birch (1982). Various authors (e.g. Birch (1982), Li (1998) and Obukhov (2000)) have estimated the global angular velocity (eq. (4.19), eq. (4.20), eq. (4.26)) but do not agree well. Additionally, the statistical significance of these findings is not undisputed.

Recently automated sky surveys (most notably the Sloan Digital Sky Survey) and the efforts of many thousand volunteers in the Galaxy Zoo projects allowed the accumulation of statistical data on a great number of galaxies. No significant anisotropy has been found in the orientation of spiral galaxies (Land et al., 2008), so it appears increasingly unlikely that the universe is actually rotating globally.

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