Cheatsheet: Lineare PDG mit konstanten Koeffizienten

Differentialgleichung

$$Au_{xx}(x,y) + Bu_{xy}(x,y) + Cu_{yy}(x,y) + F(x,y,u_x(x,y),u_y(x,y),u(x,y)) = 0$$

Ansatz

$$\begin{aligned}
u(x,y) &= v(s(x,y),t(x,y)) \\
s(x,y) &= c_1x + c_2y \\
t(x,y) &= c_3x + c_4y \\
x(s,t) &= \frac{c_4s - c_2t}{c_1c_4 - c_2c_3} \\
y(s,t) &= \frac{c_3s - c_1t}{c_1c_4 - c_2c_3}
\end{aligned}$$

Ableitungen

$$\begin{aligned}
u_x &= c_1v_s + c3v_t \\
u_y &= c_2v_s + c4v_t \\
u_{xx} &= c_1^2v_{ss} + 2c_1c_3v_{st} + c_3^2v_{tt} \\
u_{xx} &= c_2^2v_{ss} + 2c_2c_4v_{st} + c_4^2v_{tt} \\
u_{xy} &= c_1c_2v_{ss} + (c_1c_4 + c2c3)v_{st} + c_3c_4v_{tt}
\end{aligned}$$

Transformation

$$\begin{aligned}
Au_{xx} + Bu_{xy} + Cu_{yy} &+ F(x,y,u_x,u_y) \\
&= \\
&\left(Ac_1^2 + Bc_1c_2 + Cc_2^2\right)v_{ss} \\
&+ (2Ac_1c_3 + Bc_1c_4 + Bc_2c_3 + 2Cc_2c_4)v_{st} \\
&+ \left(Ac_3^2 + Bc_3c_4 + Cc_4^2\right)v_{tt} \\
&+ F\left(\frac{c_4s - c_2t}{c_1c_4 - c_2c_3}, \frac{c_3s - c_1t}{c_1c_4 - c_2c_3}, c_1v_s + c3v_t, c_2v_s + c4v_t, v(s,t)\right)
\end{aligned}$$