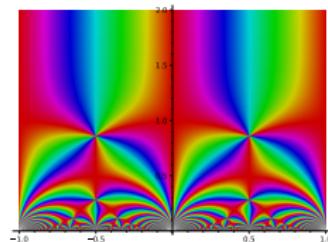


# Knots and modular forms

Matthias Storzer  
joint work with Robert Osburn  
(arXiv:2504.18060)

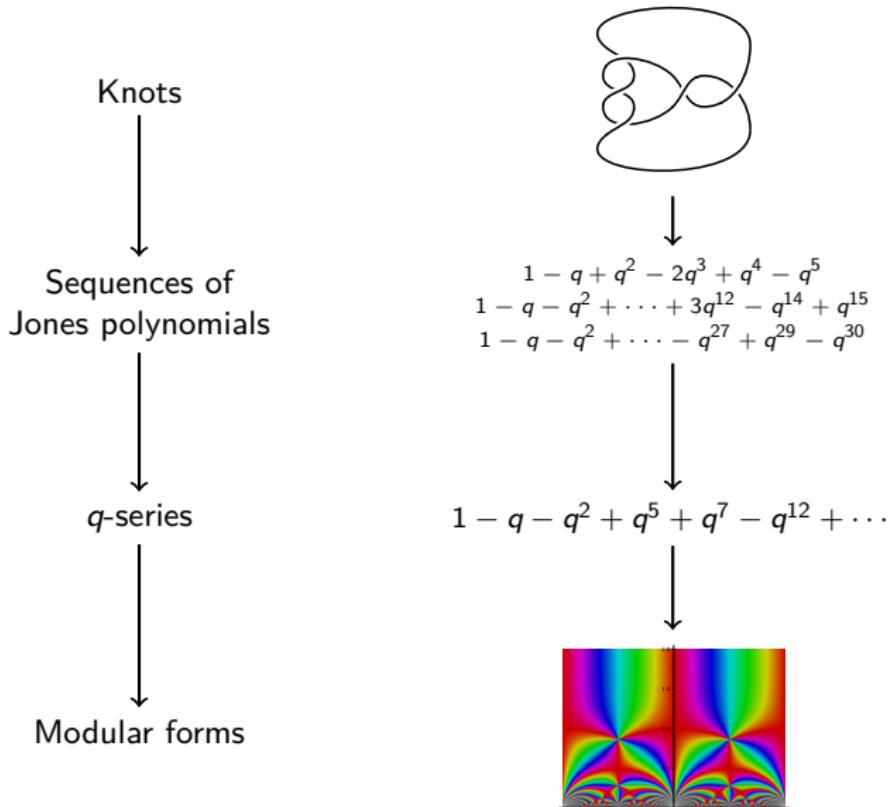
University College Cork

September 23, 2025



The slides are on my website: [matthiasstorzer.de](http://matthiasstorzer.de)

## Overview



## Knots

### Definition

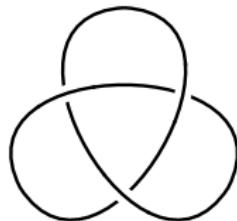
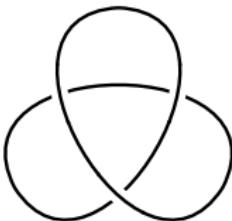
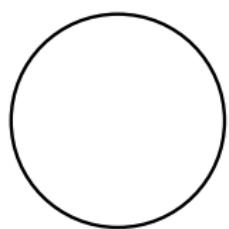
A *knot* is an embedding of  $S^1$  in  $\mathbb{R}^3$ .

A *link* of  $m$  components is an embedding of  $m$  copies of  $S^1$  in  $\mathbb{R}^3$ .

We present knots and links by two-dimensional diagrams.

### Example

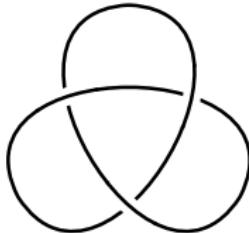
Consider the knots



We consider knots up to isotopy in  $\mathbb{R}^3$ .

## Knots invariants

**Question:** Is the third example from above the unknot?



**Answer:** No (as we will see soon). It is called the *trefoil knot* or  $3_1$ .

To study knots, e.g., detect if two knots are isotopic in  $\mathbb{R}^3$ , we use knot invariants.

# The Jones polynomial

## Definition

The *Jones polynomial*  $J_K(q) \in \mathbb{Z}[q^{\pm 1/2}]$  is an invariant for oriented links  $K$  that satisfies the following:

1.  $J_{\textcircled{O}}(q) = 1$ , where  $\textcircled{O}$  denotes the unknot.
2.  $J_K(q)$  satisfies

$$(q^{-1/2} - q^{1/2})J_{K_0}(q) = qJ_{K_-}(q) - q^{-1}J_{K_+}(q),$$

where  $K_0, K_{\pm}$  are link diagrams that differ at one crossing by

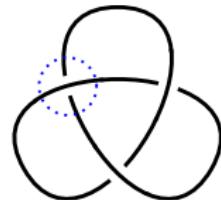


## An example

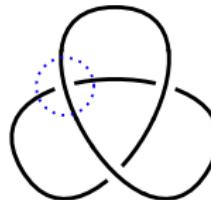
Recall the relation

$$(q^{-1/2} - q^{1/2})J_{K_0}(q) = qJ_{K_-}(q) - q^{-1}J_{K_+}(q).$$

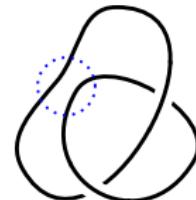
We compute the Jones polynomial for  $K = 3_1$ .



$K_+$



$K_-$



$K_0$

We have with  $K_+ = 3_1$

$$J_{K_+}(q) = q^2 J_{K_-}(q) - q(q^{-1/2} - q^{1/2})J_{K_0}(q)$$

and with

$$J_{K_-}(q) = J_{\bigcirc}(q) = 1, \quad J_{K_0}(q) = \dots = q^{1/2} + q^{5/2}$$

we obtain

$$J_{K_+}(q) = q + q^3 - q^4.$$

In particular,  $K = 3_1 \neq \bigcirc$ .

## The colored Jones polynomial (CJP)

For a knot  $K$ , denote by  $K^{(i)}$  the  $i$ -cabling of  $K$ , e.g., a diagram of  $3_1^{(2)}$  is given by



For  $N \in \mathbb{Z}_{\geq 1}$ , the  $N$ -th colored Jones polynomial  $J_K(q; N)$  is defined by

$$J_K(q; N) := \sum_{i=0}^{N-1} a_{i,N} J_{K^{(i)}}(q)$$

for explicit  $a_{i,N} \in \mathbb{Z}$ . For example, we have

$$\begin{aligned} J_K(q; 1) &= 1, \\ J_K(q; 2) &= J_K(q), \\ J_K(q; 3) &= J_{K^{(2)}}(q) - 1 \\ &\vdots \end{aligned}$$

For  $K = 3_1$ , we have

$$J_K(q; 3) = \dots = q^2 + q^5 - q^7 + q^8 - q^9 - q^{10} + q^{11}.$$

## The CJP and hyperbolic geometry

The colored Jones polynomial appears to contain geometric information about the knot.

Conjecture (“Volume Conjecture”, Kashaev (1994), Murakami–Murakami (1999))

For any knot  $K$  we have

$$2\pi \lim_{N \rightarrow \infty} \frac{\log |J_K(e^{2\pi i/N}; N)|}{N} = \text{Vol}(S^3 \setminus K),$$

where  $\text{Vol}(S^3 \setminus K)$  denotes the simplicial (hyperbolic) volume of  $S^3 \setminus K$ .

## Stability properties of the CJP



### Example

For  $K = 5_2$ , we have

$$J_K(q, 1) = 1,$$

$$q^{-1} J_K(q, 2) = 1 - q + 2q^2 - q^3 + q^4 - q^5,$$

$$q^{-2} J_K(q, 3) = 1 - q + 0q^2 + 3q^3 - 2q^4 - q^5 + 4q^6 - 3q^7 - q^8 + \cdots + q^{15},$$

$$q^{-3} J_K(q, 4) = 1 - q + 0q^2 + q^3 + 2q^4 - 2q^5 - 2q^6 + 2q^7 + 4q^8 + \cdots + q^{30},$$

$$q^{-4} J_K(q, 5) = 1 - q + 0q^2 + q^3 + 0q^4 + 2q^5 - 3q^6 - q^7 + 2q^8 + \cdots + q^{50},$$

$$q^{-5} J_K(q, 6) = 1 - q + 0q^2 + q^3 + 0q^4 + 0q^5 + q^6 - 2q^7 - q^8 + \cdots + q^{75},$$

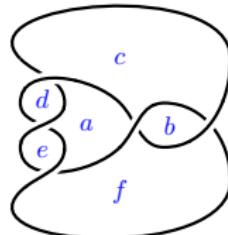
$$q^{-6} J_K(q, 7) = 1 - q + 0q^2 + q^3 + 0q^4 + 0q^5 - q^6 + 2q^7 - 2q^8 + \cdots + q^{105}$$

and the normalized colored Jones polynomials seem to stabilize to the  $q$ -series

$$1 - q + q^3 - q^6 + q^{10} - q^{15} + q^{21} - q^{28} + q^{36} - q^{45} + \cdots.$$

## The tail of the CJP

A knot is *alternating* if there exists a diagram such that the crossings alternate between over and under while traversing.



Theorem (Armond, Dasbach, Garoufalidis–Lê)

Let  $K$  be an alternating knot. Then the normalized colored Jones polynomials stabilize to a  $q$ -series  $\Phi_K(q)$ , the tail of the colored Jones polynomial. Moreover,  $\Phi_K(q)$  can be written as an explicit  $q$ -multisum.

We write  $(q)_n = \prod_{i=1}^n (1 - q^i)$  where  $n \in \mathbb{N}$ .

### Example

For  $K = 5_2$ , we have

$$\begin{aligned}\Phi_K(q) &= (q)_\infty^5 \sum_{\substack{a,b,c,d,e \geq 0 \\ f=0}} \frac{q^{4a^2/2 + a(c+d+e) + b^2 + bc + cd + de + a + c + d + e}}{(q)_a (q)_{a+c} (q)_{a+d} (q)_{a+e} (q)_b (q)_{b+c} (q)_c (q)_d (q)_e} \\ &= 1 - q + q^3 - q^6 + q^{10} - q^{15} + q^{21} - q^{28} + q^{36} - q^{45} + \dots \\ &\stackrel{?}{=} \sum_{k \geq 0} (-1)^k q^{k(k+1)/2}.\end{aligned}$$

## The GLZ conjectures

In 2011, Garoufalidis and Lê with Zagier conjectured identities for  $\Phi_K(q)$  for almost all knots with up to 8 crossings and some infinite families in terms of products of the functions

$$h_b := h_b(q) := \sum_{n \in \mathbb{Z}} \epsilon_b(n) q^{bn(n+1)/2-n}, \quad b \in \mathbb{Z}_{\geq 1},$$

where

$$\epsilon_b(n) = \begin{cases} (-1)^n & \text{if } b \text{ is odd,} \\ \text{sign}(n + \frac{1}{2}) & \text{if } b \text{ is even.} \end{cases}$$

We have

$$h_1(q) = 0, \quad h_2(q) = 1,$$

$$h_3(q) = \sum_{n \in \mathbb{Z}} (-1)^n q^{3n(n+1)/2},$$

$$h_4(q) = \sum_{k \geq 0} (-1)^k q^{k(k+1)/2} \stackrel{?}{=} \Phi_{5_2}(q).$$

The function  $q^{(2-b)^2/8b} h_b(q)$  is a

- ▶ theta function if  $b$  is odd and
- ▶ partial theta function if  $b$  is even.

The function  $q^{(2-b)^2/8b} h_b(q)$  is a *modular forms* if  $b$  is odd.

## Modular forms

Let  $\mathbb{H} := \{\tau \in \mathbb{C} : \operatorname{Im}(\tau) > 0\}$ .

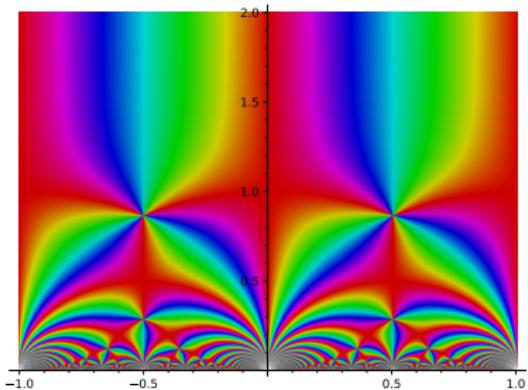
A holomorphic function  $f: \mathbb{H} \rightarrow \mathbb{C}$  is a *modular form* for

- ▶ a subgroup  $\Gamma$  of  $\operatorname{SL}_2(\mathbb{Z})$  and
- ▶ a weight  $k \in \frac{1}{2}\mathbb{Z}$

such that

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau)$$

for all  $\tau \in \mathbb{H}$  and all  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ .



A modular form

Among all holomorphic functions, modular forms are very rare and well studied.  
E.g., modular forms have a well-known asymptotic expansion.

The functions  $\tau \mapsto h_b(e^{2\pi i \tau})$ , for odd  $b$  are typical examples of modular forms.  
The (asymptotic) behavior of  $h_b$  for even  $b$  is similar.

## The GLZ conjectures

Garoufalidis–Lê–Zagier conjectured the following identities:

$K$	$\Phi_K(q)$	$\Phi_{-K}(q)$	$K$	$\Phi_K(q)$	$\Phi_{-K}(q)$
$3_1$	$h_3$	1	$7_5$	$h_3 h_4$	$h_4$
$4_1$	$h_3$	$h_3$	$7_6$	$h_3 h_4$	$h_3^2$
$5_1$	$h_5$	1	$7_7$	$h_3^3$	$h_3^2$
$5_2$	$h_4$	$h_3$	$8_1$	$h_7$	$h_3$
$6_1$	$h_5$	$h_3$	$8_2$	$h_3 h_6$	$h_3$
$6_2$	$h_3 h_4$	$h_3$	$8_3$	$h_5$	$h_5$
$6_3$	$h_3^2$	$h_3^2$	$8_4$	$h_3$	$h_4 h_5$
$7_1$	$h_7$	1	$8_5$	?	$h_3$
$7_2$	$h_6$	$h_3$	$K_p, p > 0$	$h_{2p}$	$h_3$
$7_3$	$h_5$	$h_4$	$K_p, p < 0$	$h_3$	$h_{2 p +1}$
$7_4$	$h_3$	$h_4^2$	$T(2, p), p > 0$	$h_{2p+1}$	1

These identities for  $3_1$ ,  $4_1$  and  $6_3$  have been proven by Andrews (2012) and Keilthy–Osburn (2016) proved the remaining ones using a uniform approach.

Beirne–Osburn (2017) have extended the list of identities to alternating knots with up to 10 crossings.

## Knots with 8 and 9 crossings

$K$	$\Phi_K(q)$	$\Phi_{-K}(q)$	$K$	$\Phi_K(q)$	$\Phi_{-K}(q)$	$K$	$\Phi_K(q)$	$\Phi_{-K}(q)$
8 <sub>6</sub>	$h_3 h_4$	$h_5$	9 <sub>6</sub>	$h_3 h_6$	$h_4$	9 <sub>24</sub>	$h_3 h_4$	$h_3^2 h_4$
8 <sub>7</sub>	$h_3^2$	$h_3 h_5$	9 <sub>7</sub>	$h_3 h_4$	$h_6$	9 <sub>25</sub>	$h_3^3$	?
8 <sub>8</sub>	$h_3 h_5$	$h_3^2$	9 <sub>8</sub>	$h_3 h_6$	$h_3^2$	9 <sub>26</sub>	$h_3^3$	$h_3^2 h_4$
8 <sub>9</sub>	$h_3 h_4$	$h_3 h_4$	9 <sub>9</sub>	$h_4 h_5$	$h_4$	9 <sub>27</sub>	$h_3^3$	$h_3^2 h_4$
8 <sub>10</sub>	$h_3^2$	?	9 <sub>10</sub>	$h_4^2$	$h_5$	9 <sub>28</sub>	$h_3^2 h_4$	$h_3^3$
8 <sub>11</sub>	$h_3 h_4$	$h_3 h_4$	9 <sub>11</sub>	$h_3^2$	$h_4 h_5$	9 <sub>29</sub>	?	?
8 <sub>12</sub>	$h_3 h_4$	$h_3 h_4$	9 <sub>12</sub>	$h_3 h_4$	$h_3 h_5$	9 <sub>30</sub>	$h_3^3$	?
8 <sub>13</sub>	$h_3^2 h_4$	$h_3^2$	9 <sub>13</sub>	$h_4^2$	$h_3 h_4$	9 <sub>31</sub>	$h_3^3$	$h_3^3$
8 <sub>14</sub>	$h_3^3$	$h_3 h_4$	9 <sub>14</sub>	$h_3^2 h_5$	$h_3^2$	9 <sub>32</sub>	?	?
8 <sub>15</sub>	$h_3^3$	?	9 <sub>15</sub>	$h_3 h_5$	$h_3 h_4$	9 <sub>33</sub>	?	?
8 <sub>16</sub>	?	?	9 <sub>16</sub>	$h_4$	$h_3 h_4^2$	9 <sub>34</sub>	?	?
8 <sub>17</sub>	?	?	9 <sub>17</sub>	$h_3^2 h_5$	$h_3^2$	9 <sub>35</sub>	$h_3$	?
8 <sub>18</sub>	?	?	9 <sub>18</sub>	$h_3 h_4$	$h_4^2$	9 <sub>36</sub>	$h_3^2$	?
9 <sub>1</sub>	$h_9$	1	9 <sub>19</sub>	$h_3^3$	$h_3 h_5$	9 <sub>37</sub>	$h_3^3$	?
9 <sub>2</sub>	$h_3$	$h_8$	9 <sub>20</sub>	$h_3 h_4^2$	$h_3^2$	9 <sub>38</sub>	?	?
9 <sub>3</sub>	$h_4$	$h_7$	9 <sub>21</sub>	$h_3^2 h_4$	$h_3 h_4$	9 <sub>39</sub>	?	?
9 <sub>4</sub>	$h_5$	$h_6$	9 <sub>22</sub>	$h_3^2$	?	9 <sub>40</sub>	?	?
9 <sub>5</sub>	$h_4 h_6$	$h_3$	9 <sub>23</sub>	$h_3^3$	$h_4^2$	9 <sub>41</sub>	?	?

## Knots with 10 crossings

$K$	$\Phi_K(q)$	$\Phi_{-K}(q)$	$K$	$\Phi_K(q)$	$\Phi_{-K}(q)$	$K$	$\Phi_K(q)$	$\Phi_{-K}(q)$
10 <sub>1</sub>	$h_3$	$h_9$	10 <sub>19</sub>	$h_3 h_4 h_5$	$h_3^2$	10 <sub>37</sub>	$h_3 h_5$	$h_3 h_5$
10 <sub>2</sub>	$h_3 h_8$	$h_3$	10 <sub>20</sub>	$h_3 h_4$	$h_7$	10 <sub>38</sub>	$h_3^3$	$h_4 h_5$
10 <sub>3</sub>	$h_5$	$h_7$	10 <sub>21</sub>	$h_3 h_6$	$h_3 h_4$	10 <sub>39</sub>	$h_3^2 h_5$	$h_3 h_4$
10 <sub>4</sub>	$h_4 h_7$	$h_3$	10 <sub>22</sub>	$h_4 h_5$	$h_3 h_4$	10 <sub>40</sub>	$h_3^2 h_4$	$h_3^2 h_4$
10 <sub>5</sub>	$h_3^2$	$h_3 h_7$	10 <sub>23</sub>	$h_3^2 h_4$	$h_3 h_5$	10 <sub>41</sub>	$h_3 h_4^2$	$h_3^3$
10 <sub>6</sub>	$h_3 h_6$	$h_5$	10 <sub>24</sub>	$h_3 h_4$	$h_4 h_5$	10 <sub>42</sub>	$h_3^4$	$h_3^2 h_4$
10 <sub>7</sub>	$h_3 h_4$	$h_3 h_6$	10 <sub>25</sub>	$h_3 h_4^2$	$h_3 h_4$	10 <sub>43</sub>	$h_3^2 h_4$	$h_3^2 h_4$
10 <sub>8</sub>	$h_5 h_6$	$h_3$	10 <sub>26</sub>	$h_3 h_4^2$	$h_3 h_4$	10 <sub>44</sub>	$h_3^3 h_4$	$h_3^3$
10 <sub>9</sub>	$h_3 h_4$	$h_3 h_6$	10 <sub>27</sub>	$h_3^2 h_4$	$h_3^2 h_4$	10 <sub>45</sub>	$h_3^4$	$h_3^4$
10 <sub>10</sub>	$h_3^2 h_6$	$h_3^2$	10 <sub>28</sub>	$h_3 h_4 h_5$	$h_3^2$	10 <sub>46</sub>	$h_3$	?
10 <sub>11</sub>	$h_4 h_5$	$h_5$	10 <sub>29</sub>	$h_3 h_4^2$	$h_3 h_4$	10 <sub>47</sub>	$h_3^2$	?
10 <sub>12</sub>	$h_3 h_5$	$h_3 h_5$	10 <sub>30</sub>	$h_3^3$	$h_3 h_4^2$	10 <sub>48</sub>	$h_3 h_5$	?
10 <sub>13</sub>	$h_3 h_4$	$h_4 h_5$	10 <sub>31</sub>	$h_3^2 h_4$	$h_3 h_5$	10 <sub>49</sub>	$h_3^2 h_5$	?
10 <sub>14</sub>	$h_3^2 h_5$	$h_3 h_4$	10 <sub>32</sub>	$h_3^3$	$h_3 h_4^2$	10 <sub>50</sub>	$h_3 h_4$	?
10 <sub>15</sub>	$h_3^2$	$h_5^2$	10 <sub>33</sub>	$h_3^2 h_4$	$h_3^2 h_4$	10 <sub>51</sub>	$h_3^2 h_4$	?
10 <sub>16</sub>	$h_3 h_4$	$h_4 h_5$	10 <sub>34</sub>	$h_3 h_7$	$h_3^2$	10 <sub>52</sub>	$h_3^2$	?
10 <sub>17</sub>	$h_3 h_5$	$h_3 h_5$	10 <sub>35</sub>	$h_3 h_6$	$h_3 h_4$	10 <sub>53</sub>	$h_3^3$	?
10 <sub>18</sub>	$h_3^2 h_5$	$h_3 h_4$	10 <sub>36</sub>	$h_3^3$	$h_3 h_6$	10 <sub>54</sub>	$h_3^2$	?

## Knots with 10 crossings

$K$	$\Phi_K(q)$	$\Phi_{-K}(q)$	$K$	$\Phi_K(q)$	$\Phi_{-K}(q)$
$10_{55}$	$h_3^3$	?	$10_{67}$	$h_3^3$	?
$10_{56}$	$h_3 h_4$	?	$10_{68}$	$h_3^2$	?
$10_{57}$	$h_3^2 h_4$	?	$10_{69}$	$h_3^4$	?
$10_{58}$	$h_3^3$	?	$10_{70}$	$h_3 h_4$	$h_3^2 h_4$
$10_{59}$	$h_3^3$	?	$10_{71}$	$h_3^2 h_4$	$h_3^2 h_4$
$10_{60}$	$h_3^3$	?	$10_{72}$	$h_3 h_4$	$h_3^3 h_4$
$10_{61}$	$h_3$	?	$10_{73}$	$h_3^2 h_4$	$h_3^4$
$10_{62}$	$h_3^2$	?	$10_{74}$	$h_3 h_4$	$h_3 h_4^2$
$10_{63}$	$h_3^3$	?	$10_{75}$	$h_3^3 h_4$	$h_3^3$
$10_{64}$	$h_3 h_4$	?	$10_{76}$	$h_5$	$h_3 h_4^2$
$10_{65}$	$h_3^2 h_4$	?	$10_{77}$	$h_3 h_5$	$h_3^2 h_4$
$10_{66}$	$h_3^3 h_4$	?	$10_{78}$	$h_3^2 h_5$	$h_3^3$

$$10_{79}, -10_{79}, \dots, 10_{123}, -10_{123} \rightsquigarrow ?$$

## Questions

So far, formulas for  $\Phi_K(q)$  in terms of (partial) theta series are only known for examples.

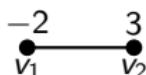
### Questions:

1. Is there a general formula?
2. What can we say about the knots, marked with "?", without known identities?

## Arborescent knots

We will construct *arborescent knots* from weighted planar trees with vertices  $\mathcal{V}$  and weights  $w: \mathcal{V} \rightarrow \mathbb{Z}$ .

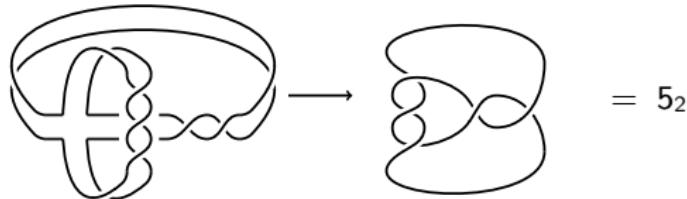
For example, consider the weighted tree



To each vertex  $v$ , we associate a ribbon with  $w(v)$  half-twists



and plumb them together according to the structure of the tree

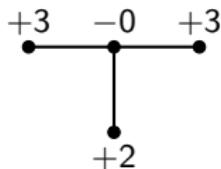


The boundary of the resulting surface forms an *arborescent knot*.

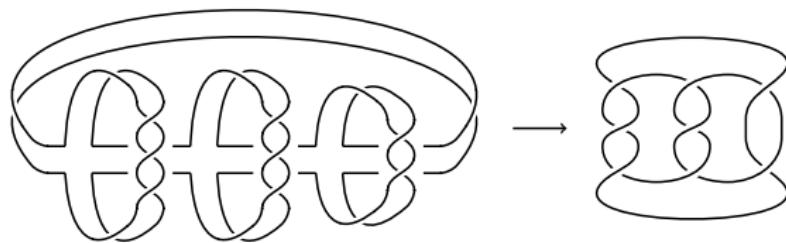
## Another arborescent knot

### Example

Consider the weighted tree



The corresponding knot is called  $8_5$  and given by



If there is a bipartition  $\mathcal{V} = \mathcal{V}_+ \cup \mathcal{V}_-$  with  $w(\mathcal{V}_+) \geq 0$  and  $w(\mathcal{V}_-) \leq 0$ , then the knot is alternating.

For example, recall that  $K = 5_2$  is constructed from the tree



## A general formula

### Theorem (Osburn-S., 2025)

Let  $K$  be an alternating arborescent knot coming from a tree with weights  $w : \mathcal{V} \rightarrow \mathbb{Z}$  and bipartition  $\mathcal{V} = \mathcal{V}_+ \dot{\cup} \mathcal{V}_-$  as above. If  $0 \notin w(\mathcal{V}_-)$ , then

$$\Phi_K(q) = \prod_{v \in \mathcal{V}_+} h_{w(v)+\text{degree}(v)}(q).$$

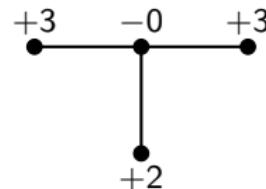
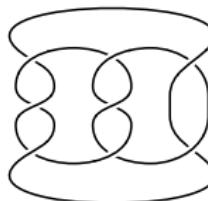
### Example

We have seen that  $K = 5_2$  is constructed from  . Hence, we have

$$\Phi_{5_2}(q) = h_{3+1}(q).$$

## Non-modularity?

Recall that the knot  $8_5$  and an associated weighted tree are given by



The theorem is not applicable, because  $0 \in w(\mathcal{V}_-)$ .

There is no known evaluation for the  $8_5$  knot in terms of  $h_b$ 's. We have

$$\begin{aligned} \Phi_{8_5}(q) &= (q)_\infty^8 \sum_{\substack{a,b,c,d,f,g \geq 0 \\ a+e, a+h, b+h, b+e \geq 0}} (-1)^b \frac{q^{\frac{b(5b+3)}{2} + a(3a+2) + ad + ae + af + ag + ah + bc + bd}}{(q)_a(q)_b(q)_c(q)_d(q)_f(q)_g(q)_{a+g}(q)_{a+f}(q)_{a+d}} \\ &\quad \times \frac{q^{be + bh + cd + de + df + eh + fg + c + d + e + f + g + h}}{(q)_{a+e}(q)_{a+h}(q)_{b+d}(q)_{b+c}(q)_{b+e}(q)_{b+h}} \\ &= (q)_\infty^2 \sum_{a,b \geq 0} \frac{q^{a^2 + a + b^2 + b}}{(q)_a^2(q)_b^2} (q)_{a+b} = 1 - 2q + q^2 - 2q^4 + 3q^5 + \dots . \end{aligned}$$

## Non-modularity

How can one prove that  $\Phi_K$  is *not* the product of  $h_b$ 's?

We compare the asymptotics of  $\Phi_{85}(q)$  with  $h_b(q)$  as  $q = e^{-h} \rightarrow 1$ .

As  $h \rightarrow 0$ , we have

$$\log(h_b(e^{-h})) \sim \frac{V}{h}$$

for some  $V \in \pi^2 \mathbb{Q}$ .

On the other hand, we have numerically

$$\log(\Phi_{85}(e^{-h})) \sim \frac{W}{h}$$

where  $W = -15.7728722388\cdots + 3.1772932786\cdots i$ .

If  $\Phi_{85}(q)$  was modular, e.g., a product of (partial) theta functions, then  $W \in \pi^2 \mathbb{Q}$ .

The number  $3.1772932786\cdots$  has a topological and geometric meaning for  $K = 85$ .

## Classification of modular tails?

Similar computations with the same outcome have been done for other “?”-cases.

This motivates the following question:

### Question

Does the theorem classify alternating knots with “modular” tails?

**Thank you!**