1 TODO

- Plot lineraization
- linearization
- linearization of abs in AD

2 ABS-Normalform

Definition 1. Smooth function
A smooth function $f(x)$ is coninuous and has a continuous derivative

Beispiel 1.

$$f(x) = x^2, f'(x) = 2x$$

is smooth.

$$\begin{aligned} f(x) &= |x| \\ g'(x) &= 1, x > 0 \\ f'(x) &= 1, x < 0 \end{aligned}$$

Therefore it is not smooth, since its derivative is not continuous.

Definition 2. A functin has a jump discontinuity at x_0 if

$$\lim_{x \to x_0^-} f(x) \neq \lim_{x->x_0^+} f(x)$$

Beispiel 2. abs(x)

Definition 3. a picewise smooth function is made up of finetly many smooth functions separated by jump discontinuities.

Haben Picewise smooth function. Picewise linearization can be achieved in the style of algorithmic differentiation. by replacing all smooth elemental functions by their tangent line or plane.

2.0.1 Assumptions

- All nonsmoothness can be cast in terms of the absolute value funtion

2.0.2 Goal

Gegben $F(X)$ wollen $F(X) = 0$. Wobei $F(X)$ picewise linear in abs NF ist.

2.0.3 Questions

- How do i get my representation?
- Finde minimum dieser funktionen?
- On Stable Picewise Linearization and Generalized Differentiation - Griewank

2.1 Linearization

Linearization is a linear approximation of a nonlinear system that is valid in a small region around an operating point.

$$https://de.mathworks.com/help/slcontrol/ug/linearizing-nonlinear-models.html$$

Linearization through tangent line at point $(a, f(a))$

$$y = f(x)$$
$$m = f'(a)$$
$$y = f(a) + f'(a)(x-a)$$
$$y = f'(a)x - f'(a)a + f(a) = f'(a)x + b$$

2.2 Anwendung

Wenn Newton nicht haut mache. Zur Minimums findung von Funktion:

- Stückweise Linearisierung in Punkt
- Berechne ABS Lösung
- Nutze Punkt für nächsten step

2.3 Representation of Picewisefunctions with MAX-MIN

Can represent every PL functin with min max functins

2.4 Representation of MAX-MIN functions with abs functions

$$min(a,b) = \frac{1}{2}(a + b - abs(a-b))$$
$$max(a,b) = \frac{1}{2}(a + b + abs(a-b))$$

2.5 Absnormalform

2.6 Example

Given is the following PL function:

$$F(x_1, x_2) = (x_2^2 - x_1^+)^+$$
$$(i)^+ = \max(0, i)$$

2.6.1 ABS-Normalization

First we rewrite the function such that picewise linearity is expressed with the aid of the abs function.

$$
\begin{aligned}
F(x_1, x_2) &= (x_2^2 - x_1^+)^+ \\
&= \max\big(0, x_2^2 - \max(0, x_1)\big) \\
&= \frac{1}{2}\Big(0 + x_2^2 - \max(0, x_1) + abs(0 - x_2^2 + \max(0, x_1))\Big) \\
&= \frac{1}{2}\Big(0 + x_2^2 - \frac{1}{2}\Big[0 + x_1 + abs(0 - x_1)\Big] + abs(0 - x_2^2 + \max(0, x_1))\Big) \\
&= \frac{1}{2}\Big(0 + x_2^2 - \frac{1}{2}\Big[0 + x_1 + abs(0 - x_1)\Big] + abs\big(0 - x_2^2 + \frac{1}{2}(0 + x_1 + abs(0 - x_1))\big)\Big) \\
&= \frac{1}{2}\Big(x_2^2 - \frac{1}{2}\Big[x_1 + abs(-x_1)\Big] + abs\big(- x_2^2 + \frac{1}{2}(x_1 + abs(-x_1))\big)\Big) \\
&= \frac{1}{2}\Big(x_2^2 - \frac{1}{2}\Big[x_1 + abs(x_1)\Big] + abs\big(- x_2^2 + \frac{1}{2}(x_1 + abs(x_1))\big)\Big)
\end{aligned}
$$

2.6.2 Straight-Line Code

Next we bring the function in straight-line code representaion:

SLC		Directional Derivatives	
w_1	$= x_1$	Δw_1	$= \Delta x_1$
w_2	$= x_2$	Δw_2	$= \Delta x_2$
w_3	$= \lvert w_1 \rvert$	Δw_3	$= \lvert \underbrace{w_1 + \Delta w_1}_{\Delta Z_1} \rvert - \lvert w_1 \rvert$
w_4	$= w_1 + w_3$	Δw_4	$= \Delta w_1 + \Delta w_3$
w_5	$= \frac{1}{2} w_4$	Δw_5	$= \frac{1}{2} \Delta w_4$
w_6	$= w_2 w_2$	Δw_6	$= 2 \Delta w_2 * w_2$
w_7	$= w_5 - w_6$	Δw_7	$= \Delta w_5 - \Delta w_6$
w_8	$= \lvert w_7 \rvert$	Δw_8	$= \lvert \underbrace{w_7 + \Delta w_7}_{\Delta Z_2} \rvert - \lvert w_7 \rvert$
w_9	$= w_6 - w_5$	Δw_9	$= \Delta w_6 - \Delta w_5$
w_{10}	$= w_9 + w_8$	Δw_{10}	$= \Delta w_9 + \Delta w_9$
w_{11}	$= \frac{1}{2} w_{10}$	Δw_{11}	$= \frac{1}{2} \Delta w_{10}$
y	$= w_{11}$	Δy	$= \Delta w_{11}$

Now we rewrite Z_i and Y_i such that they only depend on x. In the computitional graph, the equivalent operation is node contraction.

$$
\begin{aligned}
\Delta Z_1 &= w_1 + \Delta w_1 = w_1 + \Delta x_1 \\
\Delta Z_2 &= w_7 + \Delta w_7 \\
&= w_7 + \Delta w_5 - \Delta w_6 \\
&= w_7 + \frac{1}{2}\Delta w_4 - 2\Delta w_2 w_2 \\
&= w_7 + \frac{1}{2}(\Delta w_1 + \Delta w_3) - 2\Delta x_2 w_2 \\
&= w_7 + \frac{1}{2}(\Delta w_1 + |\Delta Z_1| - |w_1|) - 2\Delta x_2 w_2 \\
&= w_7 + \frac{1}{2}(\Delta x_1 + |\Delta Z_1| - |w_1|) - 2\Delta x_2 w_2 \\
&= w_7 + \frac{1}{2}\Delta x_1 + \frac{1}{2}|\Delta Z_1| - \frac{1}{2}|w_1| - 2\Delta x_2 w_2 \\
\Delta Y &= \Delta w_{11} \\
&= \frac{1}{2}\Delta w_{10} \\
&= \frac{1}{2}(\Delta w_9 + \Delta w_8) \\
&= \frac{1}{2}(\Delta w_6 - \Delta w_5 + \Delta w_8) \\
&= \frac{1}{2}(2\Delta w_2 w_2 - \frac{1}{2}\Delta w_4 + |\Delta Z_2| - |w_7|) \\
&= \frac{1}{2}(2\Delta x_2 w_2 - \frac{1}{2}(\Delta x_1 + |\Delta Z_1| - |w_1|) + |\Delta Z_2| - |w_7|) \\
&= \frac{1}{2}(2\Delta x_2 w_2 - \frac{1}{2}(\Delta x_1 + |\Delta Z_1| - |w_1|) + |\Delta Z_2| - |w_7|) \\
&= \Delta x_2 w_2 - \frac{1}{4}\Delta x_1 - \frac{1}{4}|\Delta Z_1| + \frac{1}{4}|w_1| + \frac{1}{2}|\Delta Z_2| - \frac{1}{2}|w_7|
\end{aligned}
$$

$$
\begin{aligned}
\begin{pmatrix} \Delta Z_1 \\ \Delta Z_2 \\ \hline \Delta Y \end{pmatrix} &= \begin{pmatrix} w_1 \\ w_7 - \frac{1}{2}|w_1| \\ \hline \frac{1}{4}|w_1| - \frac{1}{2}|w_7| \end{pmatrix} + \begin{pmatrix} 1\Delta x_1 + 0\Delta x_2 \\ \frac{1}{2}\Delta x_1 + 0\Delta x_2 \\ \hline -\frac{1}{4}\Delta x_1 + w_2\Delta x_2 \end{pmatrix} + \begin{pmatrix} 0|\Delta Z_1| + 0|\Delta Z_2| \\ \frac{1}{2}|\Delta Z_1| + 0\Delta Z_2 \\ \hline -\frac{1}{4}|\Delta Z_1| + \frac{1}{2}\Delta Z_2 \end{pmatrix} \\
&= \begin{pmatrix} w_1 \\ w_7 - \frac{1}{2}|w_1| \\ \hline \frac{1}{4}|w_1| - \frac{1}{2}|w_7| \end{pmatrix} + \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \hline -\frac{1}{4} & w_2 & -\frac{1}{4} & \frac{1}{2} \end{array}\right) \times \begin{pmatrix} \Delta x_1 \\ \Delta x_2 \\ \hline |\Delta Z_1| \\ |\Delta Z_2| \end{pmatrix}
\end{aligned}
$$

2.6.3 ABS-Equation-System

As a last step we bring the function in the form of a linear equation system.

$$
\begin{pmatrix} \Delta Z \\ \Delta Y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} Z & L \\ J & Y \end{pmatrix} \times \begin{pmatrix} \Delta x \\ \Delta z \end{pmatrix}
$$

3 Evaluation

Input: $a, b, Z, L, J, Y, n, m, s, \Delta x$
Output: $\Delta y, \Delta Z$

4 Gradient

5 Solve

6 Calculations

6.1 Gradient of abs()

$$\begin{aligned} f(x) &= |x| \\ \nabla f(x) &= |x + \nabla x| - |x| \\ &= \max\{x + \nabla x, -x - \nabla x\} - \max\{x, -x\} \end{aligned}$$

- $x > 0$
  - $\nabla x > 0$

    $$\nabla f(x) = x + \nabla x - x = \nabla x$$

  - $\nabla x < 0$
    - $\nabla x > -x$

      $$\nabla f(x) = x + \nabla x - x = \nabla x$$

    - $\nabla x < -x$

      $$\begin{aligned} \nabla f(x) &= \max\{x + \nabla x, -x - \nabla x\} - \max\{x, -x\} \\ &= -x - \nabla x - x = -2x - \nabla x \end{aligned}$$

    - $\nabla x = -x$

      $$\begin{aligned} \nabla f(x) &= x + \nabla x - x = \nabla x \\ &= -x - \nabla x - x = \nabla x - \nabla x + \nabla x = \nabla x \end{aligned}$$

  - $\nabla x = 0$

    $$\nabla f(x) = x + \nabla x - x = \nabla x = 0$$

- $x < 0$
  - $\nabla x > 0$
  - $\nabla x < 0$
  - $\nabla x = 0$
- $x = 0$

  ???

7 Example ABS

Let

$$F(x) = 3|x-1| = 3abs(x-1)$$

The task is to find $\Delta F(x, \Delta x)$, so that:

$$F(x + \Delta x) \approx F(x) + \Delta F(x, \Delta x)$$

therefore

$$F(x) = |x| \Rightarrow \Delta F'(x, \Delta x) = |x + \Delta x| - |x|$$

now

$$\begin{aligned} F(x) + \Delta F(x, \Delta x) &= |x| + |x + \Delta x| - |x| \\ &= |x + \Delta x| \\ &= F(x + \Delta x) \end{aligned}$$

Beispiel 3.

$$\begin{aligned} F(x) &= 3|x-1| + 4 \\ \Delta F(x, \Delta x) &= 3|(x-1) + \Delta x| - 3|x-1| \\ F(x) + \Delta F(x, \Delta x) &= 3|x-1| + 4 + 3|(x-1) + \Delta x| - 3|x-1| \\ &= 3|(x-1) + \Delta x| + 4 \\ &= F(x + \Delta x) \end{aligned}$$

8 Solve

8.0.1 ABS-Equation-System

As a last step we bring the function in the form of a linear equation system.

$$\begin{pmatrix} \Delta z \\ \Delta y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} Z & L \\ J & Y \end{pmatrix} \times \begin{pmatrix} \Delta x \\ |\Delta z| \end{pmatrix}$$

We assume:

$$\Delta y = 0$$

otherwise use

$$b' = b - \Delta y$$

$$\begin{aligned} \Delta y &= b + J\Delta x + Y|\Delta z| \\ 0 &= b + J\Delta x + Y|\Delta z| \\ -b - Y|\Delta z| &= J\Delta x \\ b + Y|\Delta z| &= J\Delta x(-1) \\ J^{-1}(b + Y|\Delta z|) &= -\Delta x \\ \Delta x = -J^{-1}(b + Y|\Delta z|) \end{aligned}$$

$$\begin{aligned} \Delta z &= a + Z\Delta x + L|\Delta z| \\ &= a + Z\Big(- J^{-1}(b + Y|\Delta z|)\Big) + L|\Delta z| \\ &= a + Z\Big(- J^{-1}b - J^{-1}Y|\Delta z|\Big) + L|\Delta z| \\ &= a - ZJ^{-1}b - ZJ^{-1}Y|\Delta z| + L|\Delta z| \\ &= a - ZJ^{-1}b - (ZJ^{-1}Y - L)|\Delta z| \end{aligned}$$

$$\begin{aligned} \Delta z &= a - ZJ^{-1}b - (ZJ^{-1}Y - L)|\Delta z| \\ &= c + S|\Delta z| \\ c &= a - ZJ^{-1}b \\ S &= (L - ZJ^{-1}Y) \end{aligned}$$

8.1 Implementation

$$\begin{aligned} c &= a - ZJ^{-1}b \\ S &= (L - ZJ^{-1}Y) \end{aligned}$$

Need the inverse of J

- Calculate Directly
- Solve SLE

$$\begin{aligned} c &= a - Z * solve(J * X = b) \\ S &= L - Z * solve(J * X = Y) \end{aligned}$$