

1 TODO

- Plot linearization
- linearization
- linearization of abs in AD

2 ABS-Normalform

Definition 1. Smooth function

A smooth function $f(x)$ is continuous and has a continuous derivative

Beispiel 1.

$$f(x) = x^2, f'(x) = 2x$$

is smooth.

$$\begin{aligned} f(x) &= |x| \\ g'(x) &= 1, x > 0 \\ f'(x) &= -1, x < 0 \end{aligned}$$

Therefore it is not smooth, since its derivative is not continuous.

Definition 2. A function has a jump discontinuity at x_0 if

$$\lim_{x \rightarrow x_0^-} f(x) \neq \lim_{x \rightarrow x_0^+} f(x)$$

Beispiel 2. $\text{abs}(x)$

Definition 3. a piecewise smooth function is made up of finitely many smooth functions separated by jump discontinuities.

Haben Piecewise smooth function. Piecewise linearization can be achieved in the style of algorithmic differentiation. by replacing all smooth elemental functions by their tangent line or plane.

2.0.1 Assumptions

- All nonsmoothness can be cast in terms of the absolute value function

2.0.2 Goal

Given $F(X)$ with $F(X) = 0$. Where $F(X)$ is piecewise linear in abs NF.

2.0.3 Questions

- How do I get my representation?
- Find minimum of these functions?
- On Stable Piecewise Linearization and Generalized Differentiation - Griewank

2.1 Linearization

Linearization is a linear approximation of a nonlinear system that is valid in a small region around an operating point.

[https : //de.mathworks.com/help/slcontrol/ug/linearizing-nonlinear-models.html](https://de.mathworks.com/help/slcontrol/ug/linearizing-nonlinear-models.html)

Linearization through tangent line at point $(a, f(a))$

$$\begin{aligned}
y &= f(x) \\
m &= f'(a) \\
y &= f(a) + f'(a)(x - a) \\
y &= f'(a)x - f'(a)a + f(a) = f'(a)x + b
\end{aligned}$$

2.2 Anwendung

Wenn Newton nicht haut mache. Zur Minimums findung von Funktion:

- Stückweise Linearisierung in Punkt
- Berechne ABS Lösung
- Nutze Punkt für nächsten step

2.3 Representation of Picewise functions with MAX-MIN

Can represent every PL functin with min max functins

2.4 Representation of MAX-MIN functions with abs functions

$$\begin{aligned}
\min(a, b) &= \frac{1}{2}(a + b - \text{abs}(a - b)) \\
\max(a, b) &= \frac{1}{2}(a + b + \text{abs}(a - b))
\end{aligned}$$

2.5 Absnormalform

2.6 Example

Given is the following PL function:

$$\begin{aligned}
F(x_1, x_2) &= (x_2^2 - x_1^+)^+ \\
(i)^+ &= \max(0, i)
\end{aligned}$$

2.6.1 ABS-Normalization

First we rewrite the function such that picewise linearity is expressed with the aid of the abs function.

$$\begin{aligned}
F(x_1, x_2) &= (x_2^2 - x_1^+)^+ \\
&= \max(0, x_2^2 - \max(0, x_1)) \\
&= \frac{1}{2} \left(0 + x_2^2 - \max(0, x_1) + \text{abs}(0 - x_2^2 + \max(0, x_1)) \right) \\
&= \frac{1}{2} \left(0 + x_2^2 - \frac{1}{2} \left[0 + x_1 + \text{abs}(0 - x_1) \right] + \text{abs}(0 - x_2^2 + \max(0, x_1)) \right) \\
&= \frac{1}{2} \left(0 + x_2^2 - \frac{1}{2} \left[0 + x_1 + \text{abs}(0 - x_1) \right] + \text{abs}(0 - x_2^2 + \frac{1}{2}(0 + x_1 + \text{abs}(0 - x_1))) \right) \\
&= \frac{1}{2} \left(x_2^2 - \frac{1}{2} \left[x_1 + \text{abs}(-x_1) \right] + \text{abs}(-x_2^2 + \frac{1}{2}(x_1 + \text{abs}(-x_1))) \right) \\
&= \frac{1}{2} \left(x_2^2 - \frac{1}{2} \left[x_1 + \text{abs}(x_1) \right] + \text{abs}(-x_2^2 + \frac{1}{2}(x_1 + \text{abs}(x_1))) \right)
\end{aligned}$$

2.6.2 Straight-Line Code

Next we bring the function in straight-line code representaion:

SLC	Directional Derivatives
$w_1 = x_1$	$\Delta w_1 = \Delta x_1$
$w_2 = x_2$	$\Delta w_2 = \Delta x_2$
$w_3 = w_1 $	$\Delta w_3 = \underbrace{ w_1 + \Delta w_1 }_{\Delta Z_1} - w_1 $
$w_4 = w_1 + w_3$	$\Delta w_4 = \Delta w_1 + \Delta w_3$
$w_5 = \frac{1}{2} w_4$	$\Delta w_5 = \frac{1}{2} \Delta w_4$
$w_6 = w_2 w_2$	$\Delta w_6 = 2 \Delta w_2 * w_2$
$w_7 = w_5 - w_6$	$\Delta w_7 = \Delta w_5 - \Delta w_6$
$w_8 = w_7 $	$\Delta w_8 = \underbrace{ w_7 + \Delta w_7 }_{\Delta Z_2} - w_7 $
$w_9 = w_6 - w_5$	$\Delta w_9 = \Delta w_6 - \Delta w_5$
$w_{10} = w_9 + w_8$	$\Delta w_{10} = \Delta w_9 + \Delta w_8$
$w_{11} = \frac{1}{2} w_{10}$	$\Delta w_{11} = \frac{1}{2} \Delta w_{10}$
$y = w_{11}$	$\Delta y = \Delta w_{11}$

Now we rewrite Z_i and Y_i such that they only depend on x . In the computational graph, the equivalent

operation is node contraction.

$$\begin{aligned}
 \Delta Z_1 &= w_1 + \Delta w_1 = w_1 + \Delta x_1 \\
 \Delta Z_2 &= w_7 + \Delta w_7 \\
 &= w_7 + \Delta w_5 - \Delta w_6 \\
 &= w_7 + \frac{1}{2} \Delta w_4 - 2 \Delta w_2 w_2 \\
 &= w_7 + \frac{1}{2} (\Delta w_1 + \Delta w_3) - 2 \Delta x_2 w_2 \\
 &= w_7 + \frac{1}{2} (\Delta w_1 + |\Delta Z_1| - |w_1|) - 2 \Delta x_2 w_2 \\
 &= w_7 + \frac{1}{2} (\Delta x_1 + |\Delta Z_1| - |w_1|) - 2 \Delta x_2 w_2 \\
 &= w_7 + \frac{1}{2} \Delta x_1 + \frac{1}{2} |\Delta Z_1| - \frac{1}{2} |w_1| - 2 \Delta x_2 w_2 \\
 \Delta Y &= \Delta w_{11} \\
 &= \frac{1}{2} \Delta w_{10} \\
 &= \frac{1}{2} (\Delta w_9 + \Delta w_8) \\
 &= \frac{1}{2} (\Delta w_6 - \Delta w_5 + \Delta w_8) \\
 &= \frac{1}{2} (2 \Delta w_2 w_2 - \frac{1}{2} \Delta w_4 + |\Delta Z_2| - |w_7|) \\
 &= \frac{1}{2} (2 \Delta x_2 w_2 - \frac{1}{2} (\Delta x_1 + |\Delta Z_1| - |w_1|) + |\Delta Z_2| - |w_7|) \\
 &= \frac{1}{2} (2 \Delta x_2 w_2 - \frac{1}{2} (\Delta x_1 + |\Delta Z_1| - |w_1|) + |\Delta Z_2| - |w_7|) \\
 &= \Delta x_2 w_2 - \frac{1}{4} \Delta x_1 - \frac{1}{4} |\Delta Z_1| + \frac{1}{4} |w_1| + \frac{1}{2} |\Delta Z_2| - \frac{1}{2} |w_7|
 \end{aligned}$$

$$\begin{aligned}
 \begin{pmatrix} \Delta Z_1 \\ \Delta Z_2 \\ \Delta Y \end{pmatrix} &= \begin{pmatrix} w_1 \\ w_7 - \frac{1}{2} |w_1| \\ \frac{1}{4} |w_1| - \frac{1}{2} |w_7| \end{pmatrix} + \begin{pmatrix} 1 \Delta x_1 + 0 \Delta x_2 \\ \frac{1}{2} \Delta x_1 + 0 \Delta x_2 \\ -\frac{1}{4} \Delta x_1 + w_2 \Delta x_2 \end{pmatrix} + \begin{pmatrix} 0 |\Delta Z_1| + 0 |\Delta Z_2| \\ \frac{1}{2} |\Delta Z_1| + 0 \Delta Z_2 \\ -\frac{1}{4} |\Delta Z_1| + \frac{1}{2} \Delta Z_2 \end{pmatrix} \\
 &= \begin{pmatrix} w_1 \\ w_7 - \frac{1}{2} |w_1| \\ \frac{1}{4} |w_1| - \frac{1}{2} |w_7| \end{pmatrix} + \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ -\frac{1}{4} & w_2 & -\frac{1}{4} & \frac{1}{2} \end{array} \right) \times \begin{pmatrix} \Delta x_1 \\ \Delta x_2 \\ |\Delta Z_1| \\ \Delta Z_2 \end{pmatrix}
 \end{aligned}$$

2.6.3 ABS-Equation-System

As a last step we bring the function in the form of a linear equation system.

$$\begin{pmatrix} \Delta Z \\ \Delta Y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} Z & L \\ J & Y \end{pmatrix} \times \begin{pmatrix} \Delta x \\ \Delta z \end{pmatrix}$$

3 Evaluation

Input: $a, b, Z, L, J, Y, n, m, s, \Delta x$

Output: $\Delta y, \Delta Z$

4 Gradient

5 Solve

6 Calculations

6.1 Gradient of abs()

$$\begin{aligned} f(x) &= |x| \\ \nabla f(x) &= |x + \nabla x| - |x| \\ &= \max\{x + \nabla x, -x - \nabla x\} - \max\{x, -x\} \end{aligned}$$

- $x > 0$

$$- \nabla x > 0$$

$$\nabla f(x) = x + \nabla x - x = \nabla x$$

$$- \nabla x < 0$$

$$* \nabla x > -x$$

$$\nabla f(x) = x + \nabla x - x = \nabla x$$

$$* \nabla x < -x$$

$$\begin{aligned} \nabla f(x) &= \max\{x + \nabla x, -x - \nabla x\} - \max\{x, -x\} \\ &= -x - \nabla x - x = -2x - \nabla x \end{aligned}$$

$$* \nabla x = -x$$

$$\begin{aligned} \nabla f(x) &= x + \nabla x - x = \nabla x \\ &= -x - \nabla x - x = \nabla x - \nabla x + \nabla x = \nabla x \end{aligned}$$

$$- \nabla x = 0$$

$$\nabla f(x) = x + \nabla x - x = \nabla x = 0$$

- $x < 0$

$$- \nabla x > 0$$

$$- \nabla x < 0$$

$$- \nabla x = 0$$

- $x = 0$

???

7 Example ABS

Let

$$F(x) = 3|x - 1| = 3abs(x - 1)$$

The task is to find $\Delta F(x, \Delta x)$, so that:

$$F(x + \Delta x) \approx F(x) + \Delta F(x, \Delta x)$$

therefore

$$F(x) = |x| \Rightarrow \Delta F'(x, \Delta x) = |x + \Delta x| - |x|$$

now

$$\begin{aligned} F(x) + \Delta F(x, \Delta x) &= |x| + |x + \Delta x| - |x| \\ &= |x + \Delta x| \\ &= F(x + \Delta x) \end{aligned}$$

Beispiel 3.

$$\begin{aligned} F(x) &= 3|x - 1| + 4 \\ \Delta F(x, \Delta x) &= 3|(x - 1) + \Delta x| - 3|x - 1| \\ F(x) + \Delta F(x, \Delta x) &= 3|x - 1| + 4 + 3|(x - 1) + \Delta x| - 3|x - 1| \\ &= 3|(x - 1) + \Delta x| + 4 \\ &= F(x + \Delta x) \end{aligned}$$

8 Solve

8.0.1 ABS-Equation-System

As a last step we bring the function in the form of a linear equation system.

$$\begin{pmatrix} \Delta z \\ \Delta y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} Z & L \\ J & Y \end{pmatrix} \times \begin{pmatrix} \Delta x \\ |\Delta z| \end{pmatrix}$$

We assume:

$$\Delta y = 0$$

otherwise use

$$b' = b - \Delta y$$

$$\Delta y = b + J\Delta x + Y|\Delta z|$$

$$0 = b + J\Delta x + Y|\Delta z|$$

$$-b - Y|\Delta z| = J\Delta x$$

$$b - Y|\Delta z| = J\Delta x(-1)$$

$$J^{-1}(b - Y|\Delta z|) = -\Delta x$$

$$\Delta x = -J^{-1}(b - Y|\Delta z|)$$

$$\Delta z = a + Z\Delta x + L|\Delta z|$$

$$= a + Z\left(-J^{-1}(b - Y|\Delta z|)\right) + L|\Delta z|$$

$$= a + Z\left(-J^{-1}b + J^{-1}Y|\Delta z|\right) + L|\Delta z|$$

$$= a - ZJ^{-1}b + ZJ^{-1}Y|\Delta z| + L|\Delta z|$$

$$= a - ZJ^{-1}b + (ZJ^{-1}Y + L)|\Delta z|$$

$$\Delta z = a - ZJ^{-1}b + (ZJ^{-1}Y + L)|\Delta z|$$

$$\Delta z = c + S|\Delta z|$$

$$c = a - ZJ^{-1}b$$

$$S = (ZJ^{-1}Y + L)$$

8.1 Useless

$$a' = a - \Delta z$$

$$0 = a' - ZJ^{-1}b + (ZJ^{-1}Y + L)|\Delta z|$$

$$-a' + ZJ^{-1}b = (ZJ^{-1}Y + L)|\Delta z|$$

$$(ZJ^{-1}Y + L)^{-1}(-a' + ZJ^{-1}b) = |\Delta z|$$