

## Problem Set 6 — Solutions (Non-Convex Optimization)

### Theoretical Exercises

**Exercise 36.** Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  twice differentiable, with  $X \subseteq \text{dom}(f)$  an open convex set, and suppose that  $f$  is smooth with parameter  $L$  over  $X$ . Prove that under these conditions, the largest eigenvalue of the Hessian  $\lambda_{\max}(\nabla^2 f(\mathbf{x})) \leq L$  for all  $\mathbf{x} \in X$ .

**Solution:** To prove that the largest eigenvalue of  $\nabla^2 f(\mathbf{x})$  is  $\leq L$  for all  $\mathbf{x} \in X$ , we need to show that  $\mathbf{y}^\top \nabla^2 f(\mathbf{x}) \mathbf{y} \leq L \|\mathbf{y}\|_2^2$  for all  $\mathbf{x} \in X$  and  $\mathbf{y} \in \mathbb{R}^d$ .

Since  $X$  is an open set, for any  $\mathbf{x} \in X$ , there exists a  $c > 0$  such that for all  $\lambda \in [0, c]$  and  $\mathbf{y} \in \mathbb{R}^d$ ,  $\mathbf{x} + \lambda \mathbf{y} \in X$ . For any such  $\mathbf{x}, \mathbf{y}$ , and  $\lambda$ , we can use Taylor's theorem and write

$$f(\mathbf{x} + \lambda \mathbf{y}) = f(\mathbf{x}) + \lambda \langle \nabla f(\mathbf{x}), \mathbf{y} \rangle + \frac{\lambda^2}{2} \mathbf{y}^\top \nabla^2 f(\mathbf{x}) \mathbf{y} + o(\lambda^2).$$

Using the smoothness of  $f$  over  $X$ , we also have that

$$f(\mathbf{x} + \lambda \mathbf{y}) \leq f(\mathbf{x}) + \lambda \langle \nabla f(\mathbf{x}), \mathbf{y} \rangle + \frac{L\lambda^2}{2} \|\mathbf{y}\|_2^2.$$

These two expressions can be combined into the observation that

$$\frac{\lambda^2}{2} \mathbf{y}^\top \nabla^2 f(\mathbf{x}) \mathbf{y} \leq \frac{L\lambda^2}{2} \|\mathbf{y}\|_2^2 - o(\lambda^2).$$

For any  $\lambda > 0$ , we can divide both sides by  $\frac{\lambda^2}{2}$  and tend  $\lambda$  to zero to obtain:

$$\mathbf{y}^\top \nabla^2 f(\mathbf{x}) \mathbf{y} \leq \lim_{\lambda \rightarrow 0} L \|\mathbf{y}\|_2^2 - 2 \frac{o(\lambda^2)}{\lambda^2} = L \|\mathbf{y}\|_2^2.$$

This proves the exercise.

**Exercise 37.** Prove that the statement of Theorem 6.2 implies that

$$\lim_{t \rightarrow \infty} \|\nabla f(\mathbf{x}_t)\|^2 = 0.$$

**Solution:** Let  $\nabla_t = \|\nabla f(\mathbf{x}_t)\|^2$ . The statement of Theorem 6.2 is that

$$S_T := \sum_{t=0}^{T-1} \nabla_t \leq K := 2L(f(\mathbf{x}_0) - f(\mathbf{x}^*)), \quad T > 0.$$

Hence,  $(S_T)_{T>0}$  is a monotone increasing and bounded sequence and as such has a limit  $K' \leq K$ . This means that for all  $\varepsilon > 0$ , there exists  $T_0$  such that  $S_T > K' - \varepsilon$  for all  $T \geq T_0$ . As  $S_{T+1} \leq K'$  for all  $T$ , we have

$$\nabla_T = S_{T+1} - S_T < \varepsilon$$

for all  $T \geq T_0$ . And this means that

$$\lim_{T \rightarrow \infty} \nabla_T = 0.$$

## Practical Exercises

The theory of non-convex optimization is unfortunately not very illuminative. However, their practical performance is usually unmatched by convex methods. In this exercise, we will use the PyTorch framework to train a small neural network on some simple datasets.

### Problem 1 (PyTorch Refresher):

Installation instructions:

[pytorch.org](https://pytorch.org)

We recommend using the following online tutorial:

[pytorch.org/tutorials/beginner/pytorch\\_with\\_examples.html](https://pytorch.org/tutorials/beginner/pytorch_with_examples.html)

You can also refer to the following exercise from the Machine Learning course:

[https://github.com/epfml/ML\\_course/blob/master/labs/ex12/](https://github.com/epfml/ML_course/blob/master/labs/ex12/)

### Problem 2 (Simple Neural Network):

Follow the notebook provided here:

[https://github.com/epfml/OptML\\_course/tree/master/labs/ex06](https://github.com/epfml/OptML_course/tree/master/labs/ex06)