Classification aggregation with neutrality*

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1. Introduction

The Classification problem which consists in mapping a set of objects into a set of categories has been broadly studied, in particular from a machine learning perspective. In Classification Aggregation, we take a voting perspective; the goal is to aggregate individual's classifications into a global one. This problem was introduced by Maniquet and Mongin [2016], which followed advances in aggregation of equivalence relations [Mirkin, 1975, Fishburn and Rubinstein, 1986] and group identification [Kasher and Rubinstein, 1997].

A popular example for classification aggregation is that of aggregating tier-lists in gaming.[1] The goal is usually to classify some video game characters or items into a predefined set of ranked categories (tiers). A tier-list that

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leaves a tier empty is usually considered inconsistent, so the number of tiers (categories) is, in general, fixed. This is very similar to the setting explored by Craven [2024] with the difference being that there is no agreement on a ranking of objects (for example, some players would prefer the axe over the sword and some would disagree).[2]

Our setting has a strong relation with preference aggregation: a well-known and broadly studied problem where individual preferences on a set of alternatives are aggregated into a societal preference. Classical works in this strand of literature start with the impossibility theorem by Arrow [1951] which shows that independence and Pareto dominance are not compatible with nondictatorial aggregation. Consequent works extensively studied related axiomatics, such as, among others, Wilson [1972], who demonstrates a similar impossibility result under independence and citizen sovereignty. Later, Moulin [1983, 1991] characterized the parameters m, n that allow for anonymous and neutral social choice functions (that assign a winner to a preference profile). Results by Moulin [1983], Campbell and Kelly [2015] show that only few aggregators can satisfy anonymity and neutrality. Following this work, Doğan and Giritligil [2022] and Bubboloni and Gori [2014] extended these results for social welfare functions (that assign a strict ordering to a preference profile).

An important part of the literature in classification aggregation studies whether these results resonate in the setting of classification aggregation. Maniquet and Mongin [2016] proved an *Arrovian* impossibility theorem in the context of classification aggregation that establishes that independent and unanimous aggregation is necessarily dictatorial. Alcantud et al. [2019] then introduced fuzzy classification aggregation (when objects have a percentage of belonging to the categories) and proved an equivalent impossibility result in this setting. Building on this, Cailloux et al. [2024] recently delivered a *Wilsonian* impossibility result in this setting, namely one that replaces unanimity with citizen sovereignty.

Fortunately, some researchers recently found escape routes to these impossibility theorems. Fioravanti [2024] relaxed the problem of fuzzy classifications and found an independent, unanimous and anonymous aggregator. Craven [2023] used the approach of domain restriction to build a non-dictatorial aggregation function that is independent and unanimous, and Craven [2024] ex-

[2] Another example of application, which gained a wide public attention in France, is a website called "De droite ou de Gauche ?" that was created by Théo Delemazure to categorize any user-inserted term as right-wing or left-wing. Initially based on GPT, it later collected user opinions to resolve classifications. See `https://dauphine.psl.eu/dauphine/media-et-communication/article/cest-de-gauche-ou-de-droite-lia-developpee-a-dauphine-fait-le-buzz`

plored the problem of classification aggregation with ranked categories and objects which makes it possible to have an independent and unanimous aggregator.

We concur that there remains much to learn about classification aggregation through the lenses of established approaches in preference or judgment aggregation. In this paper, we study what the axiomatics of Moulin [1983] imply in the context of classification aggregation. Specifically, we prove an equivalent of the Moulin impossibility result in the context of classification aggregation, and propose aggregation rules based on Doğan and Giritligil [2022], Ozkes and Sanver [2024] that are consistent aggregators for some sizes of the problem.

In our analysis we highlight the similarity between classification aggregation and preference aggregation. In particular, we prove that the problem of preference aggregation with an imposed number of equivalence classes is isomorphic to classification aggregation. Imposing a maximal number of equivalence classes for preference aggregation has been broadly studied in the literature with the examples of approval voting [Brams and Fishburn, 1978], range voting [Smith, 2000], or majority judgement [Balinski and Laraki, 2011]. The only work we know that imposes a fixed number of equivalence classes is by Maniquet and Mongin [2015], which focuses on approval voting (2 equivalence classes).

We introduce the notations and axioms in section 2, explain the relation between classification aggregation and preference aggregation in section 3, show what we can obtain from this when the number of obejcts and categories coincide in section 4. Then we prove some impossibility theorems for different numbers of objects and categories in section 5, define some interesting aggregators in section 6, and open some discussion in section 7.

2. Preliminaries

We consider a set $N = \{1, \dots, n\}$ of *individuals* with $n \geq 2$, a set $P = \{p_1, \dots, p_\rho\}$ of *categories* with $\rho \geq 2$, and a set $X = \{x_1, \dots, x_m\}$ of *objects* with $m \geq \rho$.

We define a *classification* as a surjective mapping $c : X \to P$ and denote by $\mathcal{C} \subset P^X$ the set of classifications. We write $\boldsymbol{c} = (c_1, \dots, c_n) \in \mathcal{C}^N$ for a classification profile. Given $\boldsymbol{c} \in \mathcal{C}^N$ and $x \in X$, we write $\boldsymbol{c}_x \in P^N$ for the vector of categories that object x is put into by each individual, thus $\forall i \in N, \boldsymbol{c}_x(i) = c_i(x)$. A *classification aggregation function* (CAF) is a mapping $\alpha : \mathcal{C}^N \to \mathcal{C}$.

Let S_A denote the set of permutations of the set A. Note that a classification profile $\boldsymbol{c}$ can be seen as a function from N to $\mathcal{C}$ or from X to P^N or from $N \times X$ to P. Using this fact, given $\pi \in S_P, \sigma \in S_X$ we will write $\pi \circ \boldsymbol{c}$ to denote the classification profile equal to $(\pi \circ c_i)_{i \in N}$ and $\boldsymbol{c} \circ \sigma$ to denote $(\boldsymbol{c}_{\sigma(x)})_{x \in X}$. Moreover, given $\gamma \in S_N, \boldsymbol{c} \in \mathcal{C}^N$, we let $\boldsymbol{c}^{(\gamma)}$ denote the classification profile $(c_{\gamma(i)})_{i \in N}$.

We have the following definitions that are central to our analyses.

Definition 1. A CAF α is *anonymous* if $\forall \boldsymbol{c} \in \mathcal{C}^N, \forall \pi \in S_N, \alpha(\boldsymbol{c}^{(\pi)}) = \alpha(\boldsymbol{c})$.

Definition 2. A CAF α is *object neutral* if $\forall \pi \in S_X, \boldsymbol{c} \in \mathcal{C}^N, \alpha(\boldsymbol{c} \circ \pi) = \alpha(\boldsymbol{c}) \circ \pi$.

Definition 3. A CAF α is *category neutral* if $\forall \pi \in S_P, \forall \boldsymbol{c} \in \mathcal{C}^N, \alpha(\pi \circ \boldsymbol{c}) = \pi \circ \alpha(\boldsymbol{c})$.

3. Relation between preferences and classifications

3.1. Preference aggregation

We consider a set N of voters or individuals and a set X of objects, with $m = |X| \geq 2$. We let $\mathcal{L}(X)$ the set of linear orders, i.e., complete, transitive and asymmetric binary relations on X. Given a voter $i \in N$, we write $P_i \in \mathcal{L}(X)$ for the preference of voter i on X. A n-tuple P_N in $\mathcal{L}(X)^N$ is called a preference profile.

A social welfare function (SWF) is a mapping from $\mathcal{L}(X)^N$ to $\mathcal{L}(X)$. Given a preference profile $P_N \in \mathcal{L}(X)^N$ and a permutation $\gamma \in S_N$, we write $\gamma(P_N) = (P_{\gamma(i)})_{i \in N}$ for the profile induced by the permutation of voters γ. A SWF f is said to satisfy *welfare anonymity* if $\forall \gamma \in S_N, \forall P_N \in \mathcal{L}(X)^N, f(\gamma(P_N)) = f(P_N)$.

By abuse of notation, given an individual preference $P_i \in \mathcal{L}(X)$ and a permutation $\pi \in S_X$, let $\pi(P_i)$ be such that $x\ \pi(P_i)\ y \iff x\ P_i\ y$ for all $x, y \in A$. Given a preference profile $P_N \in \mathcal{L}(X)^N$ and a permutation $\pi \in S_X$, we write $\pi(P_N) = (\pi(P_i))_{i \in N}$ for the profile induced by a permutation of objects π in P_N. A SWF f is said to satisfy *welfare neutrality* if $\forall \pi \in S_X, \forall P_N \in \mathcal{L}(X)^N, f(\pi(P_N)) = \pi(f(P_N))$.

We introduce $\mathcal{W}_\rho(X)$ to be the set of weak orders of X with exactly ρ equivalence classes. A ρ-SWF is a mapping from $\mathcal{W}_\rho(X)^N$ to $\mathcal{W}_\rho(X)$.

3.2. An isomorphism with Classification Aggregation

We claim that $\mathcal{C}$ and $\mathcal{W}_\rho(X)$ are isomorphic. Mirkin [1975] first made this remark without a formal explanation. Indeed, given $P = \{p_1, \ldots, p_m\}$ and an arbitrary strict ordering $\succ$ of P we can define $\theta_\succ : \mathcal{C} \to \mathcal{W}_\rho(X)$ as $\forall c \in \mathcal{C}, \forall x, y \in X, x \ \theta_\succ(c) \ y$ *iff* $c(x) \succ c(y)$. One can check that $\theta_\succ$ is an isomorphism. Note that in the case of $m = \rho$, we have $\mathcal{L}(X) = \mathcal{W}_\rho(X)$, so that $\mathcal{C}$ and $\mathcal{L}(X)$ are isomorphic. In what follows, we let θ be any isomorphism from $\mathcal{C}$ to $\mathcal{W}_\rho(X)$ and φ an isomorphism from $\mathcal{W}_\rho(X)$ to $\mathcal{C}$.

Remark 1. A classification c can be viewed as a function that sends each object x to a number (namely the number k corresponding to $p_k = c(x)$). Similarly, a weak order P_i having ρ equivalence classes can be viewed as a function that sends each object x to a number in $[\![1, \rho]\!]$ (namely the rank of x in P_i, corresponding to the cardinality of the number of equivalence classes in its weak upper contour set, weak meaning including itself). By doing so, we can build an isomorphism from $\mathcal{C}$ to $\mathcal{W}_\rho(X)$ by using the number that objects are mapped to. △

Note that the isomorphism θ between $\mathcal{C}$ and $\mathcal{W}_\rho(X)$ induces an isomorphism between $\mathcal{C}^N$ and $\mathcal{W}_\rho(X)^N$ and therefore between the set of CAFs and the set of SWFs with a fixed number of equivalence classes. By abuse of notation, given a CAF α, we let $\theta(\alpha)$ denote the SWF defined as $\forall \boldsymbol{c} = (c_i)_{i\in N} \in \mathcal{C}^N, \theta(\alpha)((\theta(c_i))_{i\in N}) = \theta(\alpha(\boldsymbol{c}))$. Likewise, given an SWF f, we define $\varphi(f)$ as $\forall P_N \in \mathcal{L}(X), \varphi(f)((\varphi(P_i))_{i\in N}) = \varphi(f(P_N))$.

Note that given a permutation $\pi \in S_X$, θ and π are associative, i.e. $\forall c \in \mathcal{C}, \theta(c \circ \pi) = \pi(\theta(c))$.

4. Equal number of objects and categories

In this subsection, we use the relation between Preference Aggregation and Classification Aggregation discussed in section 3 for the case $m = \rho$. By using the isomorphism between preference and classification profiles, we can extend Doğan and Giritligil's [2022] results to classification aggregation. An anonymous and neutral (object neutral or category neutral) CAF can exist iff 1 is the only divisor of n that is smaller than m.

Proposition 1. *A CAF α satisfies anonymity iff $\theta(\alpha)$ satisfies welfare anonymity.*

Proof. Let α be a CAF.

$\Rightarrow$ Suppose α satisfies anonymity, and let $\gamma \in S_N$ be a permutation of N. We have $\forall \boldsymbol{c} = (c_i)_{i\in N} \in \mathcal{C}^N, \alpha((c_{\gamma(i)})_{i\in N}) = \alpha(\boldsymbol{c})$. Let $P_N \in \mathcal{L}(X)$ be any preference profile, then $\exists \boldsymbol{c} \in \mathcal{C}, \theta(\boldsymbol{c}) = P_N$ as θ is an isomorphism. Therefore, $\theta(\alpha)(\gamma(P_N)) = \theta(\alpha)((\theta(c_{\gamma(i)}))_{i\in N}) = \theta(\alpha(\boldsymbol{c}^{(\gamma)})) = \theta(\alpha(\boldsymbol{c})) = \theta(\alpha)((\theta(c_i))_{\in N}) =$

$\theta(\alpha)(P_N)$ as α satisfies anonymity, so $\alpha(\boldsymbol{c}^{(\gamma)}) = \alpha(\boldsymbol{c})$. Then, $\theta(\alpha)$ satisfies welfare anonymity.

$\Leftarrow$ Suppose $\theta(\alpha)$ satisfies welfare anonymity, and let $\boldsymbol{c} = (c_i)_{i\in N} \in \mathcal{C}^N$ be a classification profile and $\gamma \in S_N$ be a permutation. As $\theta(\alpha)$ satisfies welfare anonymity, we have $\theta(\alpha)(\gamma((\theta(c_i))_{i\in N})) = \theta(\alpha)((\theta(c_i))_{i\in N})$.

Then $\theta(\alpha(\boldsymbol{c}^{(\gamma)})) = \theta(\alpha)(\gamma((\theta(c_i))_{i\in N})) = \theta(\alpha)((\theta(c_i))_{i\in N}) = \theta(\alpha(\boldsymbol{c}))$. As θ is an isomorphism, $\theta(\alpha(\boldsymbol{c}^{(\gamma)})) = \theta(\alpha(\boldsymbol{c}))$ implies $\alpha(\boldsymbol{c}^{(\gamma)}) = \alpha(\boldsymbol{c})$, so α satisfies anonymity. $\square$

Corollary 1. *A SWF f satisfies welfare anonymity iff $\varphi(f)$ satisfies anonymity.*

Proposition 2. *A CAF α satisfies object neutrality iff $\theta(\alpha)$ satisfies welfare neutrality.*

Proof. $\Rightarrow$ Suppose α satisfies object neutrality, and let $\pi \in S_X$ be a permutation of X.

We have that $\forall \boldsymbol{c} \in \mathcal{C}^N, \alpha((c_i \circ \pi)_{i\in N}) = \alpha(\boldsymbol{c}) \circ \pi$. Now take $P_N = (P_i)_{i\in N} \in \mathcal{L}(X)^N$, as θ is an isomorphism, $\exists \boldsymbol{c} = (c_i)_{i\in N} \in \mathcal{C}^N, \forall i \in N, \theta(c_i) = P_i$.

Now, $\theta(\alpha)(\pi(P_N)) = \theta(\alpha)(\pi((\theta(c_i)))_{i\in N}) = \theta(\alpha)(\theta((c_i \circ \pi))_{i\in N}) = \theta(\alpha((c_i \circ \pi)_{i\in N})) = \theta(\alpha(\boldsymbol{c}) \circ \pi) = \pi(\theta(\alpha)(P_N))$. Then, $\theta(\alpha)$ satisfies welfare neutrality.

$\Leftarrow$ Suppose $\theta(\alpha)$ satisfies welfare neutrality, and let π be a permutation of X.

We have that $\forall P_N \in \mathcal{L}(X)^N, \theta(\alpha)(\pi(P_N)) = \pi(\theta(\alpha)(P_N))$. Now, let $\boldsymbol{c} = (c_i)_{i\in N} \in \mathcal{C}^N$, and consider $P_N = (\theta(c_i))_{i\in N} \in \mathcal{L}(X)^N$.

We have $\theta(\alpha((c_i \circ \pi)_{i\in N})) = \theta(\alpha)((\theta(c_i \circ \pi)_{i\in N}) = \theta(\alpha)((\pi(\theta(c_i)))_{i\in N}) = \theta(\alpha)(\pi(P_N)) = \pi(\theta(\alpha)(P_N)) = \theta(\pi(\alpha(\boldsymbol{c})))$. Then, as θ is an isomorphism, $\alpha((c_i \circ \pi)_{i\in N}) = \alpha(\boldsymbol{c}) \circ \pi$, so α satisfies object neutrality. $\square$

Corollary 2. *A SWF f satisfies welfare neutrality iff $\varphi(f)$ satisfies object neutrality.*

Theorem 1 (Moulin [1983])**.** *Given $n, m \geq 2$, there exists a welfare anonymous and welfare neutral SWF iff all prime divisors of n exceed m.*

Corollary 3. *Given $n, m = \rho \geq 2$, there exists an anonymous and object neutral CAF iff all prime divisors of n exceed m.*

Doğan and Giritligil [2022] defined a SWF in their theorem 3.1 that they proved to be well defined and to satisfy welfare anonymity and welfare neutrality whenever all prime divisors of n exceed m. We refer to it as the greedy SWF.

Algorithm 1 The greedy SWF

while $\exists P_N \in \mathcal{L}(X)^N, f(P_N)$ is not defined **do**
  Pick $P_N \in \mathcal{L}(X)^N, f(P_N)$ is not defined, and $l \in \mathcal{L}(X)$
  Set $f(P_N) = l$
  for $\tilde{P_N} \in \mathcal{L}(X)^N$ s.t. $\exists \pi \in S_X, \gamma \in S_N, \forall i \in N, \tilde{P}_i = \pi(P_{\gamma(i)})$ **do**
    $f(\tilde{P_N}) = \pi(l)$
  end for
end while

We now introduce the greedy anonymous and neutral CAF which is isomorphic to the greedy SWF.

Algorithm 2 The greedy CAF

while $\exists \boldsymbol{c} \in \mathcal{C}^N, \alpha(\boldsymbol{c})$ is not defined **do**
  Pick $\boldsymbol{c} \in \mathcal{C}^N, \alpha(\boldsymbol{c})$ is not defined, and $\tilde{c} \in \mathcal{C}$
  Set $\alpha(\boldsymbol{c}) = \tilde{c}$
  for $\boldsymbol{c}' \in \mathcal{C}^N$ s.t. $\exists \pi \in S_X, \gamma \in S_N, \forall i \in N, c'_i = c_{\gamma(i)} \circ \pi$ **do**
    $\alpha(\boldsymbol{c}') = \alpha(\boldsymbol{c}) \circ \pi$
  end for
end while

Proposition 3. *The greedy anonymous and neutral CAF satisfies anonymity and neutrality iff all prime divisors of n exceed m.*

5. More objects than categories

5.1. Impossibility results

We let $\mathbb{N}$ denote the set of natural numbers and given $i, j \in \mathbb{N}$ write $i \mid j$ when i divides j (with $1 \mid i$ and $i \mid i$). Given $j, k \in \mathbb{N}$, let $[\![j, k]\!] = [j, k] \cap \mathbb{N}$ denote the set of integers from j to k (with $[\![j, k]\!] = \emptyset$ if $k < j$). Given $k \in \mathbb{N}$, we denote by $D(k) = \{l \in \mathbb{N} : l \mid k\}$ the set of divisors of k. We now state our first theorem, which resonates with Moulin [1983].

Theorem 2. *For $m > \rho > 2$, if $D(n) \cap [\![2, m]\!] \neq \emptyset$, there is no CAF that is anonymous and object neutral.*

Lemma 1. *For $m > \rho \geq 2$, if $D(n) \cap [\![2, \rho]\!] \neq \emptyset$, there is no CAF that is anonymous and object neutral.*

Proof. Let α be an anonymous and object neutral CAF. Suppose for a contradiction that some number $k \in \llbracket 2, \rho \rrbracket$ divides n. We partition N with subsets $(N_1, \dots N_k)$ of equal size. Let π be the permutation of $\llbracket 1, k \rrbracket$ such that $\forall i \in \llbracket 1, k \rrbracket, \pi(i) = (i \bmod k) + 1$. For $j \in \mathbb{N}^*$ we will use $\pi^j = \pi \circ \pi^{j-1}$ defined by induction with $\pi^0 = id$. One can see that $\pi^k = id$.

We define a classification profile $\boldsymbol{c}$ as follows. In this profile, each voter in a given set N_j, with $1 \leq j \leq k$, has the same classification. We denote this classification with c_{N_j}. Construct profile $\boldsymbol{c}$ as follows:

object	$c_{N_j}, 1 \leq j \leq k$
$x_i, 1 \leq i \leq k$	$p_{\pi^{i-1}(j)}$
$x_i, k+1 \leq i \leq \rho$	p_i
$x_i, \rho+1 \leq i \leq m$	p_ρ

Formally, $\forall i, j \in \llbracket 1, k \rrbracket, \forall l \in N_j, c_l(x_i) = p_{\pi^{i-1}(j)}, \forall i \in \llbracket k+1, \rho \rrbracket, \boldsymbol{c}_{x_i} = (p_i, \dots p_i)$, and $\forall i \in \llbracket \rho+1, m \rrbracket, \boldsymbol{c}_{x_i} = (p_\rho, \dots, p_\rho)$.

Let $\gamma \in S_X$ be the permutation defined as $\forall i \in \llbracket 1, k \rrbracket, \gamma(x_i) = x_{\pi(i)}, \forall i \in \llbracket k+1, m \rrbracket, \gamma(x_i) = x_i$. We define $\boldsymbol{c} \circ \gamma$ as the classification profile induced from applying γ to $\boldsymbol{c}$ thus, for $i \in \llbracket 1, m \rrbracket, \boldsymbol{c}^{(\gamma)}(x_i) = \boldsymbol{c}(\gamma(x_i))$. It follows that $(c \circ \gamma)_{N_j}(x_i) = c_{N_j}(x_{\pi(i)})$.

object	$(c \circ \gamma)_{N_j}, 1 \leq j \leq k$
$x_i, 1 \leq i \leq k$	$c_{N_j}(x_{\pi(i)}) = p_{\pi^{i-1}(\pi(j))}$
$x_i, k+1 \leq i \leq \rho$	p_i
$x_i, \rho+1 \leq i \leq m$	p_ρ

For $1 \leq i \leq k-1$, we have $c_{N_j}(x_{\pi(i)}) = c_{N_j}(x_{i+1}) = p_{\pi^i(j)}$. And for $i = k$, we have $c_{N_j}(x_{\pi(i)}) = c_{N_j}(x_{\pi(k)}) = c_{N_j}(x_1) = p_{\pi^0(j)} = p_{\pi^k(j)} = p_{\pi^i(j)}$ as π is a cycle of size k. Also, $p_{\pi^i(j)} = p_{\pi^{i-1}(\pi(j))}$.

It follows that $\forall j \in \llbracket 1, k \rrbracket, (c \circ \gamma)_{N_j} = c_{N_{\pi(j)}}$, thus, $\boldsymbol{c} \circ \gamma$ can be obtained by a permutation of the individuals in $\boldsymbol{c}$.

One can check that $\boldsymbol{c}, \boldsymbol{c} \circ \gamma \in \mathcal{C}^N$. By anonymity, we must have $\alpha(\boldsymbol{c}) = \alpha(\boldsymbol{c} \circ \gamma)$. Now, for $z \in X \setminus \{x_1, \dots, x_\rho\}, \boldsymbol{c}_z = \boldsymbol{c}_{x_\rho}$, thus if we swap object x_ρ and object z in $\boldsymbol{c}$, the classification profile remains the same. Therefore object neutrality imposes, $\forall z \in X \setminus \{x_1, \dots, x_\rho\}, \alpha(\boldsymbol{c})(z) = \alpha(\boldsymbol{c})(x_\rho)$. Again, by object neutrality and by definition of $\boldsymbol{c} \circ \gamma, \forall j \in \llbracket 1, k \rrbracket, \alpha(\boldsymbol{c} \circ \gamma)(x_j) = \alpha(\boldsymbol{c})(x_{\pi(j)})$.

All together, we can see that $\forall j \in \llbracket 1, k \rrbracket, \alpha(\boldsymbol{c})(x_j) = \alpha(\boldsymbol{c} \circ \gamma)(x_j) = \alpha(\boldsymbol{c})(x_{\pi(j)}) = \ldots = \alpha(\boldsymbol{c})(x_1)$. Therefore, $|\{\alpha(\boldsymbol{c})(x), x \in X\}| \leq \rho - k + 1 < \rho$ as $k > 1$, so $\alpha(\boldsymbol{c})$ is not surjective, which is a contradiction. $\square$

Lemma 2. *For $m > \rho > 2$, if $D(n) \cap [\![\rho+1, m]\!] \neq \emptyset$, there is no CAF that is anonymous and object neutral.*

Proof. Suppose for a contradiction that $[\![\rho+1, m]\!]$ contains a divisor k of n and there exists a CAF α that is anonymous and object neutral. Let $(N_1, \dots N_k)$ be any partition of N with subsets of equal size.

Let π be the permutation of $[\![1, k]\!]$ such that $\forall i \in [\![1, k]\!], \pi(i) = (i \bmod k)+1$. Also, for $j \in \mathbb{N}$, $\pi^j = \pi \circ \pi^{j-1}$ with $\pi^0 = id$.

Construct $\boldsymbol{c}$ such that:

object	$c_{N_j}, 1 \leq j \leq k$
$x_i, 1 \leq i \leq k$	$p_{\min(\pi^{i-1}(j),\rho)}$
$x_i, k+1 \leq i \leq m$	p_1

Formally, $\forall i, j \in [\![1, k]\!], \forall l \in N_j, c_l(x_i) = p_{\min(\pi^{i-1}(j),\rho)}$, and $\forall \rho < i \leq m, \boldsymbol{c}_{x_i} = (p_1, \dots, p_1)$.

We next introduce $\gamma \in S_N$ as $\forall j \in [\![1, m]\!], \gamma(N_j) = N_{\pi(j)}$ and consider the classification profile $\boldsymbol{c}^{(\gamma)}$. We have $\forall i \in [\![1, k]\!], \boldsymbol{c}^{(\gamma)}_{x_i} = \boldsymbol{c}_{x_{\pi(i)}}$. Indeed, $\forall i, \in [\![1, k]\!], c^{(\gamma)}_{N_j}(x_i) = c_{N_{\pi(j)}}(x_i) = p_{\min(\pi^{i-1}(\pi(j)),\rho)} = c_{N_j}(x_{\pi(i)})$.

object	$c^{(\pi)}_{N_j}, 1 \leq j \leq k$
$x_i, 1 \leq i \leq k$	$p_{\min(\pi^{i}(j),\rho)}$
$x_i, k+1 \leq i \leq m$	p_1

One can check that both $\boldsymbol{c}$ and $\boldsymbol{c}^{(\gamma)}$ are surjective. Then by applying anonymity, we get $\alpha(\boldsymbol{c}) = \alpha(\boldsymbol{c}^{(\gamma)})$. Moreover, object neutrality entails that $\forall z \in X \setminus \{x_1, \dots, x_k\}, \alpha(\boldsymbol{c})(x_z) = \alpha(\boldsymbol{c})(x_{k+1})$ and $\forall i \in [\![1, k]\!], \alpha(\boldsymbol{c}^{(\gamma)})(x_i) = \alpha(\boldsymbol{c})(x_{\pi(i)})$.

All together, these results imply that $\forall i \in [\![1, k]\!], \alpha(\boldsymbol{c})(x_i) = \alpha(\boldsymbol{c})(x_1)$. Moreover, object neutrality imposes that $\forall z \in X \setminus \{x_1, \dots, x_k\}, \alpha(\boldsymbol{c})(z) = \alpha(\boldsymbol{c})(x_{k+1})$. Therefore, $|\{\alpha(\boldsymbol{c}(x), x \in X\}| \leq 2 < \rho$, so $\alpha(\boldsymbol{c})$ is not surjective, a contradiction. □

Theorem 3. *For $m > \rho \geq 3$, if $D(n) \cap [\![2, \rho-1]\!] \neq \emptyset$, there is no CAF that is anonymous and category neutral.*

Proof. We prove this result towards a contradiction, suppose $[\![2, \rho-1]\!]$ contains a divisor k of n and there exists a CAF α that is anonymous and object neutral. Let $(N_1, \dots N_k)$ be any partition of N with subsets of equal size. Let π be the permutation of $[\![1, k]\!]$ such that $\forall i \in [\![1, k]\!], \pi(i) = (i \bmod k) + 1$.

We define a classification profile $\boldsymbol{c}$ as follows, $\forall i, j \in [\![1, k]\!], \forall l \in N_j, c_l(x_i) = p_{\pi^{i-1}(j)}, \forall i \in [\![k+1, \rho]\!], \boldsymbol{c}_{x_i} = (p_i, \dots p_i)$, and $\forall \rho < i \leq m, \boldsymbol{c}_{x_i} = (p_\rho, \dots, p_\rho)$. Note that for $j \in [\![1, k]\!]$ all $(c_l)_{l \in N_j}$ are the same, we denote this classification with c_{N_j}.

Now, let $\gamma \in S_P$ s.t. $\forall i > k, \gamma(p_i) = p_i$ and $\forall i \in [\![1, k]\!], \gamma(p_i) = p_{\pi(i)}$ and consider the profile $\gamma \circ \boldsymbol{c}$.

object	$(\gamma \circ c)_{N_j}, 1 \leq j \leq k$
$x_i, 1 \leq i \leq k$	$p_{\pi^i(j)}$
$x_i, k+1 \leq i \leq \rho$	$\gamma(p_i) = p_i$
$x_i, \rho+1 \leq i \leq m$	p_ρ

One can check that $\boldsymbol{c}, \gamma \circ \boldsymbol{c} \in \mathcal{C}^N$ and that $\forall i \in [\![1, k]\!], (\gamma \circ c)_{N_i} = c_{N_{\pi(i)}}$. By anonymity, we must have $\alpha(\boldsymbol{c}) = \alpha(\gamma \circ \boldsymbol{c})$. Also, by category neutrality, we have $\forall x \in X, \alpha(\gamma \circ \boldsymbol{c})(x) = \gamma(\alpha(\boldsymbol{c})(x))$.

All together, we can see that $\forall x \in X, \alpha(\boldsymbol{c})(x) = \gamma(\alpha(\boldsymbol{c})(x))$, so $\alpha(\boldsymbol{c})(x) \notin \{p_1, \dots, p_k\}$. Therefore, $\alpha(\boldsymbol{c})$ is not surjective, a contradiction. $\square$

6. Defining anonymous and neutral CAFs

We proved in the previous sections that unless 1 is the only divisor of n that is smaller than m, there cannot exist an anonymous and neutral (object neutral nor category neutral) CAF. The greedy algorithm from section 4 can define CAFs that are anonymous and neutral for all other cases if $m = \rho$. As these CAFs are very artificial and only defined when $m = \rho$, we try to define some that have some explanation and can work even when $m > \rho$. We managed to implement an anonymous and neutral CAF that is defined only under a certain condition μ (defined below). This condition implies that 1 is the only divisor of n that is smaller than m but there is no equivalence. This leaves a gap where we were not able to define such CAFs and neither to prove any impossibility.

Given $m, \rho \in \mathbb{N}, m > \rho$ we write $S_{m,\rho} = \sum\limits_{i=0}^{\rho} \binom{\rho}{i} \times (-1)^{\rho-i} \times i^m$ to denote the Stirling number of the second kind associated to (m, ρ). For work on Stirling numbers, we refer to Rennie and Dobson [1969], what is important to know here is that $S_{m,\rho}$ is the number of surjective mappings from a set of size m to a set of size ρ. Thus, $|\mathcal{C}| = \rho! \times S_{m,\rho}$.

We say that a tuple (m, n, ρ) satisfies condition μ if $\nexists(\lambda_k)_{0\leq k\leq m}$ s.t.

$$n = \sum_{k=0}^{m} k\lambda_k$$
$$\sum_{k=0}^{m} \lambda_k = m! \times S_{m,\rho}$$
$$\forall k \in [\![0, m]\!], \lambda_k \neq 1$$

Given a profile $\boldsymbol{c} \in \mathcal{C}^N$ we define a ranking of $\mathcal{C}$ as $\forall c' \in \mathcal{C}, \mathit{rank}(c') = |\{i \in N \mid c_i = c'\}|$. We define the equivalence relation $\sim$: $\forall c, c' \in \mathcal{C}, c \sim c'$ iff $\mathit{rank}(c) = \mathit{rank}(c')$. Given $c \in \mathcal{C}$, we set $K_c = \{c' \in \mathcal{C}, c \sim c'\}$ as the equivalence class of c according to $\sim$. Let $L = \underset{c\in\mathcal{C}}{\arg\min}\,|K_c|$. We define the rule $\alpha^\star$ that outputs $\underset{c\in L}{\arg\max}\,\mathit{rank}(c)$.

Theorem 4. *For $m > \rho \geq 3$, the rule $\alpha^\star$ is a CAF iff (m, n, ρ) satisfies condition μ.*

Note that if $D(n) \cap [\![2, m]\!] \neq \emptyset$, there exists a divisor k of n in $[\![2, m]\!]$ and we can set $\lambda_k = q$ where $n = kq, \lambda_0 = m! \times S_{m,\rho} - q$ and $\forall i \in [\![1, m]\!] \setminus \{k\}, \lambda_i = 0$. Then $\alpha^\star$ is not a CAF in the cases that are covered by theorem 2.

Proof. Let $\boldsymbol{c} \in \mathcal{C}^N$, and suppose $|\alpha^\star(\boldsymbol{c})| > 1$, so that $|\{c \in L : \forall c' \in L,\ \mathit{rank}(c) \geq \mathit{rank}(c')\}| > 1$. Then, $\forall c \in L, |K_c| > 1$. Given $k \in [\![0, m]\!]$, we set $\lambda_k = |\{c' \in \mathcal{C}, \mathit{rank}(c') = k\}|$. Then, we have $\forall k \in [\![0, m]\!], \lambda_k \neq 1$, also $\sum_{k=0}^{m} \lambda_k = |\mathcal{C}| = m! \times S_{m,\rho}$ and $\sum_{k=0}^{m} k\lambda_k = \sum_{c'\in\mathcal{C}} \mathit{rank}(c') = n$. Therefore, μ holds only if $\alpha^\star$ is a CAF.

If μ holds, as $\sum_{k=0}^{m} \lambda_k = |\mathcal{C}|$, and $\sum_{k=0}^{m} k\lambda_k = n$, we can define a partition $\mathcal{C}_0, \ldots, \mathcal{C}_m$ of $\mathcal{C}$ s.t. $\forall k \in [\![0, m]\!], |\mathcal{C}_k| = \lambda_k$. Then if we consider $\boldsymbol{c}$ s.t. $\forall k \in [\![0, k]\!], \forall c \in \mathcal{C}_k, \mathrm{rank}(c) = k$ then $|\alpha^\star(\boldsymbol{c})| > 1$ as $\forall k \in [\![0, k]\!], \lambda_k \neq 1$. Thus, $\alpha^\star$ is a CAF only if μ holds. $\square$

Theorem 5. *Under condition $\mu, \alpha^\star$ satisfies anonymity, unanimity, object neutrality and category neutrality.*

Proof. Let $\boldsymbol{c} \in \mathcal{C}^N$ be a classification profile and $\boldsymbol{c}' \in \mathcal{C}^N$ be any profile obtained from a permutation of the individuals in $\boldsymbol{c}$. Then $\{c_i, i \in N\} = \{c'_i, i \in N\}$, so L is the same in both cases. Then, $\alpha^\star(\boldsymbol{c}) = \alpha^\star(\boldsymbol{c}')$.

Let $c \in \mathcal{C}$, and $\boldsymbol{c} = (c, \ldots, c)$ be the profile where all individuals pick c. Then $L = \{c\}$, so $\alpha^\star(\boldsymbol{c}) = c$.

Let $\boldsymbol{c}, \boldsymbol{c}' \in \mathcal{C}^N$ s.t. $\exists \pi \in S_X, \boldsymbol{c}' = \boldsymbol{c} \circ \pi$. We denote by $L^{(\boldsymbol{c})}$ and $L^{(\boldsymbol{c}')}$ the sets of equivalence classes of smallest sizes for $\boldsymbol{c}$ and $\boldsymbol{c}'$. Let $c \in \mathcal{C}$, we must have $\mathit{rank}_{\boldsymbol{c}}(c) = \mathit{rank}_{\boldsymbol{c}'}((c_{\pi(x)})_{x\in X})$. Then, $c \in L^{(\boldsymbol{c})}$ *iff* $(c_{\pi(x)})_{x\in X} \in L^{(\boldsymbol{c}')}$.

Let $\boldsymbol{c}, \boldsymbol{c}' \in \mathcal{C}^N$ s.t. $\exists \pi \in S_P, \boldsymbol{c}' = \pi \circ \boldsymbol{c}$. We denote by $L^{(c)}$ and $L^{(c')}$ the sets of equivalence classes of smallest sizes for $\boldsymbol{c}$ and $\boldsymbol{c}'$. Let $c \in \mathcal{C}$, we must have $rank_{\boldsymbol{c}}(c) = rank_{\boldsymbol{c}'}(\pi \circ c)$. Then, $c \in L^{(\boldsymbol{c})}$ *iff* $\pi \circ c \in L^{(\boldsymbol{c}')}$. □

Table 1: Possibility of having anonymous and neutral aggregators for $\rho = 2$ (n in columns, m in lines)

	2	3	4	5	6	7	8	9	10
2	no	yes[1]	no	yes[1]	no	yes[1]	no	yes[1]	no
3	no	no	no	yes[2]	no	yes[2]	no	no	no
4	no	no	no	yes[3]	no	?	no	no	no
5	no	no	no	no	no	?	no	no	no

[1] From corollary 3 [2] From theorem 5 [3] From appendix A

Table 2: Possibility of having anonymous and neutral aggregators for $\rho = 3$ (n in columns, m in lines)

	2	3	4	5	6	7	8	9	10
3	no	no	no	yes[1]	no	yes[1]	no	no	no
4	no	no	no	?	no	?	no	no	no
5	no	no	no	no	no	?	no	no	no

[1] From corollary 3 [2] From theorem 5

As condition μ does not imply that there is a non 1 divisor of n that is smaller than m, we have a gap where $\alpha^\star$ is not a CAF and theorem 2 does not hold. Tables 1 and 2 allow us to make an estimation of the gap for small sizes of n, m and ρ. The cases where $m = \rho$ are covered by Corollary 3, $\alpha^\star$ is a CAF when μ holds (for exmple when $n = |\mathcal{C}| - 1$) and theorem 2 takes out some other some other parameter sizes (like $(m, n, \rho) = (4, 2, 10)$, but there are still lots of parameter sizes that are not covered. We illustrate the smallest in appendix A.

In order to fill the gap, we can try to define an anonymous object neutral and category neutral CAF using an algorithm similar to algorithm 2 where we also try every possible classification for the neighborhoods of remaining profile. The problem is, this algorithm is exponential in m, n and ρ so hard to compute when any parameter becomes big.

7. Conclusion

We proved that depending on the number of voters, objects and categories, an anonymous and neutral (object neutral or category neutral) CAF might not exist, obtaining a similar result to Moulin [1983]. We were able to fully characterize this impossibility when the number of objects and categories are the same by using an isomorphism between preference and classification profiles and using Doğan and Giritligil's [2022] results. When there are more objects than categories, we were only able to find an anonymous and neutral aggregator for a smaller domain than the complementary of our impossibility domain. Filling this gap would be an interesting question, it appears that some anonymous and neutral aggregators could still exist within this gap as illustrated by appendix A.

The anonymous and neutral aggregator that we found is not pareto optimal and it seems impossible to define an aggregator that would be pareto optimal, anonymous and neutral, as one could have guessed by Moulin's [1983] result. Even more, it might select classifications that are not even supported by any voter, making a similarity with Campbell and Kelly's [2015] claim that in the context on preference aggregation, anonymous and neutral aggregators might select bottom-ranked alternatives.

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A. Example of a CAF that is anonymous and neutral.

In this example, we consider $m = 4, \rho = 2$ and $n = 5$. Let $P = \{p, q\}$, given $q \in P$ we write $\|c\|_q = |\{x \in X, c_x = q\}|$, to each profile we associate the q-vector $(\|c_i\|q)_{i\in N}$.

We introduce a CAF ($\hat{\alpha}$) based on MAJ (defined in Craven [2023]):

Given a profile $\boldsymbol{c} \in \mathcal{C}^N$

- If MAJ is consistent (it outputs a surjective classification for $\boldsymbol{c}$), we take this output. (If MAJ is not consistent, we suppose wlog that $\mathrm{MAJ}(\boldsymbol{c}) = (p, p, p, p)$, then in total at most 8 objects where put in category q by some individual.)

- $\exists! i \in N, \|c_i\|_q = k$ for some $k \in [\![1, 3]\!]$:

  We output $\arg\max_{c_i, i\in N}\{k, \exists! i \in N, \|c_i\|_q = k\}$.

- The q-vector is a permutation of (1,1,2,2,2):

  We apply MAJ on the profile restricted to the individuals $\{i \in N, \|c_i\|_q = 2\}$, in this case, both p, q received 6 votes in total so MAJ must be consistent on this profile.

- The q-vector is a permutation of (1,1,1,2,2) or (1,1,1,1,1):

  In this case, there are 7 or 5 votes for q in total, we can apply the following rule: For $x \in X$, if $|\{i \in N, c_i(x) = q\}| \geq 2$ we output q and p otherwise. One can check that this is consistent.

All other cases are already covered as for MAJ not to be consistent, we need that q received at most 8 votes and each individual needs to give a surjective mapping.

Proposition 4. *$\hat{\alpha}$ is anonymous, object neutral and category neutral.*

Proof. We will prove this for each possible case, let $\boldsymbol{c} \in \mathcal{C}^N, \pi \in S_X, \sigma \in S_P$ and $\gamma \in S_N$.

- If MAJ is consistent on $\boldsymbol{c}$, it will also be consistent on $\boldsymbol{c}^{(\gamma)}, \boldsymbol{c} \circ \pi$ and $\sigma \circ \boldsymbol{c}$. As MAJ is anonymous, object neutral and category neutral, we will have $\hat{\alpha}(\boldsymbol{c}^{(\gamma)}) = \hat{\alpha}(\boldsymbol{c}), \hat{\alpha}(\boldsymbol{c} \circ \pi) = \hat{\alpha}(\boldsymbol{c}) \circ \pi$ and $\hat{\alpha}(\sigma \circ \boldsymbol{c}) = \sigma \circ \hat{\alpha}(\boldsymbol{c})$.

- $\exists! i \in N, \|c_i\|_q = k$ for some $k \in [\![1,3]\!]$:

  Any permutation on voters, object or categories will not change this property (q is defined as the category that is not represented in MAJ), then this also holds for $\boldsymbol{c}^{(\gamma)}, \boldsymbol{c} \circ \pi$ and $\sigma \circ \boldsymbol{c}$.

  It is clear that selecting $\arg\max_{c_i, i \in N}\{k, \exists! i \in N, \|c_i\|_q = k\}$ is anonymous and object neutral, and $\arg\max_{c_i, i \in N}\{k, \exists! i \in N, \|c_i\|_{\sigma(q)} = k\} = \arg\max_{c_i, i \in N}\{k, \exists! i \in N, \|c_i\|_q = k\}$.

- The q-vector is a permutation of (1,1,2,2,2): Any permutation on voters, object or categories will not change this property (q is defined as the category that is not represented in MAJ), then this also holds for $\boldsymbol{c}^{(\gamma)}, \boldsymbol{c} \circ \pi$ and $\sigma \circ \boldsymbol{c}$. We will have $\hat{\alpha}(\boldsymbol{c}^{(\gamma)}) = \hat{\alpha}(\boldsymbol{c}), \hat{\alpha}(\boldsymbol{c} \circ \pi) = \hat{\alpha}(\boldsymbol{c}) \circ \pi$ and $\hat{\alpha}(\sigma \circ \boldsymbol{c}) = \sigma \circ \hat{\alpha}(\boldsymbol{c})$ by the same arguments as for the last two cases.

- The q-vector is a permutation of (1,1,1,2,2) or (1,1,1,1,1): Any permutation on voters, object or categories will not change this property (q is defined as the category that is not represented in MAJ), then this also holds for $\boldsymbol{c}^{(\gamma)}, \boldsymbol{c} \circ \pi$ and $\sigma \circ \boldsymbol{c}$.

  Now, given $x \in X$, $|\{i \in N, (c \circ \pi)_i(x) = q\}| \geq 2$ iff $|\{i \in N, c_i(\pi(x)) = q\}| \geq 2$, $|\{i \in N, (c^{(\gamma)})_i(x) = q\}| \geq 2$ iff $|\{i \in N, c_i(x) = q\}| \geq 2$ and $|\{i \in N, (\sigma \circ c)_i(x) = q\}| \geq 2$ iff $|\{i \in N, c_i(x) = \sigma(q)\}| \geq 2$. Then we have $\hat{\alpha}(\boldsymbol{c}^{(\gamma)}) = \hat{\alpha}(\boldsymbol{c}), \hat{\alpha}(\boldsymbol{c} \circ \pi) = \hat{\alpha}(\boldsymbol{c}) \circ \pi$ and $\hat{\alpha}(\sigma \circ \boldsymbol{c}) = \sigma \circ \hat{\alpha}(\boldsymbol{c})$.

$\square$