Contents

1 Bases of a Vector Space

Definition 1

Let E be a vector space over a field K, let $n \in \mathbb{N}$, and let $\{x_1, x_2, \ldots, x_n\}$ be a family of vectors in E. We say that the family $\{x_1, x_2, \ldots, x_n\}$ is a **basis** of E if it is both **linearly independent** and **spanning** (generating) for E.

Theorem 1 (Characterization of a Basis)

Let E be a vector space over K, let $n \in \mathbb{N}$, and let $\{x_1, x_2, \ldots, x_n\}$ be a family of vectors in E. Then $B = \{x_1, x_2, \ldots, x_n\}$ is a basis of E if and only if:

$$\forall x \in E,\ \exists!(\lambda_1, \lambda_2, \ldots, \lambda_n) \in K^n \text{ such that } x = \sum_{i=1}^{n} \lambda_i x_i.$$

The scalars λ_i, $i \in \{1, 2, \ldots, n\}$ are called the **coordinates** of the vector x in the basis B.

2 Existence of a Basis

Theorem 2

Every non-zero vector space E over a field K admits at least one basis.

Theorem 3 (Incomplete Basis Theorem)

Let E be a vector space over K that admits a finite generating family. Then:

1. From every generating family of E, one can extract a basis of E.
2. Every linearly independent family of E can be extended to a basis of E.

Definition 2

In $\mathbb{R}^n$, the family $(e_1, \ldots, e_n)$ where for every $i \in \{1, \ldots, n\}$,

$$e_i = (0, \ldots, 0, \underbrace{1}_{i\text{-th coordinate}}, 0, \ldots, 0)$$

is a basis. It is called the **canonical basis** of $\mathbb{R}^n$.

Definition 3

In $\mathbb{R}_n[X]$, the family $(1, X, \ldots, X^n)$ is a basis. It is called the **canonical basis** of $\mathbb{R}_n[X]$.

3 Dimension of a Vector Space

Definition 1

Let E be a vector space over a field K. We say that E is of **finite dimension** if E admits a finite generating family.

Theorem 1

Let $E \neq \{0_E\}$ be a finite-dimensional vector space over K. Then:

1. E admits at least one basis.
2. All bases of E have the same cardinality.

Definition 2

Let E be a finite-dimensional vector space over K.

1. If $E \neq \{0_E\}$, we call the **dimension** of E, denoted $\dim(E)$, the cardinality of any basis of E.
2. If $E = \{0_E\}$, we define $\dim(E) = 0$.

Examples: $\dim(\mathbb{R}^n) = n$ and $\dim(\mathbb{R}_n[X]) = n + 1$.

Proposition

Let E a $\mathbb{K}$ vector space such that dim(E) = n, n$\in \mathbb{N}*$ and B a family of p vectors, $p \in \mathbb{N}*$.

Then :

1. B is linearly independant $\implies p \leq n$
2. B is a spanning family of E $\implies p \geq n$
3. B is a basis of E $\implies (p \leq n) \wedge (p \geq n) \Leftrightarrow p = n$

4. (p = n) $\implies$ (B is linearly independant)

Examples: We have $\{1, X-1, (X+1)^2\} = \mathbb{R}_2[X]$ and $dim(\mathbb{R}_2[X]) = 3$

Then :

1. $Card(\{1, X-1, (X+1)^2\}) = 3 = dim(\mathbb{R}_2[X])$
2. B is linearly independant

Proof that B is linearly independant :

$$\begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} : \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \implies a = b = c = 0 \implies a + b \times (X-1) + c \times (X+1)^2 = 0_{\mathbb{R}_2[X]}$$

Examples:

$F = span(u_1 =, u_2, u_3), \forall i \in [\,|1,3|\,], \forall\, u_i$ from $\mathbb{R}^{\not\models}$

Find values of t such that $dim(F) = 3$, where :

$$u_1 = (0, t^2, 1)$$
$$u_2 = (0, 0, 3)$$
$$u_3 = (1, 1, 3)$$

$dim(F) = 3 \Leftrightarrow F = E$ because F is a linear subspace of E.

We are looking for a value of B such that $span(B) = \mathbb{R}^3 \implies$ we are looking for t such that B is a basis of $\mathbb{R}^3$ since $Card(B) = 3 = dim(E) \implies$ we are looking for t such that B is linearly independant.

$$\Leftrightarrow \begin{pmatrix} 0 & 0 & 1 \\ t^2 & 0 & 1 \\ 1 & 3 & 3 \end{pmatrix} : \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} 1 & 3 & 3 \\ t^2 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} : \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} 1 & 3 & 3 \\ 0 & -3t^2 & 1-3t^2 \\ 0 & 0 & 1 \end{pmatrix} : \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Case t = 0 :
$(a_1, a_2, a_3) \neq (0, 0, 0)$

Case t $\neq$ 1 :
Kramer system : $a_1 = a_2 = a_3 = 0$

4 Linear Maps

Definition :

Let A,B two $\mathbb{K}$ vector space and

$$f : \begin{cases} A \to B \\ \alpha \mapsto f(x) \end{cases}$$

α map from A to B. Then :

$$f \text{ is linear} \Leftrightarrow \begin{cases} \forall(\alpha, u) \in \mathbb{K} \times A, f(\alpha u) = \alpha f(u) \\ \forall(u, v) \in A^2, f(u+v) = f(u) + f(v) \end{cases}$$

$$\Leftrightarrow \forall(\alpha, u, v) \in \mathbb{K} \times A^2, f(\alpha u + v) = \alpha f(u) + f(v)$$

Notation : f is a linear map from A to B $\Leftrightarrow f \in \mathcal{L}(A, B)$

Remark (necessary condition) : $f \in \mathcal{L}(A, B) \implies f(0_A) = O_B$

Contrapositive : $f(0_A) \neq O_B \implies f \notin \mathcal{L}(A, B)$

Definition :

Let $f \in \mathcal{L}(A, B)$ with A and B two $\mathbb{K}$ vector space. Then :

- We call $Ker(f) = \{X \in A, f(X) = 0_B\} \subset A$
- We call $Im(f) = \{Y \in B, f(X) = Y\} = \{f(X), X \in A\} \subset B$

Proposition :

1. $Ker(f)$ is a linear subspace of A
2. $Im(f)$ is a linear subspace of B

Definition :

Let $f \in \mathcal{L}(A, B)$.

$$\begin{cases} f \text{ is injective} \Leftrightarrow Ker(f) = \{0_A\} \\ f \text{ is surjective} \Leftrightarrow Im(f) = B \end{cases}$$

Proof :

$$f \text{ is surjective } \implies \forall Y \in B, \exists x \in A, f(X) = Y$$
$$\implies B \subset Im(f) \wedge Im(f) \subset B \text{ by definition } \implies Im(f) = B$$

5 Determinant

5.1 Determinant of order 2

A determinant of order 2 is the determinant of a matrix from $\mathbb{M}_2(\mathbb{R})$

Let $A \in \mathbb{M}_2(\mathbb{R})$,

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

We call determinant of A and denote $det(A)$ the real number :

$$det(A) = a_{11} \times a_{22} - a_{21} \times a_{12}$$

5.2 Determinant of order 3

A determinant of order 3 is the determinant of a matrix from $\mathbb{M}_3(\mathbb{R})$

Let $A \in \mathbb{M}_3(\mathbb{R})$,

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

We call determinant of A and denote $det(A)$ the real number :

$$det(A) = a_{11} \times \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \times \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} - a_{13} \times \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

5.3 General determinant for $A \in \mathbb{M}_n(\mathbb{R})$

Let $A = (a_{ij})_{1 \leq i \leq n \wedge 1 \leq j \leq n}$. We can minor of indices (i, j) the determinant Δ_{ij} of the $(n-1) \times (n-1)$matrix resulting from remaining row i and column j from A.

Then :

$$det(A) = \sum_{i=1}^{n} (-1)^{i+j} \times \Delta_{ij} \times a_{ij} \text{if we can expand it with respect to column j}$$

$$det(A) = \sum_{j=1}^{n} (-1)^{i+j} \times \Delta_{ij} \times a_{ij} \text{if we can expand it with respect to row i}$$

Example :

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 4 & 5 & 6 \\ 6 & 5 & 4 \end{pmatrix}$$

$$det(A) = (-1)^{1+1} \times 1 \times \begin{vmatrix} 5 & 6 \\ 5 & 4 \end{vmatrix} + (-1)^{1+2} \times 2 \times \begin{vmatrix} 4 & 6 \\ 6 & 4 \end{vmatrix} (-1)^{1+3} \times 0 \times \begin{vmatrix} 4 & 5 \\ 6 & 5 \end{vmatrix}$$

$$\implies det(A) = -10 - 2 \times (-20) = 30$$

5.4 Properties

1. Let $(A, B) \in \mathbb{M}_n^2(\mathbb{R}), A = (a_{ij})_{1 \leq i \leq n \wedge 1 \leq j \leq n}, B = (b_{ij})_{1 \leq i \leq n \wedge 1 \leq j \leq n}$

$$det(\lambda A) = \lambda^n det(A)$$

$$det(AB) = det(A) \times det(B)$$

2. $$det(A^T) = det(A)$$

3. When doing a row echelon for instance $\alpha R_1 + \beta R_2$:

$$det(A) = \alpha \times \beta \times det(C)$$

where C is the matrix that has been row echelon

5.5 Determinant and invertibility

Let $A \in \mathbb{M}_n(\mathbb{R})$:

$$A \text{ invertible} \Leftrightarrow \exists\, B \in \mathbb{M}_n(\mathbb{R})\ AB = BA = I_n.\ B \text{ is denoted } A^{-1}.$$

$$\implies |A \times A^{-1}| = |I_n| \implies |A||A^{-1}| = |I_n|$$

$$\implies det(A) \neq 0 \wedge det(A^{-1}) = \frac{1}{det(A)}$$

$$D : \begin{cases} \mathbb{M}_n(\mathbb{R}) \to \mathbb{R} \\ A \mapsto det(A) \end{cases}$$

where D is the mapping of the function of the determinant.

5.6 Determinant of some special matrices

Let $A \in \mathbb{M}_n(\mathbb{R})$

- A is diagonal $\Leftrightarrow (i \neq j \implies a_{ij} = 0)$

$$det(A) = \prod_{i=1}^{n} a_{ii}$$

- A is a upper-triangle matrix $\Leftrightarrow j < i \implies a_{ij} = 0$

$$det(A) = \prod_{i=1}^{n} a_{ij}$$

- A is a lower-triangle matrix $\Leftrightarrow j > i \implies a_{ij} = 0$

$$det(A) = \prod_{i=1}^{n} a_{ij}$$

Example

$$A = \begin{pmatrix} 9 & -4 & 2 \\ 21 & 8 & 3 \\ 25 & 0 & 5 \end{pmatrix}$$

$$det(A) = \begin{vmatrix} -1 & -4 & 2 \\ 6 & 8 & 3 \\ 0 & 0 & 5 \end{vmatrix} (C1{-}5C3) = \begin{vmatrix} -1 & -4 & 2 \\ 0 & -16 & 15 \\ 0 & 0 & 5 \end{vmatrix} (R2{+}6R1) = 5{\times}16 = 80$$

Or

$$det(A) = (-1)^{3+1} \times 25 \begin{vmatrix} -4 & 2 \\ 8 & 5 \end{vmatrix} + (-1)^{3+3} \times 5 \begin{vmatrix} 9 & -4 \\ 21 & 8 \end{vmatrix} = 80$$

Example

Find values of X such that A - XI is not invertible, $X \in \mathbb{R}$

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 1 & 3 & 2 \end{pmatrix} \implies A - XI = \begin{pmatrix} 1-X & 0 & 2 \\ 2 & 1-X & 0 \\ 1 & 3 & 2-X \end{pmatrix}$$

$$\left|A - XI\right| = \begin{vmatrix} 1-X & 0 & 2 \\ 2 & 1-X & 0 \\ 4-X & 4-X & 4-X \end{vmatrix} (R1{+}R2{+}R3) = (4{-}X) \begin{vmatrix} 1-X & 0 & 2 \\ 2 & 1-X & 0 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1-X & X-1 & 1+X \\ 2 & -1-X & -2 \\ 1 & 0 & 0 \end{vmatrix} = (4{-}X)((-2)(X{-}1){+}(1{+}X)^2) = (4{-}X)(X^2{+}3)$$

$$\mathbb{S} = \{4\}$$

6 Diagonalisation of endomorphism

Let $f \in \mathcal{L}(E)$ with E a $\mathbb{K}$ vector space of finite dimension :

$$Mat_{BB}(t) = A$$

$$A \in \mathbb{M}_n(\mathbb{R})$$

6.1 Definition : Eigen Value

Let $A \in \mathbb{M}_n(\mathbb{K})$ and $\lambda \in \mathbb{K}$, we call eigen value of A any scalar $\lambda \in \mathbb{K}$ such that $\exists\, U \in \mathbb{K}^n, A\vec{U} = \lambda\vec{U}$ Then we call U the eigen value associated with λ.
Notation : for a given matrix $A \in \mathbb{M}_n(\mathbb{K})$, $spectrum(A)$ is how we denote the set of eigen values of A.

6.2 Example

$\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$, $\lambda \in \mathbb{R}$ and $A = Mat_B(t), t \in \mathcal{L}(E), E = \mathbb{K}$ vector space, $dim(E) = 2$

$$\text{We look for } \begin{pmatrix}x\\ y\end{pmatrix}_B \mapsto \begin{pmatrix}x+2y\\ 3x+4y\end{pmatrix}$$

6.3 Definition : Eigen Space

Let $A \in \mathbb{M}_n\mathbb{K}$ and $\lambda \in spectrum(A)$. Then we call eigen space associated with λ : $E_\lambda = \{\vec{U} \in \mathbb{K}^n, A\vec{U} = \lambda\vec{U}\} \subset \mathbb{K}^n$

6.3.1 Proof

Let E_y a linear subspace of E :

- $(A) \times \vec{0} = \vec{0} \implies \vec{0} \in E_\lambda \implies E_y \neq \emptyset$
- $\forall(\alpha, \vec{U}, \vec{V}) \in \mathbb{K} \times E_\lambda^2, A(\alpha\vec{U} + \vec{V}) = A(\alpha\vec{U}) + A(\vec{V}) = \lambda\alpha\vec{U} + \lambda\vec{V} = \lambda(\alpha\vec{U} + \vec{V}) \implies \alpha\vec{U} + \vec{V} \in \mathbb{E}_\lambda$
- Thus, E_y is not empty and is closed by linear combination $\implies E_y$ is a linear subspace of E

Defintion 2 : Eigen Space

Let E a $\mathbb{K}$ vector space of eigen vectors and $A \in \mathbb{M}_n(\mathbb{R})$ the matrix of an endomorphism of E; λ an eigen value of A. The we call eigen space associated with λ :

$$E_\lambda = Ker(A - \lambda I)$$

7 Diagonalisability

7.1 Characteristic polynomial

7.1.1 Definition

Let $A \in \mathbb{M}_n(\mathbb{K})$. We call characteristic polynomial and denote $P_A(X)$ the following polynomial :

$$P_A(X) = det(A - XI)$$

7.1.2 Proposition

Let $A \in \mathbb{M}_n(\mathbb{K})$. We call $spectrum(A)$ the following set :

$$spectrum(A) = \{\lambda \in k, P_A(\lambda) = 0\}$$

Example

Let $A \in \mathbb{M}_n(\mathbb{K})$, such that :

$$A = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$P_A(X) = \begin{vmatrix} -X & 1 & 1 \\ -1 & 2-X & 1 \\ 0 & 0 & 1-X \end{vmatrix}$$

$$= (-1)^{3+3} \times (1-X) \times \begin{vmatrix} -X & 1 \\ -1 & 2-X \end{vmatrix}$$

$$= (1-X)(-2X + X^2 + 1) = (1-X)(X-1)^2 = (1-X)^3$$

7.1.3 Defintion

Let $f \in \mathcal{L}(E)$ and $A = Mat(f)$ in the standard basis. We say A is diagonalisable if there exists an eigen basis of f in E, i.e a basis $B_e = (u_1, ..., u_n)$ such that $\forall i \in [|1, n|], f(u_i) = \lambda_i U_i, \lambda_i \in \mathbb{K}$

7.1.4 Definition 2

Let $f \in \mathcal{L}(E)$ and $A = Mat(f)$ in the standard basis. A is diagonalisable if $\exists P \in M_n(\mathbb{R})$, **invertible** such that :

$$P^{-1}AP = D, \text{ when D is diagonal}$$

7.1.5 Theorem

Let $f \in \mathcal{L}(E)$ and $A = Mat_B(f)$.

Then the following proposition are equivalent :

- A is diagonalisable
- There exists an eigen basis for f in E
- $\sum_{\lambda \in spectrum(A)} dim(E_\lambda) = dim(E)$
- $\sum_{\lambda \in spectrum(A)} E_\lambda = E_{\lambda_1} + E_{\lambda_2} + ... + E_{\lambda_n} = E$

7.1.6 Example

Let :

$$A = \begin{pmatrix} 3 & -1 & 1 \\ 0 & 2 & 0 \\ 1 & -1 & 3 \end{pmatrix} \implies P_A(X) \begin{vmatrix} 3-X & -1 & 1 \\ 0 & 2-X & 0 \\ 1 & -1 & 3-X \end{vmatrix}$$

$$= (-1)^(2+2) \times (2-X) \times ((3-X)^2 - 1) = ((2-X)(3-X-1)(3-X+1))$$

$$= (2-X)^2(4-X) \implies spectrum(A) = \{2, 4\}$$

Thus : $E_2 = \{x \in E, AX = 2X\} = Ker(A - 2I)$

$$\implies \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 1 & -1 & 1 \end{pmatrix} : \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow x - y + z = 0 \implies \{ \begin{pmatrix} y-z \\ 0 \\ z \end{pmatrix}_B, (y,z) \in \mathbb{R}^2\}$$

$$= span(\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}_B, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}_B) \implies dim(E_2) = 2$$

Also : $E_4 = (Ker(A-4I)) = Ker \begin{pmatrix} -1 & -1 & 1 \\ 0 & -2 & 0 \\ 1 & -1 & -1 \end{pmatrix}$

$$\begin{pmatrix} 1 & 1 & -1 \\ 0 & -2 & 0 \\ 1 & -1 & -1 \end{pmatrix} : \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} 1 & 1 & -1 \\ 0 & -2 & 0 \\ 0 & -2 & 0 \end{pmatrix} : \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & -1 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix} : \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \implies \begin{cases} x = z \\ y = 0 \\ z \in \mathbb{R} \end{cases} \implies E_4 = span(\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix})$$

$\implies dim(E_1) = 1 \implies dim(E_1)+dim(E_2) = 1+2 = 3 = dim(E) \implies A$ is diagonalisable.