Mathématiques : CM

$$(\Omega, P(\Omega), \mathbb{P}) \Leftrightarrow$$

A random variable X is defined by:

$X : \Omega \to \mathbb{R}$ such that $\omega \mapsto X(\omega)$

The generative function is defined by :

$$P : \mathbb{R} \to \mathbb{R} \quad t \mapsto a_0 * t^0 + a_1 * t^1 + a_2 * t^2 + ... + a_n * t^n$$

{(X = k), k $\in X(\Omega)$ } = partition of Ω

Generating Function

Definition :

Let X a random variable such that $X(\Omega) = [\,|0, n|\,], n \in \mathbb{N}^*$.

we call generating function of X the following polynomial :

$$G_X : \begin{cases} \mathbb{R} \to \mathbb{R} \\ G \mapsto \sum_{k \in X(\Omega)} P(X = k) * t^k \end{cases}$$

Remarks :

$\alpha(\Omega)$ can be different we can have : X(Ω) $\subset [\,|\alpha, n|\,]$

Bernouilli :

$$X \sim B(P) \Leftrightarrow \begin{cases} X(\Omega) = \{0,1\} \\ P(X=0) = (1-p), P(X=1) = p \end{cases}$$

$$\Leftrightarrow G_X \begin{cases} \mathbb{R} \to \mathbb{R} \\ t \mapsto (1-p) + pt \end{cases}$$

Expected value and variance :

Theorem :

Let X a random variable and G_X its generative function :

$$\begin{cases} G_X(1) = 1 \\ \mathbb{E}(X) = G'_X(1) \\ \mathbb{V}(X) = G''_X(1) + G'_X(1) - (G'_X(1))^2 \end{cases}$$

Proof :

By definition :

$$G_X : \begin{cases} \mathbb{R} \to \mathbb{R} \\ t \mapsto \sum_{k=0}^{n} P(X=k) * t^k \end{cases}$$

$$\Rightarrow G_X(1) = \sum_{k=0}^{n} P(X=k) = 1 \text{ because } \{(X=k), k \in [\,|0,n|\,]\} \text{ is a partition of } \Omega$$

$$\Rightarrow G_X : \begin{cases} \mathbb{R} \to \mathbb{R} \\ t \mapsto \sum_{k=1}^{n} k \times P(X=k) \times t^k \end{cases}$$

$$\Rightarrow G'_X : \begin{cases} \mathbb{R} \to \mathbb{R} \\ t \mapsto \sum_{k=1}^{n} k \times P(X=k) \times t^{k-1} \end{cases} \Rightarrow G'_X = \sum_{k=1}^{n} = \sum_{k=0}^{n} k \times P(X=k) = \mathbb{E}(X)$$

$$\mathbb{V}(X) = \mathbb{E}(X) - (\mathbb{E}(X))^2 = \mathbb{E}(X^2) - E(X)^2$$

Koenig-Huygens Theorem :

$$\mathbb{V}(X) = \mathbb{E}((X - \mathbb{E}(X))^2)$$

$$\mathbb{V}(X) = \mathbb{E}(X^2) - \mathbb{E}^2(X))$$

$$G_X(t) = \sum_{k=0}^{n} P(X = k) \times t^k$$

$$G'_X(t) = \sum_{k=1}^{n} k \times P(X = k) \times t^{k-1}$$

$$G''_X(t) = \sum_{k=2}^{n} k(k-1) \times P(X = k) \times t^{k-2}$$

We have :

$$k(k-1)P(X = k) \times t^{k-2} = k^2 \times P(X = k) \times t^{k-2} - k * P(X = k) \times t^{k-2}$$

$$\Rightarrow \sum_{k=2}^{n} k^2 \times P(X = k) \times t^{k-2} - \sum_{k=2}^{n} k \times P(X = k) \times t^{k-2} = G''_X(t)$$

$$\Rightarrow G''_X(1) = \sum_{k=0}^{n} k^2 \times P(X = k) - \sum_{k=0}^{n} k \times P(X = k)$$

$$\Rightarrow G''_X(1) = \mathbb{E}(X^2) - E(X) \ (1)$$

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$$\Rightarrow G''_X(1) + G'_X(1) = \mathbb{E}(X^2)$$

$$\Rightarrow G''_X(1) + G'_x(1) - (G'_x(1))^2 = \mathbb{E}(X^2) - \mathbb{E}(X)$$

$$\Rightarrow G''_X(1) + G'_x(1) - (G'_x(1))^2 = \mathbb{V}(X)$$

X + Y :

Let X and Y two finite random variable then :

$$G_{X+Y} : \begin{cases} \mathbb{R} \to \mathbb{R} \\ t \mapsto G_X(t) \times G_Y(t) \end{cases}$$

$$\Leftrightarrow G_{X+Y} = G_X \times G_Y$$

Ex :

$$Y \sim B(n,p) \Leftrightarrow \begin{cases} Y(\Omega) = [\,|0,n|\,] \\ \forall\, k \in Y(\Omega), P(Y = k) = \binom{n}{k} \times p^k \times q^{n-k} \end{cases}$$

$$\Leftrightarrow Y = \sum_{i=1}^{n} X_i$$

$$G_Y = \prod_{i=1}^{n} G_{X_i} : t \mapsto (q + pt)^n = \sum_{k=0}^{n} \binom{n}{k} \times q^{n-k} \times (pt)^k$$

$$\sum_{k=0}^{n} \binom{n}{k} \times q^{n-k} \times p^k \times t^k$$

$$\Rightarrow \mathbb{P}(Y = t) \Leftrightarrow G_{X+Y} = G_X \times G_Y$$