**Mathématiques : CM**

# 1 Bases of a Vector Space

## Definition 1

Let $E$ be a vector space over a field $K$, let $n \in \mathbb{N}$, and let $\{x_1, x_2, \ldots, x_n\}$ be a family of vectors in $E$. We say that the family $\{x_1, x_2, \ldots, x_n\}$ is a **basis** of $E$ if it is both **linearly independent** and **spanning** (generating) for $E$.

## Theorem 1 (Characterization of a Basis)

Let $E$ be a vector space over $K$, let $n \in \mathbb{N}$, and let $\{x_1, x_2, \ldots, x_n\}$ be a family of vectors in $E$. Then $B = \{x_1, x_2, \ldots, x_n\}$ is a basis of $E$ if and only if:

$$\forall x \in E,\ \exists!(\lambda_1, \lambda_2, \ldots, \lambda_n) \in K^n \text{ such that } x = \sum_{i=1}^{n} \lambda_i x_i.$$

The scalars $\lambda_i$, $i \in \{1, 2, \ldots, n\}$ are called the **coordinates** of the vector $x$ in the basis $B$.

# 2 Existence of a Basis

## Theorem 2

Every non-zero vector space $E$ over a field $K$ admits at least one basis.

## Theorem 3 (Incomplete Basis Theorem)

Let $E$ be a vector space over $K$ that admits a finite generating family. Then:

1. From every generating family of $E$, one can extract a basis of $E$.
2. Every linearly independent family of $E$ can be extended to a basis of $E$.

## Definition 2

In $\mathbb{R}^n$, the family $(e_1, \ldots, e_n)$ where for every $i \in \{1, \ldots, n\}$,

$$e_i = (0, \ldots, 0, \underbrace{1}_{i\text{-th coordinate}}, 0, \ldots, 0)$$

is a basis. It is called the **canonical basis** of $\mathbb{R}^n$.

## Definition 3

In $\mathbb{R}_n[X]$, the family $(1, X, \ldots, X^n)$ is a basis. It is called the **canonical basis** of $\mathbb{R}_n[X]$.

# 3 Dimension of a Vector Space

## Definition 1

Let $E$ be a vector space over a field $K$. We say that $E$ is of **finite dimension** if $E$ admits a finite generating family.

## Theorem 1

Let $E \neq \{0_E\}$ be a finite-dimensional vector space over $K$. Then:

1. $E$ admits at least one basis.
2. All bases of $E$ have the same cardinality.

## Definition 2

Let $E$ be a finite-dimensional vector space over $K$.

1. If $E \neq \{0_E\}$, we call the **dimension** of $E$, denoted $\dim(E)$, the cardinality of any basis of $E$.
2. If $E = \{0_E\}$, we define $\dim(E) = 0$.

**Examples:** $\dim(\mathbb{R}^n) = n$ and $\dim(\mathbb{R}_n[X]) = n + 1$.

## Proposition

Let E a $\mathbb{K}$ vector space such that dim(E) = n, n$\in \mathbb{N}*$ and B a family of p vectors, $p \in \mathbb{N}*$.

Then :

1. B is linearly independant $\implies p \leq n$
2. B is a spanning family of E $\implies p \geq n$
3. B is a basis of E $\implies (p \leq n) \wedge (p \geq n) \Leftrightarrow p = n$

4. (p = n) $\implies$ ($B$ is linearly independant)

**Examples:** We have $\{1, X-1, (X+1)^2\} = \mathbb{R}_2[X]$ and $dim(\mathbb{R}_2[X]) = 3$

Then :

1. $Card(\{1, X-1, (X+1)^2\}) = 3 = dim(\mathbb{R}_2[X])$
2. B is linearly independant

Proof that B is linearly independant :

$$\begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} : \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \implies a = b = c = 0 \implies a + b \times (X-1) + c \times (X+1)^2 = 0_{\mathbb{R}_2[X]}$$

**Examples:**

$F = span(u_1 =, u_2, u_3), \forall i \in [\,|1, 3|\,], \forall\, u_i$ from $\mathbb{R}^{\nvDash}$

Find values of t such that $dim(F) = 3$, where :

$$u_1 = (0, t^2, 1)$$
$$u_2 = (0, 0, 3)$$
$$u_3 = (1, 1, 3)$$

$dim(F) = 3 \Leftrightarrow F = E$ because F is a linear subspace of E.

We are looking for a value of B such that $span(B) = \mathbb{R}^3 \implies$ we are looking for t such that B is a basis of $\mathbb{R}^3$ since $Card(B) = 3 = dim(E) \implies$ we are looking for t such that B is linearly independant.

$$\Leftrightarrow \begin{pmatrix} 0 & 0 & 1 \\ t^2 & 0 & 1 \\ 1 & 3 & 3 \end{pmatrix} : \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} 1 & 3 & 3 \\ t^2 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} : \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} 1 & 3 & 3 \\ 0 & -3t^2 & 1-3t^2 \\ 0 & 0 & 1 \end{pmatrix} : \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Case t = 0 :
$(a_1, a_2, a_3) \neq (0, 0, 0)$

Case t $\neq$ 1 :
Kramer system : $a_1 = a_2 = a_3 = 0$

# 4 Linear Maps

## Definition :

Let A,B two $\mathbb{K}$ vector space and

$$f : \begin{cases} A \to B \\ \alpha \mapsto f(x) \end{cases}$$

$\alpha$ map from A to B. Then :

$$f \text{ is linear} \Leftrightarrow \begin{cases} \forall(\alpha, u) \in \mathbb{K} \times A, f(\alpha u) = \alpha f(u) \\ \forall(u, v) \in A^2, f(u+v) = f(u) + f(v) \end{cases}$$

$$\Leftrightarrow \forall(\alpha, u, v) \in \mathbb{K} \times A^2, f(\alpha u + v) = \alpha f(u) + f(v)$$

Notation : f is a linear map from A to B $\Leftrightarrow f \in \mathcal{L}(A, B)$

**Remark (necessary condition) :** $f \in \mathcal{L}(A, B) \implies f(0_A) = O_B$

**Contrapositive :** $f(0_A) \neq O_B \implies f \notin \mathcal{L}(A, B)$

## Definition :

Let $f \in \mathcal{L}(A, B)$ with A and B two $\mathbb{K}$ vector space. Then :

- We call $Ker(f) = \{X \in A, f(X) = 0_B\} \subset A$
- We call $Im(f) = \{Y \in B, f(X) = Y\} = \{f(X), X \in A\} \subset B$

## Proposition :

1. $Ker(f)$ is a linear subspace of A
2. $Im(f)$ is a linear subspace of B

## Definition :

Let $f \in \mathcal{L}(A, B)$.

$$\begin{cases} f \text{ is injective} \Leftrightarrow Ker(f) = \{0_A\} \\ f \text{ is surjective} \Leftrightarrow Im(f) = B \end{cases}$$

**Proof :**

$$f \text{ is surjective} \implies \forall Y \in B, \exists\, x \in A, f(X) = Y$$
$$\implies B \subset Im(f) \wedge Im(f) \subset B \text{ by definition} \implies Im(f) = B$$