# Mathématiques : CM

# Contents

# 1 Euclidean space - scalar product

**Idea : Let E a vector space of finite dimension**

The scalar product is defined as :

$$u \in E, v \in E, < u, v >$$

$$<,> = \begin{cases} E \times E & \rightarrow \mathbb{R} \\ (u, v) & \mapsto < u, v > \end{cases}$$

When u and v are orthogonal ($\perp$) we have the following relation :

$$u \perp v \equiv < u, v >= 0$$

We also have the following relation :

$$||u|| = \sqrt{< u, u >}$$

**Standard scalar product of $\mathbb{R}^3$ :**

$$u = (x_1, y_1, z_1) v = (x_2, y_2, z_2)$$

$$< u, v >= x_1 x_2 + y_1 y_2 + z_1 z_2$$

$$\implies ||u|| = \sqrt{< u, u >} = \sqrt{x_1^2 + y_1^2 + z_1^2}$$

**Definition : symmetric bilinear form**

Let a mapping such that :

$$\phi : \begin{cases} E \times E & \rightarrow \mathbb{R} \\ (u, v) & \mapsto \phi(u, v) \end{cases}$$

$\phi$ is a symmetric bilinear form if :

1. It is bilinear : $\forall\, u_0 \in\, E, v \,\mapsto\, \phi(u_0, v)$ is a linear map
2. It is bilinear : $\forall\, v_0 \in\, E, u \,\mapsto\, \phi(u, v_0)$ is a linear map
3. It is symmetric : $\forall\, (u, v) \in\, E^2, \phi(u, v) = \phi(v, u)$ is a linear map

**Explanation :**

$$\phi = \begin{cases} E \times E \ \to \mathbb{R} \\ (u, v) \ \mapsto \phi(u, v) \end{cases}$$

Let $u_0 \in E$ :

The mapping : $\begin{cases} E \ \to \ \mathbb{R} \\ v \ \mapsto \phi(u_0, v) \end{cases}$ is linear

$$\forall\, (v, v') \in E^2, \forall \alpha \in \mathbb{R}, \phi(u_0, \alpha v + v') = \alpha\phi(u_0, v) + \phi(u_0, v')$$

If $v_0 \in E, \forall(u, u') \in E^2, \forall\alpha \in \mathbb{R}, \phi(\alpha u + u', v_0) = \alpha\phi(u, v_0) + \phi(u', v_0)$

**Example : standard scalar product of $\mathbb{R}^3$**

$$u = (x_1, y_1, z_1)v = (x_2, y_2, z_2)$$

$$\phi(u, v) = x_1x_2 + y_1y_2 + z_1z_2$$

If u is given (considered as a constant), then the mapping :

$v \,\mapsto\, x_1x_2 + y_1y_2 + z_1z_2$ is a linear map.

Note that the coordinates of u are constant numbers.

Those of v are variables.

for example if $u = (1, 2, 3)$, then $v = (x_2, 2y_2, 3z_2)$

**Similarly :**, if $v = (x_2, y_2, z_2)$ is given, then the mapping :

$u \mapsto x_1x_2 + y_1y_2 + z_1z_2$ is a linear map .

Note that the coordinates of v are being constant numbers.

Those of u are variables.

**Furthermore**, $\phi(u,v) = \phi(v,u)$

**A symmetric bilinear form $\phi$ is positive definite if :**

$$\forall u \in E, \phi(u,u) \geq 0 \wedge \phi(u,u) = 0 \implies u = 0_E$$

Example 1 : Standard scalar product of $\mathbb{R}^3$

$$\phi(u,u) = x_1^2 + y_1^2 + z_1^2 \geq 0$$

and

$$\phi(u,u) = 0 \implies \begin{cases} x_1^2 = 0 \\ y_1^2 = 0 \\ z_1^2 = 0 \end{cases} \implies u = 0_E$$

The standard scalar product is thus positive definite

Example 2 :

$$\phi(u,v) = x_1x_2 + y_1y_2 - z_1z_2 \text{ is a symmetric bilinear form}$$

If $u = (x_1, y_1, z_1)$ is given (considered as a constant), then the mapping :

$v \mapsto x_1x_2 + y_1y_2 + z_1z_2$ is a linear map.

Note that the coordinates of u are constant numbers.

Those of v are the variables.

Thus the coordinates of u are the coefficient of the linear function.

Similarly, if v is given and constant, $u \mapsto \phi(u,v)$ is linear

$$\implies \phi(u,v) = \phi(v,u)$$

Positive definite ? No

$$\phi(u,u)x_1^2 + y_1^2 - z_1^2$$

If $u = (1,0,0), \phi(u,u) = 1 > 0$

If $u = (0,0,1), \phi(u,u) = -1 < 0$

Example 3 :

$$\phi(u,v) = x_1x_2 + y_1y_2\phi(u,u) = x_1^2 + y_1^2 \geq 0$$

But $\phi(u,u) = 0 \;\not\Longrightarrow\; u = 0_E$ as for $u = (0,0,1), \phi(u,u) = 0, u \neq 0_E$

A scalar product is a symmetric bilinear form, which is positive definite.

Then we note $< u, v >$ instead of $\phi(u,v)$

We can define for $u \in E$,

$$||u|| = \sqrt{< u, v >} \implies \sqrt{< u, v >} \in \mathbb{R} \text{ because } < u, v > \geq 0$$

$$||u|| = 0 \implies u = 0_E$$

Theorem : Matrix representation of a symmetric bilinear form :

Let $\mathcal{B} = (e_1, ..., e_n)$ be a basis of E.

u has coordinates : $\begin{pmatrix} x_1 \\ y_1 \\ z_1 \\ ... \end{pmatrix}$ in $\mathcal{B}$

v has coordinates : $\begin{pmatrix} x_2 \\ y_2 \\ z_2 \\ \dots \end{pmatrix}$ in $\mathcal{B}$

Then : $\exists\, A \in \mathbb{M}_n(\mathbb{R}), \phi(u,v) = \,^{+}X_1 A X_2$ is symmetric

Examples :

Standard scalar product of $\mathbb{R}^3$ :

Example 1 :

$\mathcal{B}$ standard basis of $\mathbb{R}^3$

$u = (x_1, y_1, z_1)$ has coordinates $X_1 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$

$v = (x_2, y_2, z_2)$ has coordinates $X_2 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$

$$^{T}X_1 X_2 = (x_1, y_1, y_2) \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$$

$$= x_1x_2 + y_1y_2 + z_1z_2 \implies A = I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$^{T}X_1 A X_2 = x_1x_2 + y_1y_2 + z_1z_2 = < u, v >$$

Example 2 :

$$\phi(u,v) = x_1x_2 + y_1y_2 - z_1z_2$$

$$\text{Then : } A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$^{T}X_1AX_2 = (x_1, y_1, z_1)\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{pmatrix}\begin{pmatrix}x_2\\ y_2\\ z_2\end{pmatrix}$$

$$= (x_1, y_1, z_1)\begin{pmatrix}x_2\\ y_2\\ -z_2\end{pmatrix}$$

$$= x_1x_2 + y_1y_2 - z_1z_2$$

Example 3 :

$$\phi(u, v) = x_1x_2 + y_1y_2$$

$$\text{Then}: \ A = \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{pmatrix}$$

$$^{T}X_1AX_2 = (x_1, y_1, z_1)\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}x_2\\ y_2\\ z_2\end{pmatrix}$$

$$= (x_1, y_1, z_1)\begin{pmatrix}x_2\\ y_2\\ 0\end{pmatrix}$$

$$= x_1x_2 + y_1y_2$$

## 1.1 Spectral Theorem

**Spectral Theorem :** all symmetric matrix A is diagonalizable.

We can choose an orthonormal eigenbasis $(\epsilon_1, ..., \epsilon_n)$.

Then the transition matrix P satisfies to :

$$^{T}PP = P \times {}^{T}P = I \Leftrightarrow P^{-1} = {}^{T}P$$

Note : if the eigenbasis is orthonormal, then :

$$< \epsilon_i, \epsilon_j >= \begin{cases} 1 \text{ if } i = j \\ 0 \text{ otherwise} \end{cases}$$

## 1.2 Theorem

**Theorem :** Let E = $\mathbb{R}^n$, let A be a symmetric matrix in $\mathcal{M}_n(\mathbb{R})$.

Let us define $\phi(u, v) = {}^{T}X_1 A X_2$.

$\phi$ is a scalar product $\Leftrightarrow$ All eigenvalues of A are ¿ 0.

Explanation in a an orthogonal eigenbasis $\mathcal{B}'$, $\phi(u, v) = {}^{T}X_1' D X_2'$

Where D is a diagonal matrix.

# 2 Orthogonal Projection

Let $V_0$ be the orthogonal projection of $V$ onto $F$.

$V_0$ is the element of $F$ the closest to $V$.

## 2.1 Motivation

Assume you must find $\lambda \in \mathbb{R}$ such that :

$$v = \lambda u$$

The "best solution" is $\lambda$ such that $v_0 = \lambda u$

Ex: $u = (1, 1), v = (2, 1)$

Solving $v = \lambda u$ : $\begin{cases} \lambda = 2 \\ \lambda = 1 \end{cases}$

$$v_0 = \alpha u = (\alpha, \alpha)$$
$$(v - v_0) \perp u$$
$$(2 - \alpha, 1 - \alpha) \perp (1, 1)$$
$$\implies (2 - \alpha) \times 1 + (1 - \alpha) \times 1 = 0$$
$$\implies 3 - 2\alpha = 0 \implies \alpha = \frac{3}{2}$$
$$\implies v_0 = \frac{3}{2}(1, 1) = (\frac{3}{2}, \frac{3}{2})$$

The solution does not have a solution.

However, we can determine $v_0$.

$v_0$ represent the solution with the samellest range of error.

## 2.2 Definition

Let $(E, <,>)$ be an euclidean space.

Let $F$ be a linear subspace of E.

$$F^{\perp} = \{u \in E, \forall\, v \in F, < u, v >= 0\}$$

## 2.3 Theorem

If $F$ admits a basis $\mathcal{B} = (e_1, e_2, ..., e_n)$, then $F^{\perp} = \mathcal{B}^{\perp}$

$$F^{\perp} = \{u \in E, \begin{cases} < u, e_1 >= 0 \\ < u, e_2 >= 0 \\ ... \\ < u, e_n >= 0 \end{cases} \}$$

## 2.4 Theorem

If $F$ is finite-dimensional, then $F \oplus F^{\perp} = E$

$$\Leftrightarrow \forall\, u \in E, \exists!(v, w) \in F \times F^{\perp}, u = v + w$$

Then $v$ is the orthogonal projection of $u$ onto $F$.

## 2.5 Property

Let $u \in E$ and let $v_0 = P_F(u)$.

$P_F$ is the orthogonal projection onto $F$.

$v_0$ is the element of $F$ which minimizes $||u - v||^2$.

It is the solution of the optimization :

$$Min(||u - v||),\ \text{constraint}:\ v \in F$$

## 2.6 Proof

Let $u = v_0 + w$ where $(v_0, w) \in F \times F^\perp$.

Then $\forall\, v \in F$ :

$$||u - v||^2 = < u - v, u - v > \implies u - v = (u - v_0) + (v_0 - v)$$

Indeed, $(u - v_0) = w \in F^\perp$ and $(v_0 - v) \in F$ as $(v_0, v) \in F^2$

$$\implies ||u - v||^2 - < u - v, u - v > = < (u - v_0) + (v_0 - v), (u - v_0) + (v_0 - v) >$$

$$= < (u - v_0), (u - v_0) > + 2 < (u - v_0), (v_0 - v) > + < (v_0 - v), (v_0 - v) >$$

$$= ||u - v_0||^2 + ||v_0 - v||^2 \geq ||u - v_0||^2$$

$$P_F(u) \in F \Leftrightarrow \exists (\lambda_1, ..., \lambda_n) \in \mathbb{R}^n, u = \lambda_1 e_1 + ... + \lambda_n e_n$$

$$u - P_F(u) \in F^\perp \Leftrightarrow \begin{cases} < (u - \lambda_1 e_1 + ... + \lambda_n e_n), e_1 > = 0 \\ ... \\ < (u - \lambda_1 e_1 + ... + \lambda_n e_n), e_n > = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} < (\lambda_1 < e_1, e_1 > + ... + \lambda_n < e_n, e_n >), e_1 > = < u, e_1 > \\ ... \\ < (\lambda_1 < e_1, e_1 > + ... + \lambda_n < e_n, e_n >), e_n > = < u, e_n > \end{cases}$$

## 2.7 Finding $P_F(u)$

Give $F$ admits a basis $\mathcal{B}_\mathcal{F} = (\epsilon_1, ..., \epsilon_n)$

$$\text{We write } P_F(u) \in F \implies u - P_F(u) \in F^\perp$$

$$\implies u = P_F(u) + (u - P_F(u))$$

$$P_F(u) \in F \Leftrightarrow \exists(\alpha_1, ..., \alpha_n) \in \mathbb{R}^n, P_F(u) = \alpha_1\epsilon_1 + ... + \alpha_n\epsilon_n$$

$$u - P_F(u) \in F^\perp \Leftrightarrow \begin{cases} < u - P_F(u), \epsilon_1 >= 0 \\ ... \\ < u - P_F(u), \epsilon_n >= 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} < u - \alpha_1\epsilon_1 + ... + \alpha_n\epsilon_n, \epsilon_1 >= 0 \\ ... \\ < u - \alpha_1\epsilon_1 + ... + \alpha_n\epsilon_n, \epsilon_n >= 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} < \alpha_1 < \epsilon_1, \epsilon_1 > +... + \alpha_n < \epsilon_n, \epsilon_n >, \epsilon_1 >=< u, \epsilon_1 > \\ ... \\ < \alpha_1 < \epsilon_1, \epsilon_1 > +... + \alpha_n < \epsilon_n, \epsilon_n >, \epsilon_n >=< u, \epsilon_n > \end{cases}$$