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1 Bases of a Vector Space

Definition 1

Let E be a vector space over a field K , let $n \in \mathbb{N}$, and let $\{x_1, x_2, \dots, x_n\}$ be a family of vectors in E . We say that the family $\{x_1, x_2, \dots, x_n\}$ is a **basis** of E if it is both **linearly independent** and **spanning** (generating) for E .

Theorem 1 (Characterization of a Basis)

Let E be a vector space over K , let $n \in \mathbb{N}$, and let $\{x_1, x_2, \dots, x_n\}$ be a family of vectors in E . Then $B = \{x_1, x_2, \dots, x_n\}$ is a basis of E if and only if:

$$\forall x \in E, \exists (\lambda_1, \lambda_2, \dots, \lambda_n) \in K^n \text{ such that } x = \sum_{i=1}^n \lambda_i x_i.$$

The scalars λ_i , $i \in \{1, 2, \dots, n\}$ are called the **coordinates** of the vector x in the basis B .

2 Existence of a Basis

Theorem 2

Every non-zero vector space E over a field K admits at least one basis.

Theorem 3 (Incomplete Basis Theorem)

Let E be a vector space over K that admits a finite generating family. Then:

1. From every generating family of E , one can extract a basis of E .
2. Every linearly independent family of E can be extended to a basis of E .

Definition 2

In \mathbb{R}^n , the family (e_1, \dots, e_n) where for every $i \in \{1, \dots, n\}$,

$$e_i = (0, \dots, 0, \underbrace{1}_{i\text{-th coordinate}}, 0, \dots, 0)$$

is a basis. It is called the **canonical basis** of \mathbb{R}^n .

Definition 3

In $\mathbb{R}_n[X]$, the family $(1, X, \dots, X^n)$ is a basis. It is called the **canonical basis** of $\mathbb{R}_n[X]$.

3 Dimension of a Vector Space

Definition 1

Let E be a vector space over a field K . We say that E is of **finite dimension** if E admits a finite generating family.

Theorem 1

Let $E \neq \{0_E\}$ be a finite-dimensional vector space over K . Then:

1. E admits at least one basis.
2. All bases of E have the same cardinality.

Definition 2

Let E be a finite-dimensional vector space over K .

1. If $E \neq \{0_E\}$, we call the **dimension** of E , denoted $\dim(E)$, the cardinality of any basis of E .
2. If $E = \{0_E\}$, we define $\dim(E) = 0$.

Examples: $\dim(\mathbb{R}^n) = n$ and $\dim(\mathbb{R}_n[X]) = n + 1$.

Proposition

Let E a \mathbb{K} vector space such that $\dim(E) = n$, $n \in \mathbb{N}^*$ and B a family of p vectors, $p \in \mathbb{N}^*$.

Then :

1. B is linearly independent $\implies p \leq n$
2. B is a spanning family of $E \implies p \geq n$
3. B is a basis of $E \implies (p \leq n) \wedge (p \geq n) \Leftrightarrow p = n$

4. $(p = n) \implies (B \text{ is linearly independent})$

Examples: We have $\{1, X - 1, (X + 1)^2\} = \mathbb{R}_2[X]$ and $\dim(\mathbb{R}_2[X]) = 3$

Then :

1. $\text{Card}(\{1, X - 1, (X + 1)^2\}) = 3 = \dim(\mathbb{R}_2[X])$

2. B is linearly independent

Proof that B is linearly independent :

$$\begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} : \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \implies a = b = c = 0 \implies a + b \times (X-1) + c \times (X+1)^2 = 0_{\mathbb{R}_2[X]}$$

Examples:

$$F = \text{span}(u_1, u_2, u_3), \forall i \in [1, 3], \forall u_i \text{ from } \mathbb{R}^3$$

Find values of t such that $\dim(F) = 3$, where :

$$u_1 = (0, t^2, 1)$$

$$u_2 = (0, 0, 3)$$

$$u_3 = (1, 1, 3)$$

$$\dim(F) = 3 \Leftrightarrow F = E \text{ because } F \text{ is a linear subspace of } E.$$

We are looking for a value of B such that $\text{span}(B) = \mathbb{R}^3 \implies$ we are looking for t such that B is a basis of \mathbb{R}^3 since $\text{Card}(B) = 3 = \dim(E) \implies$ we are looking for t such that B is linearly independent.

$$\Leftrightarrow \begin{pmatrix} 0 & 0 & 1 \\ t^2 & 0 & 1 \\ 1 & 3 & 3 \end{pmatrix} : \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} 1 & 3 & 3 \\ t^2 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} : \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} 1 & 3 & 3 \\ 0 & -3t^2 & 1-3t^2 \\ 0 & 0 & 1 \end{pmatrix} : \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Case $t = 0$:

$$(a_1, a_2, a_3) \neq (0, 0, 0)$$

Case $t \neq 1$:

$$\text{Kramer system : } a_1 = a_2 = a_3 = 0$$

4 Linear Maps

Definition :

Let A,B two \mathbb{K} vector space and

$$f : \begin{cases} A \rightarrow B \\ \alpha \mapsto f(x) \end{cases}$$

α map from A to B. Then :

$$\begin{aligned} f \text{ is linear} &\Leftrightarrow \begin{cases} \forall(\alpha, u) \in \mathbb{K} \times A, f(\alpha u) = \alpha f(u) \\ \forall(u, v) \in A^2, f(u + v) = f(u) + f(v) \end{cases} \\ &\Leftrightarrow \forall(\alpha, u, v) \in \mathbb{K} \times A^2, f(\alpha u + v) = \alpha f(u) + f(v) \end{aligned}$$

Notation : f is a linear map from A to B $\Leftrightarrow f \in \mathcal{L}(A, B)$

Remark (necessary condition) : $f \in \mathcal{L}(A, B) \implies f(0_A) = 0_B$

Contrapositive : $f(0_A) \neq 0_B \implies f \notin \mathcal{L}(A, B)$

Definition :

Let $f \in \mathcal{L}(A, B)$ with A and B two \mathbb{K} vector space. Then :

- We call $Ker(f) = \{X \in A, f(X) = 0_B\} \subset A$
- We call $Im(f) = \{Y \in B, f(X) = Y\} = \{f(X), X \in A\} \subset B$

Proposition :

1. $Ker(f)$ is a linear subspace of A
2. $Im(f)$ is a linear subspace of B

Definition :

Let $f \in \mathcal{L}(A, B)$.

$$\begin{cases} f \text{ is injective} \Leftrightarrow Ker(f) = \{0_A\} \\ f \text{ is surjective} \Leftrightarrow Im(f) = B \end{cases}$$

Proof :

$$\begin{aligned} f \text{ is surjective} &\implies \forall Y \in B, \exists x \in A, f(x) = Y \\ &\implies B \subset \text{Im}(f) \wedge \text{Im}(f) \subset B \text{ by definition} \implies \text{Im}(f) = B \end{aligned}$$

5 Determinant

5.1 Determinant of order 2

A determinant of order 2 is the determinant of a matrix from $\mathbb{M}_2(\mathbb{R})$

Let $A \in \mathbb{M}_2(\mathbb{R})$,

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

We call determinant of A and denote $\det(A)$ the real number :

$$\det(A) = a_{11} \times a_{22} - a_{21} \times a_{12}$$

5.2 Determinant of order 3

A determinant of order 3 is the determinant of a matrix from $\mathbb{M}_3(\mathbb{R})$

Let $A \in \mathbb{M}_3(\mathbb{R})$,

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

We call determinant of A and denote $\det(A)$ the real number :

$$\det(A) = a_{11} \times \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \times \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \times \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

5.3 General determinant for $A \in \mathbb{M}_n(\mathbb{R})$

Let $A = (a_{ij})_{1 \leq i \leq n \wedge 1 \leq j \leq n}$. We can minor of indices (i, j) the determinant Δ_{ij} of the $(n-1) \times (n-1)$ matrix resulting from remaining row i and column j from A.

Then :

$$\det(A) = \sum_{i=1}^n (-1)^{i+j} \times \Delta_{ij} \times a_{ij} \text{ if we can expand it with respect to column j}$$

$$\det(A) = \sum_{j=1}^n (-1)^{i+j} \times \Delta_{ij} \times a_{ij} \text{ if we can expand it with respect to row i}$$

Example :

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 4 & 5 & 6 \\ 6 & 5 & 4 \end{pmatrix}$$

$$\begin{aligned} \det(A) &= (-1)^{1+1} \times 1 \times \begin{vmatrix} 5 & 6 \\ 5 & 4 \end{vmatrix} + (-1)^{1+2} \times 2 \times \begin{vmatrix} 4 & 6 \\ 6 & 4 \end{vmatrix} + (-1)^{1+3} \times 0 \times \begin{vmatrix} 4 & 5 \\ 6 & 5 \end{vmatrix} \\ &\implies \det(A) = -10 - 2 \times (-20) = 30 \end{aligned}$$

5.4 Properties

1. Let $(A, B) \in \mathbb{M}_n^2(\mathbb{R})$, $A = (a_{ij})_{1 \leq i \leq n \wedge 1 \leq j \leq n}$, $B = (b_{ij})_{1 \leq i \leq n \wedge 1 \leq j \leq n}$

$$\det(\lambda A) = \lambda^n \det(A)$$

$$\det(AB) = \det(A) \times \det(B)$$

- 2.

$$\det(A^T) = \det(A)$$

3. When doing a row echelon for instance $\alpha R_1 + \beta R_2$:

$$\det(A) = \alpha \times \beta \times \det(C)$$

where C is the matrix that has been row echelon

5.5 Determinant and invertibility

Let $A \in \mathbb{M}_n(\mathbb{R})$:

A invertible $\Leftrightarrow \exists B \in \mathbb{M}_n(\mathbb{R})$ $AB = BA = I_n$. B is denoted A^{-1} .

$$\begin{aligned} \Rightarrow |A \times A^{-1}| &= |I_n| \Rightarrow |A||A^{-1}| = |I_n| \\ \Rightarrow \det(A) &\neq 0 \wedge \det(A^{-1}) = \frac{1}{\det(A)} \end{aligned}$$

$$D : \begin{cases} \mathbb{M}_n(\mathbb{R}) \rightarrow \mathbb{R} \\ A \mapsto \det(A) \end{cases}$$

where D is the mapping of the function of the determinant.

5.6 Determinant of some special matrices

Let $A \in \mathbb{M}_n(\mathbb{R})$

- A is diagonal $\Leftrightarrow (i \neq j \Rightarrow a_{ij} = 0)$

$$\det(A) = \prod_{i=1}^n a_{ii}$$

- A is a upper-triangle matrix $\Leftrightarrow j < i \Rightarrow a_{ij} = 0$

$$\det(A) = \prod_{i=1}^n a_{ij}$$

- A is a lower-triangle matrix $\Leftrightarrow j > i \Rightarrow a_{ij} = 0$

$$\det(A) = \prod_{i=1}^n a_{ij}$$

Example

$$A = \begin{pmatrix} 9 & -4 & 2 \\ 21 & 8 & 3 \\ 25 & 0 & 5 \end{pmatrix}$$

$$\det(A) = \begin{vmatrix} -1 & -4 & 2 \\ 6 & 8 & 3 \\ 0 & 0 & 5 \end{vmatrix} (C1-5C3) = \begin{vmatrix} -1 & -4 & 2 \\ 0 & -16 & 15 \\ 0 & 0 & 5 \end{vmatrix} (R2+6R1) = 5 \times 16 = 80$$

Or

$$\det(A) = (-1)^{3+1} \times 25 \begin{vmatrix} -4 & 2 \\ 8 & 5 \end{vmatrix} + (-1)^{3+3} \times 5 \begin{vmatrix} 9 & -4 \\ 21 & 8 \end{vmatrix} = 80$$

Example

Find values of X such that $A - XI$ is not invertible, $X \in \mathbb{R}$

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 1 & 3 & 2 \end{pmatrix} \implies A - XI = \begin{pmatrix} 1-X & 0 & 2 \\ 2 & 1-X & 0 \\ 1 & 3 & 2-X \end{pmatrix}$$

$$|A - XI| = \begin{vmatrix} 1-X & 0 & 2 \\ 2 & 1-X & 0 \\ 4-X & 4-X & 4-X \end{vmatrix} (R1+R2+R3) = (4-X) \begin{vmatrix} 1-X & 0 & 2 \\ 2 & 1-X & 0 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1-X & X-1 & 1+X \\ 2 & -1-X & -2 \\ 1 & 0 & 0 \end{vmatrix} = (4-X)((-2)(X-1)+(1+X)^2) = (4-X)(X^2+3)$$

$$\mathbb{S} = \{4\}$$