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Mathématiques : CM

1 Euclidean space - scalar product

Idea : Let E a vector space of finite dimension

The scalar product is defined as :

$$u \in E, v \in E, \langle u, v \rangle$$

$$\langle, \rangle = \begin{cases} E \times E \rightarrow \mathbb{R} \\ (u, v) \mapsto \langle u, v \rangle \end{cases}$$

When u and v are orthogonal (\perp) we have the following relation :

$$u \perp v \equiv \langle u, v \rangle = 0$$

We also have the following relation :

$$\|u\| = \sqrt{\langle u, u \rangle}$$

Standard scalar product of \mathbb{R}^3 :

$$u = (x_1, y_1, z_1), v = (x_2, y_2, z_2)$$

$$\langle u, v \rangle = x_1x_2 + y_1y_2 + z_1z_2$$

$$\Rightarrow \|u\| = \sqrt{\langle u, u \rangle} = \sqrt{x_1^2 + y_1^2 + z_1^2}$$

Definition : symmetric bilinear form

Let a mapping such that :

$$\phi : \begin{cases} E \times E \rightarrow \mathbb{R} \\ (u, v) \mapsto \phi(u, v) \end{cases}$$

ϕ is a symmetric bilinear form if :

1. It is bilinear : $\forall u_0 \in E, v \mapsto \phi(u_0, v)$ is a linear map
2. It is bilinear : $\forall v_0 \in E, u \mapsto \phi(u, v_0)$ is a linear map
3. It is symmetric : $\forall (u, v) \in E^2, \phi(u, v) = \phi(v, u)$ is a linear map

Explanation :

$$\phi = \begin{cases} E \times E \rightarrow \mathbb{R} \\ (u, v) \mapsto \phi(u, v) \end{cases}$$

Let $u_0 \in E$:

The mapping : $\begin{cases} E \rightarrow \mathbb{R} \\ v \mapsto \phi(u_0, v) \end{cases}$ is linear

$$\forall (v, v') \in E^2, \forall \alpha \in \mathbb{R}, \phi(u_0, \alpha v + v') = \alpha \phi(u_0, v) + \phi(u_0, v')$$

$$\text{If } v_0 \in E, \forall (u, u') \in E^2, \forall \alpha \in \mathbb{R}, \phi(\alpha u + u', v_0) = \alpha \phi(u, v_0) + \phi(u', v_0)$$

Example : standard scalar product of \mathbb{R}^3

$$u = (x_1, y_1, z_1) v = (x_2, y_2, z_2)$$

$$\phi(u, v) = x_1 x_2 + y_1 y_2 + z_1 z_2$$

If u is given (considered as a constant), then the mapping :

$v \mapsto x_1 x_2 + y_1 y_2 + z_1 z_2$ is a linear map.

Note that the coordinates of u are constant numbers.

Those of v are variables.

for example if $u = (1, 2, 3)$, then $v = (x_2, 2y_2, 3z_2)$

Similarly : if $v = (x_2, y_2, z_2)$ is given, then the mapping :

$u \mapsto x_1x_2 + y_1y_2 + z_1z_2$ is a linear map .

Note that the coordinates of v are being constant numbers.

Those of u are variables.

Furthermore, $\phi(u, v) = \phi(v, u)$

A symmetric bilinear form ϕ is positive definite if :

$$\forall u \in E, \phi(u, u) \geq 0 \wedge \phi(u, u) = 0 \implies u = 0_E$$

Example 1 : Standard scalar product of \mathbb{R}^3

$$\phi(u, u) = x_1^2 + y_1^2 + z_1^2 \geq 0$$

and

$$\phi(u, u) = 0 \implies \begin{cases} x_1^2 = 0 \\ y_1^2 = 0 \\ z_1^2 = 0 \end{cases} \implies u = 0_E$$

The standard scalar product is thus positive definite

Example 2 :

$\phi(u, v) = x_1x_2 + y_1y_2 - z_1z_2$ is a symmetric bilinear form

If $u = (x_1, y_1, z_1)$ is given (considered as a constant), then the mapping :

$v \mapsto x_1x_2 + y_1y_2 + z_1z_2$ is a linear map.

Note that the coordinates of u are constant numbers.

Those of v are the variables.

Thus the coordinates of u are the coefficient of the linear function.

Similarly, if v is given and constant, $u \mapsto \phi(u, v)$ is linear

$$\implies \phi(u, v) = \phi(v, u)$$

Positive definite ? No

$$\phi(u, u)x_1^2 + y_1^2 - z_1^2$$

If $u = (1, 0, 0)$, $\phi(u, u) = 1 > 0$

If $u = (0, 0, 1)$, $\phi(u, u) = -1 < 0$

Example 3 :

$$\phi(u, v) = x_1x_2 + y_1y_2\phi(u, u) = x_1^2 + y_1^2 \geq 0$$

But $\phi(u, u) = 0 \not\Rightarrow u = 0_E$ as for $u = (0, 0, 1)$, $\phi(u, u) = 0$, $u \neq 0_E$

A scalar product is a symmetric bilinear form, which is positive definite.

Then we note $\langle u, v \rangle$ instead of $\phi(u, v)$

We can define for $u \in E$,

$$\|u\| = \sqrt{\langle u, u \rangle} \implies \sqrt{\langle u, u \rangle} \in \mathbb{R} \text{ because } \langle u, u \rangle \geq 0$$

$$\|u\| = 0 \implies u = 0_E$$

Theorem : Matrix representation of a symmetric bilinear form :

Let $\mathcal{B} = (e_1, \dots, e_n)$ be a basis of E .

u has coordinates : $\begin{pmatrix} x_1 \\ y_1 \\ z_1 \\ \dots \end{pmatrix}$ in \mathcal{B}

v has coordinates : $\begin{pmatrix} x_2 \\ y_2 \\ z_2 \\ \dots \end{pmatrix}$ in \mathcal{B}

Then : $\exists A \in \mathbb{M}_n(\mathbb{R}), \phi(u, v) = {}^+X_1 A X_2$ is symmetric

Examples :

Standard scalar product of \mathbb{R}^3 :

Example 1 :

\mathcal{B} standard basis of \mathbb{R}^3

$u = (x_1, y_1, z_1)$ has coordinates $X_1 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$

$v = (x_2, y_2, z_2)$ has coordinates $X_2 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$

$${}^T X_1 X_2 = (x_1, y_1, z_1) \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$$

$$= x_1 x_2 + y_1 y_2 + z_1 z_2 \implies A = I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$${}^T X_1 A X_2 = x_1 x_2 + y_1 y_2 + z_1 z_2 = \langle u, v \rangle$$

Example 2 :

$$\phi(u, v) = x_1 x_2 + y_1 y_2 - z_1 z_2$$

$$\text{Then : } A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$${}^T X_1 A X_2 = (x_1, y_1, z_1) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$$

$$= (x_1, y_1, z_1) \begin{pmatrix} x_2 \\ y_2 \\ -z_2 \end{pmatrix}$$

$$= x_1 x_2 + y_1 y_2 - z_1 z_2$$

Example 3 :

$$\phi(u, v) = x_1 x_2 + y_1 y_2$$

$$\text{Then : } A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$${}^T X_1 A X_2 = (x_1, y_1, z_1) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$$

$$= (x_1, y_1, z_1) \begin{pmatrix} x_2 \\ y_2 \\ 0 \end{pmatrix}$$

$$= x_1 x_2 + y_1 y_2$$