Mathématiques : CM

$$\sum_{n=0}^{+\infty} u_n \in \mathbb{R} \quad \text{if } (u_n) \text{ converges } \Leftrightarrow u_0 + u_1 + u_2 + \cdots \Leftrightarrow \lim_{n\to\infty} \sum_{k=0}^{n} u_k$$

$$\sum_{k\geq 0} u_k \in \mathbb{R}^{\mathbb{N}} \quad \Leftrightarrow \quad \sum_{k=0}^{n} u_k = S_n \in \mathbb{R}$$

The notation $\sum u_n$ if we don't know or don't care about the first terms.

The two previous ones are named Series as they are equal to a sequence.

Series

Definitions:

Convergence and divergence:

Let $(u_n) \in \mathbb{R}^{\mathbb{N}}$ be a sequence and $(S_n) \in \mathbb{R}^{\mathbb{N}}$ the sequence of its partial sums, i.e.

$$S_n = \sum_{k=0}^{n} u_k.$$

We call series of the sequence (u_n) the general term S_n and denote

$$\sum u_n.$$

We say $\sum u_n$ is convergent if and only if (S_n) converges, else $\sum u_n$ diverges.

Example: $\sum q^n$: geometric series of general term q^n.

Let $(S_n) \in \mathbb{R}^{\mathbb{N}}$ (the sequence of partial sums). We have:

For $n \in \mathbb{N}$,

$$S_n = \sum_{k=0}^{n} q^k = 1 + q + q^2 + \cdots + q^n.$$

$$\Rightarrow q \neq 1 \quad \Rightarrow \quad S_n = \frac{1 - q^{n+1}}{1 - q}.$$

$$\lim_{n \to +\infty} S_n = \begin{cases} \frac{1}{1-q}, & \text{if } |q| < 1, \\ \infty, & \text{if } |q| \geq 1. \end{cases}$$

$\Rightarrow$ diverges if else.

Conclusion

$\sum q^n$ converges when $|q| < 1$ with

$$\sum_{n=0}^{+\infty} q^n = \frac{1}{1 - q},$$

and diverges when $|q| \geq 1$.

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Proposition:

Let $\sum u_n$ and $\sum v_n$ be two numerical series and $\lambda \in \mathbb{R}$.Then we have:

1. If $\sum u_n$ and $\sum v_n$ converge, then $\sum(u_n + v_n)$ converges.
2. If $\sum u_n$ converges, then $\sum \lambda u_n$ converges.
3. If $\sum u_n$ converges and $\sum v_n$ diverges, then $\sum(u_n + v_n)$ diverges.
4. If $\sum u_n$ diverges and $\sum v_n$ diverges, then <u>we can't tell anything</u>.

Sum and Remainders of a Series

Let $\sum u_n$ be a numerical series. We call:

<u>Sum of $\sum u_n$</u> the following limit (in case of convergence):

$$\lim_{n\to+\infty} \sum_{k=0}^{n} u_k = S_n,$$

where S_n is the sum of a convergent series.

Example : $\sum q^n$ converges if and only if $|q| < 1$ and in that case:

$$S_n = \lim_{n\to+\infty} S_n = \frac{1}{1-q}$$

<u>Necessary condition of convergence:</u>

$$\sum u_n \text{ convergent } \Rightarrow \lim_{n\to+\infty} u_n = 0.$$

Contrapositive: If $\lim_{n\to+\infty} u_n \neq 0 \Rightarrow \sum u_n$ diverges.

Positive Term Series:

We call PTS (positive term series) any series $\sum u_n$ such that $\forall$ n $\in \mathbb{N}$ $\quad u_n \geq 0$.

Proposition: Let $\sum u_k$ be a PTS and (S_n) its sequence of partial sums.

Then: $\sum u_k$ is convergent $\Leftrightarrow (S_n)$ is bounded.

Let (u_n), (v_n) be two positive sequences such that: $\forall n \in \mathbb{N}$, $0 \leq u_n \leq v_n$,

Then :

$$\begin{cases} \sum v_n \text{ is convergent } \Rightarrow \sum u_n \text{ is convergent} \\ \sum u_n \text{ is divergent } \Rightarrow \sum v_n \text{ is divergent} \end{cases}$$

Ex :

$$\sum \frac{1}{n(sin(n))} = u_n$$

$$\forall\, n \in \mathbb{N}^*, 0 \leq |sin(n)| \leq 1 \Rightarrow u_n \geq \frac{1}{n} \text{ but} \sum \frac{1}{n} \text{ diverges} \Rightarrow \sum u_n \text{ diverges}$$

Riemann Series Theorem

Definition :

We call Riemann series any series of general term :

$$\frac{1}{n^\alpha}, \alpha \in \mathbb{R}$$

Ex :

$$\sum \frac{1}{n}, \sum \frac{1}{n^2}, \sum \frac{1}{n^{\frac{9}{2}}}$$

Riemann theorem :

$$\forall\, \alpha \in \mathbb{R}, \sum \frac{1}{n^\alpha} converges \Leftrightarrow \alpha > 1$$

Ex :

$$u_n = \frac{2 + cos(n)}{n^4}, \forall n \in \mathbb{N}^*; 0 \leq \frac{2 + cos(n)}{n^4} \leq \frac{3}{n^4}$$

$$\text{As } \sum \frac{2 + cos(n)}{n^4} \text{ is a PTS :}$$

$$\begin{cases} \sum u_n \text{ is a PTS} \\ \sum \frac{1}{n^4} \text{ converges (Riemann } \alpha = 4 > 1) \Rightarrow \frac{3}{n^4} \text{ converges} \end{cases}$$

Thus :

$$\sum \frac{2+cos(n)}{n^4} \text{ converges}$$

Proposition :

①

$$\text{Let } (u_n) \text{ and } (v_n) \text{ two sequences such that :}$$

$$u_n = u_n = o(v_n) \quad (\frac{u_n}{v_n} \to 0)$$

then :

$$\sum v_n \text{ converges} \Rightarrow u_n \text{ converges}$$

②

$$\text{if } u_n \underset{n\to+\infty}{\sim} v_n \text{ then :} \sum u_n \text{ same nature as } \sum v_n$$

③

$$\text{if } u_n = O(v_n) \text{ then :}$$

$$\sum v_n \text{ converges} \Rightarrow u_n \text{ converges}$$

①

Ex :

$$\sum e^{-\sqrt{n}} \text{ (PTS)}$$

$$n^2 \times e^{-\sqrt{n}} \underset{n\to+\infty}{\to} 0$$

$$\frac{e^{-\sqrt{n}}}{\frac{1}{n^2}} \Rightarrow e^{-\sqrt{n}} = o(\frac{1}{n^2})$$

$$\Rightarrow [\sum \frac{1}{n^2} \text{ converges (Riemann series where } \alpha > 1) \Rightarrow \sum e^{-\sqrt{n}} \text{ converges]}$$

②

Ex :

$$\sum ln(\frac{n+1}{n}) = \sum ln(1+\frac{1}{n}) \Rightarrow ln(1+\frac{1}{n}) \underset{n\to+\infty}{\sim} \frac{1}{n}$$

$$1n \Rightarrow \sum ln(1+\frac{1}{n}) \text{ of same nature as } \sum \frac{1}{n} \Rightarrow \sum ln(1+\frac{1}{n}) \text{ diverges}$$

Definition :

$$u_n \underset{+\infty}{\sim} v_n \Leftrightarrow \frac{u_n}{v_n} \underset{n\to+\infty}{\to} 1 \Rightarrow \exists\, n_0 \in \mathbb{N}, [n \geq n_0 \Rightarrow \text{sign of } u_n = \text{sign of } v_n]$$

$$\Leftrightarrow \exists n_0 \in \mathbb{N}, [n \geq n_0 \Rightarrow u_n \times v_n \geq 0]$$

Proposition :

Let $(u_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$.Then : $\sum(u_{n+1} - u_n)$ converges $\Leftrightarrow$ $(u_n)_{n\in\mathbb{N}}$ converges

Proof :

Let $(S_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ such that : $\forall n \in \mathbb{N}, S_n = \sum_{k=0}^{n}(u_{k+1} - u_k) = \sum_{k=0}^{n} u_{k+1} - \sum_{k=0}^{n} u_k$

$$= \sum_{k=1}^{n+1} u_k - \sum_{k=0}^{n} u_k = u_{n+1} - u_0 \Leftrightarrow \underset{n\to+\infty}{lim} S_n = \underset{n\to+\infty}{lim}(u_{n+1} - u_0)$$

$$\Longrightarrow [S_n \underset{n\to+\infty}{\to} l \Leftrightarrow u_n \underset{n\to+\infty}{\to} l + u_0]$$

Ex :

$$\forall\, n \in \mathbb{N}^*, u_n = 1 + \frac{1}{2} + ... + \frac{1}{n} - ln(n) \hookleftarrow \text{nature of } (u_n)?$$

$$= \sum_{k=1}^{n} \frac{1}{k} - ln(n)$$

$$u_{n+1} - u_n = \sum_{k=1}^{n+1} \frac{1}{k} - ln(n+1) - \sum_{k=1}^{n} \frac{1}{k} - ln(n)$$

$$= \frac{1}{n+1} + ln(\frac{n}{n+1} = \frac{1}{n+1} + ln(1 - \frac{1}{n+1})$$

$$\frac{1}{n+1} - \frac{1}{n+1} - \frac{1}{2} \times \frac{1}{(n+1)^2} + o(\frac{1}{n^2}) = -\frac{1}{2} \times \frac{1}{(n+1)^2} + o(\frac{1}{n^2}) \underset{+\infty}{\sim} -\frac{1}{2} \times \frac{1}{n^2}$$

$$\Longrightarrow \sum u_{n+1} - u_n \text{ converges} \Leftrightarrow u_n \text{ converges}$$

The Rules :

Riemann's rule :

Theorem :

Let (u_n) a positive term series :

If $\exists\, \alpha \in \mathbb{R},\ \alpha > 1,\ n^\alpha \times u_n \underset{n \to +\infty}{\rightarrow} 0$ then : $\sum u_n$ converges

Proof :

$$n^\alpha \times u_n \underset{n \to +\infty}{\rightarrow} 0 \Leftrightarrow \frac{u_n}{\frac{1}{n^\alpha}} \underset{n \to +\infty}{\rightarrow} 0 \Leftrightarrow u_n = o(\frac{1}{n^\alpha})$$

Thus : [$\sum \frac{1}{n^\alpha}$ converges (Riemann $\alpha > 1$) $\Longrightarrow \sum u_n$ converges]

D'Alembert's rule :

Theorem :

Let (u_n) a strictly positive sequence such that :

$\frac{u_{n+1}}{u_n} \underset{n\to+\infty}{\rightarrow} l$ where $l \in \mathbb{R} \cup \{+\infty\}$

$$\text{Then : } \begin{cases} l < 1 \Rightarrow \sum u_n \text{ converge} \\ l > 1 \Rightarrow \sum u_n \text{ diverge} \end{cases}$$

Ex : $\forall n \in \mathbb{N}, u_n = \frac{1}{n!}$

$\frac{u_{n+1}}{u_n} = \frac{1}{(n+1)!} \times n! = \frac{1}{n+1} \underset{n\to+\infty}{\rightarrow} 0 < 1 \implies \sum u_n$ converges

Cauchy's rule :

Theorem :

Let u_n a strictly positive sequence such that :

$$\sqrt[n]{u_n} \underset{n\to+\infty}{\rightarrow} l \text{ where l } \in \mathbb{R}_+ \cup \{+\infty\}$$

$$\text{Then : } \begin{cases} l < 1 \implies \sum u_n \text{ converges} \\ l < 1 \implies \sum u_n \text{ diverges} \end{cases}$$

Ex :

$$\forall n \in \mathbb{N}^*, u_n = (\frac{n}{n+1})^{n^2}$$

$$\sqrt[n]{u_n} = (u_n)^{\frac{1}{n}} = (\frac{n}{n+1})^n \text{ and we know that } (\frac{n}{n+1}) \underset{n\to+\infty}{\rightarrow} 1$$

$$\text{Moreover, } e^{nln(\frac{n}{n+1})} = e^{nln(1-\frac{1}{n+1})} \text{we have } n \times ln(1 - \frac{1}{n+1}) \underset{+\infty}{\sim} -1$$

$$\implies e^{nln(\frac{n}{n+1})} \underset{+\infty}{\sim} e^{-1}$$

Series with Arbitrary terms :

Alternating Series

Definition :

Let (u_n) be a real sequence. We say that (u_n) is alternating if there exist

a positive real sequence (a_n) such that $\forall\, n \in \mathbb{N}$:

$$u_n = (-1)^n a_n \text{ or } u_n = (-1)^{n+1} a_n$$

We say that a numerical series $\sum u_n$ is alternating if the sequence (u_n) is alternating

Special criterion for alternating series :

Leibniz Criterion :

Let (u_n) be an alternating real sequence.

If $(|u_n|)$ is decreasing and converges to 0, then :

$$\begin{cases} \text{The series } \sum u_n \text{ converges} \\ \forall\, n \in \mathbb{N}, |R_n| \leq |u_{n+1}| \end{cases}$$

Where (R_n) is the sequence of remainders associated with $\sum u_n$

Ex :

Let $\alpha \in \mathbb{R}$. Then :

$$\sum \frac{(-1)^n}{n^\alpha} \text{ converges if and only if } \alpha > 0$$

Absolute Convergence

Definition :

A numerical series $\sum u_n$ is said to converge absolutely if the series $\sum |u_n|$ converges.

Theorem :

If $\sum u_n$ is a numerical series that converges absolutely, then $\sum u_n$ also converges

Definition :

A convergent series that is not absolutely convergent is called conditionally convergent (or semi-convergent).