Mathématiques : CM

$$(\Omega, P(\Omega), \mathbb{P}) \Leftrightarrow$$

A random variable X is defined by:

$$X : \Omega \to \mathbb{R} \quad \text{such that} \quad \Omega \mapsto X(\Omega)$$

The generative function is defined by :

$$P : \mathbb{R} \to \mathbb{R} \quad t \mapsto a_0 * t^0 + a_1 * t^1 + a_2 * t^2 + ... + a_n * t^n$$

$$\{(\text{X} = \text{k}),\ \text{k}\ \in X(\Omega)\ \} = \text{partition of } \Omega$$

Generating Function

Definition :

Let X a random variable such that $X(\Omega) = [\,|0, n|\,], n \in \mathbb{N}^*$.

we call generating function of X the following polynomial :

$$G_X : \begin{cases} \mathbb{R} \to \mathbb{R} \\ G \mapsto \sum_{k \in X(\Omega)} P(X = k) * t^k \end{cases}$$

Remarks :

$\alpha(\Omega)$ can be different we can have : X$(\Omega) \subset [\,|\alpha, n|\,]$

Bernouilli :

$$X \sim B(P) \Leftrightarrow \begin{cases} X(\Omega) = \{0,1\} \\ P(X=0) = (1-p), P(X=1) = p \end{cases}$$

$$\Leftrightarrow G_X \begin{cases} \mathbb{R} \to \mathbb{R} \\ t \mapsto (1-p) + pt \end{cases}$$

Expected value and variance :

Theorem :

Let X a random variable and G_X its generative function :

$$\begin{cases} G_X(1) = 1 \\ \mathbb{E}(X) = G'_X(1) \\ \mathbb{V}(X) = G''_X(1) + G'_X(1) - (G'_X(1))^2 \end{cases}$$

Proof :

By definition :

$$G_X : \begin{cases} \mathbb{R} \to \mathbb{R} \\ t \mapsto \sum_{k=0}^{n} P(X=k) * t^k \end{cases}$$

$$\Rightarrow G_X(1) = \sum_{k=0}^{n} P(X=k) = 1 \text{ because } \{(X=k), k \in [|0,n|]\} \text{ is a partition of } \Omega$$

$$\Rightarrow G_X : \begin{cases} \mathbb{R} \to \mathbb{R} \\ t \mapsto \sum_{k=1}^{n} k \times P(X=k) \times t^k \end{cases}$$

$$\Rightarrow G'_X : \begin{cases} \mathbb{R} \to \mathbb{R} \\ t \mapsto \sum_{k=1}^{n} k \times P(X=k) \times t^{k-1} \end{cases} \Rightarrow G'_X = \sum_{k=1}^{n} = \sum_{k=0}^{n} k \times P(X=k) = \mathbb{E}(X)$$

$$\mathbb{V}(X) = \mathbb{E}(X) - (\mathbb{E}(X))^2 = \mathbb{E}(X^2) - E(X)^2$$

Koenig-Huygens Theorem :

$$\mathbb{V}(X) = \mathbb{E}((X - \mathbb{E}(X))^2)$$

$$\mathbb{V}(X) = \mathbb{E}(X^2) - \mathbb{E}^2(X))$$

$$G_X(t) = \sum_{k=0}^{n} P(X = k) \times t^k$$

$$G'_X(t) = \sum_{k=1}^{n} k \times P(X = k) \times t^{k-1}$$

$$G''_X(t) = \sum_{k=2}^{n} k(k-1) \times P(X = k) \times t^{k-2}$$

We have :

$$k(k-1)P(X = k) \times t^{k-2} = k^2 \times P(X = k) \times t^{k-2} - k * P(X = k) \times t^{k-2}$$

$$\Rightarrow \sum_{k=2}^{n} k^2 \times P(X = k) \times t^{k-2} - \sum_{k=2}^{n} k \times P(X = k) \times t^{k-2} = G''_X(t)$$

$$\Rightarrow G''_X(1) = \sum_{k=0}^{n} k^2 \times P(X = k) - \sum_{k=0}^{n} k \times P(X = k)$$

$$\Rightarrow G''_X(1) = \mathbb{E}(X^2) - E(X) \ (1)$$

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$$\Rightarrow G''_X(1) + G'_X(1) = \mathbb{E}(X^2)$$

$$\Rightarrow G''_X(1) + G'_x(1) - (G'_x(1))^2 = \mathbb{E}(X^2) - \mathbb{E}(X)$$

$$\Rightarrow G''_X(1) + G'_x(1) - (G'_x(1))^2 = \mathbb{V}(X)$$

X + Y :

Let X and Y two finite random variable then :

$$G_{X+Y} : \begin{cases} \mathbb{R} \to \mathbb{R} \\ t \mapsto G_X(t) \times G_Y(t) \end{cases}$$

$$\Leftrightarrow G_{X+Y} = G_X \times G_Y$$

Ex :

$$Y \sim B(n,p) \Leftrightarrow \begin{cases} Y(\Omega) = [\,|0,n|\,] \\ \forall\, k \in Y(\Omega), P(Y=k) = \binom{n}{k} \times p^k \times q^{n-k} \end{cases}$$

$$\Leftrightarrow Y = \sum_{i=1}^{n} X_i$$

$$G_Y = \prod_{i=1}^{n} G_{X_i} : t \mapsto (q+pt)^n = \sum_{k=0}^{n} \binom{n}{k} \times q^{n-k} \times (pt)^k$$

$$\sum_{k=0}^{n} \binom{n}{k} \times q^{n-k} \times p^k \times t^k$$

$$\Rightarrow \mathbb{P}(Y=t) \Leftrightarrow G_{X+Y} = G_X \times G_Y$$

Power Series :

Definition :

Let $\sum u_n(x)$ a series such that the general form $u_n(x)$ is of the following form :

$\forall n \in \mathbb{N}, u_n(x) = a_n x^n$ with$(a_n)_{n \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$. a_n should not depend on x.

Ex :

$$\sum n^2 x^n \ , \ \sum \frac{x^n}{n!} = e^x$$

Remark :

Power series are series of functions. In case of convergence at $x \in I \subset \mathbb{R}$,

we can define a function by the sum of the series. $\forall x \in I, f(x) = S(x) = \sum P_n(x)$

$\forall (a_n) \in \mathbb{R}^{\mathbb{N}}$, at x = 0, we have : $n = 0 \implies a_0 \times 0^0 = a_1 n_1 \implies a_1 \times 0^1 = 0$

$\implies \sum a_n x^n$ is convergent at x = 0 and $f(0) = \sum a_n x^n = a_0$

Convergence Radius :

Definition :

Let $(a_n) \in \mathbb{R}^{\mathbb{N}}$, and $\sum a_n x^n$ the power series associated with a_n : $\sum a_n x^n$

we call R the radius of convergence of the series.Then $\exists\, R \in \mathbb{R}_+ \cup \{+\infty\}$ such that:

$$\begin{cases} \forall\, x \in \mathbb{R}, |x| < R, \text{ the series converges absolutely.} \\ \forall\, x \in \mathbb{R}, |x| > R, \ \sum a_n x^n \text{ diverges.} \end{cases}$$

Definition :

We call open disc of convergence of $\sum a_n x^n$ an open interval $(-R, R)$.

Remark :

This open disc of convergence is the included domain of function f :

$$f : \begin{cases} (-R, R) \to \mathbb{R} \\ x \mapsto \sum a_n x^n, \text{ with } R \neq 0 \end{cases}$$

If R $= 0$, then $\sum a_n x^n$ is only convergent at x $= 0 \implies D_f = 0$ and

$f(0) = a_0$ and is divergent otherwise.

If $R \in \mathbb{R}_+^*$, then :

$$\begin{cases} (-R, R) \cap D_f = (-R, R) \\ (-\infty, -R) \cap D_f = \emptyset \\ (R, +\infty) \cap D_f = \emptyset \end{cases}$$

How to define R :

D'Alembert's criteria for power series :

Let $\sum a_n x^n$ a power series, $\exists\, n_0 \in \mathbb{N}, (n \geq n_0) \implies a_n \neq 0$.

If $\exists\, P \in \mathbb{R}_+ \cup \{+\infty\}$ such that $|\frac{a_{n+1}}{a_n}| \to P$, then :

$$\begin{cases} P = 0 \implies R = +\infty \\ P \in \mathbb{R}_+^* \implies R = \frac{1}{P} \\ P = +\infty \implies R = 0 \end{cases}$$

- $\forall n \in \mathbb{N}, \sum a_n x^n : |\frac{a_{n+1} x^{n+1}}{a_n x^n}| = |\frac{a_{n+1}}{a_n}||x| \underset{n \to +\infty}{\to} P|x|$

Then :

- If $P|x| < 1, \sum a_n x^n$ converges, where $|x| < \frac{1}{P}$. Thus $R \geq \frac{1}{P}$

- If $P|x| > 1, \sum a_n x^n$ diverges, where $|x| > \frac{1}{P}$. Thus $R \leq \frac{1}{P}$

Hence R $= \frac{1}{P}$

- If P $= 0$, then $\forall x \in \mathbb{R}^*$, $\frac{a_{n+1}}{a_n}|x| \underset{n \to +\infty}{\to} 0$, thus $\sum a_n x^n$ diverges hence R $= +\infty$

- If P $= +\infty$, then $\forall x \in \mathbb{R}^*$, $\frac{a_{n+1}}{a_n}|x| \underset{n \to +\infty}{\to} +\infty$, thus $\sum a_n x^n$ diverges hence R ≤ 0

- At x $= 0$, $\sum a_n x^n converges \implies R \leq 0 \implies R = 0$

Ex : $\sum x^n, \forall n \in \mathbb{N}, a_n = 1; \frac{a_{n+1}}{a_n} \underset{n \to +\infty}{\to} 1 \implies R = 1$

At x $= 1 \implies \sum 1$ diverges, $D_f = (-1, 1) \implies S$

$$\begin{cases} (-1,1) \to \mathbb{R} \\ x \mapsto S(x) = \sum x^k = \frac{1}{1-x} \end{cases}$$

At x = -1 $\implies \sum (-1)^n$ diverges

$$\sum \frac{x^n}{n!}, \forall n \in \mathbb{N}, a_n = \frac{1}{n!}, \frac{a_{n+1}}{a_n} \implies \frac{n!}{(n+1)!} = \frac{1}{n+1} \underset{n \to +\infty}{0} \implies R = +\infty$$

$$D_f = (-\infty, +\infty), S : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto \sum \frac{x^n}{n!} = e^x \end{cases}$$

$$\sum \frac{x^n}{n!}$$

Definition

Let $(\Omega, P(\Omega), \mathbb{P})$ be a probability space. We call a **Discrete Infinite Random Variable** a random variable X such that $X(\Omega)$ is indexable by $\mathbb{N}$, i.e., there exists a bijection from $\mathbb{N}$ to $X(\Omega)$. We then denote:

$$X(\Omega) = \{x_k \mid k \in \mathbb{N}\}.$$

Hence, we will only consider the cases when $X(\Omega) \subset \mathbb{N}$, i.e., X takes only integer values.

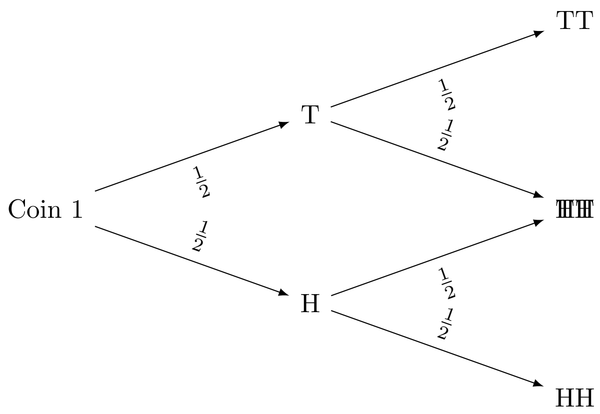
Distribution of X

We define the distribution of X with $P(X = k)$, $k \in X(\Omega)$ where $X(\Omega) \subset \mathbb{N}$. Then:

$$\sum_{k=0}^{+\infty} P(X = k) = P(\Omega) = 1.$$

Note that if $k = 0$, $X(\Omega) = \mathbb{N}$.

Geometric Distribution



Infinite number of trials.

X : the rank of first success, or the number of trials until the first success.

- $X(\Omega) = \{1, 2, 3, \dots\} = \mathbb{N}^*$
- $\forall k \in \mathbb{N}^*, \quad P(X = k) = q^{k-1} \cdot p \quad \Leftrightarrow \quad X \sim \text{Geom}(p)$

Consider the series with general term $P(X = k) = q^{k-1}p$:

$$\sum_{k \in X(\Omega)} q^{k-1}p = (1-q) \sum_{k \geq 1} q^{k-1}.$$

This geometric series converges for $q \in (0, 1)$:

$$\sum_{k=1}^{+\infty} q^{k-1} = \sum_{k=0}^{+\infty} q^k = \frac{1}{1-q} \quad \Longrightarrow \quad (1-q)\sum_{k=1}^{+\infty} q^{k-1} = 1.$$

Expectation and Variance

Let X be a discrete infinite random variable with $X(\Omega) = \mathbb{N}$. Then:

- The expected value of X is

$$\mathbb{E}(X) = \sum_{k \in X(\Omega)} k\, P(X = k),$$

  provided the series converges.

- The variance of X is

$$\mathbb{V}(X) = \sum_{k \in X(\Omega)} (k - \mathbb{E}(X))^2 P(X = k),$$

  provided $\mathbb{E}(X)$ exists and $\sum k^2 P(X = k)$ converges.

Remark: Under the existence conditions, we have

$$\mathbb{V}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2.$$

Properties of $\mathbb{E}$ and $\mathbb{V}$

Let X and Y two Infinite Discrete Random Value with $\mathbb{E}$ and $\mathbb{V}$ and $(a, b) \in \mathbb{R}^2$
Then :

$$\mathbb{E}(X+Y) = \mathbb{E}(X)+\mathbb{E}(Y)\mathbb{E}(aX+bY) = a\mathbb{E}(X)+b\mathbb{E}(Y)\mathbb{E}(aX+b) = a\mathbb{E}(X)+b\mathbb{V}(aX) = a^2\mathbb{V}(X)$$

Generating function of Infinite Discrete Random Variable :

Let X and Y two Infinite Discrete Random Variable such that :

$X(\Omega) = \mathbb{N}$, then : The generating function of X is $G_X : t \mapsto \mathbb{E}(t^x)$, i.e the sum of the series :

$$\sum P(X = k) \times t^k \implies G_X : t \mapsto \sum_{k=0}^{+\infty} P(X = k) \times t^k$$

Here, P(X = k) could be named a_n which is a sequence.

We also talk about "the generating series of X"

$$t = 1 \implies \sum_{k=0}^{+\infty} P(X = k) \times t^k \text{converges and} G_X(1) = 1 \implies R \geq 1$$

Properties of G_X

(-1, 1) and -1, 1 excluded :

$$\begin{cases} G_X \in C([-1, 1]) \\ G_X(1) = 1 \end{cases}$$

If Y another I.D.R.V, $Y(\Omega) = \mathbb{N}$, $G_{X+Y} = G_X + G_Y$ over $[-1, 1]$ where -1 and 1 are excluded
And :

$$\begin{cases} \mathbb{E}(X) = G'_X(1) \\ \mathbb{V}(X) = G''_X(1) + G'_X(1) - (G'_X(1))^2 \end{cases}$$

$$\sum P(X = k \times (-1)^k)$$

$$\Leftrightarrow \sum k \times (q)^{k-1} \times p$$

$$\Leftrightarrow p \sum_{k \geq 1} k \times (q)^{k-1} \times pisconvergenton] - 1, 1[and \mathbb{E}(X) = p \times (\frac{1}{1-q})$$

$$\frac{p}{(1-q)^2} = \frac{1}{1-q} = \frac{1}{p}$$