Mathématiques : CM

$$(\Omega, P(\Omega), \mathbb{P}) \Leftrightarrow$$

A random variable X is defined by:

$$X : \Omega \to \mathbb{R} \quad \text{such that} \quad \Omega \mapsto X(\Omega)$$

The generative function is defined by :

$$P : \mathbb{R} \to \mathbb{R} \quad t \mapsto a_0 * t^0 + a_1 * t^1 + a_2 * t^2 + ... + a_n * t^n$$

$\{$(X = k), k $\in X(\Omega)$ $\}$ = partition of Ω

Generating Function

Definition :

Let X a random variable such that $X(\Omega) = [\,|0, n|\,], n \in \mathbb{N}^*$.

we call generating function of X the following polynomial :

$$G_X : \begin{cases} \mathbb{R} \to \mathbb{R} \\ G \mapsto \sum_{k \in X(\Omega)} P(X = k) * t^k \end{cases}$$

Remarks :

$\alpha(\Omega)$ can be different we can have : X(Ω) $\subset [\,|\alpha, n|\,]$

Bernouilli :

$$X \sim B(P) \Leftrightarrow \begin{cases} X(\Omega) = \{0, 1\} \\ P(X = 0) = (1 - p), P(X = 1) = p \end{cases}$$

$$\Leftrightarrow G_X \begin{cases} \mathbb{R} \to \mathbb{R} \\ t \mapsto (1 - p) + pt \end{cases}$$

Expected value and variance :

Theorem :

Let X a random variable and G_X its generative function :

$$\begin{cases} G_X(1) = 1 \\ \mathbb{E}(X) = G'_X(1) \\ \mathbb{V}(X) = G''_X(1) + G'_X(1) - (G'_X(1))^2 \end{cases}$$

Proof :

By definition :

$$G_X : \begin{cases} \mathbb{R} \to \mathbb{R} \\ t \mapsto \sum_{k=0}^{n} P(X = k) * t^k \end{cases}$$

$\Rightarrow G_X(1) = \sum_{k=0}^{n} P(X = k) = 1$ because $\{(X = k), k \in [\,|0, n|\,]\}$ is a partition of Ω

$$\Rightarrow G_X : \begin{cases} \mathbb{R} \to \mathbb{R} \\ t \mapsto \sum_{k=1}^{n} k \times P(X = k) \times t^k \end{cases}$$

$$\Rightarrow G'_X : \begin{cases} \mathbb{R} \to \mathbb{R} \\ t \mapsto \sum_{k=1}^{n} k \times P(X = k) \times t^{k-1} \Rightarrow G'_X = \sum_{k=1}^{n} = \sum_{k=0}^{n} k \times P(X = k) = \mathbb{E}(X) \end{cases}$$

$$\mathbb{V}(X) = \mathbb{E}(X) - (\mathbb{E}(X))^2 = \mathbb{E}(X^2) - E(X)^2$$

Koenig-Huygens Theorem :

$$\mathbb{V}(X) = \mathbb{E}((X - \mathbb{E}(X))^2)$$

$$\mathbb{V}(X) = \mathbb{E}(X^2) - \mathbb{E}^2(X))$$

$$G_X(t) = \sum_{k=0}^{n} P(X = k) \times t^k$$

$$G'_X(t) = \sum_{k=1}^{n} k \times P(X = k) \times t^{k-1}$$

$$G''_X(t) = \sum_{k=2}^{n} k(k-1) \times P(X = k) \times t^{k-2}$$

We have :

$$k(k-1)P(X = k) \times t^{k-2} = k^2 \times P(X = k) \times t^{k-2} - k * P(X = k) \times t^{k-2}$$

$$\Rightarrow \sum_{k=2}^{n} k^2 \times P(X = k) \times t^{k-2} - \sum_{k=2}^{n} k \times P(X = k) \times t^{k-2} = G''_X(t)$$

$$\Rightarrow G''_X(1) = \sum_{k=0}^{n} k^2 \times P(X = k) - \sum_{k=0}^{n} k \times P(X = k)$$

$$\Rightarrow G''_X(1) = \mathbb{E}(X^2) - E(X) \ (1)$$

(1)

$$\Rightarrow G''_X(1) + G'_X(1) = \mathbb{E}(X^2)$$

$$\Rightarrow G''_X(1) + G'_x(1) - (G'_x(1))^2 = \mathbb{E}(X^2) - \mathbb{E}(X)$$

$$\Rightarrow G''_X(1) + G'_x(1) - (G'_x(1))^2 = \mathbb{V}(X)$$

X + Y :

Let X and Y two finite random variable then :

$$G_{X+Y} : \begin{cases} \mathbb{R} \to \mathbb{R} \\ t \mapsto G_X(t) \times G_Y(t) \end{cases}$$

$$\Leftrightarrow G_{X+Y} = G_X \times G_Y$$

Ex :

$$Y \sim B(n,p) \Leftrightarrow \begin{cases} Y(\Omega) = [\,|0,n|\,] \\ \forall\, k \in Y(\Omega), P(Y=k) = \binom{n}{k} \times p^k \times q^{n-k} \end{cases}$$

$$\Leftrightarrow Y = \sum_{i=1}^{n} X_i$$

$$G_Y = \prod_{i=1}^{n} G_{X_i} : t \mapsto (q+pt)^n = \sum_{k=0}^{n} \binom{n}{k} \times q^{n-k} \times (pt)^k$$

$$\sum_{k=0}^{n} \binom{n}{k} \times q^{n-k} \times p^k \times t^k$$

$$\Rightarrow \mathbb{P}(Y=t) \Leftrightarrow G_{X+Y} = G_X \times G_Y$$

Definition

Let $(\Omega, P(\Omega), \mathbb{P})$ be a probability space. We call a **Discrete Infinite Random Variable** a random variable X such that $X(\Omega)$ is indexable by $\mathbb{N}$, i.e., there exists a bijection from $\mathbb{N}$ to $X(\Omega)$. We then denote:

$$X(\Omega) = \{x_k \mid k \in \mathbb{N}\}.$$

Hence, we will only consider the cases when $X(\Omega) \subset \mathbb{N}$, i.e., X takes only integer values.

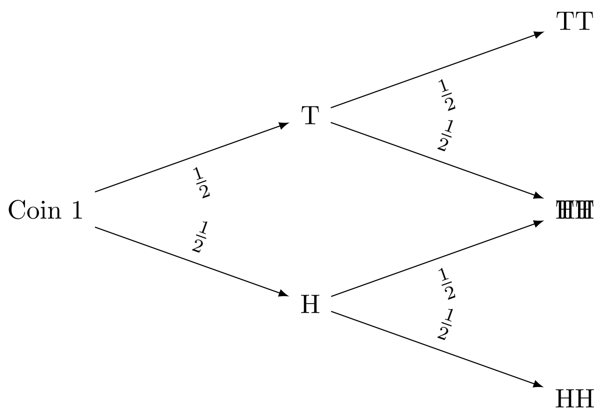
Distribution of X

We define the distribution of X with $P(X = k)$, $k \in X(\Omega)$ where $X(\Omega) \subset \mathbb{N}$. Then:

$$\sum_{k=0}^{+\infty} P(X = k) = P(\Omega) = 1.$$

Note that if $k = 0$, $X(\Omega) = \mathbb{N}$.

Geometric Distribution



Infinite number of trials.

$X:$ the rank of first success, or the number of trials until the first success.

- $X(\Omega) = \{1, 2, 3, \dots\} = \mathbb{N}^*$
- $\forall k \in \mathbb{N}^*, \quad P(X = k) = q^{k-1} \cdot p \quad \Leftrightarrow \quad X \sim \text{Geom}(p)$

Consider the series with general term $P(X = k) = q^{k-1}p$:

$$\sum_{k \in X(\Omega)} q^{k-1}p = (1 - q) \sum_{k \geq 1} q^{k-1}.$$

This geometric series converges for $q \in (0,1)$:

$$\sum_{k=1}^{+\infty} q^{k-1} = \sum_{k=0}^{+\infty} q^k = \frac{1}{1-q} \quad \Longrightarrow \quad (1-q)\sum_{k=1}^{+\infty} q^{k-1} = 1.$$

Expectation and Variance

Let X be a discrete infinite random variable with $X(\Omega) = \mathbb{N}$. Then:

- The expected value of X is

  $$\mathbb{E}(X) = \sum_{k \in X(\Omega)} k\, P(X = k),$$

  provided the series converges.

- The variance of X is

  $$\mathbb{V}(X) = \sum_{k \in X(\Omega)} (k - \mathbb{E}(X))^2 P(X = k),$$

  provided $\mathbb{E}(X)$ exists and $\sum k^2 P(X = k)$ converges.

Remark: Under the existence conditions, we have

$$\mathbb{V}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2.$$

Properties of $\mathbb{E}$ and $\mathbb{V}$

Let X and Y two Infinite Discrete Random Value with $\mathbb{E}$ and $\mathbb{V}$ and $(a, b) \in \mathbb{R}^2$
Then :

$$\mathbb{E}(X+Y) = \mathbb{E}(X)+\mathbb{E}(Y)\mathbb{E}(aX+bY) = a\mathbb{E}(X)+b\mathbb{E}(Y)\mathbb{E}(aX+b) = a\mathbb{E}(X)+b\mathbb{V}(aX) = a^2\mathbb{V}(X)$$

Generating function of Infinite Discrete Random Variable :

Let X and Y two Infinite Discrete Random Variable such that :

$X(\Omega) = \mathbb{N}$, then : The generating function of X is $G_X : t \mapsto \mathbb{E}(t^x)$, i.e the sum of the series :

$$\sum P(X = k) \times t^k \implies G_X : t \mapsto \sum_{k=0}^{+\infty} P(X = k) \times t^k$$

Here, P(X = k) could be named a_n which is a sequence.

We also talk about "the generating series of X"

$$t = 1 \implies \sum_{k=0}^{+\infty} P(X = k) \times t^k \text{converges and} G_X(1) = 1 \implies R \geq 1$$

Properties of G_X

(-1, 1) and -1, 1 excluded : $\begin{cases} G_X is continuous over [-1, 1], with G_X(1) = 1 \\ G_X \in C^{\infty} \end{cases}$

If Y another I.D.R.V, $Y(\Omega) = \mathbb{N}$, $G_{X+Y} = G_X + G_Y$ over $[-1, 1]$ where -1 and 1 are excluded

And : $\begin{cases} \mathbb{E}(X) = G'_X(1) \\ \mathbb{V}(X) = G''_X(1) + G'_X(1) - G'_X(1)^2 \end{cases}$

$$\sum P(X = k \times (-1)^k)$$

$$\Leftrightarrow \sum k \times (q)^{k-1} \times p$$

$$\Leftrightarrow p \sum_{k1} k \times (q)^{k-1} \times p is convergent on]-1, 1[and \mathbb{E}(X) = p \times (\frac{1}{1-q})$$

$$\frac{p}{(1-q)^2} = \frac{1}{1-q} = \frac{1}{p}$$