

# Mathématiques : CM

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# 1 Euclidean space - scalar product

**Idea : Let  $E$  a vector space of finite dimension**

The scalar product is defined as :

$$u \in E, v \in E, \langle u, v \rangle$$

$$\langle, \rangle = \begin{cases} E \times E \rightarrow \mathbb{R} \\ (u, v) \mapsto \langle u, v \rangle \end{cases}$$

When  $u$  and  $v$  are orthogonal ( $\perp$ ) we have the following relation :

$$u \perp v \equiv \langle u, v \rangle = 0$$

We also have the following relation :

$$\|u\| = \sqrt{\langle u, u \rangle}$$

**Standard scalar product of  $\mathbb{R}^3$  :**

$$u = (x_1, y_1, z_1), v = (x_2, y_2, z_2)$$

$$\langle u, v \rangle = x_1x_2 + y_1y_2 + z_1z_2$$

$$\implies \|u\| = \sqrt{\langle u, u \rangle} = \sqrt{x_1^2 + y_1^2 + z_1^2}$$

**Definition : symmetric bilinear form**

Let a mapping such that :

$$\phi : \begin{cases} E \times E \rightarrow \mathbb{R} \\ (u, v) \mapsto \phi(u, v) \end{cases}$$

$\phi$  is a symmetric bilinear form if :

1. It is bilinear :  $\forall u_0 \in E, v \mapsto \phi(u_0, v)$  is a linear map
2. It is bilinear :  $\forall v_0 \in E, u \mapsto \phi(u, v_0)$  is a linear map
3. It is symmetric :  $\forall (u, v) \in E^2, \phi(u, v) = \phi(v, u)$  is a linear map

**Explanation :**

$$\phi = \begin{cases} E \times E \rightarrow \mathbb{R} \\ (u, v) \mapsto \phi(u, v) \end{cases}$$

Let  $u_0 \in E$  :

$$\text{The mapping : } \begin{cases} E \rightarrow \mathbb{R} \\ v \mapsto \phi(u_0, v) \end{cases} \text{ is linear}$$

$$\forall (v, v') \in E^2, \forall \alpha \in \mathbb{R}, \phi(u_0, \alpha v + v') = \alpha \phi(u_0, v) + \phi(u_0, v')$$

$$\text{If } v_0 \in E, \forall (u, u') \in E^2, \forall \alpha \in \mathbb{R}, \phi(\alpha u + u', v_0) = \alpha \phi(u, v_0) + \phi(u', v_0)$$

**Example : standard scalar product of  $\mathbb{R}^3$**

$$u = (x_1, y_1, z_1)v = (x_2, y_2, z_2)$$

$$\phi(u, v) = x_1x_2 + y_1y_2 + z_1z_2$$

If u is given (considered as a constant), then the mapping :

$v \mapsto x_1x_2 + y_1y_2 + z_1z_2$  is a linear map.

Note that the coordinates of u are constant numbers.

Those of v are variables.

for example if  $u = (1, 2, 3)$ , then  $v = (x_2, 2y_2, 3z_2)$

**Similarly :** if  $v = (x_2, y_2, z_2)$  is given, then the mapping :

$u \mapsto x_1x_2 + y_1y_2 + z_1z_2$  is a linear map .

Note that the coordinates of  $v$  are being constant numbers.

Those of  $u$  are variables.

**Furthermore,**  $\phi(u, v) = \phi(v, u)$

**A symmetric bilinear form  $\phi$  is positive definite if :**

$$\forall u \in E, \phi(u, u) \geq 0 \wedge \phi(u, u) = 0 \implies u = 0_E$$

Example 1 : Standard scalar product of  $\mathbb{R}^3$

$$\phi(u, u) = x_1^2 + y_1^2 + z_1^2 \geq 0$$

and

$$\phi(u, u) = 0 \implies \begin{cases} x_1^2 = 0 \\ y_1^2 = 0 \\ z_1^2 = 0 \end{cases} \implies u = 0_E$$

The standard scalar product is thus positive definite

Example 2 :

$\phi(u, v) = x_1x_2 + y_1y_2 - z_1z_2$  is a symmetric bilinear form

If  $u = (x_1, y_1, z_1)$  is given (considered as a constant), then the mapping :

$v \mapsto x_1x_2 + y_1y_2 + z_1z_2$  is a linear map.

Note that the coordinates of  $u$  are constant numbers.

Those of  $v$  are the variables.

Thus the coordinates of  $u$  are the coefficient of the linear function.

Similarly, if  $v$  is given and constant,  $u \mapsto \phi(u, v)$  is linear

$$\implies \phi(u, v) = \phi(v, u)$$

Positive definite ? No

$$\phi(u, u)x_1^2 + y_1^2 - z_1^2$$

If  $u = (1, 0, 0)$ ,  $\phi(u, u) = 1 > 0$

If  $u = (0, 0, 1)$ ,  $\phi(u, u) = -1 < 0$

Example 3 :

$$\phi(u, v) = x_1x_2 + y_1y_2\phi(u, u) = x_1^2 + y_1^2 \geq 0$$

But  $\phi(u, u) = 0 \not\Rightarrow u = 0_E$  as for  $u = (0, 0, 1)$ ,  $\phi(u, u) = 0, u \neq 0_E$

A scalar product is a symmetric bilinear form, which is positive definite.

Then we note  $\langle u, v \rangle$  instead of  $\phi(u, v)$

We can define for  $u \in E$ ,

$$\|u\| = \sqrt{\langle u, u \rangle} \implies \sqrt{\langle u, u \rangle} \in \mathbb{R} \text{ because } \langle u, u \rangle \geq 0$$

$$\|u\| = 0 \implies u = 0_E$$

Theorem : Matrix representation of a symmetric bilinear form :

Let  $\mathcal{B} = (e_1, \dots, e_n)$  be a basis of  $E$ .

$u$  has coordinates :  $\begin{pmatrix} x_1 \\ y_1 \\ z_1 \\ \dots \end{pmatrix}$  in  $\mathcal{B}$

$v$  has coordinates :  $\begin{pmatrix} x_2 \\ y_2 \\ z_2 \\ \dots \end{pmatrix}$  in  $\mathcal{B}$

Then :  $\exists A \in \mathbb{M}_n(\mathbb{R}), \phi(u, v) = {}^+X_1AX_2$  is symmetric

Examples :

Standard scalar product of  $\mathbb{R}^3$  :

Example 1 :

$\mathcal{B}$  standard basis of  $\mathbb{R}^3$

$u = (x_1, y_1, z_1)$  has coordinates  $X_1 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$

$v = (x_2, y_2, z_2)$  has coordinates  $X_2 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$

$${}^TX_1X_2 = (x_1, y_1, z_1) \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$$

$$= x_1x_2 + y_1y_2 + z_1z_2 \implies A = I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$${}^TX_1AX_2 = x_1x_2 + y_1y_2 + z_1z_2 = \langle u, v \rangle$$

Example 2 :

$$\phi(u, v) = x_1x_2 + y_1y_2 - z_1z_2$$

$$\text{Then : } A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$${}^T X_1 A X_2 = (x_1, y_1, z_1) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$$

$$= (x_1, y_1, z_1) \begin{pmatrix} x_2 \\ y_2 \\ -z_2 \end{pmatrix}$$

$$= x_1 x_2 + y_1 y_2 - z_1 z_2$$

Example 3 :

$$\phi(u, v) = x_1 x_2 + y_1 y_2$$

$$\text{Then : } A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$${}^T X_1 A X_2 = (x_1, y_1, z_1) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$$

$$= (x_1, y_1, z_1) \begin{pmatrix} x_2 \\ y_2 \\ 0 \end{pmatrix}$$

$$= x_1 x_2 + y_1 y_2$$



## 1.1 Spectral Theorem

**Spectral Theorem :** all symmetric matrix A is diagonalizable.

We can choose an orthonormal eigenbasis  $(\epsilon_1, \dots, \epsilon_n)$ .

Then the transition matrix P satisfies to :

$${}^T P P = P \times {}^T P = I \Leftrightarrow P^{-1} = {}^T P$$

Note : if the eigenbasis is orthonormal, then :

$$\langle \epsilon_i, \epsilon_j \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

## 1.2 Theorem

**Theorem :** Let  $E = \mathbb{R}^n$ , let A be a symmetric matrix in  $\mathcal{M}_n(\mathbb{R})$ .

Let us define  $\phi(u, v) = {}^T X_1 A X_2$ .

$\phi$  is a scalar product  $\Leftrightarrow$  All eigenvalues of A are  $\neq 0$ .

Explanation in a an orthogonal eigenbasis  $\mathcal{B}'$ ,  $\phi(u, v) = {}^T X'_1 D X'_2$

Where D is a diagonal matrix.

## 2 Orthogonal Projection

Let  $V_0$  be the orthogonal projection of  $V$  onto  $F$ .

$V_0$  is the element of  $F$  the closest to  $V$ .

### 2.1 Motivation

Assume you must find  $\lambda \in \mathbb{R}$  such that :

$$v = \lambda u$$

The "best solution" is  $\lambda$  such that  $v_0 = \lambda u$

Ex:  $u = (1, 1), v = (2, 1)$

$$\text{Solving } v = \lambda u : \begin{cases} \lambda = 2 \\ \lambda = 1 \end{cases}$$

$$v_0 = \alpha u = (\alpha, \alpha)$$

$$(v - v_0) \perp u$$

$$(2 - \alpha, 1 - \alpha) \perp (1, 1)$$

$$\implies (2 - \alpha) \times 1 + (1 - \alpha) \times 1 = 0$$

$$\implies 3 - 2\alpha = 0 \implies \alpha = \frac{3}{2}$$

$$\implies v_0 = \frac{3}{2}(1, 1) = \left(\frac{3}{2}, \frac{3}{2}\right)$$

The solution does not have a solution.

However, we can determine  $v_0$ .

$v_0$  represent the solution with the smallest range of error.

## 2.2 Definition

Let  $(E, <, >)$  be an euclidean space.

Let  $F$  be a linear subspace of  $E$ .

$$F^\perp = \{u \in E, \forall v \in F, < u, v > = 0\}$$

## 2.3 Theorem

If  $F$  admits a basis  $\mathcal{B} = (e_1, e_2, \dots, e_n)$ , then  $F^\perp = \mathcal{B}^\perp$

$$F^\perp = \{u \in E, \left\{ \begin{array}{l} < u, e_1 > = 0 \\ < u, e_2 > = 0 \\ \dots \\ < u, e_n > = 0 \end{array} \right. \}$$

## 2.4 Theorem

If  $F$  is finite-dimensional, then  $F \oplus F^\perp = E$

$$\Leftrightarrow \forall u \in E, \exists!(v, w) \in F \times F^\perp, u = v + w$$

Then  $v$  is the orthogonal projection of  $u$  onto  $F$ .

## 2.5 Property

Let  $u \in E$  and let  $v_0 = P_F(u)$ .

$P_F$  is the orthogonal projection onto  $F$ .

$v_0$  is the element of  $F$  which minimizes  $\|u - v\|^2$ .

It is the solution of the optimization :

$$\text{Min}(\|u - v\|), \text{ constraint : } v \in F$$

## 2.6 Proof

Let  $u = v_0 + w$  where  $(v_0, w) \in F \times F^\perp$ .

Then  $\forall v \in F$  :

$$\|u - v\|^2 = \langle u - v, u - v \rangle \implies u - v = (u - v_0) + (v_0 - v)$$

Indeed,  $(u - v_0) = w \in F^\perp$  and  $(v_0 - v) \in F$  as  $(v_0, v) \in F^2$

$$\implies \|u - v\|^2 = \langle u - v, u - v \rangle = \langle (u - v_0) + (v_0 - v), (u - v_0) + (v_0 - v) \rangle$$

$$= \langle (u - v_0), (u - v_0) \rangle + 2 \langle (u - v_0), (v_0 - v) \rangle + \langle (v_0 - v), (v_0 - v) \rangle$$

$$= \|u - v_0\|^2 + \|v_0 - v\|^2 \geq \|u - v_0\|^2$$

$$P_F(u) \in F \Leftrightarrow \exists (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n, u = \lambda_1 e_1 + \dots + \lambda_n e_n$$

$$u - P_F(u) \in F^\perp \Leftrightarrow \begin{cases} \langle (u - \lambda_1 e_1 - \dots - \lambda_n e_n), e_1 \rangle = 0 \\ \dots \\ \langle (u - \lambda_1 e_1 - \dots - \lambda_n e_n), e_n \rangle = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \langle (\lambda_1 \langle e_1, e_1 \rangle + \dots + \lambda_n \langle e_n, e_n \rangle), e_1 \rangle = \langle u, e_1 \rangle \\ \dots \\ \langle (\lambda_1 \langle e_1, e_1 \rangle + \dots + \lambda_n \langle e_n, e_n \rangle), e_n \rangle = \langle u, e_n \rangle \end{cases}$$

## 2.7 Finding $P_F(u)$

Give  $F$  admits a basis  $\mathcal{B}_F = (\epsilon_1, \dots, \epsilon_n)$

We write  $P_F(u) \in F \implies u - P_F(u) \in F^\perp$

$$\implies u = P_F(u) + (u - P_F(u))$$

$$P_F(u) \in F \Leftrightarrow \exists(\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n, P_F(u) = \alpha_1 \epsilon_1 + \dots + \alpha_n \epsilon_n$$

$$u - P_F(u) \in F^\perp \Leftrightarrow \begin{cases} \langle u - P_F(u), \epsilon_1 \rangle = 0 \\ \dots \\ \langle u - P_F(u), \epsilon_n \rangle = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \langle u - \alpha_1 \epsilon_1 + \dots + \alpha_n \epsilon_n, \epsilon_1 \rangle = 0 \\ \dots \\ \langle u - \alpha_1 \epsilon_1 + \dots + \alpha_n \epsilon_n, \epsilon_n \rangle = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \langle \alpha_1 \langle \epsilon_1, \epsilon_1 \rangle + \dots + \alpha_n \langle \epsilon_n, \epsilon_n \rangle, \epsilon_1 \rangle = \langle u, \epsilon_1 \rangle \\ \dots \\ \langle \alpha_1 \langle \epsilon_1, \epsilon_1 \rangle + \dots + \alpha_n \langle \epsilon_n, \epsilon_n \rangle, \epsilon_n \rangle = \langle u, \epsilon_n \rangle \end{cases}$$

### 3 Data Analysis

Example of a dataset :

Student	Class	Average	Sport	French	Math	Physics	English
Student1	A1	1	SP1	FR1	MATH1	PHY1	ENG1
Student2	B2	1	SP2	FR2	MATH2	PHY2	ENG2
Student3	C1	1	SP3	FR3	MATH3	PHY3	ENG3
Student4	B2	1	SP4	FR4	MATH4	PHY4	ENG4

Table 1: Table of marks average in a high school.

The question that we adress:

- Consider the student good in maths. Are they good in physics ?

We try to find a relation :

$$PHY = \alpha MAT + \beta + Err_1$$

And compare the error with :

$$PHY = \beta + Err_2$$

The "best"  $\beta$  is the Average of PHY column.

Is  $||Err_1|| < ||Err_2||$  ?

If yes, is  $\alpha > 0$  ?

For the model :

$$PHY = \alpha MAT + \beta + Err_1$$

we look for the best solution of :

$$(S) : \begin{cases} PHY_1 = \alpha MAT_1 + \beta \\ PHY_2 = \alpha MAT_2 + \beta \\ \dots \end{cases}$$

$$\Leftrightarrow PHY = \alpha MAT + \beta \begin{pmatrix} 1 \\ 1 \\ \dots \\ 1 \end{pmatrix}$$

Where  $PHY$  and  $MAT$  are the whole column.

Optimal values :  $\alpha_0$  and  $\beta_0$  such that  $\alpha_0 MAT + \beta_0 \begin{pmatrix} 1 \\ 1 \\ \dots \\ 1 \end{pmatrix}$  is the orthogonal

projection of  $PHY$  on  $Span(Mat, \begin{pmatrix} 1 \\ 1 \\ \dots \\ 1 \end{pmatrix})$

- The students good at maths and sport, do they tend to be even better than those good only at maths ?

$$PHY = \alpha MAT + \beta \begin{pmatrix} 1 \\ 1 \\ \dots \\ 1 \end{pmatrix} + \gamma SP + Err_3$$

Find  $\alpha_0, \beta_0, \gamma_0$  optimal :

Is  $||Err_3|| < ||Err_1||$  ?

If yes, sign of  $\beta$  ?

Let  $X$  be a column of the table.

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix}$$

$X$  is an identically distributed random variable and independant.

Empirical expectation of  $X$  :

$$\hat{\mathbb{E}}(X) = \frac{x_1 + x_2 + \dots + x_n}{n} = \frac{{}^T \begin{pmatrix} 1 \\ \dots \\ 1 \end{pmatrix} \cdot X}{n}$$

Empirical variance of X :

$$\begin{aligned} \widehat{\text{Var}}(X) &= \hat{\mathbb{E}}((X - \hat{\mathbb{E}}(X))^2) \\ &= \frac{(x_1 - \hat{\mathbb{E}}(X))^2 + (x_2 - \hat{\mathbb{E}}(X))^2 + \dots + (x_n - \hat{\mathbb{E}}(X))^2}{n} \end{aligned}$$

$$\text{Let } X_c = X - \hat{\mathbb{E}}(X) \begin{pmatrix} 1 \\ \dots \\ 1 \end{pmatrix}$$

$$X_c = \begin{pmatrix} x_1 - \hat{\mathbb{E}}(X) \\ x_2 - \hat{\mathbb{E}}(X) \\ \dots x_n - \hat{\mathbb{E}}(X) \end{pmatrix}$$

$X_c$  is the centered version of X.

$$\text{Thus } \widehat{\text{Var}} = \frac{(x_1 - \hat{\mathbb{E}}(X))^2 + (x_2 - \hat{\mathbb{E}}(X))^2 + \dots + (x_n - \hat{\mathbb{E}}(X))^2}{n}$$

$$= \frac{1}{n} \begin{pmatrix} x_1 - \hat{\mathbb{E}}(X) & x_2 - \hat{\mathbb{E}}(X) & \dots & x_n - \hat{\mathbb{E}}(X) \end{pmatrix} \begin{pmatrix} x_1 - \hat{\mathbb{E}}(X) \\ x_2 - \hat{\mathbb{E}}(X) \\ \dots \\ x_n - \hat{\mathbb{E}}(X) \end{pmatrix}$$

$$= \frac{{}^T X_c X_c}{n} = \frac{||X_c||^2}{n}$$



Empirical standard deviation of X :

$$\hat{\sigma}(X) = \sqrt{\widehat{\text{Var}}(X)} = \sqrt{\frac{^T X_c X_c}{n}} = \frac{\|X_c\|}{\sqrt{n}}$$

Consider two columns  $X = \begin{pmatrix} x_1 \\ \dots \\ x_n \end{pmatrix}$  and  $Y = \begin{pmatrix} y_1 \\ \dots \\ y_n \end{pmatrix}$

Empirical covariance of X and Y :

$$\begin{aligned} \widehat{\text{Cov}}(X, Y) &= \hat{\mathbb{E}}((X - \hat{\mathbb{E}}(X)) \times (Y - \hat{\mathbb{E}}(Y))) \\ &= \frac{1}{n} (x_1 - \hat{\mathbb{E}}(X))(y_1 - \hat{\mathbb{E}}(Y)) + \dots + (x_n - \hat{\mathbb{E}}(X))(y_n - \hat{\mathbb{E}}(Y)) \\ &= \frac{1}{n} \begin{pmatrix} x_1 - \hat{\mathbb{E}}(X) & x_2 - \hat{\mathbb{E}}(X) & \dots & x_n - \hat{\mathbb{E}}(X) \end{pmatrix} \begin{pmatrix} y_1 - \hat{\mathbb{E}}(Y) \\ y_2 - \hat{\mathbb{E}}(Y) \\ \dots \\ y_n - \hat{\mathbb{E}}(Y) \end{pmatrix} \\ &= \frac{^T X_c Y_c}{n} \\ &= \frac{1}{n} \langle X_c, Y_c \rangle \end{aligned}$$

Empirical correlation coefficient of X and Y :

$$\hat{c}(X, Y) = \frac{\widehat{\text{Cov}}(X, Y)}{\hat{\sigma}(X)\hat{\sigma}(Y)}$$

Note that  $\widehat{\text{Cov}}(X, X) = \widehat{\text{Var}}(X)$  and  $\widehat{\text{Cov}}(Y, Y) = \widehat{\text{Var}}(Y)$

Let  $D = \text{Dataset} = \begin{pmatrix} X & Y \end{pmatrix} = \begin{pmatrix} x_1 & y_1 \\ \dots & \dots \\ x_n & y_n \end{pmatrix}$

$D_c = \text{centered data set} = \begin{pmatrix} X_c & Y_c \end{pmatrix}$

The Empirical covariance matrix of D is :

$$\widehat{\text{Cov}}(D) = \begin{pmatrix} \widehat{\text{Var}}(X) & \widehat{\text{Cov}}(X, Y) \\ \widehat{\text{Cov}}(X, Y) & \widehat{\text{Var}}(Y) \end{pmatrix} = \frac{{}^T D_c D_c}{n}$$

$$\begin{aligned} {}^T D_c D_c &= \begin{pmatrix} x_1 - \hat{\mathbb{E}}(X) & \dots & x_n - \hat{\mathbb{E}}(X) \\ y_1 - \hat{\mathbb{E}}(Y) & \dots & y_n - \hat{\mathbb{E}}(Y) \end{pmatrix} \begin{pmatrix} x_1 - \hat{\mathbb{E}}(X) & y_1 - \hat{\mathbb{E}}(Y) \\ \dots & \dots \\ x_n - \hat{\mathbb{E}}(X) & y_n - \hat{\mathbb{E}}(Y) \end{pmatrix} \\ &= \begin{pmatrix} {}^T X_c X_c & {}^T X_c Y_c \\ {}^T Y_c X_c & {}^T Y_c Y_c \end{pmatrix} \end{aligned}$$

Ex : find  $\widehat{\text{Var}}(aX + bY)$  where  $aX + bY = D \begin{pmatrix} a \\ b \end{pmatrix}$

$$D \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \dots & \dots \\ x_n & y_n \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} ax_1 + by_1 \\ ax_2 + by_2 \\ \dots \\ ax_n + by_n \end{pmatrix} = aX + bY$$

$$\widehat{\text{Var}}(X) = {}^T X_c X_c$$

$$\begin{aligned} \widehat{\text{Var}}\left(D \begin{pmatrix} a \\ b \end{pmatrix}\right) &= (D_c \begin{pmatrix} a \\ b \end{pmatrix})(D_c \begin{pmatrix} a \\ b \end{pmatrix}) \times \frac{1}{n} \\ &= (a \ b) {}^T D_c D_c \begin{pmatrix} a \\ b \end{pmatrix} \times \frac{1}{n} \\ &= (a \ b) \widehat{\text{Cov}}(D) \begin{pmatrix} a \\ b \end{pmatrix} \end{aligned}$$

### 3.1 Correlation coefficient

**Definition :**

$$\hat{r}(X, Y) = \frac{\widehat{\text{Cov}}(X, Y)}{\hat{\sigma}(X)\hat{\sigma}(Y)}$$

$$\widehat{\text{Cov}}(X, Y) = \frac{1}{n} {}^T X_c Y_c = \frac{1}{n} \langle X_c, Y_c \rangle$$

$$\hat{\sigma}(X) = \sqrt{\frac{1}{n} {}^T X_c X_c}$$

$$= \frac{1}{\sqrt{n}} \sqrt{X_c X_c} = \frac{1}{\sqrt{n}} \|X_c\|$$

$$\hat{\sigma}(Y) = \frac{1}{\sqrt{n}} \|Y_c\|$$

$$\Rightarrow \hat{r}(X, Y) = \frac{\langle X_c, Y_c \rangle}{\|X_c\| \cdot \|Y_c\|}$$

In all euclidean space,

$$-\|u\| \cdot \|v\| \leq \langle u, v \rangle \leq \|u\| \cdot \|v\|$$

Explanation :

$$\begin{cases} u = u_1 + u_2 \\ u_1 = \lambda v \\ \langle u_2, v \rangle = 0 \\ \lambda \in \mathbb{R} \end{cases}$$

$$\begin{aligned} \langle u, v \rangle &= \langle u_1 + u_2, v \rangle = \langle u_1, v \rangle + \langle u_2, v \rangle \\ &= \lambda \langle v, v \rangle = \lambda \|v\|^2 \end{aligned}$$

$$\lambda \|v\|^2 = \lambda \|v\| \cdot \|v\|$$

$$\Rightarrow -\|u\|^2 \leq \lambda \|v\| \leq \|u\|$$

$$(\lambda \|v\|)^2 = \lambda^2 \langle v, v \rangle$$

$$\langle \lambda v, \lambda v \rangle = \langle u_1, u_1 \rangle$$

$$= \|u_1\|^2 \leq \|u\|^2$$

Because  $\|u\|^2 = \|u_1\|^2 + \|u_2\|^2$

Consequence :  $-1 \leq \hat{r}(X, Y) \leq 1$

$$\widehat{\text{Cov}}(aX + bY, a'X + b'Y) = \begin{pmatrix} a & b \end{pmatrix} \widehat{\text{Cov}}(D) \begin{pmatrix} a' \\ b' \end{pmatrix}$$