Mathématiques : CM

$$(\Omega, P(\Omega), \mathbb{P}) \Leftrightarrow$$

A random variable X is defined by:

$$X : \Omega \to \mathbb{R} \quad \text{such that} \quad \omega \mapsto X(\omega)$$

The generative function is defined by :

$$P : \mathbb{R} \to \mathbb{R} \quad t \mapsto a_0 * t^0 + a_1 * t^1 + a_2 * t^2 + ... + a_n * t^n$$

$$\{(\text{X} = \text{k}),\ \text{k} \ \in X(\Omega)\ \} = \text{partition of } \Omega$$

Generating Function

0.1 Definition :

Let X a random variable such that $X(\Omega) = [\,|0, n|\,]\,, n \in \mathbb{N}^*$.

we call generating function of X the following polynomial :

$$G_X : \begin{cases} \mathbb{R} \to \mathbb{R} \\ G \mapsto \sum_{k \in X(\Omega)} P(X = k) * t^k \end{cases}$$

Remarks

$\alpha(\Omega)$ can be different we can have : $\text{X}(\Omega) \subset [\,|\alpha, n|\,]$

0.2 Bernouilli :

$$X \sim B(P) \Leftrightarrow \begin{cases} X(\Omega) = \{0, 1\} \\ P(X = 0) = (1 - p), P(X = 1) = p \end{cases}$$

$$\Leftrightarrow G_X \begin{cases} \mathbb{R} \to \mathbb{R} \\ t \mapsto (1 - p) + pt \end{cases}$$

0.2.1 Expected value and variance :

Theorem :

Let X a random variable and G_X its generative function :

$$\begin{cases} G_X(1) = 1 \\ \mathbb{E}(X) = G'_X(1) \\ \mathbb{V}(X) = G''_X(1) + G'_X(1) - (G'_X(1))^2 \end{cases}$$

0.2.2 Proof :

By definition :

$$G_X : \begin{cases} \mathbb{R} \to \mathbb{R} \\ t \mapsto \sum_{k=0}^{n} P(X = k) * t^k \end{cases}$$

$\Rightarrow G_X(1) = \displaystyle\sum_{k=0}^{n} P(X = k) = 1$ because $\{(X = k), k \in [\,|0, n|\,]\}$ is a partition of Ω

$$\Rightarrow G_X : \begin{cases} \mathbb{R} \to \mathbb{R} \\ t \mapsto \sum_{k=1}^{n} k \times P(X = k) \times t^k \end{cases}$$

$$\Rightarrow G'_X : \begin{cases} \mathbb{R} \to \mathbb{R} \\ t \mapsto \sum_{k=1}^{n} k \times P(X = k) \times t^{k-1} \Rightarrow G'_X = \sum_{k=1}^{n} = \sum_{k=0}^{n} k \times P(X = k) = \mathbb{E}(X) \end{cases}$$

$$\mathbb{V}(X) = \mathbb{E}(X) - (\mathbb{E}(X))^2 = \mathbb{E}(X^2) - E(X)^2$$

0.2.3 Koenij-Huygens Theorem :

$$\mathbb{V}(X) = \mathbb{E}((X - \mathbb{E}(X))^2)$$

$$\mathbb{V}(X) = \mathbb{E}(X^2) - \mathbb{E}^2(X))$$

$$G_X(t) = \sum_{k=0}^{n} P(X = k) \times t^k$$

$$G'_X(t) = \sum_{k=1}^{n} k \times P(X = k) \times t^{k-1}$$

$$G''_X(t) = \sum_{k=2}^{n} k(k-1) \times P(X=k) \times t^{k-2}$$

We have :

$$k(k-1)P(X=k) \times t^{k-2} = k^2 \times P(X=k) \times t^{k-2} - k * P(X=k) \times t^{k-2}$$

$$\Rightarrow \sum_{k=2}^{n} k^2 \times P(X=k) \times t^{k-2} - \sum_{k=2}^{n} k \times P(X=k) \times t^{k-2} = G''_X(t)$$

$$\Rightarrow G''_X(1) = \sum_{k=0}^{n} k^2 \times P(X=k) - \sum_{k=0}^{n} k \times P(X=k)$$

$$\Rightarrow G''_X(1) = \mathbb{E}(X^2) - E(X) \ (1)$$

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$$\Rightarrow G''_X(1) + G'_X(1) = \mathbb{E}(X^2)$$

$$\Rightarrow G''_X(1) + G'_x(1) - (G'_x(1))^2 = \mathbb{E}(X^2) - \mathbb{E}(X)$$

$$\Rightarrow G''_X(1) + G'_x(1) - (G'_x(1))^2 = \mathbb{V}(X)$$

0.3 X + Y :

Let X and Y two finite random variable then :

$$G_{X+Y} : \begin{cases} \mathbb{R} \to \mathbb{R} \\ t \mapsto G_X(t) \times G_Y(t) \end{cases}$$

$$\Leftrightarrow G_{X+Y} = G_X \times G_Y$$

Ex :

$$Y \sim B(n,p) \Leftrightarrow \begin{cases} Y(\Omega) = [\,|0,n|\,] \\ \forall k \in Y(\Omega), P(Y=k) = \binom{n}{k} \times p^k \times q^{n-k} \end{cases}$$

$$\Leftrightarrow Y = \sum_{i=1}^{n} X_i$$

$$G_Y = \prod_{i=1}^{n} G_{X_i} : t \mapsto (q+pt)^n = \sum_{k=0}^{n} \binom{n}{k} \times q^{n-k} \times (pt)^k$$

$$\sum_{k=0}^{n} \binom{n}{k} \times q^{n-k} \times p^k \times t^k$$

$$\Rightarrow \mathbb{P}(Y=t) \Leftrightarrow G_{X+Y} = G_X \times G_Y$$