Mathématiques : CM

Contents

1 Euclidean space - scalar product

Idea : Let E a vector space of finite dimension

The scalar product is defined as :

$$u \in E, v \in E, < u, v >$$

$$<, > = \begin{cases} E \times E & \to \mathbb{R} \\ (u, v) & \mapsto < u, v > \end{cases}$$

When u and v are orthogonal ($\perp$) we have the following relation :

$$u \perp v \equiv < u, v >= 0$$

We also have the following relation :

$$||u|| = \sqrt{< u, u >}$$

Standard scalar product of $\mathbb{R}^3$:

$$u = (x_1, y_1, z_1) v = (x_2, y_2, z_2)$$

$$< u, v >= x_1 x_2 + y_1 y_2 + z_1 z_2$$
$$\implies ||u|| = \sqrt{< u, u >} = \sqrt{x_1^2 + y_1^2 + z_1^2}$$

Definition : symmetric bilinear form

Let a mapping such that :

$$\phi : \begin{cases} E \times E & \to \mathbb{R} \\ (u, v) & \mapsto \phi(u, v) \end{cases}$$

ϕ is a symmetric bilinear form if :

1. It is bilinear : $\forall\, u_0 \in\, E, v \,\mapsto\, \phi(u_0, v)$ is a linear map
2. It is bilinear : $\forall\, v_0 \in\, E, u \,\mapsto\, \phi(u, v_0)$ is a linear map
3. It is symmetric : $\forall\, (u, v) \in\, E^2, \phi(u, v) = \phi(v, u)$ is a linear map

Explanation :

$$\phi = \begin{cases} E \times E & \rightarrow \mathbb{R} \\ (u, v) & \mapsto \phi(u, v) \end{cases}$$

Let $u_0 \in E$:

The mapping : $\begin{cases} E & \rightarrow \mathbb{R} \\ v & \mapsto \phi(u_0, v) \end{cases}$ is linear

$$\forall\, (v, v') \in E^2, \forall \alpha \in \mathbb{R}, \phi(u_0, \alpha v + v') = \alpha \phi(u_0, v) + \phi(u_0, v')$$

If $v_0 \in E, \forall (u, u') \in E^2, \forall \alpha \in \mathbb{R}, \phi(\alpha u + u', v_0) = \alpha \phi(u, v_0) + \phi(u', v_0)$

Example : standard scalar product of $\mathbb{R}^3$

$$u = (x_1, y_1, z_1) v = (x_2, y_2, z_2)$$

$$\phi(u, v) = x_1 x_2 + y_1 y_2 + z_1 z_2$$

If u is given (considered as a constant), then the mapping :

$v \,\mapsto\, x_1 x_2 + y_1 y_2 + z_1 z_2$ is a linear map.

Note that the coordinates of u are constant numbers.

Those of v are variables.

for example if $u = (1, 2, 3)$, then $v = (x_2, 2y_2, 3z_2)$

Similarly :, if $v = (x_2, y_2, z_2)$ is given, then the mapping :

$u \mapsto x_1x_2 + y_1y_2 + z_1z_2$ is a linear map .

Note that the coordinates of v are being constant numbers.

Those of u are variables.

Furthermore, $\phi(u, v) = \phi(v, u)$

A symmetric bilinear form ϕ is positive definite if :

$$\forall u \in E, \phi(u, u) \geq 0 \wedge \phi(u, u) = 0 \implies u = 0_E$$

Example 1 : Standard scalar product of $\mathbb{R}^3$

$$\phi(u, u) = x_1^2 + y_1^2 + z_1^2 \geq 0$$

and

$$\phi(u, u) = 0 \implies \begin{cases} x_1^2 = 0 \\ y_1^2 = 0 \\ z_1^2 = 0 \end{cases} \implies u = 0_E$$

The standard scalar product is thus positive definite

Example 2 :

$$\phi(u, v) = x_1x_2 + y_1y_2 - z_1z_2 \text{ is a symmetric bilinear form}$$

If $u = (x_1, y_1, z_1)$ is given (considered as a constant), then the mapping :

$v \mapsto x_1x_2 + y_1y_2 + z_1z_2$ is a linear map.

Note that the coordinates of u are constant numbers.

Those of v are the variables.

Thus the coordinates of u are the coefficient of the linear function.

Similarly, if v is given and constant, $u \mapsto \phi(u, v)$ is linear

$$\implies \phi(u, v) = \phi(v, u)$$

Positive definite ? No

$$\phi(u,u)x_1^2 + y_1^2 - z_1^2$$

If $u = (1,0,0), \phi(u,u) = 1 > 0$

If $u = (0,0,1), \phi(u,u) = -1 < 0$

Example 3 :

$$\phi(u,v) = x_1x_2 + y_1y_2\phi(u,u) = x_1^2 + y_1^2 \geq 0$$

But $\phi(u,u) = 0 \not\Longrightarrow u = 0_E$ as for $u = (0,0,1), \phi(u,u) = 0, u \neq 0_E$

A scalar product is a symmetric bilinear form, which is positive definite.

Then we note $< u, v >$ instead of $\phi(u,v)$

We can define for $u \in E$,

$$||u|| = \sqrt{< u, v >} \implies \sqrt{< u, v >} \in \mathbb{R} \text{ because } < u, v > \geq 0$$

$$||u|| = 0 \implies u = 0_E$$

Theorem : Matrix representation of a symmetric bilinear form :

Let $\mathcal{B} = (e_1, ..., e_n)$ be a basis of E.

u has coordinates : $\begin{pmatrix} x_1 \\ y_1 \\ z_1 \\ ... \end{pmatrix}$ in $\mathcal{B}$

v has coordinates : $\begin{pmatrix} x_2 \\ y_2 \\ z_2 \\ \dots \end{pmatrix}$ in $\mathcal{B}$

Then : $\exists\, A \in \mathbb{M}_n(\mathbb{R}), \phi(u, v) = {}^{+}X_1 A X_2$ is symmetric

Examples :

Standard scalar product of $\mathbb{R}^3$:

Example 1 :

$\mathcal{B}$ standard basis of $\mathbb{R}^3$

$u = (x_1, y_1, z_1)$ has coordinates $X_1 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$

$v = (x_2, y_2, z_2)$ has coordinates $X_2 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$

$$ {}^{T}X_1 X_2 = (x_1, y_1, y_2) \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} $$

$$ = x_1 x_2 + y_1 y_2 + z_1 z_2 \implies A = I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} $$

$$ {}^{T}X_1 A X_2 = x_1 x_2 + y_1 y_2 + z_1 z_2 = < u, v > $$

Example 2 :

$$ \phi(u, v) = x_1 x_2 + y_1 y_2 - z_1 z_2 $$

$$ \text{Then : } A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} $$

$$^{T}X_1AX_2 = (x_1, y_1, z_1)\begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & -1\end{pmatrix}\begin{pmatrix}x_2\\y_2\\z_2\end{pmatrix}$$

$$= (x_1, y_1, z_1)\begin{pmatrix}x_2\\y_2\\-z_2\end{pmatrix}$$

$$= x_1x_2 + y_1y_2 - z_1z_2$$

Example 3 :

$$\phi(u, v) = x_1x_2 + y_1y_2$$

$$\text{Then : } A = \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 0\end{pmatrix}$$

$$^{T}X_1AX_2 = (x_1, y_1, z_1)\begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 0\end{pmatrix}\begin{pmatrix}x_2\\y_2\\z_2\end{pmatrix}$$

$$= (x_1, y_1, z_1)\begin{pmatrix}x_2\\y_2\\0\end{pmatrix}$$

$$= x_1x_2 + y_1y_2$$

2 Spectral Theorem

Spectral Theorem : all symmetric matrix A is diagonalizable.

We can choose an orthonormal eigenbasis $(\epsilon_1, ..., \epsilon_n)$.

Then the transition matrix P satisfies to :

$$^{T}PP = P \times {}^{T}P = I \Leftrightarrow P^{-1} = {}^{T}P$$

Note : if the eigenbasis is orthonormal, then :

$$< \epsilon_i, \epsilon_j >= \begin{cases} 1 \text{ if } i = j \\ 0 \text{ otherwise} \end{cases}$$

3 Theorem

Theorem : Let E = $\mathbb{R}^n$, let A be a symmetric matrix in $\mathcal{M}_n(\mathbb{R})$.

Let us define $\phi(u, v) = {}^{T}X_1AX_2$.

ϕ is a scalar product $\Leftrightarrow$ All eigenvalues of A are ¿ 0.

Explanation in a an orthogonal eigenbasis $\mathcal{B}'$, $\phi(u, v) = {}^{T}X_1'DX_2'$

Where D is a diagonal matrix.