Topic 6: Random matrix models (part I)

5354ACMT6Y ACMT

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Outline

Conservative systems

- CM: symmetries $\Leftrightarrow$ constants of motion
- QM: symmetries $\Leftrightarrow$ quantum numbers

Definition: Conservative system

Invariant w.r.t. time translation, i.e. $H(t+\tau) = H(t)$.

- If system conservative:
  1. Solve Schrödinger equation $i\hbar\frac{\partial\psi}{\partial t} = H\psi$ by separation of variables
  2. Ansatz $\psi_n(x,t) = \psi_n(x)\exp(iE_n t/\hbar)$, with $H\psi_n = E_n\psi_n$
  3. Thus, energy is conserved quantity, with quantum number n
- Completely chaotic system: No other conserved quantities

Solving (finite-dimensional) conservative systems

- How to solve Schrödinger eq. for finite-dimensional conservative systems?
  1. Expand eigenfunctions $\psi_n(x)$ into set of mutually orthogonal basis functions $\{\phi_n(x)\}$:

$$\psi_n(x) = \sum_m a_{nm}\phi_m(x)\,, \qquad \langle\psi_n|\psi_m\rangle = \int \phi_n^*(x)\psi_m(x)\mathrm{d}x = \delta_{nm}\,.$$

  2. After substitution into Schrödinger eq., we find the matrix representation

$$\sum_m H_{nm}a_m = E_n a_n\,,$$

$$H_{nm} = \langle\phi_n|\,H_{nm}\,|\phi_m\rangle = \int \phi_n^*(x)H\phi_m(x)\mathrm{d}x\,.$$

- Solving Schrödinger equation $\rightarrow$ diagonalizing H

Exploiting symmetries (I)

- Let R be an operator representing an additional symmetry
- Then

$$[H, R] = 0\,,$$

  implying we can diagonalize H in eigenbasis of R, given by $\{\phi_{n,\alpha}\}$ with $R\phi_{n,\alpha} = r_n\phi_{n,\alpha}$
- In this basis, rewrite $[H, R] = 0$ as

$$0 = \langle\phi_{n,\alpha}|\, RH - HR\,|\phi_{m,\beta}\rangle = (r_n - r_m)\,\langle\phi_{n,\alpha}|\, H\,|\phi_{m,\beta}\rangle\ .$$

- Since we assumed r_n to be different,

$$\langle\phi_{n,\alpha}|\, H\,|\phi_{m,\beta}\rangle = \delta_{nm}H^{(n)}_{\alpha\beta}\,, \qquad \text{with}\ \ H^{(n)}_{\alpha\beta} = \langle\phi_{n,\alpha}|\, H\,|\phi_{n,\beta}\rangle\ .$$

Exploiting symmetries (II)

- Hence, the matrix representation of H has been reduced to block-diagonal form:

$$H = \begin{pmatrix} H^{(1)} & 0 & \cdots \\ 0 & H^{(2)} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix},$$

  with $H^{(n)} = H^{(n)}_{\alpha\beta} = \langle \phi_{n,\alpha} | \, H \, | \phi_{n,\beta} \rangle$.
- Repeat until all symmetries have been exhausted.
- **Conclusion**: symmetries make H 'closer to diagonal'
- Completely integrable systems ($\#$ conserved quantities $= \#$ d.o.f.): H completely diagonal

Wigner and Poisson statistics in (non)-integrable systems

Recap:

integrable systems (non-chaotic) $\Rightarrow$ 'many' conserved quantities
non-integrable systems (chaotic) $\Rightarrow$ 'few' conserved quantities

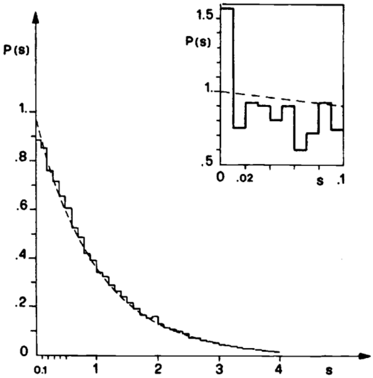
Examples:

- Integrable: rectangular billiard (2 d.o.f.)
  - Symmetries: parity $(\times 2) \to p_x^2,\ p_y^2$
- Non-integrable: atomic nuclei (many d.o.f.)
  - Symmetries: time-translation, rotation, parity $\to E,\ I^\pi\ (\pi = \pm)$ and m_I

How to do statistics on QM systems?

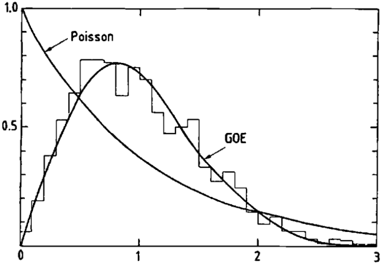
1. Arrange energy levels into subspectra in the same Hilbert space sector
2. Normalize spectra such that average d.o.s. is one

Popular measure: distribution $p(s)$ of spacings $s_n = E_n - E_{n-1}$



(a) Level spacing distr. for rectangular billiard follows **Poisson** [5, 2]:

$$\boxed{p(s) = \exp(-s)}$$



(b) Level spacing distr. for nuclear data ensemble follows **Wigner-Dyson** [5, 1]:

$$\boxed{p(s) = \frac{\pi}{2} s \exp\left(-\frac{\pi}{4} s^2\right)}$$

Universality classes

Pertains to symmetries and interactions of the Hamiltonian.[1]

- **UUC**: Unitary universality class
- **OUC**: Orthogonal universality class
- **SUC**: Symplectic universality class

class	Properties of system		Representations of H
UUC	no time-reversal symmetry		Hermitian, and $H = UHU^\dagger\,,\quad UU^\dagger = \mathbb{1}$
OUC	time-reversal symmetry	no spin-$\frac{1}{2}$ interactions	real symmetric, and $H = OHO^T\,,\quad OO^T = \mathbb{1}$
SUC		spin-$\frac{1}{2}$ interactions	quaternion real, and $H = SHS^R\,,\quad SS^R = \mathbb{1}$

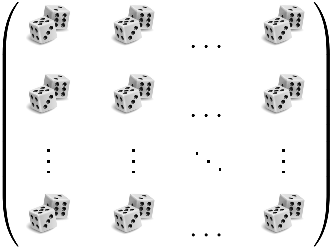
Overwhelming majority of systems fall into OUC.

[1] Note: Not (directly?) connected to scale-invariant limits under RG-flow.

Random matrices

- We have seen: Wigner distributions arise in many chaotic systems (atomic nucleus, billiards etc.)
- Suggestion: Details of the interaction *don't matter that much*
- Idea: Replace matrix elements of H with random numbers!

$$H = \begin{pmatrix} H_{11} & H_{12} & \cdots & H_{1N} \\ H_{21} & H_{22} & \cdots & H_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ H_{N1} & H_{N2} & \cdots & H_{NN} \end{pmatrix} \longrightarrow$$



- However, we still need to satisfy the constraints from the universality class

Gaussian ensembles (I)

- Focus on OUC, because of its ubiquity
- Real-symmetric matrices, with $N(N+1)/2$ independent entries
- Assumption 1:

$$p(H_{11}, \ldots, H_{NN}) = p(H'_{11}, \ldots, H'_{NN}),$$

where $H' = OHO^T$, with $OO^T = \mathbb{1}$.

- Observation 1: Trace of powers of H invariant under orthogonal transformations:

$$\mathrm{Tr}(H') = \mathrm{Tr}\left(OHO^T\right) = \mathrm{Tr}\left(HO^TO\right) = \mathrm{Tr}(H),$$
$$\mathrm{Tr}(H'^2) = \mathrm{Tr}\left(OHO^TOHO^T\right) = \mathrm{Tr}\left(OHO^T\right) = \mathrm{Tr}(H^2), \ldots$$

and therefore we restrict $p(H_{11}, \ldots, H_{NN}) = f[\mathrm{Tr}(H), \mathrm{Tr}(H^2), \ldots]$.

Gaussian ensembles (II)

- Assumption 2:

$$p(H_{11},\ldots,H_{NN}) = p(H_{11})p(H_{12})\cdots p(H_{NN})\,.$$

- Assumption 1 + Observation 1 + Assumption 2:

$$\boxed{p(H_{11},\ldots,H_{NN}) = C\exp\left[-B\,\mathrm{Tr}(H) - A\,\mathrm{Tr}\left(H^2\right)\right]\,.}$$

- We can set $B = 0$ (shift energy)

Gaussian ensembles (III)

$$\boxed{p(H_{11},\ldots,H_{NN}) = C\exp\left[-A\,\mathrm{Tr}\left(H^2\right)\right].}$$

- We can fix C by imposing normalization:

$$C\int \exp\left[-A\,\mathrm{Tr}\left(H^2\right)\right]\mathrm{d}H_{11}\cdots\mathrm{d}H_{NN} = 1\,,$$

$$C\int \exp\left[-AH_{11}^2\right]\mathrm{d}H_{11}\int \exp\left[-AH_{12}^2\right]\mathrm{d}H_{12}\cdots\int \exp\left[-AH_{NN}^2\right]\mathrm{d}H_{NN} = 1\,,$$

and then performing Gaussian integration:

$$C\underbrace{\prod_n^N \int \mathrm{d}H_{nn}\exp\left[-AH_{nn}^2\right]}_{\text{diagonal}}\underbrace{\prod_{n\neq m}\int \mathrm{d}H_{nm}\exp\left[-2AH_{nm}^2\right]}_{\text{off-diagonal}} = 1\,,$$

$$C\left(\sqrt{\frac{\pi}{A}}\right)^N\left(\sqrt{\frac{\pi}{2A}}\right)^{N(N-1)} = 1\,.$$

Gaussian ensembles (IV)

- We found:

$$\boxed{p(H_{11},\ldots,H_{NN}) = \left(\frac{A}{\pi}\right)^{N/2}\left(\frac{2A}{\pi}\right)^{N(N-1)/2} \exp\left[-A\sum_{n,m} H_{nm}^2\right].}$$

- The set of real random matrices with matrix elements obeyeing this distribution defines the **Gaussian Orthogonal Ensemble** (GOE).
- In an analogous way, we could have derived the *Gaussian Unitary Ensemble* (GUE) and the *Gaussian Symplectic Ensemble* (GSE).

Correlated eigenenergy distribution (I)

- Knowing H_{nm} not super useful...
- Instead, we would like a distribution of eigenenergies (measurable!)
- This amounts to a change of variables, so based on previous result:

$$p(H_{11},\ldots,H_{NN})\mathrm{d}H_{nm} \sim \exp\left(-A\sum_k E_k^2\right)|J|\mathrm{d}E_k\mathrm{d}p_\alpha\,,$$

where we used basis-independence of the trace to write $\sum_{n,m} H_{nm}^2 = \mathrm{Tr}\big(H^2\big) = \sum_k E_k^2$, and the Jacobian is

$$|J| = \left|\frac{\partial(H_{nm})}{\partial(E_k,p_\alpha)}\right|.$$

Correlated eigenenergy distribution (II)

- Use the fact that every symmetric matrix can be diagonalized by means of orthogonal transformation:

$$H = OH_D O^T \quad \Longrightarrow \quad H_{nm} = \sum_{k,l} O_{nk}(H_D)_{kl} O^T_{lm} = \sum_{k,l} O_{nk} E_k \delta_{kl} O_{ml}$$
$$= \sum_{k} O_{nk} E_k O_{mk} \,.$$

- Then the components of $|J|$ are:

$$\frac{\partial H_{nm}}{\partial E_k} = \sum_{a,b} O_{na} \frac{\partial (H_D)_{ab}}{\partial E_k} O^T_{bm} = \sum_{a,b} O_{na} \frac{\partial E_a \delta_{ab}}{\partial E_k} O_{mb} = \sum_{a,b} O_{na} \delta_{ak} \delta_{ab} O_{mb}$$
$$= \boxed{\sum_{k} O_{nk} O_{mk}} \,,$$
$$\frac{\partial H_{nm}}{\partial p_\alpha} = \sum_{i,j} O_{ni} O_{mj} (S_\alpha H_D - H_D S_\alpha)_{ij} = \sum_{i,j} O_{ni} O_{mj} \Big((S_\alpha)_{ik} (H_D)_{kj} - (H_D)_{ik} (S_\alpha)_{kj} \Big)$$
$$= \sum_{i,j} O_{ni} O_{mj} \Big((S_\alpha)_{ik} \delta_{kj} E_k - \delta_{ik} E_k (S_\alpha)_{kj} \Big) = \boxed{\sum_{i,j} O_{ni} O_{mj} (S_\alpha)_{ij} \Big(E_j - E_i \Big)} \,.$$

Correlated eigenenergy distribution (III)

- Then we can write:

$$J_{nm,k\alpha} = \sum_{i,j} O_{ni} O_{mj} \begin{pmatrix} \delta_{ij} & 0 \\ 0 & (S_\alpha)_{ij}(E_j - E_i) \end{pmatrix} \equiv \sum_{i,j} (\hat{O})_{nm,ij} M_{ij,k\alpha} \,,$$

- And because $\det(AB) = \det(A)\det(B)$, we obtain:

$$|J| = |\hat{O}| \cdot |M| = |\hat{O}| \cdot |S| \cdot \prod_{i>j} (E_i - E_j) \,.$$

- Finally, plug this back into

$$p(H_{11}, \ldots, H_{NN}) \mathrm{d}H_{nm} \sim \exp\left(-A \sum_k E_k^2 \right) |J| \mathrm{d}E_k \mathrm{d}p_\alpha \,,$$

and perform the integration over p_α.

Correlated eigenenergy distribution (IV)

- Result: correlated distribution function for the GOE:

$$P(E_1, \ldots, E_N) \sim \prod_{n>m} (E_n - E_m) \exp\left(-A \sum_n E_n^2\right).$$

- For the GUE and GSE, the ensembles can also be derived, see for instance Mehta [4]. The results can be cast into a single expression:

$$\boxed{P(E_1, \ldots, E_N) \sim \prod_{n>m} (E_n - E_m)^{\nu} \exp\left(-A \sum_n E_n^2\right)},$$

with *universality index* ν:

  - $\nu = 1$: GOE
  - $\nu = 2$: GUE
  - $\nu = 4$: GSE
  - $\nu = 0$: E_i uncorrelated (Poisson ensemble)

Nearest neighbour distance distribution

- Restrict to Gaussian ensembles of 2×2 matrices
- Starting from correlated distribution function $P(E_1, E_2)$, calculate

$$
\begin{aligned}
p(s) &= \int_{-\infty}^{+\infty} \mathrm{d}E_1 \int_{-\infty}^{+\infty} \mathrm{d}E_2 P(E_1, E_2)\delta(s - |E_1 - E_2|) \\
&= C \int_{-\infty}^{+\infty} \mathrm{d}E_1 \int_{-\infty}^{+\infty} \mathrm{d}E_2 |E_1 - E_2|^{\nu} \exp\left(-A \sum_n E_n^2\right)\delta(s - |E_1 - E_2|)\,,
\end{aligned}
$$

  and fix A, C by normalizing both the mean and first moment of the level spacing s:

$$
p(s) = \begin{cases} \frac{\pi}{2} s \exp\left(-\frac{\pi}{4} s^2\right)\,, & \nu = 1 \;\; (\mathrm{GOE})\,, \\ \frac{32}{\pi^2} s^2 \exp\left(-\frac{4}{\pi} s^2\right)\,, & \nu = 2 \;\; (\mathrm{GUE})\,, \\ \frac{2^{18}}{3^6 \pi^3} s^4 \exp\left(-\frac{64}{9\pi} s^2\right)\,, & \nu = 4 \;\; (\mathrm{GSE})\,. \end{cases}
$$

- These are again Wigner distributions
- Turn out to accurately describe $p(s)$ for *arbitrary* rank matrices too

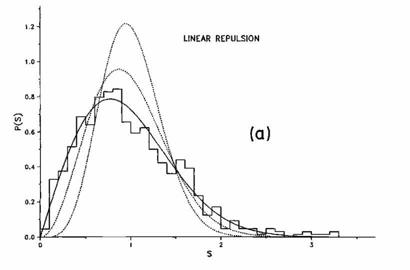
Example: spin-1/2 in 3D anharmonic oscillator potential

- Hamiltonian:

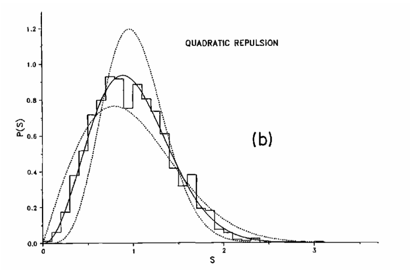
$$H = \frac{1}{2}(p_x^2 + p_y^2 + p_z^2) + x^4 + \frac{1}{2}y^4 + \frac{1}{10}z^4 + 12x^2y^2 + 14x^2z^2 + 16y^2z^2 + r^2 z(ax + by) + cr\mathbf{L}\cdot\mathbf{S}\,,$$

  with $r = (x^2 + y^2 + z^2)^{1/2}$.

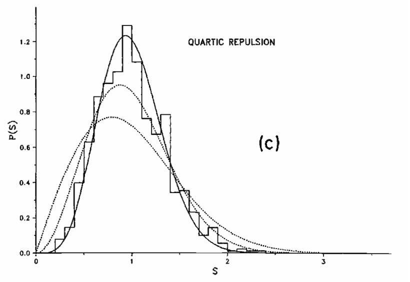
- Three cases [3]:
  - $a = b = 0$: three reflection symmetries (in xy-, xz- and yz-planes); GOE class
  - $a \neq 0$ or $b \neq 0$: one reflection symmetry destroyed; GUE class
  - $a, b \neq 0$: no reflection symmetries; GSE class



(a) $a = b = 0$, and GOE ($\nu = 1$) prediction.



(b) $a = 0$, $b \neq 0$, and GUE ($\nu = 2$) prediction.



(c) $a \neq 0$, $b \neq 0$, and GSE ($\nu = 4$) prediction.

Teaser: Bohani-Giannoni-Schmit (BGS) conjecture

BGS conjecture

Spectra of time-reversal-invariant systems whose classical analogs are K-systems follow the same statistical properties as those of random matrices from the GOE.

- K-mixing: all parts of classical phase space show chaotic dynamics
- Overwhelming experimental evidence

References

[1] Oriol Bohigas, RU Haq, and Ahkilesh Pandey. "Fluctuation properties of nuclear energy levels and widths: comparison of theory with experiment". In: *Nuclear Data for Science and Technology: Proceedings of the International Conference Antwerp 6–10 September 1982*. Springer. 1983, pp. 809–813.

[2] Giulio Casati, BV Chirikov, and Italo Guarneri. "Energy-level statistics of integrable quantum systems". In: *Physical review letters* 54.13 (1985), p. 1350.

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[4] Madan Lal Mehta. *Random matrices*. Vol. 142. Elsevier, 2004.

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