Integrability, Bethe Ansatz and open quantum systems

Master's thesis presentation

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Introduction

Idea of the thesis: Showcase a hands-on view on integrable systems through the lens of condensed matter theory. Explore integrable systems with dissipative terms.

Setup of the presentation:

- Selection of three interesting topics
- Focus on ‘new’ stuff, limited background information (see thesis)
- Interrupt me anytime

Outline

1. The open Lieb-Liniger model
2. Algebraic Bethe Ansatz and Richardson-Gaudin models in a nutshell
3. Efficient computation of rapidities for BA-solvable models

Section 1

The open Lieb-Liniger model

Lieb-Liniger model: Introduction (I)

Bosons 'on the line'.

$$H_{LL} = \int \mathrm{d}x [\partial_x \Psi^\dagger(x) \partial_x \Psi(x) + c \Psi^\dagger(x) \Psi^\dagger(x) \Psi(x) \Psi(x)] \qquad \text{(QFT)}$$

$$\Updownarrow$$

$$H_{LL,N} = -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + 2c \sum_{1 \leq k < j \leq N} \delta(x_k - x_j) \qquad \text{(QM)}$$

Applications in the description of (ultra)cold atomic gasses (e.g. [7]).

Lieb-Liniger model: Introduction (II)

Solvable by coordinate Bethe Ansatz:

$$\Psi(x_1, \ldots, x_N; \mathcal{Q}) = \sum_{\mathcal{P}} A_{\mathcal{P}}(\mathcal{Q}) e^{i \sum_j k_{\mathcal{P}_j} x_j} .$$

Bethe equations:

$$\boxed{k_j L = 2\pi I_j + (N-1)\pi + \sum_{l=1}^{N} \underbrace{-2 \arctan\left(\frac{k_j - k_l}{c}\right)}_{\equiv \theta(k_j - k_l)},}$$

with $j = 1, \ldots, N$ and $I_j = -(N+1)/2 + j$ in the ground state.

Lieb-Liniger model: Introduction (III)

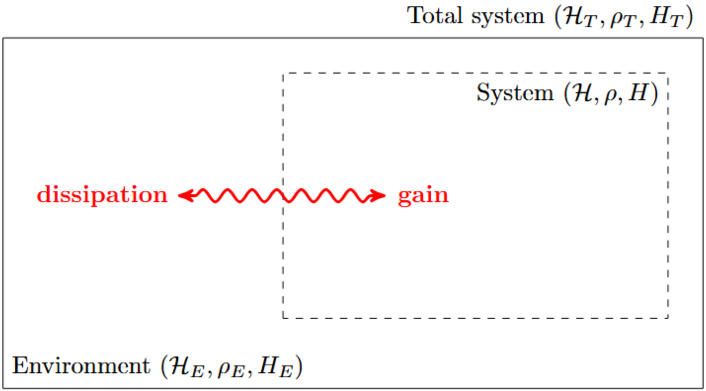
Table 1: Comparison between the repulsive and attractive Lieb-Liniger model.

	repulsive	attractive
coupling	$c > 0$	$c < 0$
rapidities	real	complex-valued
eqs. for rapidities	Bethe	Bethe-Takahashi
bound states	no	yes
ground state	equally spaced particles	single string centered at 0
limit $\|c\| \to \infty$	Tonks-Girardeau	super Tonks-Girardeau
bosonic description	hard-core	strongly attractive
fermionic descripton	free	repulsive, long-range interactions
thermodynamic limit	$N \to \infty$, $\rho = N/L$ fixed	unstable

Lindbladian (I)

Lindblad master equation (LME):

$$\boxed{\mathcal{L}(\hat{\rho}) \equiv \partial_t \hat{\rho} = \frac{1}{i\hbar}[\hat{H}, \hat{\rho}] + \sum_{k=1}^{K} \left(\hat{L}_k \hat{\rho} \hat{L}_k^\dagger - \frac{1}{2}\{\hat{L}_k^\dagger \hat{L}_k, \hat{\rho}\} \right)}$$



Lindbladian (II)

Theorem (CPTP)

Every Lindbladian generates a completely positive trace-preserving map (in the sense that $e^{\mathcal{L}t}$ is CPTP).

Can be used to prove the following useful theorem:

Theorem (non-positive real part)

For a Markovian system with a finite-dimensional Hilbert space, the real part of the spectrum of $\mathcal{L}$ is non-positive.

Discussion of work by Torres [8]

Split up LME in excitation-preserving and de-excitation part:

$$\mathcal{L}\rho = \mathcal{K}\rho + \mathcal{A}\rho\,,$$

with

$$\mathcal{K}\rho = \frac{1}{i\hbar}(K\rho - \rho K^\dagger)\,, \qquad \mathcal{A}\rho = \sum_s \gamma_s A_s \rho A_s^\dagger\,,$$

and the non-Hermitian Hamiltonian

$$K = H - i\hbar \sum_s \frac{\gamma_s}{2} A_s^\dagger A_s\,.$$

Key point of the paper: The eigenvalues λ of $\mathcal{K}$ are given in terms of the eigenvalues ϵ of K by

$$\boxed{\lambda_{j,k}^{(l,n)} = \frac{1}{i\hbar}\left[\epsilon_j^{(n+l)} - \epsilon_k^{*(n)}\right]}\,, \quad \text{with} \quad \begin{array}{l} n+l, n = 0, \ldots, N \\ j = 1, \ldots, d_{n+l} \\ k = 1, \ldots, d_n \end{array}$$

and (under certain assumptions) $\mathcal{L}$ has the same spectrum!

Finding the spectrum of the non-Hermitian Lieb-Liniger model (I)

Lieb-Liniger model with dissipative term:

$$\begin{aligned}\tilde{H}_{LL} &\equiv \int \mathrm{d}x[\partial_x \Psi^\dagger(x)\partial_x \Psi(x) + (\gamma + i\omega)\Psi^\dagger(x)\Psi^\dagger(x)\Psi(x)\Psi(x)] \\ &= H_{LL} + i\omega \int \mathrm{d}x \Psi^\dagger(x)\Psi^\dagger(x)\Psi(x)\Psi(x)\,,\end{aligned}$$

Identify the Lindbladian

$$\begin{aligned}\mathcal{L}_{LL}\rho = \frac{1}{i\hbar}[H_{LL}, \rho] + \frac{i\omega}{2}\int \mathrm{d}x[2\Psi(x)\Psi(x)\rho\Psi^\dagger(x)\Psi^\dagger(x) \\ - \Psi^\dagger(x)\Psi^\dagger(x)\Psi(x)\Psi(x)\rho - \rho\Psi^\dagger(x)\Psi^\dagger(x)\Psi(x)\Psi(x)]\,,\end{aligned}$$

with the double-field jump operators $L = \Psi(x)\Psi(x)$. The operator $\mathcal{K}_{LL}$ is identified with

$$\mathcal{K}_{LL}\rho = \frac{1}{i\hbar}(\tilde{H}_{LL}\rho - \rho\tilde{H}^\dagger_{LL})\,.$$

Finding the spectrum of the non-Hermitian Lieb-Liniger model (II)

Three step approach:

Approach

1. Numerically solve the Bethe equations for $\tilde{H}_{LL}$ to find the set of Bethe roots $\{k_j\}$, $j = 1, \ldots, N$.
2. Build the energies $E = \sum_i^N k_i^2$, which are the eigenvalues of $\tilde{H}_{LL}$.
3. Build the eigenvalues of $\mathcal{K}_{LL}$. We have then also found the spectrum of $\mathcal{L}_{LL}$ 'for free'.

Implementation (I)

For step 1, use an optimized Newton-Raphson solver with damping and LU-decomposition to achieve required stability.
The update scheme is:

$$\vec{k}^{(n+1)} = \vec{k}^{(n)} - J^{-1} F\left(\vec{k}^{(n)}\right),$$

with $F(\vec{k}^{(n)})$ the vector of Bethe equations

$$F_j(k_1, k_2, \ldots, k_N) = k_J L - 2\pi I_j - \sum_{j=1}^{N} \theta(k_j - k_l) = 0\,, \quad j = 1, 2, \ldots, N\,,$$

and Jacobian

$$J_{jl} = \frac{\partial F_j}{\partial k_l} = L\delta_{jl} - \frac{\mathrm{d}\theta(k_j - k_l)}{\mathrm{d}k_j}$$

(recall that the scattering phase is given by $\theta(k) \equiv -2\arctan(k/c)$).

Implementation (II)

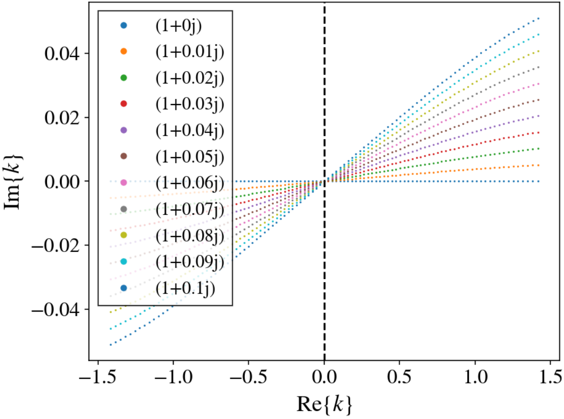


Figure 1: $\rho = 1$.

Step 2 and 3 are a matter of assembling the rapidities in the correct way.

Results (I)

Vary coupling

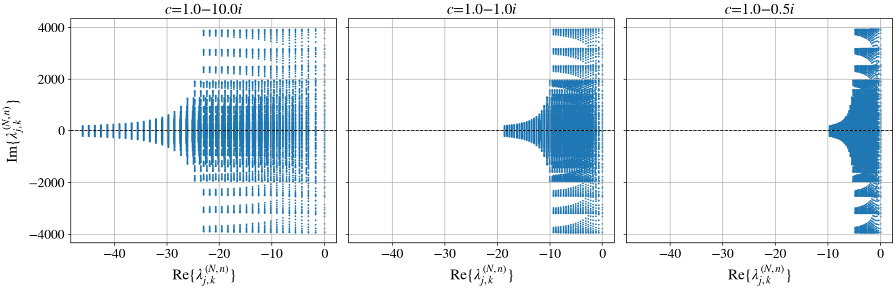


Figure 2: $\rho = 1$, 10 excitations.

Results (II)

Vary density

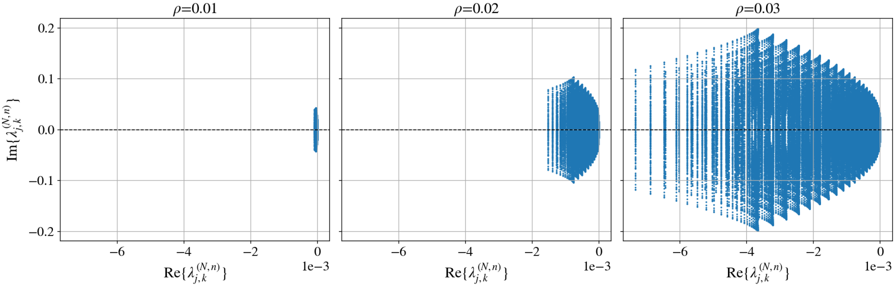
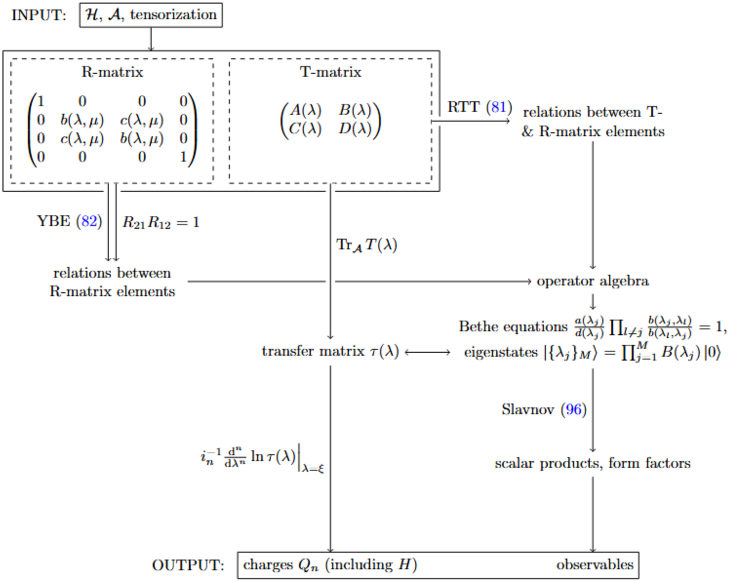


Figure 3: $c = 1 - 0.5i$, 10 excitations.

Section 2

Algebraic Bethe Ansatz and Richardson-Gaudin models in a nutshell

Schematic overview



ABA in a nutshell

- 'Constructive approach': build integrable systems from scratch.
- Yang-Baxter equation.
- Bethe equations emerge organically as constraints on the rapdities.
- Slavnov's theorem bridges the gap between theory and experiment.

RG-models in a nutshell

- ‘Rewriting’ of ABA.
- Produces the same output: the Bethe equations.
- Starting point is the generalized Gaudin algebra.
- We will be interested in the XXZ-parametrization of the algebra, for which the Bethe equations take a particular form.

Section 3

Efficient computation of rapidities for BA-solvable models

The BA/ODE correspondence (I) [3, 2]

Problem: A generic set of Bethe eqs.,

$$F(\lambda_i) = \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j},$$

will have a diverging rhs if two rapidities coincide.

Workaround: Introduce a 'clever substitution' defined in terms of the polynomials $P(z)$,

$$\Lambda(z) = \sum_{k=1}^{M} \frac{1}{z - \lambda_k} = \frac{P(z)}{P'(z)}, \qquad P(z) = \prod_{k=1}^{M} (z - \lambda_k).$$

If the set $\{\lambda_i\}$ is a solution to the Bethe eqs., it is easy to show that

$$\Lambda'(z) + \Lambda^2(z) - \sum_{\alpha} \frac{2F(\lambda_\alpha)}{z - \lambda_\alpha} = 0.$$

The BA/ODE correspondence (II) [3, 2]

By taking repeated derivatives and the limit $z \to \epsilon_j$, it can be shown that for XXZ Gaudin models, with

$$F(\lambda_\alpha) = -\sum_{i=1}^{N} \frac{A_i}{(\epsilon_i - \lambda_\alpha)} + \frac{B}{2g}\lambda_\alpha + \frac{C}{2g}\,, \qquad \text{with} \;\; A_i = |s_i|\Omega\,,$$

the following system of ODEs holds:

$$\begin{cases} (1-2A_j)\Lambda'(\epsilon_j) + \Lambda^2(\epsilon_j) + \frac{B}{g}M - \frac{B\epsilon_j + C}{g}\Lambda(\epsilon_j) + \sum_{i\neq j} 2A_i \frac{\Lambda(\epsilon_j) - \Lambda(\epsilon_i)}{\epsilon_i - \epsilon_j} & = 0\,, \\ (1-A_j)\Lambda''(\epsilon_j) + 2\Lambda(\epsilon_j)\Lambda'(\epsilon_j) - \frac{B}{g}\Lambda(\epsilon_j) - \frac{B\epsilon_j + C}{g}\Lambda'(\epsilon_j) & \\ \quad + \sum_{i\neq j} 2A_i \frac{\Lambda(\epsilon_j) - \Lambda(\epsilon_i)}{(\epsilon_i - \epsilon_j)^2} + \Lambda'(\epsilon_j)\sum_{i\neq j}\frac{2A_j}{\epsilon_i - \epsilon_j} & = 0\,, \\ (1-\frac{2}{3}A_j)\Lambda'''(\epsilon_j) + 2\Lambda(\epsilon_j)\Lambda''(\epsilon_j) + 2\Lambda'(\epsilon_j)^2 - 2\frac{B}{g}\Lambda'(\epsilon_j) - \frac{B\epsilon_j + C}{g}\Lambda''(\epsilon_j) & \\ \quad + \sum_{i\neq j} 4A_i \frac{\Lambda(\epsilon_j) - \Lambda(\epsilon_i)}{(\epsilon_i - \epsilon_j)^3} + \sum_{i\neq j}\frac{4A_i\Lambda'(\epsilon_j)}{(\epsilon_i - \epsilon_j)^2} + 2\sum_{i\neq j}\frac{2A_i\Lambda''(\epsilon_j)}{\epsilon_i - \epsilon_j} & = 0\,, \\ \ldots \end{cases}$$

Root extraction (I) [6, 4]

How to go from $\{\Lambda_i\} \to \{\lambda_i\}$? Recall that we found

$$\Lambda'(z) + \Lambda^2(z) - \sum_\alpha \frac{2F(\lambda_\alpha)}{z - \lambda_\alpha} = 0\,.$$

Using the same form for $F(\lambda_\alpha)$ as before and setting $A_i = \frac{1}{2}d_i$ for convenience, we obtain (after massaging) the following condition on the polynomials:

$$\boxed{P''(z) - F(z)P'(z) + G(z)P(z) = 0}\,,$$

$$\text{with}\;\; F(z) = \frac{C}{g} + \frac{Bz}{g} + \sum_{j=1}^{N} \frac{d_j}{z - \epsilon_j}\,, \quad G(z) = \frac{MB}{g} + \sum_{j=1}^{N} \frac{d_j \Lambda(\epsilon_j)}{z - \epsilon_j}\,.$$

Root extraction (II) [6, 4]

Barycentric representation: Decompose the $P(z)$ onto the basis of Lagrange polynomials in first barycentric form:

$$P(z) = \ell(z) \sum_{i=1}^{M+1} \frac{w_i}{z - z_i} P(z_i) \equiv \ell(z) \sum_{i=1}^{M+1} \frac{u_i}{z - z_i}\,,$$
$$\text{with}\ \ \ell(z) \equiv \prod_{i=1}^{M+1} (z - z_i)\,, \quad w_i = \prod_{j=1, j\neq i}^{M+1} (z_i - z_j)^{-1}\,, \quad u_i = w_i P(z_i)\,.$$

System of linear eqs. for $\{u_i\}$: For a general gridpoint z_l, subbing the barycentric representation into $P''(z) - F(z)P'(z) + G(z)P(z) = 0$ and demanding residues sum to zero as $z \to z_l$ yields:

$$\boxed{\begin{aligned}\sum_{j\neq k(\neq i)}^{M+1} \frac{u_i}{(z_i - z_j)(z_i - z_k)} &+ 2 \sum_{j\neq k(\neq i)}^{M+1} \frac{u_j}{(z_i - z_j)(z_i - z_k)} \\ &- F(z_i)\left(\sum_{j\neq i}^{M+1} \frac{u_i}{z_i - z_j} + \sum_{j\neq i}^{M+1} \frac{u_j}{z_i - z_j} \right) + G(z_i) u_i = 0\,.\end{aligned}}$$

Root extraction (III) [6, 4]

Key point: For any grid $\{z_i\}$, we can obtain the $\{u_i\}$ by solving a set of linear eqs.

Implication: Once we have the $\{u_i\}$, we have a representation of the polynomial. We can then use a standard root-finding algorithm (e.g. Laguerre with polynomial deflation) to extract the $\{\lambda_i\}$.

Remarks:

- The eqs. simplify if grid is chosen such that $\{z_i\} = \{\epsilon_i\}$:

$$\sum_{j(\neq i)}^{M+1} \frac{u_i}{\epsilon_i - z_j} + \sum_{j(\neq i)}^{M+1} \frac{u_j}{\epsilon_i - z_j} = \Lambda(\epsilon_i) u_i \,.$$

- However, better stability is achieved for complex-valued couplings with a dynamical grid.

Two convenient identities (I)

We can avoid BA/ODE by directly finding a quadratic equation for the eigenvalues of the conserved charges:

Identity 1: Dimo and Faribault [1]

For an RG-model with charges $\hat{Q}_i$ of the form

$$\hat{Q}_i = \vec{B}_i \cdot \vec{\sigma}_i + \sum_{k \neq i}^{N} \sum_{\alpha = x,y,z} \Gamma^{\alpha}_{i,k} \hat{\sigma}^{\alpha}_i \hat{\sigma}^{\alpha}_k \,,$$

the square of the eigenvalues K_i of $\hat{Q}_i$ are

$$K_i^2 = -2 \sum_{j \neq i} \frac{\Gamma^{\alpha}_{ij} \Gamma^{\gamma}_{ij}}{\Gamma^{\beta}_{ji}} K_j + \sum_{\alpha} (B^{\alpha}_i)^2 + \sum_{\alpha} \sum_{k \neq i} (\Gamma^{\alpha}_{ik})^2 \,.$$

Two convenient identities (II)

For the central spin model with

$$H = \hat{S}_j^z + g \sum_{k=1\neq j}^{N} \frac{1}{\epsilon_j - \epsilon_k} \vec{S}_j \cdot \vec{S}_k \,,$$

and $\hat{Q}_j = \hat{S}_j^z + g \sum_{k=1\neq j}^{N} \frac{1}{\epsilon_j - \epsilon_k} \vec{S}_j \cdot \vec{S}_k$, after making the correct identifications, we find the quadratic equation

$$F_j(K_1, K_2, \ldots, K_N; g) \equiv K_j^2 - \frac{g}{2} \sum_{i\neq j}^{N} \frac{K_j}{\epsilon_j - \epsilon_i} - \frac{3}{16} g^2 \sum_{i\neq j}^{N} \frac{1}{(\epsilon_j - \epsilon_i)^2} - \frac{1}{4} = 0 \,,$$

and we can again use Newton-Raphson to find the $\vec{K}$.

Two convenient identities (III)

Identity 2: Faribault and Schuricht [5]

In RG-models realized in terms of finite magnitude (pseudo-)spins, the following relation holds between the $\{K_i\}$ and auxiliary variables $\{\Lambda_i\}$:

$$\Lambda(\epsilon_i) = \frac{2}{g}\left[K_i - \frac{g}{4}\sum_{j\neq i}^{N}\frac{1}{\epsilon_i-\epsilon_j} + \frac{1}{2}\right].$$

We can thus rewrite the eqs. for the $\{u_i\}$,

$$\left[\sum_{j\neq i}^{M+1}\frac{1}{\epsilon_i-\epsilon_j} - \Lambda(\epsilon_i)\right]u_i + \sum_{j\neq i}^{M+1}\frac{1}{\epsilon_i-\epsilon_j}u_j = 0 \iff \mathrm{Mat}\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_M \\ u_{M+1}\end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1\end{pmatrix},$$

in terms of the $\{K_i\}$. Here, red (blue) terms represent (off-)diagonal entries of Mat.

Conclusions

Fast, efficient method to find rapidities of a class of XXZ RG-models.

We can avoid BA/ODE because of identity 1, and write everything in terms of the $\{K_i\}$ by using identity 2.

Drawbacks/limitations:

- Identity 1 holds for a particular form of conserved charges;
- Identity 2 holds for Gaudin models with finite magnitude (pseudo-)spins, i.e. excludes any model with bosonic degrees of freedom (Jaynes-Cummings-Dicke).

Example computation with complex-valued coupling (I)

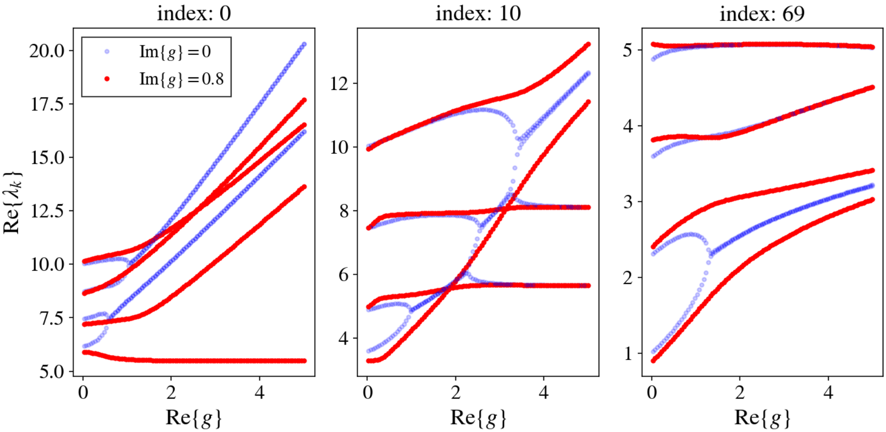


Figure 4: Central spin model. Real part of $\{\lambda_i\}$, $N = 8$, $M = 4$.

Example computation with complex-valued coupling (II)

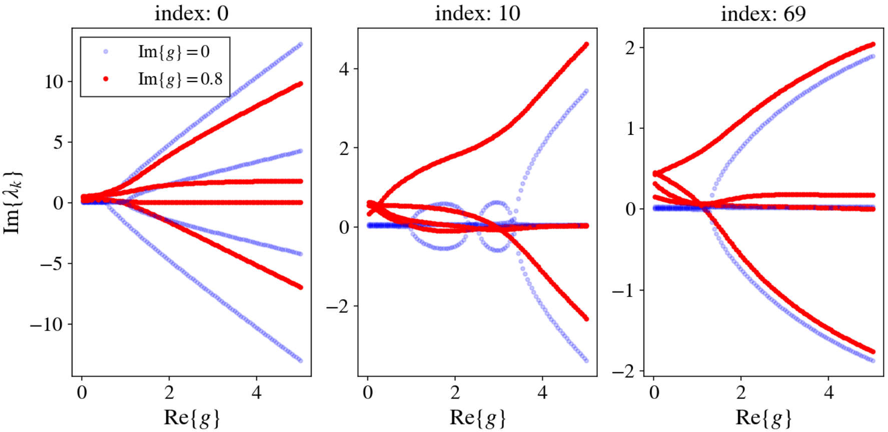


Figure 5: Central spin model. Imaginary part of $\{\lambda_i\}$, $N = 8$, $M = 4$.

References I

[1] C. Dimo and A. Faribault. "Quadratic operator relations and Bethe equations for spin-1/2 Richardson–Gaudin models". In: *Journal of Physics A: Mathematical and Theoretical* 51.32 (2018), p. 325202.

[2] P. Dorey, C. Dunning, and R. Tateo. "The ode/im correspondence". In: *Journal of Physics A: Mathematical and Theoretical* 40.32 (2007), R205.

[3] P. Dorey and R. Tateo. "Anharmonic oscillators, the thermodynamic Bethe Ansatz and nonlinear integral equations". In: *Journal of Physics A: Mathematical and General* 32.38 (1999), p. L419.

[4] O. El Araby, V. Gritsev, and A. Faribault. "Bethe Ansatz and ordinary differential equation correspondence for degenerate Gaudin models". In: *Physical Review B—Condensed Matter and Materials Physics* 85.11 (2012), p. 115130.

[5] A. Faribault and D. Schuricht. "On the determinant representations of Gaudin models' scalar products and form factors". In: *Journal of Physics A: Mathematical and Theoretical* 45.48 (2012), p. 485202.

References II

[6] A. Faribault et al. "Gaudin models solver based on the correspondence between Bethe Ansatz and ordinary differential equations". In: *Physical Review B—Condensed Matter and Materials Physics* 83.23 (2011), p. 235124.

[7] T. Kinoshita, T. Wenger, and D. S. Weiss. "Observation of a one-dimensional Tonks-Girardeau gas". In: *Science* 305.5687 (2004), pp. 1125–1128.

[8] J. M. Torres. "Closed-form solution of Lindblad master equations without gain". In: *Physical Review A* 89.5 (2014), p. 052133.