Integrable systems & the Lieb-Liniger model

5354SSTP3Y Student Seminar Theoretical Physics

Matthijs Wanrooij

University of Amsterdam

November 27, 2024

Outline

Integrability: The big picture

- What 'system'?
  - A set of differential equations (ODE's or PDE's)

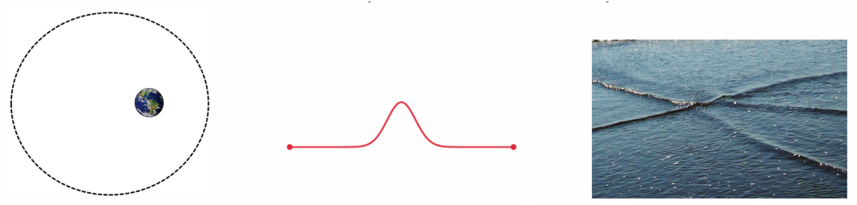


Figure 1: Examples of integrable systems (M. Vermeeren, 2019).

  - A set of difference equations
  - Quantum integrable systems: spin chains
- Classical & quantum integrability don't work the same way

Integrability: Classical mechanics

- Hamiltonian systems

Definition: Liouville integrability

A system with n degrees of freedom is called *Liouville integrable* if it has n conserved quantities that are in involution (Poisson brackets vanish).

- Liouville-Arnold theorem:
  - Phase space diffeomorphic to n-torus
  - Permits action-angle variables
  - Conserved quantities + involution $\longrightarrow$ hidden linear structure!

Heuristic (KAM theorem)

A nonlinear system is integrable if it behaves *almost* like a linear system.

- Historic meaning:
  - 19th century: exactly solvable (‘solvable by quadrature’)
  - nowadays: surprising structure (that can help solve the system)

Integrability: Quantum mechanics (I)

- What if we just generalize?

(Naive but common) definition

A system is quantum integrable if it possesses a maximal set of independent commuting quantum operators $\mathcal{Q}_\alpha$, $\alpha = 1, \ldots, \dim(\mathcal{H})$.

- (J.-S. Caux & J. Mossel, 2011): Try again

Integrability: Quantum mechanics (I)

- What if we just generalize?

(Naive but common) definition

A system is quantum integrable if it possesses a maximal set of independent commuting quantum operators $\mathcal{Q}_\alpha$, $\alpha = 1, \ldots, \dim(\mathcal{H})$.

- (J.-S. Caux & J. Mossel, 2011): Try again
  - Spectral theorem: Hermitian Hamiltonians are diagonalizable

Integrability: Quantum mechanics (I)

- What if we just generalize?

(Naive but common) definition

A system is quantum integrable if it possesses a maximal set of independent commuting quantum operators $\mathcal{Q}_\alpha$, $\alpha = 1, \ldots, \dim(\mathcal{H})$.

- (J.-S. Caux & J. Mossel, 2011): Try again
  - Spectral theorem: Hermitian Hamiltonians are diagonalizable
  - $\Rightarrow$ Find $\dim(\mathcal{H})$ orthog. state vectors $|\Psi_\alpha\rangle$ and build $\mathcal{Q}_\alpha = |\Psi_\alpha\rangle \langle\Psi_\alpha|$

Integrability: Quantum mechanics (I)

- What if we just generalize?

(Naive but common) definition

A system is quantum integrable if it possesses a maximal set of independent commuting quantum operators $\mathcal{Q}_\alpha$, $\alpha = 1, \ldots, \dim(\mathcal{H})$.

- (J.-S. Caux & J. Mossel, 2011): Try again
  - Spectral theorem: Hermitian Hamiltonians are diagonalizable
  - $\Rightarrow$ Find $\dim(\mathcal{H})$ orthog. state vectors $|\Psi_\alpha\rangle$ and build $\mathcal{Q}_\alpha = |\Psi_\alpha\rangle \langle\Psi_\alpha|$
  - This is a maximal independent commuting set.

Integrability: Quantum mechanics (I)

- What if we just generalize?

(Naive but common) definition

A system is quantum integrable if it possesses a maximal set of independent commuting quantum operators $\mathcal{Q}_\alpha$, $\alpha = 1, \ldots, \dim(\mathcal{H})$.

- (J.-S. Caux & J. Mossel, 2011): Try again
  - Spectral theorem: Hermitian Hamiltonians are diagonalizable
  - $\Rightarrow$ Find $\dim(\mathcal{H})$ orthog. state vectors $|\Psi_\alpha\rangle$ and build $\mathcal{Q}_\alpha = |\Psi_\alpha\rangle \langle\Psi_\alpha|$
  - This is a maximal independent commuting set.
  - $\Rightarrow$ All quantum models with $\dim(\mathcal{H}) < \infty$ are integrable!

Integrability: Quantum mechanics (II)

- We need more restrictions on the conserved quantities.
- Another common definition:
  - Integrability $= \infty$ many local conserved quantities in therm. limit
  - Non-integrability $=$ no (nontrivial) local conserved quantity
- Presence of conserved quantities strongly connected to solvability:
  - Integrable systems: eigenstates labelled by quantum numbers corresponding to the conserved quantities

Integrability: Quantum mechanics (II)

- We need more restrictions on the conserved quantities.
- Another common definition:
  - Integrability $= \infty$ many local conserved quantities in therm. limit
  - Non-integrability $=$ no (nontrivial) local conserved quantity
- Presence of conserved quantities strongly connected to solvability:
  - Integrable systems: eigenstates labelled by quantum numbers corresponding to the conserved quantities

Tentative definition

A quantum integrable system has an extensive number of local conserved charges.

Integrability: Quantum mechanics (II)

- We need more restrictions on the conserved quantities.
- Another common definition:
  - Integrability $= \infty$ many local conserved quantities in therm. limit
  - Non-integrability $=$ no (nontrivial) local conserved quantity
- Presence of conserved quantities strongly connected to solvability:
  - Integrable systems: eigenstates labelled by quantum numbers corresponding to the conserved quantities

Tentative definition

A quantum integrable system has an extensive number of local conserved charges.

- *Conjecture*: Energy levels in integrable (non-integrable) systems are Poisson (Wigner-Dyson) distributed

Integrability: Quantum mechanics (III)

- Yang-Baxter equation sufficient *but too strong* condition:

$$R_{12}(u)R_{13}(u+v)R_{23}(v) = R_{23}(v)R_{13}(u+v)R_{12}(u)$$

- Physical interpretation: factorization of particle scattering:

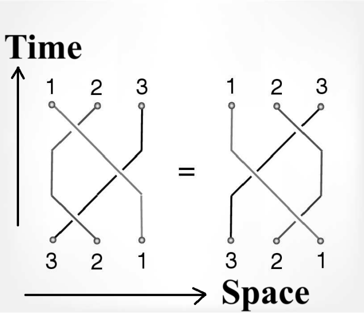


Figure 2: Graphical YBE (A. Marzuoli & M. Rasetti, 2005).

Integrability: Quantum mechanics (III')

Some people really like Yang-Baxter...

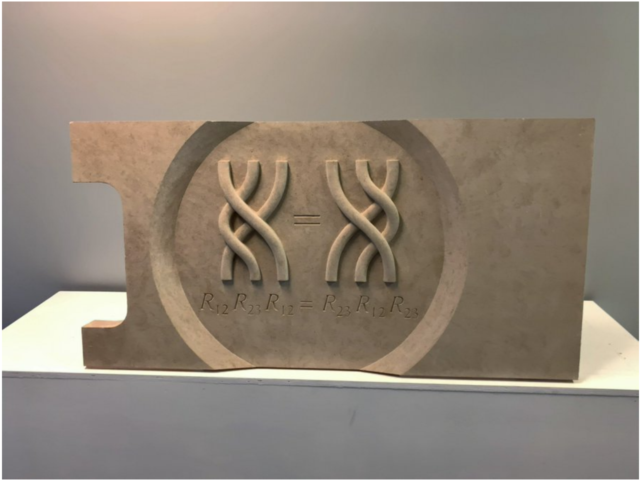


Figure 3: YBE sculpture (M. Yamazaki, source: *X.com*).

Lieb-Liniger model: Introduction (I)

- Bosons in one dimension
- Why? Simple example of exactly solvable quant. many-particle syst.
  - Seminal paper by M. Girardeau, 1960
  - Exact analysis by H. Lieb & W. Liniger, 1963

JOURNAL OF MATHEMATICAL PHYSICS VOLUME 1, NUMBER 6 NOVEMBER–DECEMBER, 1960

Relationship between Systems of Impenetrable Bosons and Fermions in One Dimension*†

M. GIRARDEAU‡
Brandeis University, Waltham, Massachusetts
(Received March 3, 1960)

- Rich physics
- Stepping stone for more intricate quantum integrable models

Lieb-Liniger model: Introduction (II)

- Hamiltonian:

$$H = \int \mathrm{d}x [\partial_x \Psi^\dagger(x) \partial_x \Psi(x) + c \Psi^\dagger(x) \Psi^\dagger(x) \Psi(x) \Psi(x)] \quad \text{(QFT)}$$

$$\Updownarrow$$

$$H_N = -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + 2c \sum_{1 \le k < j \le N} \delta(x_k - x_j) \quad \text{(QM)}$$

- Big difference between repulsive ($c > 0$) and attractive ($c < 0$) case
- Applications:
  - cold atom systems
  - nonlinear quantum optics (Kerr effect)

Repulsive case ($c > 0$): Coordinate Bethe ansatz

$$
\begin{aligned}
&\Psi_N(x_1, \ldots, x_N | \lambda_1, \ldots, \lambda_N) \\
&\qquad = \frac{\prod_{j>k}(\lambda_j - \lambda_k)}{\sqrt{N! \prod_{j>k}[(\lambda_j - \lambda_k)^2 + c^2]}} \\
&\qquad \times \sum_{\mathcal{P}} \exp\{i \sum_{n=1}^{N} x_n \lambda_{\mathcal{P}_n}\} \prod_{j>k} \left[1 - ic \frac{\operatorname{sign}(x_j - x_k)}{\lambda_{\mathcal{P}_j} - \lambda_{\mathcal{P}_k}} \right] \\
&= \sum_{\mathcal{P}} a(\mathcal{P}) e^{i \sum_{j=1}^{N} \lambda_{\mathcal{P}_j} x_j}
\end{aligned}
$$

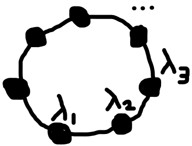
- $\{x_i\}$: positions, $\{\lambda_i\}$: rapidities (quasi-momenta)
- "Pauli principle for bosons" (in momentum space)

Intermezzo: Incarnations of the Bethe ansatz

- Coordinate Bethe ansatz
- Algebraic Bethe ansatz
- Thermodynamic Bethe ansatz
  - Lieb-Liniger: Yang-Yang equation
- Nested Bethe ansatz
- ...

Repulsive case ($c > 0$): Bethe equations

- Impose periodicity: $\Psi(x_1, x_2, \ldots, x_N) = \Psi(x_2, \ldots, x_N, x_1 + L)$



$$\boxed{e^{i\lambda_j L} = \prod_{k \neq j} \frac{\lambda_j - \lambda_k + ic}{\lambda_j - \lambda_k - ic}}$$

Theorem (Yang & Yang, 1969)

The solutions exist and are uniquely parametrized by a set of integers.

Theorem

All the solutions are real numbers.

- Tonks-Girardeau regime: $c \to +\infty$
  - strongly repulsive bosons = free fermions

Experimental realization

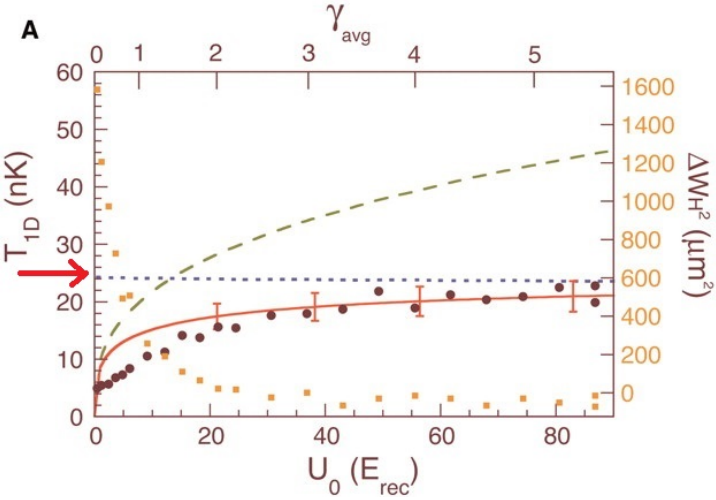
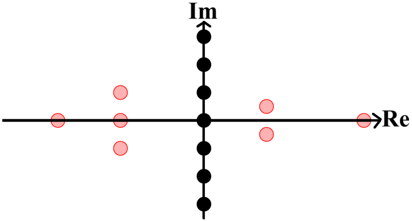


Figure 4: Experimental observation of Tonks-Girardeau gas. T: temperature. U_0: horizontal confinement.

Attractive case ($c < 0$): Bound states

- Bound states
- String hypothesis:

$$\lambda_\alpha^{j,a} = \lambda_\alpha^j - i\frac{c}{2}(j+1-2a) + i\delta_\alpha^{j,a}$$



- Bethe-Takahashi equations:

$$\boxed{j\lambda_\alpha^j - \sum_{(k,\beta)} \Phi_{jk}(\lambda_\alpha^j - \lambda_\beta^k) = 2\pi I_\alpha^j},$$

$$\Phi_{jk}(\lambda) = (1-\delta_{jk})\phi_{|j-k|}(\lambda) \qquad \text{(scattering phase shifts)}$$

$$+ 2\phi_{|j-k|+2}(\lambda) + \ldots + 2\phi_{j+k-2}(\lambda) + \phi_{j+k}(\lambda),$$

$$\phi_j(\lambda) = 2\arctan(-2\lambda/cj)$$

- Super Tonks-Girardeau regime: $c \to -\infty$
  - strongly attractive bosons = repulsive fermions

Complex-valued coupling ($c = \gamma + i\omega$)

- Non-Hermitian Hamiltonian
- Rapidities will be complex-valued, in general
- My guess for string hypothesis:

$$\lambda_\alpha^{j,a} = \lambda_\alpha^j + \frac{\omega}{2}(j+1-2a) - i\frac{\gamma}{2}(j+1-2a) + \text{deviations}$$

Lindblad master equation (I): background

$$\boxed{\dot{\rho} = \mathcal{L}\rho = \frac{1}{i\hbar}[H,\rho] + \sum_s \frac{\gamma_s}{2}\left(2A_s\rho A_s^\dagger - A_s^\dagger A_s\rho - \rho A_s^\dagger A_s\right)}$$

- Interaction with environment
  - Dissipation (Markovian!)
- state vectors $\longrightarrow$ density matrix ρ
- $\mathcal{L}$: *Lindbladian* (superoperator)

Lindblad master equation (II): origin

$$\boxed{\dot{\rho} = \mathcal{L}\rho = \frac{1}{i\hbar}[H,\rho] + \sum_s \frac{\gamma_s}{2}\left(2A_s\rho A_s^\dagger - A_s^\dagger A_s\rho - \rho A_s^\dagger A_s\right)}$$

- We require a CPTP map:
  - completely positive
  - trace-preserving

Definition: Positive map

A map Φ is *positive* iff for a given $X \geq 0$ ($X \in \mathcal{B}(\mathcal{H})$), $\Phi(X) \geq 0$.

Definition: Completely positive map

A map Φ is *completely positive* iff $\forall n \in \mathbb{N}$, $\Phi \otimes \mathbb{1}_n$ is positive.

Why is this interesting

(Torres, 2014): Find closed form solution for eigenvalue problem of dissipative quantum systems without gain.

- Define a non-Hermitian Hamiltonian

$$K = H - i\hbar \sum_s \frac{\gamma_s}{2} A_s^\dagger A_s$$

- Lindblad equation 'decomposes' into excitation-conserving part and de-excitation part

$$\mathcal{K}\rho = \frac{1}{i\hbar}(K\rho - \rho K^\dagger), \qquad \mathcal{A}\rho = \sum_s \gamma_s A_s \rho A_s^\dagger$$

- Spectrum is found.

Recommended reading

Classical integrability:

- Dunajski, M. (2012). Integrable systems. University of Cambridge Lecture Notes.

Quantum integrability:

- Caux, Jean-Sébastien, and Jorn Mossel. "Remarks on the notion of quantum integrability." Journal of Statistical Mechanics: Theory and Experiment 2011.02 (2011): P02023.

Lieb-Liniger model:

- Franchini, F. (2017). An introduction to integrable techniques for one-dimensional quantum systems, (arXiv preprint). (Chapter III)

Lindblad master equation:

- Manzano, D. (2020). A short introduction to the Lindblad master equation. Aip advances, 10(2).