MATH 3QC3 Assignment 2

Matthew Yu — 400322243 — Yum77

February 11, 2025

QUESTION 1

Pure State on the Bloch Sphere.
Consider the single-qubit pure state

$$|\psi\rangle = \cos\left(\frac{0}{2}\right)|0\rangle + e^{i\varphi}\sin\left(\frac{0}{2}\right)|1\rangle, 0 \leq \theta \leq \pi, 0 \leq \varphi \leq 2\pi.$$

a) Write down the corresponding density operator $\rho = |\psi\rangle\langle\psi|$.

b) Show that ρ can be expressed in the form

$$\rho = \frac{1}{2}\left[I + \vec{n}\cdot\vec{\sigma}\right],$$

where $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ is the vector of Pauli matrices, and $\vec{n} = (n_x, n_y, n_z)$ is a real 3D unit vector (i.e. $\|\vec{n}\| = 1$). Derive the coordinates n_x, n_y, n_z in terms of the angles θ, φ.

c) Interpret $\vec{n}$ as a point on the Bloch sphere (of radius 1). Briefly explain the geometric meaning of θ, φ in spherical conditions

QUESTION 2

General Single-Qubit Density Operator.
A general 2×2 density matrix (mixed state) can be written as

$$\rho = \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} \qquad \text{with } \rho^\dagger = \rho,\ tr(\rho) = 1,\ p \geq 0.$$

a) Show that any valid single-qubit, ρ admits the *Bloch parametrization:*

$$\rho = \frac{1}{2}\left[I + \vec{r}\cdot\vec{\sigma}\right], \quad \|\vec{r}\| \leq 1.$$

Indicate how the vector $\vec{r}$ relates tot he off-diagonal and diagonal elements of ρ.

b) Suppose

$$\rho = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

Verify that ρ is a valid density matrix (positive semi-definite, trace 1). Then find the Bloch vector $\vec{r} = (r_x, r_y, r_z)$ such that $\rho = \frac{1}{2}(I + r_x\sigma_x + r_y\sigma_y + r_z\sigma_z)$. nCheck that $\|\vec{r}\| \leq 1$. Does ρ correspond to a pure state or a mixed state? How do you know?

QUESTION 3

Let ρ_i be a density operator on $\mathbb{H}_i$ for $i = 1, 2$. Let $\rho_1 \otimes \rho_2$ be the tensor product, which you may assume is represented by the Kronecker product relative to some choice of ONBs. Show that $\rho_1 \otimes \rho_2$ is a density operator on $\mathbb{H}_1 \otimes \mathbb{H}_2$.

QUESTION 4

a) Let A_i be an observable on $\mathbb{H}_i$ for $i = 1, 2$. Show that $A_1 \otimes A_2$ is an observable on $\mathbb{H}_1 \otimes \mathbb{H}_2$.

b) (Measuring Entangled States of Composite Systems) Let $\mathbb{H}_A$ and $\mathbb{H}_B$ represent two quantum systems. Let O be an observable on $\mathbb{H}_A$ and let 1_B be the identity operator on $\mathbb{H}_B$. Then $O \otimes 1_B$ is an observable on $\mathbb{H}_A \otimes \mathbb{H}_B$ by an earlier question. Define

$$|\psi\rangle = \frac{1}{\sqrt{3}}(|00\rangle + |01\rangle + |10\rangle)$$

and define

$$O = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Compute the eigenvalues/eigenstates of $\sigma_z \otimes 1_B$ and the probabilities associated to each observation. Find the collapsed state upon each of the possible observations.

QUESTION 5

a) Let U_i be a unitary operator on $\mathbb{H}_i$ for $i = 1, 2$. Show that $U_1 \otimes U_2$ is a unitary operator on $H_1 \otimes H_2$.

b) Give an example of a unitary operator on ${}^{\P}\mathbb{H} \otimes {}^{\P}\mathbb{H}$ that is *entangled*, i.e. not of the form $U_1 \otimes U_2$.

QUESTION 6

Consider the two-qubit Hilbert space $\mathbb{H}_A \otimes \mathbb{H}_B$, each of the dimension 2, with the standard computational basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$. Define

$$|\psi\rangle = \frac{1}{\sqrt{3}}(|00\rangle + |01\rangle + |10\rangle).$$

Let

$$\rho_{AB} = |\psi\rangle\langle\psi|$$

be the corresponding 4×4 density operator.

a) **Show it is Entangled.** Verify that $|\psi\rangle$ cannot be written as a product state $(\alpha|0\rangle + \beta|1\rangle) \otimes (\gamma|0\rangle + \delta|1\rangle)$ with fixed complex scalars $\alpha, \beta, \gamma, \delta$ satisfying $|\alpha|^2 + |\beta|^2 = 1$. Conclude that $|\psi\rangle$ is entangled.
Hint: Attempt to match coefficients of basis states and show there's no consistent solution unless one of them is zero–which doesn't match $|\psi\rangle$.

b) **Compute the Partial Trace Over B.** Write ρ_{AB} explicitly in the basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$, then compute

$$\rho_A = tr_B(\rho_{AB}).$$

You may group the 4×4 matrix ρ_{AB} into 2×2 blocks corresponding to subsystem B.

c) **Is the Reduced State Pure or Mixed?** Check whether ρ_A is a rank-1 projector (i.e. pure). If not, it must be a mixed state. Comment on how ρ_A being mixed reflects the entanglement of the global state $|\psi\rangle$.

QUESTION 7

We have four qubits A, B, C, D, with two disjoint pairs (A, B) and (C, D) initially prepared in the Bell state

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle).$$

Hence the entire 4-qubit system is

$$|\Psi\rangle_{ABCD} = (|\Phi^+\rangle_{AB}) \otimes (|\Phi^+\rangle_{CD}).$$

We will *measure only qubits* B, C, with a Hermitian observable M_{BC}. Recall that in the full space, this is represented by

$$\underbrace{\mathbb{I}_A}_{2\times 2} \otimes \underbrace{M_{BC}}_{4\times 4} \otimes \underbrace{\mathbb{I}_D}_{2\times 2},$$

a 16×16 operator which acts trivially on qubits A and D.

a) **Defining M_{BC} as a Bell-measurement operator.**
let the four Bell states for qubits (B,C) be

$$|\Phi^\pm\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle), \quad |\Psi\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle).$$

Define the Hermitian operator on B, C:

$$M_{BC} = 0\Phi^+\Phi^+ + 1\Phi^-\Phi^- + 2\Psi^+\Psi^+ + 3\Psi^-\Psi^-.$$

Explain why each $|\Phi^\pm\rangle, |\Psi^\pm\rangle$ is an eigenvector of M_{BC}, and why this operator has distinct eigenvalues 0,1,2,3.

b) **Measuring $\mathbb{I}_A \otimes M_{BC} \otimes \mathbb{I}_D$ on the state $|\Psi\rangle_{ABCD}$.**

i) Expand $|\Psi\rangle_{ABCD}$ in the Bell basis of (B, C). Show each Bell outcome $\{\Phi^+, \Phi^-, \Psi^+\Psi^-\}$ occurs with the probability $1/4$ when measuring the observable $\mathbb{I}_A \otimes M_{BC} \otimes \mathbb{I}_D$.

ii) Determine the post-measurement staet of (A,D) for each outcome. Conclude that (A,D) end up in one of the four Bell states (thus entangled) even though A was initially entangled only with B, and C was entangled only with D.

c) **Discussion.**
Briedly discuss how $\mathbb{I}_{\mathbb{A}} \otimes M_{BC} \otimes \mathbb{I}_{\mathbb{D}}$ is formally a 16×16 operator acting on A, B, C, D. However, because it factorizes as the identity on A and D, it "probes" only the degrees of freedom in B, C. In this way, we effectively *measure qubits* B, C (and not A or D). This measurement "swaps" entanglement so that (A, D) end up entangled.