

Solution of Homework #1

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1 Problem 3.2

Given a Z channel with probabilities $p_{Y|X}(0|0) = 1$, $p_{Y|X}(1|0) = 0$, $p_{Y|X}(1|1) = p_{Y|X}(0|1) = \frac{1}{2}$, we want to find the channel capacity C . We know that the channel capacity for a discrete memory channel is defined as follows.

$$C = \max_{p_X} I(X; Y) = H(Y) - H(Y|X) \quad (1)$$

First we compute $H(Y)$ in function of $p = p_X(1)$.

$$\begin{aligned} H(Y) &= -p_Y(1) \cdot \log(p_Y(1)) - p_Y(0) \cdot \log(p_Y(0)) \\ &= -(p_{Y|X}(1|0) \cdot p_X(0) + p_{Y|X}(1|1) \cdot p_X(1)) \cdot \log(p_{Y|X}(1|0) \cdot p_X(0) + p_{Y|X}(1|1) \cdot p_X(1)) \\ &\quad - (p_{Y|X}(0|0) \cdot p_X(0) + p_{Y|X}(0|1) \cdot p_X(1)) \cdot \log(p_{Y|X}(0|0) \cdot p_X(0) + p_{Y|X}(0|1) \cdot p_X(1)) \\ &= -\frac{1}{2} \cdot p_X(1) \log\left(\frac{1}{2} \cdot p_X(1)\right) - \left(p_X(0) + \frac{1}{2} \cdot p_X(1)\right) \cdot \log\left(p_X(0) + \frac{1}{2} \cdot p_X(1)\right) \\ &= -\frac{1}{2} \cdot p \cdot \log\left(\frac{1}{2} \cdot p\right) - \left(1 - \frac{1}{2} \cdot p\right) \cdot \log\left(1 - \frac{1}{2} \cdot p\right) \end{aligned} \quad (2)$$

Then we compute $H(Y|X)$ always in function of $p = p_X(1)$.

$$\begin{aligned} H(Y|X) &= -p_{YX}(0, 0) \cdot \log(p_{Y|X}(0|0)) - p_{YX}(1, 0) \cdot \log(p_{Y|X}(1|0)) \\ &\quad - p_{YX}(0, 1) \cdot \log(p_{Y|X}(0|1)) - p_{YX}(1, 1) \cdot \log(p_{Y|X}(1|1)) \\ &= -p_{Y|X}(0|0) \cdot p_X(0) \cdot \log(p_{Y|X}(0|0)) - p_{Y|X}(1|0) \cdot p_X(0) \cdot \log(p_{Y|X}(1|0)) \\ &\quad - p_{Y|X}(0|1) \cdot p_X(1) \cdot \log(p_{Y|X}(0|1)) - p_{Y|X}(1|1) \cdot p_X(1) \cdot \log(p_{Y|X}(1|1)) \\ &= 0 + 0 + \frac{1}{2} \cdot p + \frac{1}{2} \cdot p = p \end{aligned} \quad (3)$$

Now we are able to write $I(X; Y)$ in function of p as we can see from (4).

$$I(X; Y) = -\frac{1}{2} \cdot p \cdot \log\left(\frac{1}{2} \cdot p\right) - \left(1 - \frac{1}{2} \cdot p\right) \cdot \log\left(1 - \frac{1}{2} \cdot p\right) - p \quad (4)$$

In 1 we show the function's trend for all possible values of p .

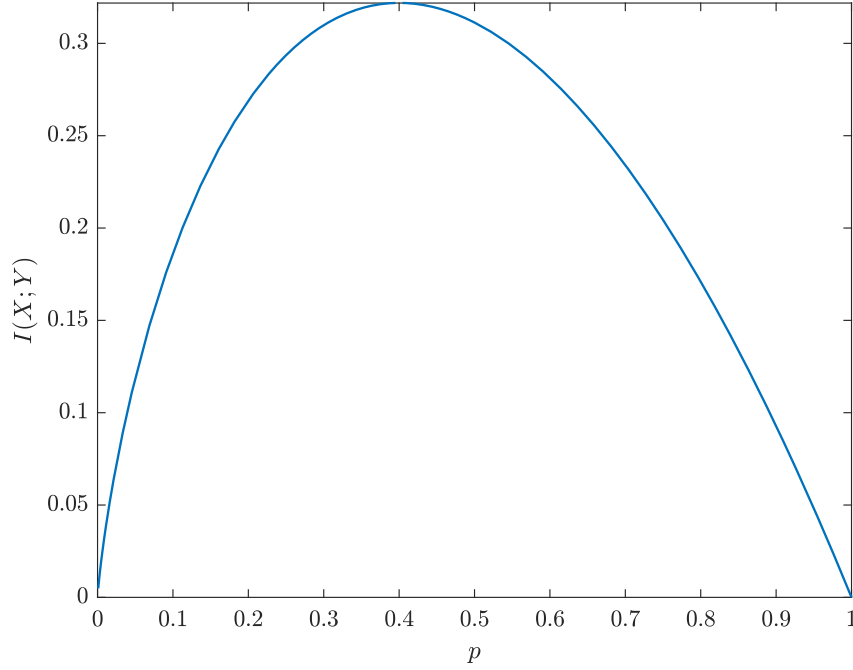


Figure 1: Mutual information as a function of $p \triangleq p_X(1)$

Now we want to find the value of p for which (4) is maximize. With this aim we compute the first and the second derivate of $I(X; Y)$.

$$\begin{aligned} \frac{dI(X; Y)}{dp} &= \frac{1}{2} \cdot \log\left(1 - \frac{p}{2}\right) + \frac{1}{2} \cdot \frac{1 - \frac{p}{2}}{1 - \frac{p}{2}} - \frac{1}{2} \cdot \log\left(\frac{p}{2}\right) - \frac{p}{2} \cdot \frac{2}{p} \cdot \frac{1}{2} - 1 \\ &= \frac{1}{2} \cdot \log\left(\frac{2}{p} - 1\right) - 1 \end{aligned} \quad (5)$$

$$\frac{d^2 I(X; Y)}{dp^2} = \frac{1}{2} \cdot \frac{p}{2 - p} \cdot \frac{1}{2} = \frac{1}{4} \cdot \frac{p}{2 - p} \quad (6)$$

We notice that $\frac{d^2 I(X; Y)}{dp^2}$ is always equal or greater than zero for every possibli value of p . This means that $I(X; Y)$ is a convex function and then the value of p for which $\frac{dI(X; Y)}{dp}$ is zero coincides with the maximum point of $I(X; Y)$. Therefore we compute the values for which we have $\frac{dI(X; Y)}{dp} = 0$.

$$\begin{aligned} \frac{1}{2} \cdot \log\left(\frac{2}{p} - 1\right) - 1 &= 0 \\ \log\left(\frac{2}{p} - 1\right) &= 2 \\ \frac{2}{p} - 1 &= 2^2 \\ p &= \frac{2}{5} \end{aligned} \quad (7)$$

So for $p = \frac{2}{5}$ we maximize $I(X; Y)$ and therefore we obtain the value of the channel capacity C . The value of C is reported in (8).

$$\begin{aligned}
 C &= I(X; Y)|_{p=\frac{2}{5}} \\
 &= -\frac{4}{5} \cdot \log\left(\frac{4}{5}\right) - \frac{1}{5} \cdot \log\left(\frac{1}{5}\right) - \frac{2}{5} \\
 &= \frac{1}{5} \cdot \log\left(\frac{3125}{256}\right) - \frac{2}{5}
 \end{aligned} \tag{8}$$

2 Problem 3.13

Is given a Gaussian product channel $Y_j = g_j \cdot X_j + Z_j$ $j \in \{1, 2\}$ with $g_1 < g_2$ and average power constraint P . We want to know above what power P^* we start to use both the channels and what are the features of the *energy-per-bit-rate function* $E_b(R)$.

2.1 Optimal power allocation

We notice that to achieve the maximum capacity from the channel we have to solve the following optimization problem.

$$\begin{aligned}
 \max_{P_j} \sum_{j=1}^2 \frac{1}{2} \cdot \log(1 + g_j^2 \cdot P_j) \quad \text{subject to} \\
 -P_j \neq 0 \quad j \in \{1, 2\} \\
 \sum_{j=1}^2 P_j - P = 0
 \end{aligned} \tag{9}$$

In order to solve (9) we build the following equation.

$$\nabla_{P_j} \left\{ -\sum_{j=1}^2 \frac{1}{2} \cdot \log(1 + g_j^2 \cdot P_j) + \sum_{j=1}^2 \lambda_j \cdot (-P_j) + \nu \cdot \left(\sum_{j=1}^2 P_j - P \right) \right\} = 0 \tag{10}$$

Solving (10) we obtain what follows.

$$\begin{aligned}
 \frac{1}{2} \cdot \frac{1}{P_j + \frac{1}{g_j^2}} + \lambda_j - \nu &= 0 \\
 \Rightarrow \frac{1}{2} \cdot \frac{1}{P_j + \frac{1}{g_j^2}} &\leq \nu = \frac{1}{2\mu}
 \end{aligned} \tag{11}$$

$$\begin{aligned}
 \sum_{j=1}^2 \max \left\{ \mu - \frac{1}{g_j^2}, 0 \right\} &= P \\
 \Rightarrow \mu &= \frac{P}{2} + \frac{1}{2g_1^2} + \frac{1}{2g_2^2}
 \end{aligned} \tag{12}$$

$$\begin{aligned}
 P_j &= \max \left\{ \mu - \frac{1}{g_j^2}, 0 \right\} \\
 \Rightarrow P_1 &= \max \left\{ \frac{P}{2} + \frac{1}{2g_2^2} - \frac{1}{2g_1^2}, 0 \right\} = \begin{cases} \frac{P}{2} + \frac{1}{2g_2^2} - \frac{1}{2g_1^2} & P \geq \frac{1}{g_1^2} - \frac{1}{g_2^2} \\ 0 & P < \frac{1}{g_1^2} - \frac{1}{g_2^2} \end{cases} \quad (13) \\
 \Rightarrow P_2 &= \max \left\{ \frac{P}{2} + \frac{1}{2g_1^2} - \frac{1}{2g_2^2}, 0 \right\} = \frac{P}{2} + \frac{1}{2g_1^2} - \frac{1}{2g_2^2} \quad \forall \quad P \geq 0
 \end{aligned}$$

We conclude that the second channel is opened for every amount of allocated power P while the first channel is opened only if and only if (14) is verified.

$$P \geq \frac{1}{g_1^2} - \frac{1}{g_2^2} \quad (14)$$

2.2 Energy-per-bit-rate function computation

Now we want to compute the *energy-per-bit-rate function* $E_b(R)$ in the scenario previously described. We know that $P = R \cdot E$ where R is the *bit-rate* of the channel and E is the used energy for every bit. In addition we know that the *bit-rate* is bounded by channel capacity as shown in (15) and (16).

$$R \leq \frac{1}{2} \cdot \log(1 + g_2^2 P) \Leftrightarrow P < \frac{1}{g_1^2} - \frac{1}{g_2^2} \quad (15)$$

$$R \leq \frac{1}{2} \cdot \sum_{j=1}^2 \log \left(\frac{g_j^2}{2} \cdot \left(P + \frac{1}{g_1^2} + \frac{1}{g_2^2} \right) \right) = \frac{1}{2} \cdot \log \left(\frac{g_1^2 g_2^2}{4} \cdot \left(P + \frac{1}{g_1^2} + \frac{1}{g_2^2} \right)^2 \right) \Leftrightarrow P \geq \frac{1}{g_1^2} - \frac{1}{g_2^2} \quad (16)$$

Let's consider (15) which is the case where only one channel is opened. Writing R in function of P we get what follows.

$$P \geq \left(\frac{2^{2R} - 1}{g_2^2} \right) \Leftrightarrow R < \frac{1}{2} \log \left(\frac{g_2^2}{g_1^2} \right) \quad (17)$$

Let's consider (16) which is the case where only one channel is opened. Writing R in function of P we get what follows.

$$P \geq \left(\frac{2^{R+1}}{g_1 g_2} - \frac{1}{g_1^2} - \frac{1}{g_2^2} \right) \Leftrightarrow R \geq \frac{1}{2} \log \left(\frac{g_2^2}{g_1^2} \right) \quad (18)$$

Knowing that $P = E \cdot R$ we define the *energy-per-bit-rate function* $E_b(R)$ as shown in (19).

$$E_b(R) = \begin{cases} \frac{1}{R} \left(\frac{2^{2R} - 1}{g_2^2} \right) & R < \frac{1}{2} \log \left(\frac{g_2^2}{g_1^2} \right) \\ \frac{1}{R} \left(\frac{2^{R+1}}{g_1 g_2} - \frac{1}{g_1^2} - \frac{1}{g_2^2} \right) & R \geq \frac{1}{2} \log \left(\frac{g_2^2}{g_1^2} \right) \end{cases} \quad (19)$$

Now we want to compute the minimum energy-per-bit as $R \rightarrow 0$. We notice that for low value of R we find ourself in the case where only one of the two channels is opened.

Then for compute $\lim_{R \rightarrow 0} E_b(R)$ we proceed as follows.

$$\lim_{R \rightarrow 0} E_b(R) = \lim_{R \rightarrow 0} \left(\frac{2^{2R} - 1}{g_2^2 R} \right) = \lim_{R \rightarrow 0} \frac{\frac{d(2^{2R} - 1)}{dR}}{\frac{dg_2^2 R}{dR}} = \frac{2 \ln 2}{g_2^2} \quad (20)$$

We conclude that the minimum energy-per-bit as $R \rightarrow 0$ is equal to (21).

$$\lim_{R \rightarrow 0} E_b(R) = \frac{2 \ln 2}{g_2^2} \quad (21)$$