Solution of Homework #1

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1 Problem 3.2

Given a Z channel with probabilities $p_{Y|X}(0|0) = 1$, $p_{Y|X}(1|0) = 0$, $p_{Y|X}(1|1) = p_{Y|X}(0|1) = \frac{1}{2}$, we want to find the channel capacity C. We know that the channel capacity for a discrete memory channel is defined as follows.

$$C = \max_{p_X} I(X;Y) = H(Y) - H(Y|X) \tag{1}$$

First we compute $H(Y)$ in function of $p = p_X(1)$.

$$\begin{aligned}
H(Y) =& - p_Y(1) \cdot \log(p_Y(1)) - p_Y(0) \cdot \log(p_Y(0)) \\
=& - (p_{Y|X}(1|0) \cdot p_X(0) + p_{Y|X}(1|1) \cdot p_X(1)) \cdot \log(p_{Y|X}(1|0) \cdot p_X(0) + p_{Y|X}(1|1) \cdot p_X(1)) \\
& - (p_{Y|X}(0|0) \cdot p_X(0) + p_{Y|X}(0|1) \cdot p_X(1)) \cdot \log(p_{Y|X}(0|0) \cdot p_X(0) + p_{Y|X}(0|1) \cdot p_X(1)) \\
=& - \frac{1}{2} \cdot p_X(1) \log\left(\frac{1}{2} \cdot p_X(1)\right) - \left(p_X(0) + \frac{1}{2} \cdot p_X(1)\right) \cdot \log\left(p_X(0) + \frac{1}{2} \cdot p_X(1)\right) \\
=& - \frac{1}{2} \cdot p \cdot \log\left(\frac{1}{2} \cdot p\right) - \left(1 - \frac{1}{2} \cdot p\right) \cdot \log\left(1 - \frac{1}{2} \cdot p\right)
\end{aligned} \tag{2}$$

Then we compute $H(Y|X)$ always in funcion of $p = p_X(1)$.

$$\begin{aligned}
H(Y|X) =& - p_{YX}(0,0) \cdot \log(p_{Y|X}(0|0)) - p_{YX}(1,0) \cdot \log(p_{Y|X}(1|0)) \\
& - p_{YX}(0,1) \cdot \log(p_{Y|X}(0|1)) - p_{YX}(1,1) \cdot \log(p_{Y|X}(1|1)) \\
=& - p_{Y|X}(0|0) \cdot p_X(0) \cdot \log(p_{Y|X}(0|0)) - p_{Y|X}(1|0) \cdot p_X(0) \cdot \log(p_{Y|X}(1|0)) \\
& - p_{Y|X}(0|1) \cdot p_X(1) \cdot \log(p_{Y|X}(0|1)) - p_{Y|X}(1|1) \cdot p_X(1) \cdot \log(p_{Y|X}(1|1)) \\
=& 0 + 0 + \frac{1}{2} \cdot p + \frac{1}{2} \cdot p = p
\end{aligned} \tag{3}$$

Now we are able to write $I(X;Y)$ in function of p as we can see from (4).

$$I(X;Y) = -\frac{1}{2} \cdot p \cdot \log\left(\frac{1}{2} \cdot p\right) - \left(1 - \frac{1}{2} \cdot p\right) \cdot \log\left(1 - \frac{1}{2} \cdot p\right) - p \tag{4}$$

In 1 we show the function's trend for all possible values of p.

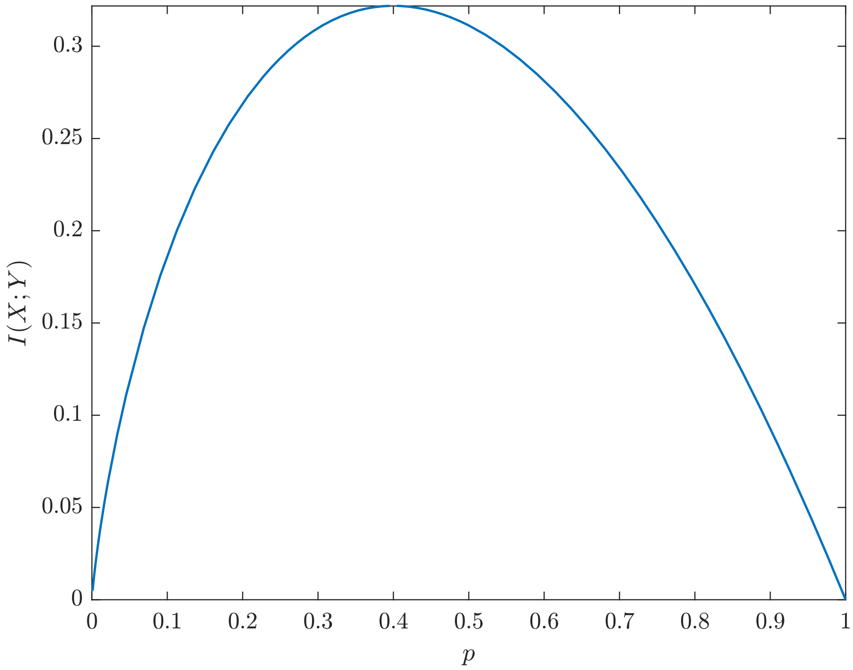

Figure 1: Mutual information as a function of $p \triangleq p_X(1)$

Now we want to find the value of p for which (4) is maximize. With this aim we compute the first and the second derivates of $I(X;Y)$.

$$\begin{aligned} \frac{dI(X;Y)}{dp} &= \frac{1}{2} \cdot \log\left(1 - \frac{p}{2}\right) + \frac{1}{2} \cdot \frac{1 - \frac{p}{2}}{1 - \frac{p}{2}} - \frac{1}{2} \cdot \log\left(\frac{p}{2}\right) - \frac{p}{2} \cdot \frac{2}{p} \cdot \frac{1}{2} - 1 \\ &= \frac{1}{2} \cdot \log\left(\frac{2}{p} - 1\right) - 1 \end{aligned} \tag{5}$$

$$\frac{d^2I(X;Y)}{dp^2} = \frac{1}{2} \cdot \frac{p}{2-p} \cdot \frac{1}{2} = \frac{1}{4} \cdot \frac{p}{2-p} \tag{6}$$

We notice that $\frac{d^2I(X;Y)}{dp^2}$ is always equal or greater than zero for every possibli value of p. This means that $I(X;Y)$ is a convex function and then the value of p for which $\frac{dI(X;Y)}{dp}$ is zero coincides with the maximum point of $I(X;Y)$. Therefore we compute the values for which we have $\frac{dI(X;Y)}{dp} = 0$.

$$\begin{aligned} \frac{1}{2} \cdot \log\left(\frac{2}{p} - 1\right) - 1 &= 0 \\ \log\left(\frac{2}{p} - 1\right) &= 2 \\ \frac{2}{p} - 1 &= 2^2 \\ p &= \frac{2}{5} \end{aligned} \tag{7}$$

So for $p = \frac{2}{5}$ we maximize $I(X;Y)$ and therefore we obtain the value of the channel capacity C. The value of C is reported in (8).

$$\begin{aligned} C =& I(X;Y)\big|_{p=\frac{2}{5}} \\ =& -\frac{4}{5} \cdot \log\left(\frac{4}{5}\right) - \frac{1}{5} \cdot \log\left(\frac{1}{5}\right) - \frac{2}{5} \\ =& \frac{1}{5} \cdot \log\left(\frac{3125}{256}\right) - \frac{2}{5} \end{aligned} \tag{8}$$

2 Problem 3.13

Is given a Gaussian product channel $Y_j = g_j \cdot X_j + Z_j$ $j \in \{1,2\}$ with $g_1 < g_2$ and average power constraint P. We want to know above what power P^* we start to use both the channels and what are the features of the *energy-per-bit-rate function* $E_b(R)$.

2.1 Optimal power allocation

We notiche that to achieve the maximum capacity from the channel we have to solve the following optimization problem.

$$\begin{gathered} \max_{P_j} \sum_{j=1}^{2} \frac{1}{2} \cdot \log(1 + g_j^2 \cdot P_j) \quad \textit{subject} \quad \textit{to} \\ -P_j \neq 0 \quad j \in \{1,2\} \\ \sum_{j=1}^{2} P_j - P = 0 \end{gathered} \tag{9}$$

In order to solve (9) we build the following equation.

$$\nabla_{P_j}\left\{ -\sum_{j=1}^{2} \frac{1}{2} \cdot \log(1 + g_j^2 \cdot P_j) + \sum_{j=1}^{2} \lambda_j \cdot (-P_j) + \nu \cdot \left(\sum_{j=1}^{2} P_j - P \right) \right\} = 0 \tag{10}$$

Solving (10) we obtain what follows.

$$\begin{aligned} &\frac{1}{2} \cdot \frac{1}{P_j + \frac{1}{g_j^2}} + \lambda_j - \nu = 0 \\ &\Rightarrow \frac{1}{2} \cdot \frac{1}{P_j + \frac{1}{g_j^2}} \leq \nu = \frac{1}{2\mu} \end{aligned} \tag{11}$$

$$\begin{aligned} &\sum_{j=1}^{2} \max\left\{ \mu - \frac{1}{g_j^2}, 0 \right\} = P \\ &\Rightarrow \mu = \frac{P}{2} + \frac{1}{2g_1^2} + \frac{1}{2g_2^2} \end{aligned} \tag{12}$$

$$
\begin{aligned}
P_j &= \max\left\{\mu - \frac{1}{g_j^2}, 0\right\} \\
\Rightarrow P_1 = \max\left\{\frac{P}{2} + \frac{1}{2g_2^2} - \frac{1}{2g_1^2}, 0\right\} &= \begin{cases} \frac{P}{2} + \frac{1}{2g_2^2} - \frac{1}{2g_1^2} & P \geq \frac{1}{g_1^2} - \frac{1}{g_2^2} \\ 0 & P < \frac{1}{g_1^2} - \frac{1}{g_2^2} \end{cases} \\
\Rightarrow P_2 = \max\left\{\frac{P}{2} + \frac{1}{2g_1^2} - \frac{1}{2g_2^2}, 0\right\} &= \frac{P}{2} + \frac{1}{2g_1^2} - \frac{1}{2g_2^2} \quad \forall \quad P \geq 0
\end{aligned}
\tag{13}
$$

We conclude that the second channel is opened for every amount of allocated power P while the first channel is opened only if and only if (14) is verified.

$$
P \geq \frac{1}{g_1^2} - \frac{1}{g_2^2} \tag{14}
$$

2.2 Energy-per-bit-rate function computation

Now we want to compute the *energy-per-bit-rate function* $E_b(R)$ in the scenario previously described. We know that $P = R \cdot E$ where R is the *bit-rate* of the channel and E is the used energy for every bit. In addiction we know that depending of what amount of power is avaliable, the *bit-rate* is restricted as shown in (15) and (16).

$$
R \leq \frac{1}{2} \cdot \log\left(1 + g_2^2 P\right) \Leftrightarrow P < \frac{1}{g_1^2} - \frac{1}{g_2^2} \tag{15}
$$

$$
R \leq \frac{1}{2} \cdot \sum_{j=1}^{2} \log\left(\frac{g_j^2}{2} \cdot \left(P + \frac{1}{g_1^2} + \frac{1}{g_2^2}\right)\right) \Leftrightarrow P \geq \frac{1}{g_1^2} - \frac{1}{g_2^2} \tag{16}
$$

Supponing to achieve the maximum rate in both (15) and (16) we can compute R in function of P and then write again the boundaries condition as follows.

$$
\begin{aligned}
R &\leq \frac{1}{2} \cdot \log\left(1 + g_2^2 P\right) \\
\Rightarrow \frac{2^{2R} - 1}{g_2^2} &\leq P < \frac{1}{g_1^2} - \frac{1}{g_2^2} \\
\Rightarrow R &< \frac{1}{2} \log\left(\frac{g_2^2}{g_1^2}\right)
\end{aligned}
\tag{17}
$$

$$
\begin{aligned}
R &\leq \frac{1}{2} \cdot \sum_{j=1}^{2} \log\left(\frac{g_j^2}{2} \cdot \left(P + \frac{1}{g_1^2} + \frac{1}{g_2^2}\right)\right) \\
\Rightarrow P &\geq \frac{2^{R+1}}{g_1 g_2} - \frac{1}{g_1^2} - \frac{1}{g_2^2}
\end{aligned}
\tag{18}
$$

Let's first consider the case where only one channel is opened. Sobstituting $P = E \cdot R$ in (15) we get what follows.

$$
\begin{aligned}
R &\leq \frac{1}{2} \cdot \log\left(1 + g_2^2 E R\right) \\
\Rightarrow E &\geq \frac{1}{R}\left(\frac{2^{2R} - 1}{g_2^2}\right)
\end{aligned}
\tag{19}
$$

Let's now consider the case where both the channels are opened. Sobstituiting $P = E \cdot R$ in (16) we get what follows.

$$\begin{aligned} R &\leq \frac{1}{2} \cdot \sum_{j=1}^{2} \log\Big(\frac{g_j^2}{2} \cdot \Big(ER + \frac{1}{g_1^2} + \frac{1}{g_2^2}\Big)\Big) \\ \Rightarrow R &\leq \frac{1}{2} \cdot \log\Big(\frac{g_1^2 g_2^2}{4} \cdot \Big(ER + \frac{1}{g_1^2} + \frac{1}{g_2^2}\Big)^2\Big) \\ \Rightarrow E &\geq \frac{1}{R}\Big(\frac{2^{R+1}}{g_1 g_2} - \frac{1}{g_1^2} - \frac{1}{g_2^2}\Big) \end{aligned} \tag{20}$$

We conclude that *energy-per-bit rate function* is equal to the minimum energy-per-bit required for every value of R as we can see from (21).

$$E_b(R) = \begin{cases} \frac{1}{R}\Big(\frac{2^{2R}-1}{g_2^2}\Big) & P < \frac{1}{g_1^2} - \frac{1}{g_2^2} \\ \frac{1}{R}\Big(\frac{2^{R+1}}{g_1 g_2} - \frac{1}{g_1^2} - \frac{1}{g_2^2}\Big) & P \geq \frac{1}{g_1^2} - \frac{1}{g_2^2} \end{cases} \tag{21}$$

Now we want to compute the minimum energy-per-bit as $R \to 0$.

$$\lim_{R\to 0} E_b(R) = \lim_{R\to 0}\Big(\frac{2^{2R}-1}{g_2^2 R}\Big) = \lim_{R\to 0} \frac{\frac{d(2^{2R}-1)}{dR}}{\frac{dg_2^2 R}{dR}} = \frac{2\ln 2}{g_2^2} \quad P < \frac{1}{g_1^2} - \frac{1}{g_2^2} \tag{22}$$

$$\lim_{R\to 0} E_b(R) = \lim_{R\to 0} \frac{1}{R}\Big(\frac{2^{R+1}}{g_1 g_2} - \frac{1}{g_1^2} - \frac{1}{g_2^2}\Big) = \lim_{R\to 0} \frac{\frac{d(2^{R+1})}{dR}}{\frac{dg_1 g_2 R}{dR}} = \frac{2\ln 2}{g_1 g_2} \quad P \geq \frac{1}{g_1^2} - \frac{1}{g_2^2} \tag{23}$$

We conclude that the minimum energy-per-bit as $R \to 0$ is equal to (24).

$$\lim_{R\to 0} E_b(R) = \begin{cases} \frac{2\ln 2}{g_2^2} & P < \frac{1}{g_1^2} - \frac{1}{g_2^2} \\ \frac{2\ln 2}{g_1 g_2} & P \geq \frac{1}{g_1^2} - \frac{1}{g_2^2} \end{cases} \tag{24}$$