# Solution of Homework #1

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## 1 Problem 3.2

Given a Z channel with probabilities $p_{Y|X}(0|0) = 1$, $p_{Y|X}(1|0) = 0$, $p_{Y|X}(1|1) = p_{Y|X}(0|1) = \frac{1}{2}$, we want to find the channel capacity $C$. We know that the channel capacity for a discrete memory channel is defined as follows.

$$C = \max_{p_X} I(X;Y) = H(Y) - H(Y|X) \tag{1}$$

First we compute $H(Y)$ in function of $p = p_X(1)$.

$$
\begin{aligned}
H(Y) = & - p_Y(1) \cdot \log(p_Y(1)) - p_Y(0) \cdot \log(p_Y(0)) \\
= & - (p_{Y|X}(1|0) \cdot p_X(0) + p_{Y|X}(1|1) \cdot p_X(1)) \cdot \log(p_{Y|X}(1|0) \cdot p_X(0) + p_{Y|X}(1|1) \cdot p_X(1)) \\
& - (p_{Y|X}(0|0) \cdot p_X(0) + p_{Y|X}(0|1) \cdot p_X(1)) \cdot \log(p_{Y|X}(0|0) \cdot p_X(0) + p_{Y|X}(0|1) \cdot p_X(1)) \\
= & - \frac{1}{2} \cdot p_X(1) \log\left(\frac{1}{2} \cdot p_X(1)\right) - \left(p_X(0) + \frac{1}{2} \cdot p_X(1)\right) \cdot \log\left(p_X(0) + \frac{1}{2} \cdot p_X(1)\right) \\
= & - \frac{1}{2} \cdot p \cdot \log\left(\frac{1}{2} \cdot p\right) - \left(1 - \frac{1}{2} \cdot p\right) \cdot \log\left(1 - \frac{1}{2} \cdot p\right)
\end{aligned}
\tag{2}
$$

Then we compute $H(Y|X)$ always in funcion of $p = p_X(1)$.

$$
\begin{aligned}
H(Y|X) = & - p_{YX}(0,0) \cdot \log(p_{Y|X}(0|0)) - p_{YX}(1,0) \cdot \log(p_{Y|X}(1|0)) \\
& - p_{YX}(0,1) \cdot \log(p_{Y|X}(0|1)) - p_{YX}(1,1) \cdot \log(p_{Y|X}(1|1)) \\
= & - p_{Y|X}(0|0) \cdot p_X(0) \cdot \log(p_{Y|X}(0|0)) - p_{Y|X}(1|0) \cdot p_X(0) \cdot \log(p_{Y|X}(1|0)) \\
& - p_{Y|X}(0|1) \cdot p_X(1) \cdot \log(p_{Y|X}(0|1)) - p_{Y|X}(1|1) \cdot p_X(1) \cdot \log(p_{Y|X}(1|1)) \\
= & 0 + 0 + \frac{1}{2} \cdot p + \frac{1}{2} \cdot p = p
\end{aligned}
\tag{3}
$$

Now we are able to write $I(X;Y)$ in function of $p$ as we can see from (4).

$$I(X;Y) = -\frac{1}{2} \cdot p \cdot \log\left(\frac{1}{2} \cdot p\right) - \left(1 - \frac{1}{2} \cdot p\right) \cdot \log\left(1 - \frac{1}{2} \cdot p\right) - p \tag{4}$$

In 1 we show the function's trend for all possible values of $p$.
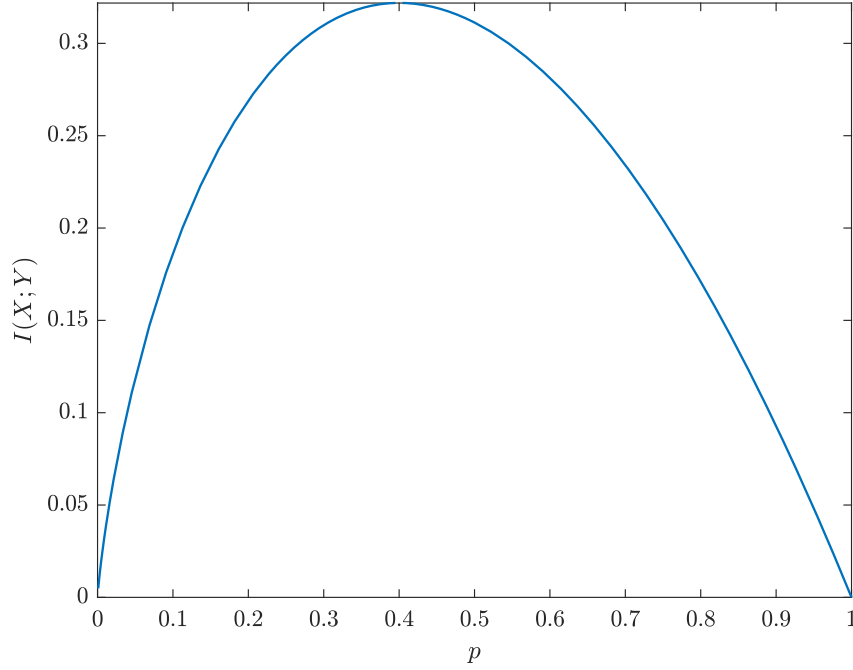


Figure 1: Mutual information as a function of $p \triangleq p_X(1)$

Now we want to find the value of $p$ for which (4) is maximize. With this aim we compute the first and the second derivates of $I(X;Y)$.

$$
\begin{aligned}
\frac{dI(X;Y)}{dp} &= \frac{1}{2} \cdot \log\left(1 - \frac{p}{2}\right) + \frac{1}{2} \cdot \frac{1 - \frac{p}{2}}{1 - \frac{p}{2}} - \frac{1}{2} \cdot \log\left(\frac{p}{2}\right) - \frac{p}{2} \cdot \frac{2}{p} \cdot \frac{1}{2} - 1 \\
&= \frac{1}{2} \cdot \log\left(\frac{2}{p} - 1\right) - 1
\end{aligned}
\tag{5}
$$

$$
\frac{d^2 I(X;Y)}{dp^2} = \frac{1}{2} \cdot \frac{p}{2 - p} \cdot \frac{1}{2} = \frac{1}{4} \cdot \frac{p}{2 - p}
\tag{6}
$$

We notice that $\frac{d^2 I(X;Y)}{dp^2}$ is always equal or greater than zero for every possibli value of p. This means that $I(X;Y)$ is a convex function and then the value of $p$ for which $\frac{dI(X;Y)}{dp}$ is zero coincides with the maximum point of $I(X;Y)$. Therefore we compute the values for which we have $\frac{dI(X;Y)}{dp} = 0$.

$$
\begin{aligned}
\frac{1}{2} \cdot \log\left(\frac{2}{p} - 1\right) - 1 &= 0 \\
\log\left(\frac{2}{p} - 1\right) &= 2 \\
\frac{2}{p} - 1 &= 2^2 \\
p &= \frac{2}{5}
\end{aligned}
\tag{7}
$$

So for $p = \frac{2}{5}$ we maximize $I(X;Y)$ and therefore we obtain the value of the channel capacity $C$. The value of $C$ is reported in (8).

$$
\begin{aligned}
C &= I(X;Y)|_{p=\frac{2}{5}} \\
&= -\frac{4}{5} \cdot \log\left(\frac{4}{5}\right) - \frac{1}{5} \cdot \log\left(\frac{1}{5}\right) - \frac{2}{5} \\
&= \frac{1}{5} \cdot \log\left(\frac{3125}{256}\right) - \frac{2}{5}
\end{aligned}
\tag{8}
$$

# 2 Problem 3.13

Is given a Gaussian product channel $Y_j = g_j \cdot X_j + Z_j \ j \in \{1, 2\}$ with $g_1 < g_2$ and average power constraint $P$. We want to know above what power $P^*$ we start to use both the channels and what are the features of the *energy-per-bit-rate function* $E_b(R)$.

## 2.1 Optimal power allocation

We notiche that to achieve the maximum capacity from the channel we have to solve the following optimization problem.

$$
\max_{P_j} \sum_{j=1}^{2} \frac{1}{2} \cdot \log(1 + g_j^2 \cdot P_j) \quad subject \quad to
$$

$$
-P_j \neq 0 \quad j \in \{1, 2\}
\tag{9}
$$

$$
\sum_{j=1}^{2} P_j - P = 0
$$

In order to solve (9) we build the following equation.

$$
\nabla_{P_j}\left\{ -\sum_{j=1}^{2} \frac{1}{2} \cdot \log(1 + g_j^2 \cdot P_j) + \sum_{j=1}^{2} \lambda_j \cdot (-P_j) + \nu \cdot \left(\sum_{j=1}^{2} P_j - P\right) \right\} = 0
\tag{10}
$$

Solving (10) we obtain what follows.

$$
\frac{1}{2} \cdot \frac{1}{P_j + \frac{1}{g_j^2}} + \lambda_j - \nu = 0
$$

$$
\Rightarrow \frac{1}{2} \cdot \frac{1}{P_j + \frac{1}{g_j^2}} \leq \nu = \frac{1}{2\mu}
\tag{11}
$$

$$
\sum_{j=1}^{2} \max\left\{\mu - \frac{1}{g_j^2}, 0\right\} = P
$$

$$
\Rightarrow \mu = \frac{P}{2} + \frac{1}{2g_1^2} + \frac{1}{2g_2^2}
\tag{12}
$$

$$P_j = \max\left\{\mu - \frac{1}{g_j^2}, 0\right\}$$

$$\Rightarrow P_1 = \max\left\{\frac{P}{2} + \frac{1}{2g_2^2} - \frac{1}{2g_1^2}, 0\right\} = \begin{cases} \frac{P}{2} + \frac{1}{2g_2^2} - \frac{1}{2g_1^2} & P \geq \frac{1}{g_1^2} - \frac{1}{g_2^2} \\ 0 & P < \frac{1}{g_1^2} - \frac{1}{g_2^2} \end{cases} \tag{13}$$

$$\Rightarrow P_2 = \max\left\{\frac{P}{2} + \frac{1}{2g_1^2} - \frac{1}{2g_2^2}, 0\right\} = \frac{P}{2} + \frac{1}{2g_1^2} - \frac{1}{2g_2^2} \quad \forall \quad P \geq 0$$

We conclude that the second channel is opened for every amount of allocated power $P$ while the first channel is opened only if and only if (14) is verified.

$$P \geq \frac{1}{g_1^2} - \frac{1}{g_2^2} \tag{14}$$

## 2.2 Energy-per-bit-rate function computation

Now we want to compute the *energy-per-bit-rate function* $E_b(R)$ in the scenario previously described. We know that $P = R \cdot E$ where $R$ is the *bit-rate* of the channel and $E$ is the used energy for every bit. In addiction we know that depending of what amount of power is avaliable, the *bit-rate* is restricted as shown in (15) and (16).

$$R \leq \frac{1}{2} \cdot \log\left(1 + g_2^2 P\right) \Leftrightarrow P < \frac{1}{g_1^2} - \frac{1}{g_2^2} \tag{15}$$

$$R \leq \frac{1}{2} \cdot \sum_{j=1}^{2} \log\left(\frac{g_j^2}{2} \cdot \left(P + \frac{1}{g_1^2} + \frac{1}{g_2^2}\right)\right) \Leftrightarrow P \geq \frac{1}{g_1^2} - \frac{1}{g_2^2} \tag{16}$$

Let's first consider the case where only one channel is opened. Sobstituting $P = E \cdot R$ in (15) we get what follows.

$$R \leq \frac{1}{2} \cdot \log\left(1 + g_2^2 ER\right)$$
$$\Rightarrow E \geq \frac{1}{R}\left(\frac{2^{2R} - 1}{g_2^2}\right) \tag{17}$$

Let's now consider the case where both the channels are opened. Sobstituiting $P = E \cdot R$ in (16) we get what follows.

$$R \leq \frac{1}{2} \cdot \sum_{j=1}^{2} \log\left(\frac{g_j^2}{2} \cdot \left(ER + \frac{1}{g_1^2} + \frac{1}{g_2^2}\right)\right)$$
$$\Rightarrow R \leq \frac{1}{2} \cdot \log\left(\frac{g_1^2 g_2^2}{4} \cdot \left(ER + \frac{1}{g_1^2} + \frac{1}{g_2^2}\right)^2\right) \tag{18}$$
$$\Rightarrow E \geq \frac{1}{R}\left(\frac{2^{R+1}}{g_1 g_2} - \frac{1}{g_1^2} - \frac{1}{g_2^2}\right)$$

We conclude that *energy-per-bit rate function* is equal to the minimum energy-per-bit required for every value of R as we can see from (19).

$$E_b(R) = \begin{cases} \frac{1}{R}\left(\frac{2^{2R}-1}{g_2^2}\right) & P < \frac{1}{g_1^2} - \frac{1}{g_2^2} \\ \frac{1}{R}\left(\frac{2^{R+1}}{g_1 g_2} - \frac{1}{g_1^2} - \frac{1}{g_2^2}\right) & P \geq \frac{1}{g_1^2} - \frac{1}{g_2^2} \end{cases} \tag{19}$$

Now we want to compute the minimum energy-per-bit as $R \to 0$.

$$\lim_{R \to 0} E_b(R) = \lim_{R \to 0} \left(\frac{2^{2R}-1}{g_2^2 R}\right) = \lim_{R \to 0} \frac{\frac{d(2^{2R}-1)}{dR}}{\frac{dg_2^2 R}{dR}} = \frac{2\ln 2}{g_2^2} \quad P < \frac{1}{g_1^2} - \frac{1}{g_2^2} \tag{20}$$

$$\lim_{R \to 0} E_b(R) = \lim_{R \to 0} \frac{1}{R}\left(\frac{2^{R+1}}{g_1 g_2} - \frac{1}{g_1^2} - \frac{1}{g_2^2}\right) = \lim_{R \to 0} \frac{\frac{d(2^{R+1})}{dR}}{\frac{dg_1 g_2 R}{dR}} = \frac{2\ln 2}{g_1 g_2} \quad P \geq \frac{1}{g_1^2} - \frac{1}{g_2^2} \tag{21}$$

We conclude that the minimum energy-per-bit as $R \to 0$ is equal to (22).

$$\lim_{R \to 0} E_b(R) = \begin{cases} \frac{2\ln 2}{g_2^2} & P < \frac{1}{g_1^2} - \frac{1}{g_2^2} \\ \frac{2\ln 2}{g_1 g_2} & P \geq \frac{1}{g_1^2} - \frac{1}{g_2^2} \end{cases} \tag{22}$$