Solution of Homework #1

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1 Problem 3.2

Given a Z channel with probabilities $p_{Y|X}(0|0) = 1$, $p_{Y|X}(1|0) = 0$, $p_{Y|X}(1|1) = p_{Y|X}(0|1) = \frac{1}{2}$, we want to find the channel capacity C. We know that the channel capacity for a discrete memory channel is defined as follows.

$$C = \max_{p_X} I(X;Y) = H(Y) - H(Y|X) \tag{1}$$

First we compute $H(Y)$ in function of $p = p_X(1)$.

$$\begin{aligned}
H(Y) =& - p_Y(1) \cdot \log(p_Y(1)) - p_Y(0) \cdot \log(p_Y(0)) \\
=& - (p_{Y|X}(1|0) \cdot p_X(0) + p_{Y|X}(1|1) \cdot p_X(1)) \cdot \log(p_{Y|X}(1|0) \cdot p_X(0) + p_{Y|X}(1|1) \cdot p_X(1)) \\
& - (p_{Y|X}(0|0) \cdot p_X(0) + p_{Y|X}(0|1) \cdot p_X(1)) \cdot \log(p_{Y|X}(0|0) \cdot p_X(0) + p_{Y|X}(0|1) \cdot p_X(1)) \\
=& - \frac{1}{2} \cdot p_X(1) \log\left(\frac{1}{2} \cdot p_X(1)\right) - \left(p_X(0) + \frac{1}{2} \cdot p_X(1)\right) \cdot \log\left(p_X(0) + \frac{1}{2} \cdot p_X(1)\right) \\
=& - \frac{1}{2} \cdot p \cdot \log\left(\frac{1}{2} \cdot p\right) - \left(1 - \frac{1}{2} \cdot p\right) \cdot \log\left(1 - \frac{1}{2} \cdot p\right)
\end{aligned}$$

Then we compute $H(Y|X)$ always in funcion of $p = p_X(1)$.

$$\begin{aligned}
H(Y|X) =& - p_{YX}(0,0) \cdot \log(p_{Y|X}(0|0)) - p_{YX}(1,0) \cdot \log(p_{Y|X}(1|0)) \\
& - p_{YX}(0,1) \cdot \log(p_{Y|X}(0|1)) - p_{YX}(1,1) \cdot \log(p_{Y|X}(1|1)) \\
=& - p_{Y|X}(0|0) \cdot p_X(0) \cdot \log(p_{Y|X}(0|0)) - p_{Y|X}(1|0) \cdot p_X(0) \cdot \log(p_{Y|X}(1|0)) \\
& - p_{Y|X}(0|1) \cdot p_X(1) \cdot \log(p_{Y|X}(0|1)) - p_{Y|X}(1|1) \cdot p_X(1) \cdot \log(p_{Y|X}(1|1)) \\
=& 0 + 0 + \frac{1}{2} \cdot p + \frac{1}{2} \cdot p = p
\end{aligned}$$

Now we are able to write $I(X;Y)$ in function of p as we can see from (2).

$$I(X;Y) = -\frac{1}{2} \cdot p \cdot \log\left(\frac{1}{2} \cdot p\right) - \left(1 - \frac{1}{2} \cdot p\right) \cdot \log\left(1 - \frac{1}{2} \cdot p\right) - p \tag{2}$$

In 1 we show the function's trend for all possible values of p.

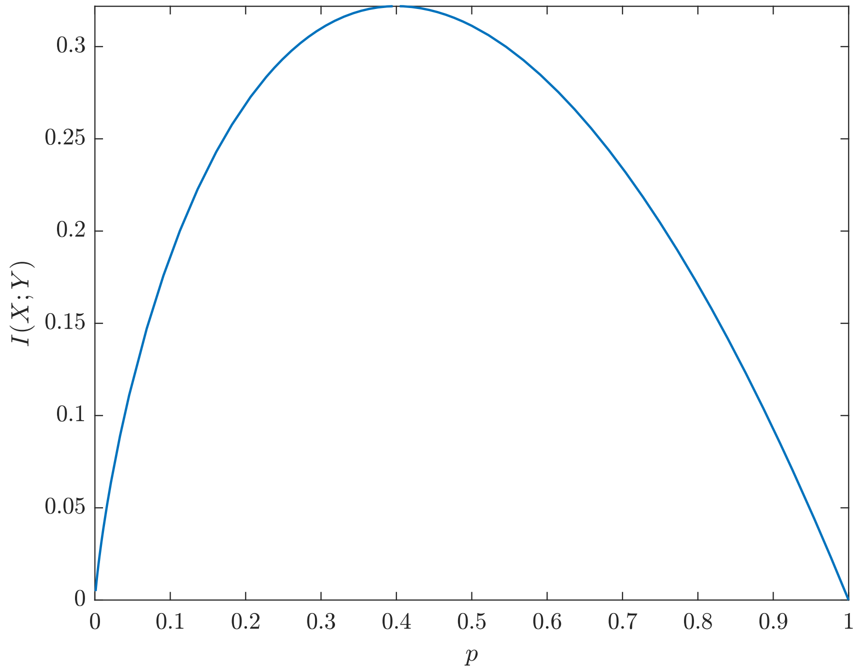

Figure 1: Mutual information as a function of $p \triangleq p_X(1)$

Now we want to find the value of p for which (2) is maximize. With this aim we compute the first and the second derivates of $I(X;Y)$.

$$\begin{aligned}\frac{dI(X;Y)}{dp} =& \frac{1}{2} \cdot \log\left(1 - \frac{p}{2}\right) + \frac{1}{2} \cdot \frac{1 - \frac{p}{2}}{1 - \frac{p}{2}} - \frac{1}{2} \cdot \log\left(\frac{p}{2}\right) - \frac{p}{2} \cdot \frac{2}{p} \cdot \frac{1}{2} - 1 \\ =& \frac{1}{2} \cdot \log\left(\frac{2}{p} - 1\right) - 1\end{aligned}$$

$$\frac{d^2I(X;Y)}{dp^2} = \frac{1}{2} \cdot \frac{p}{2-p} \cdot \frac{1}{2} = \frac{1}{4} \cdot \frac{p}{2-p}$$

We notice that $\frac{d^2I(X;Y)}{dp^2}$ is always equal or greater than zero for every possibli value of p. This means that $I(X;Y)$ is a convex function and then the value of p for which $\frac{dI(X;Y)}{dp}$ is zero coincides with the maximum point of $I(X;Y)$. Therefore we compute the values for which we have $\frac{dI(X;Y)}{dp} = 0$.

$$\frac{1}{2} \cdot \log\left(\frac{2}{p} - 1\right) - 1 = 0$$

$$\log\left(\frac{2}{p} - 1\right) = 2$$

$$\frac{2}{p} - 1 = 2^2$$

$$p = \frac{2}{5}$$

So for $p = \frac{2}{5}$ we maximize $I(X;Y)$ and therefore we obtain the value of the channel capacity C.

$$\begin{aligned}
C =& I(X;Y)\big|_{p=\frac{2}{5}} \\
=& -\frac{4}{5} \cdot \log\left(\frac{4}{5}\right) - \frac{1}{5} \cdot \log\left(\frac{1}{5}\right) - \frac{2}{5} \\
=& \frac{1}{5} \cdot \log\left(\frac{3125}{256}\right) - \frac{2}{5}
\end{aligned}$$