Solution of Homework #1

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1 Problem 3.2

Given a Z channel with probabilities $p_{Y|X}(0|0) = 1$, $p_{Y|X}(1|0) = 0$, $p_{Y|X}(1|1) = p_{Y|X}(0|1) = \frac{1}{2}$, we want to find the channel capacity C. We know that the channel capacity for a discrete memory channel is defined as follows.

$$C = \max_{p_X} I(X;Y) = H(Y) - H(Y|X) \tag{1.1}$$

First we compute $H(Y)$ in function of $p = p_X(1)$.

$$\begin{aligned}
H(Y) =& - p_Y(1) \cdot \log(p_Y(1)) - p_Y(0) \cdot \log(p_Y(0)) \\
=& - (p_{Y|X}(1|0) \cdot p_X(0) + p_{Y|X}(1|1) \cdot p_X(1)) \cdot \log(p_{Y|X}(1|0) \cdot p_X(0) + p_{Y|X}(1|1) \cdot p_X(1)) \\
& - (p_{Y|X}(0|0) \cdot p_X(0) + p_{Y|X}(0|1) \cdot p_X(1)) \cdot \log(p_{Y|X}(0|0) \cdot p_X(0) + p_{Y|X}(0|1) \cdot p_X(1)) \\
=& - \frac{1}{2} \cdot p_X(1) \log\left(\frac{1}{2} \cdot p_X(1)\right) - \left(p_X(0) + \frac{1}{2} \cdot p_X(1)\right) \cdot \log\left(p_X(0) + \frac{1}{2} \cdot p_X(1)\right) \\
=& - \frac{1}{2} \cdot p \cdot \log\left(\frac{1}{2} \cdot p\right) - \left(1 - \frac{1}{2} \cdot p\right) \cdot \log\left(1 - \frac{1}{2} \cdot p\right)
\end{aligned} \tag{1.2}$$

Then we compute $H(Y|X)$ always in funcion of $p = p_X(1)$.

$$\begin{aligned}
H(Y|X) =& - p_{YX}(0,0) \cdot \log(p_{Y|X}(0|0)) - p_{YX}(1,0) \cdot \log(p_{Y|X}(1|0)) \\
& - p_{YX}(0,1) \cdot \log(p_{Y|X}(0|1)) - p_{YX}(1,1) \cdot \log(p_{Y|X}(1|1)) \\
=& - p_{Y|X}(0|0) \cdot p_X(0) \cdot \log(p_{Y|X}(0|0)) - p_{Y|X}(1|0) \cdot p_X(0) \cdot \log(p_{Y|X}(1|0)) \\
& - p_{Y|X}(0|1) \cdot p_X(1) \cdot \log(p_{Y|X}(0|1)) - p_{Y|X}(1|1) \cdot p_X(1) \cdot \log(p_{Y|X}(1|1)) \\
=& 0 + 0 + \frac{1}{2} \cdot p + \frac{1}{2} \cdot p = p
\end{aligned} \tag{1.3}$$

Now we are able to write $I(X;Y)$ in function of p as we can see from (1.4).

$$I(X;Y) = -\frac{1}{2} \cdot p \cdot \log\left(\frac{1}{2} \cdot p\right) - \left(1 - \frac{1}{2} \cdot p\right) \cdot \log\left(1 - \frac{1}{2} \cdot p\right) - p \tag{1.4}$$

In 1 we show the function's trend for all possible values of p.

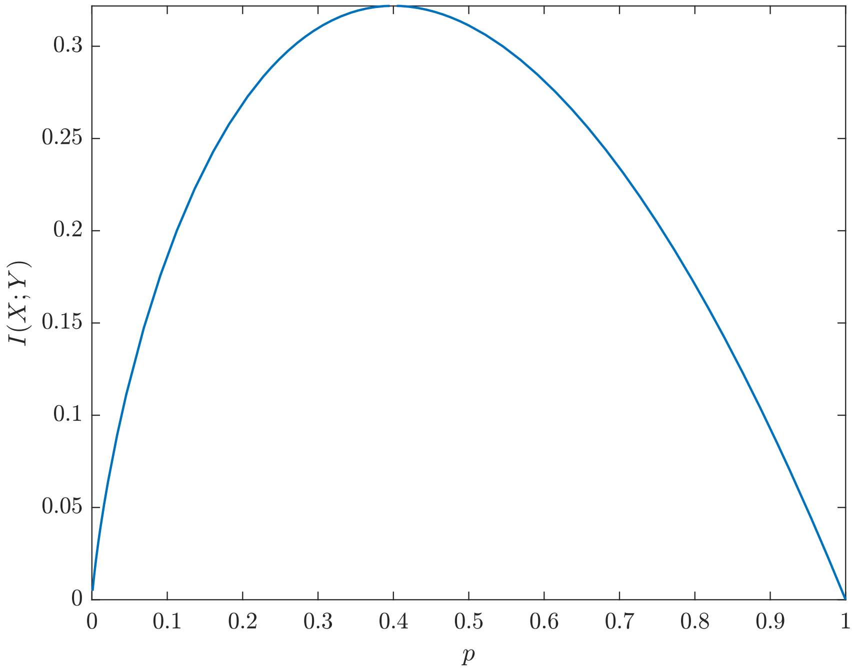

Figure 1: Mutual information as a function of $p \triangleq p_X(1)$

Now we want to find the value of p for which (1.4) is maximize. With this aim we compute the first and the second derivates of $I(X;Y)$.

$$\begin{aligned}\frac{dI(X;Y)}{dp} =& \frac{1}{2}\cdot\log\left(1-\frac{p}{2}\right)+\frac{1}{2}\cdot\frac{1-\frac{p}{2}}{1-\frac{p}{2}}-\frac{1}{2}\cdot\log\left(\frac{p}{2}\right)-\frac{p}{2}\cdot\frac{2}{p}\cdot\frac{1}{2}-1\\ =& \frac{1}{2}\cdot\log\left(\frac{2}{p}-1\right)-1\end{aligned} \tag{1.5}$$

$$\frac{d^2I(X;Y)}{dp^2}=\frac{1}{2}\cdot\frac{p}{2-p}\cdot\frac{1}{2}=\frac{1}{4}\cdot\frac{p}{2-p} \tag{1.6}$$

We notice that $\frac{d^2I(X;Y)}{dp^2}$ is always equal or greater than zero for every possibli value of p. This means that $I(X;Y)$ is a convex function and then the value of p for which $\frac{dI(X;Y)}{dp}$ is zero coincides with the maximum point of $I(X;Y)$. Therefore we compute the values for which we have $\frac{dI(X;Y)}{dp}=0$.

$$\begin{aligned}\frac{1}{2}\cdot\log\left(\frac{2}{p}-1\right)-1=0\\ \log\left(\frac{2}{p}-1\right)=2\\ \frac{2}{p}-1=2^2\\ p=\frac{2}{5}\end{aligned} \tag{1.7}$$

So for $p = \frac{2}{5}$ we maximize $I(X;Y)$ and therefore we obtain the value of the channel capacity C. The value of C is reported in (1.8).

$$\begin{aligned} C =& I(X;Y)\big|_{p=\frac{2}{5}} \\ =& -\frac{4}{5} \cdot \log\left(\frac{4}{5}\right) - \frac{1}{5} \cdot \log\left(\frac{1}{5}\right) - \frac{2}{5} \\ =& \frac{1}{5} \cdot \log\left(\frac{3125}{256}\right) - \frac{2}{5} \end{aligned} \tag{1.8}$$

2 Problem 3.5

2.1 3.5(a)

Let's define

$$N(m) \triangleq |\{i : y_i = x_i(m)\}| \tag{2.1}$$

Note that it is strictly related to to the hamming distance, in fact
$d(x^n(m), y^n) = n - N(m)$.
In the case of a memoryless channel, the MLD works as follows:

$$\hat{m} = \arg\max_m \prod_{i=1}^{n} p_{Y|X}(y_i|x_i(m)) = \arg\max_m \,(1-p)^{N(m)} p^{n-N(m)} \tag{2.2}$$

where the last equality comes from the fact that the Binary Symmetric Channel (BSC) only cares about whether a symbol is correctly or incorrectly transmitted. Assuming that $x^n(m)$ is sent and y^n is received, there will be $N(m)$ correctly transmitted bits (each w.p. $(1-p)$ and $n - N(m)$ errors (each w.p. p).

3 Problem 3.13

Is given a Gaussian product channel $Y_j = g_j \cdot X_j + Z_j$ $j \in \{1, 2\}$ with $g_1 < g_2$ and average power constraint P. We want to know above what power P^* we start to use both the channels and what are the features of the *energy-per-bit-rate function* $E_b(R)$.

3.1 Optimal power allocation

We notiche that to achieve the maximum capacity from the channel we have to solve the following optimization problem.

$$\begin{gathered} \max_{P_j} \sum_{j=1}^{2} \frac{1}{2} \cdot \log(1 + g_j^2 \cdot P_j) \quad subject \quad to \\ -P_j \neq 0 \quad j \in \{1, 2\} \\ \sum_{j=1}^{2} P_j - P = 0 \end{gathered} \tag{3.1}$$

In order to solve (3.1) we build the following equation.

$$\nabla_{P_j}\Big\{-\sum_{j=1}^{2}\frac{1}{2}\cdot\log(1+g_j^2\cdot P_j)+\sum_{j=1}^{2}\lambda_j\cdot(-P_j)+\nu\cdot\Big(\sum_{j=1}^{2}P_j-P\Big)\Big\}=0 \tag{3.2}$$

Solving (3.2) we obtain what follows.

$$\begin{aligned}&\frac{1}{2}\cdot\frac{1}{P_j+\frac{1}{g_j^2}}+\lambda_j-\nu=0\\ &\Rightarrow\frac{1}{2}\cdot\frac{1}{P_j+\frac{1}{g_j^2}}\le\nu=\frac{1}{2\mu}\end{aligned} \tag{3.3}$$

$$\begin{aligned}&\sum_{j=1}^{2}\max\Big\{\mu-\frac{1}{g_j^2},0\Big\}=P\\ &\Rightarrow\mu=\frac{P}{2}+\frac{1}{2g_1^2}+\frac{1}{2g_2^2}\end{aligned} \tag{3.4}$$

$$\begin{aligned}&P_j=\max\Big\{\mu-\frac{1}{g_j^2},0\Big\}\\ &\Rightarrow P_1=\max\Big\{\frac{P}{2}+\frac{1}{2g_2^2}-\frac{1}{2g_1^2},0\Big\}=\begin{cases}\frac{P}{2}+\frac{1}{2g_2^2}-\frac{1}{2g_1^2} & P\ge\frac{1}{g_1^2}-\frac{1}{g_2^2}\\ 0 \quad P<\frac{1}{g_1^2}-\frac{1}{g_2^2}\end{cases}\\ &\Rightarrow P_2=\max\Big\{\frac{P}{2}+\frac{1}{2g_1^2}-\frac{1}{2g_2^2},0\Big\}=\frac{P}{2}+\frac{1}{2g_1^2}-\frac{1}{2g_2^2}\quad\forall\quad P\ge 0\end{aligned} \tag{3.5}$$

We conclude that the second channel is opened for every amount of allocated power P while the first channel is opened only if and only if (3.6) is verified.

$$P\ge\frac{1}{g_1^2}-\frac{1}{g_2^2} \tag{3.6}$$

3.2 Energy-per-bit-rate function computation

Now we want to compute the *energy-per-bit-rate function* $E_b(R)$ in the scenario previously described. We know that $P=R\cdot E$ where R is the *bit-rate* of the channel and E is the used energy for every bit. In addiction we know that the *bit-rate* is bounded by channel capacity as shown in (3.7) and (3.8).

$$R\le\frac{1}{2}\cdot\log\big(1+g_2^2P\big)\Leftrightarrow P<\frac{1}{g_1^2}-\frac{1}{g_2^2} \tag{3.7}$$

$$R\le\frac{1}{2}\cdot\sum_{j=1}^{2}\log\Big(\frac{g_j^2}{2}\cdot\Big(P+\frac{1}{g_1^2}+\frac{1}{g_2^2}\Big)\Big)=\frac{1}{2}\cdot\log\Big(\frac{g_1^2g_2^2}{4}\cdot\Big(P+\frac{1}{g_1^2}+\frac{1}{g_2^2}\Big)^2\Big)\Leftrightarrow P\ge\frac{1}{g_1^2}-\frac{1}{g_2^2} \tag{3.8}$$

Let's consider (3.7) which is the case where only one channel is opened. Writing R in function of P we get what follows.

$$P\ge\Big(\frac{2^{2R}-1}{g_2^2}\Big)\Leftrightarrow R<\frac{1}{2}\log\Big(\frac{g_2^2}{g_1^2}\Big) \tag{3.9}$$

Let's consider (3.8) which is the case where only one channel is opened. Writing R in function of P we get what follows.

$$P \geq \left(\frac{2^{R+1}}{g_1 g_2} - \frac{1}{g_1^2} - \frac{1}{g_2^2}\right) \Leftrightarrow R \geq \frac{1}{2}\log\left(\frac{g_2^2}{g_1^2}\right) \tag{3.10}$$

Knowing that $P = E \cdot R$ we define the *energy-per-bit-rate function* $E_b(R)$ as shown in (3.11).

$$E_b(R) = \begin{cases} \frac{1}{R}\left(\frac{2^{2R}-1}{g_2^2}\right) & R < \frac{1}{2}\log\left(\frac{g_2^2}{g_1^2}\right) \\ \frac{1}{R}\left(\frac{2^{R+1}}{g_1 g_2} - \frac{1}{g_1^2} - \frac{1}{g_2^2}\right) & R \geq \frac{1}{2}\log\left(\frac{g_2^2}{g_1^2}\right) \end{cases} \tag{3.11}$$

Now we want to show that the function (3.11) is strictly monotonically increasing and convex on R. To do so we first compute the first and the second derivates of the function itself.

$$E_b'(R) = \begin{cases} \frac{1}{R^2 g_2^2} \cdot \left(2^{2R}(2R\ln(2) - 1) + 1\right) & R < \frac{1}{2}\log\left(\frac{g_2^2}{g_1^2}\right) \\ \frac{1}{R^2 g_1^2 g_2^2}\left(2^{R+1} g_1 g_2 (R\ln(2) - 1) + g_1^2 + g_2^2\right) & R \geq \frac{1}{2}\log\left(\frac{g_2^2}{g_1^2}\right) \end{cases} \tag{3.12}$$

$$E_b''(R) = \begin{cases} \frac{1}{R^3 g_2^2}\left(2^{2R}(4R^2\ln(2)^2 - 2R\ln(2) + 1) - 1\right) & R < \frac{1}{2}\log\left(\frac{g_2^2}{g_1^2}\right) \\ \frac{1}{R^3 g_1 g_2}\left(2^{R+1} g_1 g_2 (R^2\ln(2)^2 - R\ln(2) + 1) - g_1^2 - g_2^2\right) & R \geq \frac{1}{2}\log\left(\frac{g_2^2}{g_1^2}\right) \end{cases} \tag{3.13}$$

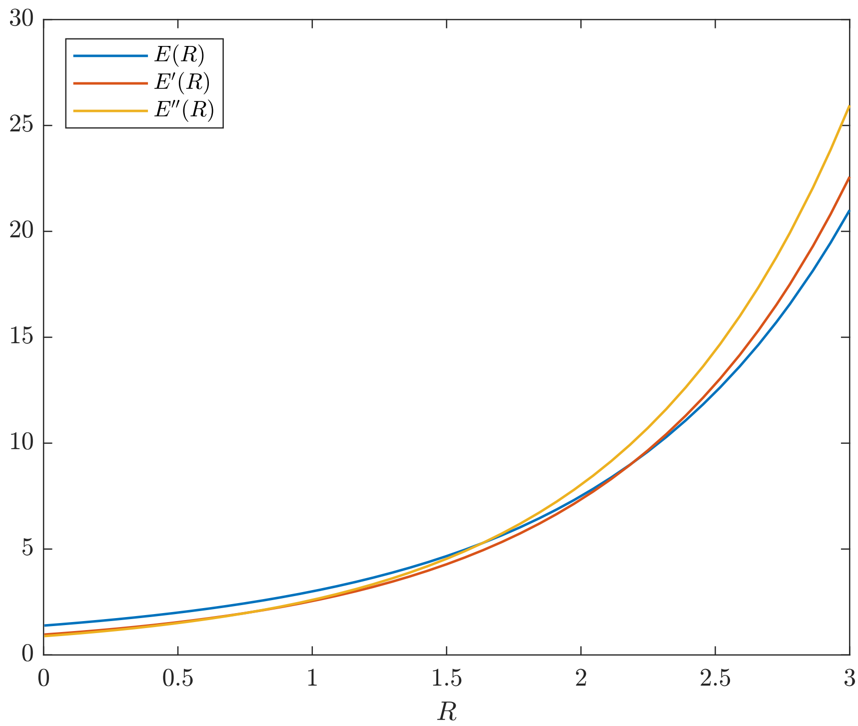

Figure 2: $E_b(R)$, $E_b'(R)$, $E_b''(R)$ for $R < \frac{1}{2}\log\left(\frac{g_2^2}{g_1^2}\right)$

Now we want to compute the minimum energy-per-bit as $R \to 0$. We notice that for low value of R we find ourself in the case where only one of the two channels id opened. Then for compute $\lim_{R\to 0} E_b(R)$ we procede as follows.

$$\lim_{R\to 0} E_b(R) = \lim_{R\to 0} \left(\frac{2^{2R} - 1}{g_2^2 R} \right) = \lim_{R\to 0} \frac{\frac{d(2^{2R}-1)}{dR}}{\frac{dg_2^2 R}{dR}} = \frac{2\ln 2}{g_2^2} \tag{3.14}$$

We conclude that the minimum energy-per-bit as $R \to 0$ is equal to (3.15).

$$\lim_{R\to 0} E_b(R) = \frac{2\ln 2}{g_2^2} \tag{3.15}$$

4 Problem 3.14

5 Problem 3.20