Branch and bound algorithm for SCP with Conflict Sets

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1 Introduction

The project tackle a variant of the Set Covering Problem (SCP) introducing penalties if certain couples of sets are selected. Formally, let $E = \{1, ..., n\}$ be a set of elements and let $\mathcal{S} = \{S_j \subseteq E : j \in M\}$ be the set of subsets, where $M = \{1, ..., m\}$. The subsets composition is defined by a matrix $A \in \mathbb{R}^{n \times m}$:

$$A_{ij} = \begin{cases} 1 & \text{if } E_i \in S_j \\ 0 & \text{otherwise} \end{cases}$$

The vector $c \in \mathbb{R}^m$ defines the costs on each subset, while the matrix $P \in \mathbb{R}^{m \times m}$ defines the penalties for each couple of subsets, i.e. P_{ij} is the penalty paid if $S_i, S_j \in \mathcal{S}$ are both selected. A MILP formulation for the SCP with conflict sets is the following:

$$\min \sum_{i \in S} x_i c_i + \sum_{i \in S} \sum_{j \in S} y_{ij} P_{ij} \tag{1}$$

$$\text{s.t.} \sum_{i \in S} A_{ik} x_i \geq 1 \qquad \forall k \in E \tag{2}$$

$$x_i + x_j \leq 1 + y_{ij} \qquad \forall i, j \in M \tag{3}$$

$$x_i \in \{0, 1\} \qquad \forall i \in M \tag{4}$$

$$y_{ij} \in \{0, 1\} \qquad \forall i, j \in M \tag{5}$$

$$\tag{6}$$

The chosen approach is to use a branch and bound algorithm with a lagrangean relaxation used for dual bound computing.

2 Lagrangean relaxation

We chose to relax the covering constraints, obtaining the following lagrangean relaxation:

$$
\begin{aligned}
\min \sum_{i \in S} x_i c_i + \sum_{i \in S} \sum_{j \in S} y_{ij} P_{ij} + \lambda_i \left(\sum_{i \in S} A_{ik} x_i - 1 \right) & & \quad (7) \\
\text{s.t. } x_i + x_j \leq 1 + y_{ij} & \quad \forall i, j \in M & \quad (8) \\
x_i \in \{0, 1\} & \quad \forall i \in M & \quad (9) \\
y_{ij} \in \{0, 1\} & \quad \forall i, j \in M & \quad (10) \\
& & \quad (11)
\end{aligned}
$$