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THE METHOD OF VIRTUAL POWER IN CONTINUUM MECHANICS. PART 2: MICROSTRUCTURE*

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Abstract. The advantages of a systematic application of the method of virtual power are exemplified by the study of continuous media with microstructure. Results on micromorphic media of order one are easily found and the equations of motion for the general micromorphic medium are established for the first time. Various interesting special cases, which have been previously considered in the literature, may be derived upon imposing some convenient constraints. The paper ends with some comparisons with other related work.

1. Introduction. In a recent note (Germain (1972)), it was emphasized how the classical method of virtual work may be used in a more systematic way than it has been in the past in order to derive the fundamental equations of a given theory in continuum mechanics. As an example, first and second gradient theories have been considered with some details in a more recent paper (Germain (1973a)). It is not claimed of course that the idea is new. Nevertheless, it may give new insight into some questions which have been treated so far by more conventional tools. More important, it provides the shortest way to obtain the required equations and to avoid any ambiguity or error. Recent papers from Casal (1972) and Breuneval (1973) give new illustrations of the relevance of this proposition.

The present contribution is related to continuous media inside which microstructure may be taken into account. This topic has been considered by many authors, for instance, Mindlin (1964) and Eringen and his coworkers in many papers published since 1964—some of them may be found in the References. These media are called "micromorphic" in Eringen's terminology. The present paper is quite limited in scope. The main steps of the application of the method are briefly reviewed in § 2. Then classical equations for equilibrium in the most simple case are derived in § 3. The dynamical equations require special attention (§ 4). Generalizations when a more refined description of the kinematics and of the stresses is desired follow in § 5. Special cases and comparison with other theories are finally considered in §§ 6 and 7. We do not intend in this paper to introduce constitutive relations which of course must be taken into account in order to be able to apply a micromorphic theory to specific physical problems. Numerous papers in the literature deal with these applications. Our aim here is just to give a new illustration of the simplicity and usefulness of the method of virtual power.

The author is specially pleased to be able to publish this work in the volume dedicated to Professor Prager and to join his friends and colleagues in this tribute of admiration and gratitude to this great scientist who, in a quite different context, has shown on many occasions how d'Alembert's principle of virtual work is a very precious tool in various problems of continuum mechanics.

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The following notation is used:

U_i	—velocity (components).
D_{ij}	—strain rate tensor.
χ_{ij}	—gradient of the relative velocities—symmetric part: microstrain rate tensor, antisymmetric part: microrotation rate tensor.
η_{ij}	—relative microvelocity gradient.
$\varkappa_{ijk} = \chi_{ij,k}$	—second gradient of the relative velocities.
$\sigma_{ij} = \sigma_{ji}$	—intrinsic stress tensor.
s_{ij}	—intrinsic microstress tensor.
v_{ijk}	—intrinsic second microstress tensor.
$\tau_{ij} = \sigma_{ij} + s_{ij}$	—stress tensor.
f_i	—volumic force.
Ψ_{ij}	—volumic double force.
T_i	—surface traction.
M_{ij}	—surface double traction.
γ_i	—acceleration.
γ_{ij}	—microacceleration.

2. Description of the method. Without going into too many details, it is nevertheless worthwhile to review the main ideas involved in the application of the method. Forces and stresses are not introduced directly but by the value of the virtual power they produce for a given class of virtual motions. Then, roughly speaking, if one extends the class of virtual motions which are considered, one refines the description of forces and stresses.

More precisely, for a given system S and at a given time, a virtual motion is defined by a vector field V on S: V is the field of virtual velocities. A class of virtual motions is a vector space $\mathscr{V}$ whose elements are V. The "system of forces" one wants to consider is defined by a linear continuous application $\mathscr{V} \to R$ or

$$\mathscr{P} = \mathscr{L}(V). \tag{1}$$

$\mathscr{P}$, a real number, is the virtual power produced by "this system of forces" in the virtual motion V. Obviously, $\mathscr{V}$ must be a topological vector space in order to be able to define the continuity of $\mathscr{L}(V)$.

In fact, virtual velocities like real velocities are defined with respect to a given frame. The velocity fields of the *same* virtual motion in two different frames differ only by a rigid body velocity field C. Rigid body velocity fields, sometimes called *distributor*, define a vector space $\mathscr{C}$. It will always be assumed in what follows that $\mathscr{C}$ is a subspace of $\mathscr{V}$.

The various "forces" which act on the mechanical system are divided in a very classical way into two classes: *external forces* which represent the dynamical effects on S due to the interaction with other systems which have no common part with S, and *internal forces* which represent the mutual dynamical effects of subsystems of S; for instance, if S_i and S_j are two disjoint subsystems of S, the action of S_i on S_j represents "external forces" acting on S_j, but "internal forces" if one considers the system S itself. Let us stretch the meaning of the word "forces" for the time being and let it be taken in the vague sense it may have in the common language and not in any precise mathematical conception it may receive (and will eventually receive below).

It is crucial to notice that the definition of the virtual power of "internal forces" is subject to a limitation which may be expressed as the following axiom.

AXIOM OF POWER OF INTERNAL FORCES. *The virtual power of the "internal forces" acting on a system S for a given virtual motion is an objective quantity; i.e., it has the same value whatever be the frame in which the motion is observed.*

It is obviously equivalent to state:

The virtual power of the "internal forces" acting on a system S is null for any distributor.

It is quite clear that this axiom is the counterpart in this presentation of dynamics of the "principle of superposed rigid body motion" or of the "principle of objectivity" (material indifference). "External forces" are not subjected to a similar limitation. It is barely necessary to recall d'Alembert's famous statement:

PRINCIPLE OF VIRTUAL POWER. *Let a system S be in equilibrium with respect to a given Galilean frame; then, in any virtual motion, the virtual power of all the "internal forces" and "external forces" acting on S is null.*

Looking at the definition of the virtual power, one sees that the natural mathematical frame for such a theory would be the frame of *generalized functions or distributions.* In fact we will not use it, first for simplicity and second because, as we are dealing with physical phenomena, one may reasonably assume that the distributions involved are "regular enough" in order to be represented by functions (or densities). To be more explicit, let us close this section by recalling briefly the simplest situation which leads to the classical theory.

Assume that S is a simple connected open domain in equilibrium in a given Galilean frame. The crucial point in order to build a continuum theory is to decide on the kinematical description which will be used, i.e., to choose $\mathscr{V}$ and V, and the linear application $V \rightarrow \mathscr{P}$. Choose V as a velocity field U_i defined on S (U_i are the components of a velocity vector in an orthonormal Cartesian frame) and state that the definition of $\mathscr{P}$ will involve the value of U_i and of its first derivatives $U_{i,j}$. The corresponding theory is the first gradient theory (Germain (1973a)). The "internal forces" acting on a given subdomain of S have a virtual power which may be written[1]

$$\mathscr{P}_{(i)} = -\int_{\mathscr{D}} p\, dv, \tag{2}$$

with p a linear form on U_i and $U_{i,j}$. But with the axiom, only the symmetric part of the gradient can be involved and then

$$p = \sigma_{ij} D_{ij}, \qquad D_{ij} = \tfrac{1}{2}(U_{i,j} + U_{j,i}). \tag{3}$$

Here σ_{ij} defines a symmetric second order tensor called the *intrinsic stress tensor*. Following the classical assumptions, "external forces" acting on $\mathscr{D}$ will

[1] We assume here that p is defined at every point M of S independently of the subdomain $\mathscr{D}$. This assumption means that "internal forces" are "short range forces" and excludes, as is usually done in continuum mechanics, "long range internal forces". This more general case, which may also be investigated by the present method, is not considered in this paper.

be of two types: first long range volume forces, produced on $\mathscr{D}$ by systems located outside S, giving the virtual power

$$\mathscr{P}_{(d)} = \int_{\mathscr{D}} (f_i U_i + C_{ij}\Omega_{ij} + \Phi_{ij}D_{ij})\, dv, \tag{4}$$

where Ω_{ij} is the antisymmetric part of $U_{i,j}$. Formula (4) shows that f_i is the volumic force, C_{ij} and Φ_{ij} the antisymmetric and symmetric volumic double force—C_{ij} is in fact a volumic couple; usually all these quantities f_i, C_{ij}, Φ_{ij} are considered as given (some of them may be zero of course). Second, contact forces are considered, produced on $\mathscr{D}$ by the parts of S located outside $\mathscr{D}$, giving the virtual power

$$\mathscr{P}_{(c)} = \int_{\partial\mathscr{D}} T_i U_i\, d\sigma \tag{5}$$

(it is not necessary to write here the contributions of Ω_{ij} and D_{ij} because it may be shown that they must be zero); T_i defines the usual traction acting on the boundary of $\mathscr{D}$. Using (2), (3), (4), (5) and applying the principle of virtual power, one finds after some integrations by parts that

$$\int_{\mathscr{D}} (f_i + \sigma_{ij,j} - C_{ij,j} - \Phi_{ij,j})U_i\, dv + \int_{\partial\mathscr{D}} \{T_i - (\sigma_{ij} - C_{ij} - \Phi_{ij})n_j\}U_i\, d\sigma = 0$$

which must be satisfied for any motion U_i. It is easy to show that, if one introduces the stress tensor τ_{ij},

$$\tau_{ij} = \sigma_{ij} - C_{ij} - \Phi_{ij}, \tag{6}$$

one must have

$$f_i + \tau_{ij,j} = 0, \qquad T_i = \tau_{ij}n_j. \tag{7}$$

Formulas (6) and (7) are the fundamental equations of the first gradient theory, which reduces to the classical theory if one assumes that $C_{ij} = \Phi_{ij} = 0$.

3. Equilibrium of a micromorphic continuum of degree 1. The main point in the application of the method, as reviewed above, is to define the kinematics of the virtual motions which are being considered. In the classical description, a continuum is a continuous distribution of *particles*, each of them being represented *geometrically* by a point M and characterized *kinematically* by a velocity U_i. In a theory which takes microstructure into account, from the macroscopic point of view (which is the point of view of a continuum theory), each particle is still represented by a point M, but its kinematical properties are defined in a more refined way. In order to discover how to define the kinematical properties of a particle, one has to look at it from a microscopic point of view. At this level of observation, a particle appears itself as a continuum $P(M)$ of small extent. Let us call M its center of mass, M' a point of $P(M)$, U_i the velocity of M, x_i' the coordinates of M' in a Cartesian frame parallel to the given frame x_i with M as the origin, U_i' the velocity of M' with respect to the given frame, and x_i the coordinates of M in the given frame. If one assumes for the moment that the x_i are given, i.e., if one looks at a given particle, the field U_i' appears as a function of the x_i' defined on $P(M)$. As $P(M)$ is of small extent, it is natural to look at the Taylor expansion of U_i' with respect to the x_j' and also, as a first approximation, to stop this expansion with the terms of degree 1. Then, with this assumption, one can write

$$U'_i = U_i + \chi_{ij}x'_j. \tag{8}$$

The physical significance of this assumption is clear: one postulates that one can get a sufficient description of the relative motion of the various points of the particle if one assumes that this relative motion is a homogeneous deformation. With such an assumption, the macroscopic kinematical description of our continuum will be completely defined if one knows two fields on S, the *velocity field* U_i, and the field χ_{ij} of the gradients of the relative velocities of P (M), also called the *microvelocity* gradient, χ_{ij} obviously being a second order tensor. The *symmetric part of* χ_{ij} *will be called the microstrain rate tensor, the antisymmetric part—the microrotation rate tensor.* With such a choice of the vector space $\mathscr{V}$ of the virtual motions, we say that we are dealing with a micromorphic continuum of degree 1.

The next step is to choose our linear application $\mathscr{L}$. To be specific and to stick to a simple case, we shall be satisfied with a first gradient theory. Let us note that

$$\eta_{ij} = U_{i,j} - \chi_{ij}, \qquad \varkappa_{ijk} = \chi_{ij,k}. \tag{9}$$

The power of the internal forces will be written as in (2) with p a linear combination of U_i, $U_{i,j}$, χ_{ij}, $\varkappa_{ijk}$ or equivalently of U_i, $U_{i,j}$, η_{ij}, $\varkappa_{ijk}$:

$$p = K_iU_i + \sigma_{ij}U_{i,j} + s_{ij}\eta_{ij} + v_{ijk}\varkappa_{ijk}. \tag{10}$$

But as $\mathscr{D}$ is an arbitrary subdomain of S, and p is independent of $\mathscr{D}$—see footnote 1—the axiom on "internal forces" tells us that p itself must remain unchanged when computed in a different frame. Obviously, such a change of frame implies that U_i is to be replaced by $U_i + \bar{U}_i + \bar{\Omega}_{ij}x_j$, where $\bar{U}_i$ and $\bar{\Omega}_{ij}$ are a vector and a second order tensor respectively, both constant in S. A similar change is to be applied to U'_i, the velocity field of the subparticle, which leads to the conclusion that χ_{ij} has to be replaced by $\chi_{ij} + \bar{\Omega}_{ij}$. As a result, η_{ij} defined by (9) *remains invariant* by this change of frame, a quite natural conclusion because, obviously, η_{ij} is the *relative microvelocity gradient* and then is related to the local relative motion of the particle with respect to the continuum. Then, one must have

$$K_i\bar{U}_i + \sigma_{ij}\bar{\Omega}_{ij} = 0$$

identically at every point of S, which implies that in (10), K_i is zero and σ_{ij} is a symmetric tensor.

Finally, the virtual power of the "internal forces" in $\mathscr{D}$ may be written

$$\mathscr{P}_{(i)} = -\int_{\mathscr{D}} \{(\sigma_{ij} + s_{ij})U_{i,j} - s_{ij}\chi_{ij} + v_{ijk}\varkappa_{ijk}\}\, dv,$$

or, after an integration by parts,

$$\begin{aligned}\mathscr{P}_{(i)} = &\int_{\mathscr{D}} \{(\sigma_{ij,j} + s_{ij,j})U_i + (s_{ij} + v_{ijk,k})\chi_{ij}\}\, dv \\ &- \int_{\partial\mathscr{D}} \{(\sigma_{ij} + s_{ij})n_jU_i + v_{ijk}\chi_{ij}n_k\}\, d\sigma.\end{aligned} \tag{11}$$

The following step is to introduce the virtual power of the "external forces" applied to $\mathscr{D}$. We can generalize what has been reviewed in § 2 by considering long range volumic forces with virtual power

$$\mathscr{P}_{(d)} = \int_{\mathscr{D}} (f_i U_i + \Psi_{ij}\chi_{ij})\, dv, \tag{12}$$

and contact forces with virtual power

$$\mathscr{P}_{(c)} = \int_{\partial\mathscr{D}} (T_i U_i + M_{ij}\chi_{ij})\, d\sigma. \tag{13}$$

Expressions (12) and (13) are not as general as a systematic first gradient theory, but will be quite sufficient for the main applications.

Note that Ψ_{ij} is a volumic double force, and M_{ij} a double surface traction, both of them including in the general case a symmetric and an antisymmetric part.

The last step is to apply the principle of virtual power. First assume that U_i and χ_{ij} are chosen in such a way that U_i and χ_{ij} are zero outside a compact subset interior to $\mathscr{D}$. In the sum $\mathscr{P}_{(i)} + \mathscr{P}_{(d)} + \mathscr{P}_{(c)}$, only the volume integral remains, which must be zero for any choice of U_i and χ_{ij}. As an obvious consequence, if one assumes that all the quantities which have been written are continuous in S, one must have, when putting

$$\tau_{ij} = \sigma_{ij} + s_{ij}, \tag{14}$$

the two equations of equilibrium

$$\tau_{ij,j} + f_i = 0, \tag{15}$$

$$s_{ij} + v_{ijk,k} + \Psi_{ij} = 0. \tag{16}$$

It is natural to call τ_{ij} the (Cauchy) *stress tensor*, σ_{ij} the *intrinsic part of the stress tensor*, s_{ij} the *microstress tensor*, v_{ijk} the *second microstress tensor*. With this partial result one may write

$$0 = \int_{\partial\mathscr{D}} \{(T_i - \tau_{ij}n_j)U_i + (M_i - v_{ijk}n_k)\chi_{ij}\}\, d\sigma,$$

and, again, as U_i and χ_{ij} may take arbitrary values in every point of $\partial\mathscr{D}$, one obtains the boundary conditions

$$T_i = \tau_{ij}n_j, \qquad M_{ij} = v_{ijk}n_k, \tag{17}$$

which express the traction T_i (stress vector) and the double surface traction M_{ij} as linear functions of the unit normal n_i, the linear application so defined being the stress tensor and the second microstress tensor respectively.

These results are summarized in the following theorem.

THEOREM 1. *The state of stress in a micromorphic medium of degree* 1 *is completely defined by the intrinsic stress tensor* σ_{ij}*—a symmetric tensor—the microstress tensor* s_{ij}*, and the second microstress tensor* v_{ijk}*. Let* f_i *and* Ψ_{ij} *be the volumic force and double force,* T_i *and* M_{ij} *the surface traction and double surface traction respectively. Define the stress tensor by* $\tau_{ij} = \sigma_{ij} + s_{ij}$*. Then, the equations* *in S of equilibrium are given by* (15) *and* (16), *the boundary conditions by* (17) *and the volumic energy of stresses by*

$$p = \tau_{ij}U_{i,j} - s_{ij}\chi_{ij} + \nu_{ijk}\varkappa_{ijk}, \tag{18}$$

where U_i *and* χ_{ij} *are the velocity and the gradient of the relative velocities of a particle, and* $\varkappa_{ijk} = \chi_{ij,k}$, *the second gradient of the relative velocities.*

The above results were announced in a previous note (Germain (1972)). Their development here is much easier than the way previously proposed in the literature.

4. Dynamical equations for a micromorphic medium of order 1. Inertia effects must now be taken into account. Before giving the convenient definitions for a micromorphic medium and the generalized statement of the principle of virtual power, it is worthwhile to derive some preliminary results for a homogeneous deformation.

4.1. Homogeneous deformation. Let us consider a body V—which later on will represent a particle—and call μ its mass, ρ' its mass density, x_i the position of its center of mass M, x_i' the position of a point M' of V with respect to its center of mass M. Then

$$\int_V \rho'\,dv' = \mu, \qquad \int_V \rho' x_j'\,dv' = 0. \tag{19}$$

One also introduces a reduced inertia tensor I_{ij} defined by

$$\mu I_{ij} = \int_V \rho' x_i' x_j'\,dv'. \tag{20}$$

In this section, one has to distinguish between real kinematic concepts and their virtual counterparts; the latter will be denoted with a $\hat{}$—for example, U_i is the real velocity of M, $\hat{U}_i$ is the virtual velocity of M, U_i' is the real velocity of M', $\hat{U}_i'$ the virtual velocity of M'.

It is assumed that *any time* the velocity field on V is defined by (8), where χ_{ij} is independent of the x_i'. Accordingly, the vector space of virtual motions is defined by a vector $\hat{U}_i$ and a second order tensor $\hat{\chi}_{ij}$ in such a way that the virtual velocity field of the body V is given by

$$\hat{U}_i' = \hat{U}_i + \hat{\chi}_{ij}x_j'. \tag{21}$$

The (real) acceleration field of V is given by

$$\gamma_i' = \gamma_i + \dot{\chi}_{ij}x_j' + \chi_{ij}\dot{x}_j',$$

where a dot means the time derivative (material derivative, more precisely) and where $\gamma_i = \dot{U}_i$ is the acceleration of the center of mass M. But $\dot{x}_j'$ is the relative velocity of a point M' with respect to the frame of the center of mass M and then, as $\dot{x}_j' = U_j' - U_j$, one may write

$$\gamma_i' = \gamma_i + (\dot{\chi}_{ij} + \chi_{ip}\chi_{pj})x_j'. \tag{22}$$

Now, the kinetic energy of V, the virtual power of mass-velocity or of momentum of V, and the virtual power of mass-acceleration of V are given respectively by

$$\mu k = \frac{1}{2}\int_V \rho' U_i' U_i' \, dv', \tag{23}$$

$$\mu \hat{p}_K = \int_V \rho' U_i' \hat{U}_i' \, dv', \tag{24}$$

$$\mu \hat{p}_A = \int_V \rho' \gamma_i' U_i' \, dv'. \tag{25}$$

Thus, a straightforward calculation shows the following result.

LEMMA 1. *The quantities k, $\hat{p}_K$, $\hat{p}_A$ defined by* (23), (24), *and* (25) *are given by*

$$k = \tfrac{1}{2}(U_i U_i + \chi_{ip}\chi_{iq} I_{pq}), \tag{26}$$

$$\hat{p}_K = \hat{U}_i U_i + \hat{\chi}_{ij}\chi_{ip} I_{pj}, \tag{27}$$

$$\hat{p}_A = \hat{U}_i \gamma_i + \hat{\chi}_{ij}\gamma_{ip} I_{pj}, \tag{28}$$

with

$$\gamma_{ip} = \dot{\chi}_{ip} + \chi_{iq}\chi_{qp}. \tag{29}$$

It is also easy to prove the next lemma.

LEMMA 2. *When t varies, then*

$$\dot{I}_{ij} = \chi_{ip} I_{pj} + \chi_{jp} I_{pi}. \tag{30}$$

For, as μ is a constant, one has by derivation of (20):

$$\mu \dot{I}_{ij} = \int_V \rho'(x_i'\dot{x}_j' + x_j'\dot{x}_i') \, dv' = \int_V \rho'(\chi_{jp} x_i' x_p' + \chi_{ip} x_j' x_p') \, dv'.$$

4.2. Dynamics of a micromorphic medium. As in previous sections, but here with different notations, a virtual motion $\hat{\mathscr{V}}$ of such a medium will be defined by the two fields $\hat{U}_i$ and $\hat{\chi}_{ij}$, both functions of the point x_i in S at a given time. The inertia properties are defined by the mass density and the field of the reduced particle inertia tensor I_{ij} given by (20). Conservation of mass implies, for the real motion (see in particular (30)):

$$\dot{\rho} + \rho U_{i,i} = 0, \qquad \dot{I}_{ij} = \chi_{ip} I_{pj} + \chi_{jp} I_{pi}. \tag{31}$$

Note that the mass of a volume $\Delta(M)$ around M, small with respect to the dimensions of S, but large with respect to the particle size, is approximately given by

$$\rho(M)\Delta v = \sum_i \mu_i,$$

where Δv is the volume of $\Delta(M)$, and μ_i is the mass of the (great number of) particles included in $\Delta(M)$. With this remark, it is clear that one is led to the following definitions:

Kinetic energy of S:

$$K = \int_S \rho k \, dv = \frac{1}{2}\int_S \rho(U_i U_i + \chi_{ip}\chi_{iq} I_{pq}) \, dv. \tag{32}$$

Virtual power of mass velocity of S:

$$\hat{\mathscr{P}}_K = \int_S \rho \hat{p}_K \, dv = \int_S \rho(\hat{U}_i U_i + \hat{\chi}_{ij}\chi_{ip}I_{pj}) \, dv. \tag{33}$$

Virtual power of mass acceleration of S:

$$\hat{\mathscr{P}}_A = \int_S \rho \hat{p}_A \, dv = \int_S \rho(\hat{U}_i \gamma_i + \hat{\chi}_{ij}\gamma_{ip}I_{pj}) \, dv. \tag{34}$$

Obviously, similar definitions may be given for any subdomain $\mathscr{D}$ of S. Let us denote by $\mathscr{V}$, $\mathscr{A}$ the real velocity field (defined by U_i and χ_{ij} on S) and the real acceleration field (defined by γ_i and γ_{ij} on S—the acceleration and microacceleration) and by $\hat{\mathscr{V}}$ the virtual velocity field; then $\hat{\mathscr{P}}_K$ is a symmetric bilinear form of $\mathscr{V}$ and $\hat{\mathscr{V}}$—say $\mathscr{Q}$—and one may write

$$\hat{\mathscr{P}}_K = \mathscr{Q}(\hat{\mathscr{V}}, \mathscr{V}), \quad 2K = \mathscr{Q}(\mathscr{V}, \mathscr{V}), \quad \hat{\mathscr{P}}_A = \mathscr{Q}(\hat{\mathscr{V}}, \mathscr{A}). \tag{35}$$

Moreover, it is possible to differentiate $\hat{\mathscr{P}}_K$ with respect to time, provided we supplement the definition of virtual motion with the following precision: *the virtual motion of each point of the system is a uniform translation.* For a micromorphic medium, this means that the motion of each point M' of any particle $P\,(M)$ is a uniform translation. Thus, one must have

$$\dot{\hat{U}}'_i = \dot{\hat{U}}_i + (\dot{\hat{\chi}}_{ij})x'_j + \hat{\chi}_{ij}\dot{x}'_j = 0$$

and, as $\dot{x}'_j = U'_j - U_j = \chi_{jk}x'_k$, for any point M' of $P\,(M)$, one must write

$$\dot{\hat{U}}_i + (\dot{\hat{\chi}}_{ij} + \hat{\chi}_{ik}\chi_{kj})x'_j = 0.$$

As a consequence, one can state the following lemma.

LEMMA 3. *Differentiation of a virtual velocity field must be computed with the following formulas*:

$$\dot{\hat{U}}_i = 0, \qquad (\dot{\hat{\chi}}_{ij}) + \hat{\chi}_{ik}\chi_{kj} = 0. \tag{36}$$

With such a result one may relate $\hat{\mathscr{P}}_A$ to $\hat{\mathscr{P}}_K$.

LEMMA 4. *If all the quantities involved are continuously differentiable on S, then*

$$\hat{\mathscr{P}}_A = \dot{\hat{\mathscr{P}}}_K. \tag{37}$$

The proof is quite easy; starting with (33), one has just to compute

$$\frac{d}{dt}(\hat{\chi}_{ij}\chi_{ip}I_{pj})$$

using (36), (30) and of course the symmetry of I_{ij}.

It is time now to recall the statement of the principle of virtual power when dynamical effects have to be taken into account.

Principle of virtual power (dynamics). *Let a system S be in motion with respect to a given Galilean frame; at any time, and for any virtual motion, the material derivative of the virtual power of mass-velocity of S is equal to the virtual power of all "the internal" forces and "external" forces acting on S.*

Disregarding first the case when discontinuities exist on S, using Lemma 4 one immediately obtains the following theorem.

THEOREM 2. *The equations of the dynamics of a micromorphic medium of degree* 1 *are obtained by replacing, in the equations of the statics* (*see Theorem* 1), f_i *by* $f_i - \rho\gamma_i$, Ψ_{ij} *by* $\Psi_{ij} - \rho\gamma_{ip}I_{pj}$, *where* γ_i *and* γ_{ij} *are the acceleration and the microacceleration of a particle* (*the latter being given by* (29)).

It is obviously quite easy to supplement Theorem 2 with the statement giving the jump relations across any surface of discontinuity. If Σ is the surface across which discontinuities may arise, $\mathbf{W}$ the velocity of a point of Σ, $\mathbf{V} = \mathbf{U} - \mathbf{W}$ the relative velocity of the medium with respect to Σ, one knows that

$$\frac{d}{dt}\int_S \rho B\, dv = \int_S \rho \frac{dB}{dt}\, dv + \int_\Sigma [\![\rho v B]\!]\, d\sigma,$$

where v is the normal component of $\mathbf{V}$ (with respect to Σ), $[h]$ the jump of h across Σ, provided that ρ and B are piecewise continuously differentiable in S. By conservation of mass, it is seen that $m = \rho v$ is continuous across Σ. Then by classical reasoning, the following theorem is obtained (see, for instance, Germain (1973b)).

THEOREM 3. *If* $m = \rho v$ *is the mass flux density across any discontinuity surface* Σ, *then*

$$[\![T_i]\!] = m[\![V_i]\!], \qquad M_{ij} = m[\![\chi_{ip}I_{pj}]\!]. \tag{38}$$

4.3. Energy equation. The problem is to give significance to the concepts involved in the classical statement of the first principle of thermodynamics: *the material derivative of the total energy of a system is the sum of the power of "external forces" acting on this system and of the rate of heat supplied into S—say* $\dot{Q}$.

Assuming the total energy to be the sum of the kinetic energy K and the internal energy E—this latter being defined by a specific internal energy e—one may write

$$\dot{E} + \dot{K} = \mathscr{P}_{(d)} + \mathscr{P}_{(c)} + \dot{Q}.$$

But the theorem of energy in dymamics, i.e., the statement of the principle of virtual power applied with the real motion, gives

$$\dot{K} = \mathscr{P}_{(d)} + \mathscr{P}_{(c)} + \mathscr{P}_{(i)};$$

and then one obtains

$$\dot{E} = -\mathscr{P}_{(i)} + \dot{Q}. \tag{39}$$

If, as usual, we apply this result to a subsystem $\mathscr{D}$ of S and write

$$\dot{Q} = \int_{\mathscr{D}} r\, dv - \int_{\partial\mathscr{D}} q_i n_i\, d\sigma$$

(r the rate of volumic heat, q_i the heat flow vector), we can derive, by the application of the divergence theorem and the introduction of the expression of $\mathscr{P}_{(i)}$ given in § 3,

$$\rho\dot{e} = \tau_{ij}U_{i,j} - s_{ij}\chi_{ij} + \nu_{ijk}\varkappa_{ijk} + r - q_{i,i}. \tag{40}$$

It is outside the scope of the present paper to put more emphasis on this result and to continue the theory to considerations of constitutive relations—for this, the reader is referred to the literature and, in particular, to the papers of Eringen and his coworkers. We want only to point out the advantage of the present method which, starting directly with the expression of the rate of energy of internal forces, leads immediately to the expression (40) of the equation of energy (39).

5. General micromorphic medium. In the last two sections a new method has been proposed in order to derive quite simply results which were obtained previously by different and, according to the author's opinion, more complicated methods. This theory for the time being is rich enough to fit various physical situations. The principal difficulty indeed is to discover the practical significance of some of the concepts which have been introduced, to design a method in order to exhibit their physical validity (if any!), and to measure them in some specific physical situations. Thus, the generalization proposed in this section is not given with the idea of providing a new schematization which can be used at the present time, but just for the sake of showing the possibilities of the method and for eventual further comparisons with other theories which may be found in the literature.

5.1. Statics. Starting again with the model of a continuum with a microstructure, a natural generalization is to describe more accurately the relative motion of the particle with respect to its center of mass by replacing (8) by a finite or infinite expansion

$$U_i' = U_i + \chi_{ij}x_j' + \chi_{ijk}x_j'x_k' + \chi_{ijkl}x_j'x_k'x_l' \cdots. \tag{41}$$

We begin here with the formal theory, when the full infinite expansion is written. Without loss of generality, the χ_{ijkl} may be assumed to be fully symmetric with respect to the indices j, k, l.

As in § 3, a virtual motion is defined here by a sequence of fields

$$U_i, \chi_{ij}, \chi_{ijk}, \chi_{ijkl}, \cdots.$$

Writing the power of "internal forces" as in (2), and restricting ourselves to a first gradient theory, one is led to define p by

$$p = (\sigma_{ij} + s_{ij})U_{i,j} - (s_{ij}\chi_{ij} + s_{ijk}\chi_{ijk} + \cdots) + (v_{ijk}\varkappa_{ijk} + v_{ijkl}\varkappa_{ijkl} + \cdots), \tag{42}$$

where

$$\varkappa_{ijk} = \chi_{ij,k}, \qquad \varkappa_{ijkl} = \chi_{ijk,l}.$$

As above, σ_{ij} is symmetric; without loss of generality, the various tensors which have been introduced in order to describe the state of stress may be assumed to satisfy some symmetry properties—for instance, s_{ijk}, v_{ijk} are symmetric with respect to the indices j and k. Note also that definition (42) fulfills the condition arising from the axiom of virtual power of internal forces.

Similarly, formulas (12) and (13) have to be generalized by

$$\mathscr{P}_{(d)} = \int_{\mathscr{D}} (f_iU_i + \Psi_{ij}\chi_{ij} + \Psi_{ijk}\chi_{ijk} + \cdots)\,dv, \tag{43}$$

$$\mathscr{P}_{(c)} = \int_{\partial\mathscr{D}} (T_iU_i + M_{ij}\chi_{ij} + M_{ijk}\chi_{ijk} + \cdots)\,d\sigma. \tag{44}$$

Note again that, for instance, Ψ_{ijk} and M_{ijk} are symmetric with respect to the indices j and k. Introducing the stress tensor τ_{ij} by (14), one gets by application of the principle of virtual work the *equations of equilibrium for the general micromorphic medium*:

$$
\begin{aligned}
&\tau_{ij,j} + f_i = 0, && T_i = \tau_{ij}n_j,\\
&s_{ij} + v_{ijk,k} + \Psi_{ij} = 0, && M_{ij} = v_{ijk}n_k,\\
&s_{ijk} + v_{ijkl,l} + \Psi_{ijk} = 0, && M_{ijk} = v_{ijkl}n_l,\\
&\cdots. && \cdots.
\end{aligned} \tag{45}
$$

5.2. Dynamics. If one wants to take into account the motion of the system, one has again to distinguish between the real motion defined by $\mathscr{V}: U_i, \chi_{ij}, \cdots$ and a virtual motion which will be denoted by $\hat{\mathscr{V}} = \hat{U}_i, \hat{\chi}_{ij}, \cdots$. It is not difficult to generalize the above analysis of § 5, even if the final expressions are evidently more involved.

In order to describe inertia properties of the medium, one must introduce the generalized reduced inertia tensors $I_{ij}, I_{ijk}, I_{ijkl}$, fully symmetric (with respect to their indices) and defined for any particle $V = P\,(M)$ by formulas generalizing (20), such that

$$
\mu I_{ijk} = \int_V \rho' x_i' x_j' x_k' \, dv'. \tag{46}
$$

It is not difficult to generalize (30) also. The result may be written

$$
\begin{aligned}
\dot{I}_{ij} &= 2(\chi_{ip}I_{pj} + \chi_{ipq}I_{pqj} + \cdots)_{(ij)},\\
\dot{I}_{ijk} &= 3(\chi_{ip}I_{pjk} + \chi_{ipq}I_{pqjk} + \cdots)_{(ijk)},\\
\dot{I}_{ijkl} &= 4(\chi_{ip}I_{pjkl} + \chi_{ipq}I_{pqjkl} + \cdots)_{(ijkl)},\\
&\cdots.
\end{aligned} \tag{47}
$$

if one introduces the following notation:

$$
3!(A_{ijkl})_{(jkl)} = A_{ijkl} + A_{iklj} + A_{iljk} + A_{ijlk} + A_{ilkj} + A_{ikjl}. \tag{48}
$$

Note that formulas (19) are still valid. Accordingly, differentiation with respect to time gives the following identity (because $\dot{x}_j' = U_j' - U_j$) which is satisfied by the real motion $\mathscr{V}$:

$$
\mathscr{L}(\mathscr{V}) = \chi_{ijk}I_{jk} + \chi_{ijkl}I_{jkl} + \cdots = 0.
$$

In what follows, it will be assumed also that the virtual motion $\hat{\mathscr{V}}$ satisfies the same relation $\mathscr{L}(\hat{\mathscr{V}}) = 0$.

The kinetic energy K is still a quadratic form of $\mathscr{V}$—say $2K = \mathscr{Q}(\mathscr{V}, \mathscr{V})$—where $\mathscr{Q}(\hat{\mathscr{V}}, \mathscr{V})$ is the expression of the virtual power of mass velocity of S, a bilinear symmetric form which may be expressed with the inertia components I_{ijk}. The integrand of (24) is

$$
\rho'\left[\hat{U}_i + \sum_{n=1}^{\infty} \hat{\chi}_{i\underbrace{p\cdots q}_{n}} x_p' \cdots x_q'\right]\left[U_i + \sum_{m=1}^{\infty} \chi_{i\underbrace{j\cdots k}_{m}} x_j' \cdots x_k'\right],
$$

and then we may write

$$\mathscr{Q}(\hat{\mathscr{V}}, \mathscr{V}) = \int_S \rho \left\{ \hat{U}_i U_i + \sum_{n=1}^{\infty} \hat{\chi}_{i\underbrace{p\cdots q}_{n}} \left(\sum_{m=1}^{\infty} \chi_{i\underbrace{j\cdots k}_{m}} I_{j\cdots kp\cdots q} \right) \right\} dv, \tag{49}$$

The terms inside the braces—say $\hat{p}_k$—may also be written as

$$\hat{p}_K = \hat{U}_i U_i + \sum_{n=1}^{\infty} \hat{\chi}_{i\underbrace{p\cdots q}_{n}} K_{i\underbrace{p\cdots q}_{n}}, \tag{50}$$

with

$$K_{ip\cdots q} = \chi_{ij} I_{jp\cdots q} + \chi_{ijk} I_{jkp\cdots q} + \cdots . \tag{51}$$

The above result gives the generalization of (33) and consequently of (32).

We now have to deal with accelerations. First we obtain the generalization of (22) by differentiation of (41):

$$\gamma_i' = \gamma_i + \sum_{m=1}^{\infty} \dot{\chi}_{i\underbrace{jk\cdots l}_{m}} x_j' \cdots x_l' + \sum_{m=1}^{\infty} m \chi_{i\underbrace{jk\cdots l}_{m}} (\dot{x}_j' x_k' \cdots x_l')_{(jk\cdots l)}, \tag{52}$$

if one uses notation (48), taking account of the symmetry with respect to the indices $j, k, \cdots, l$. But

$$\dot{x}_j' = U_j' - U_j = \sum_{m'=1}^{\infty} \chi_{j\underbrace{r\cdots s}_{m'}} x_r' \cdots x_s'. \tag{53}$$

Then, we may write

$$\gamma_i' = \gamma_i + \sum_{m=1}^{\infty} \gamma_{i\underbrace{jk\cdots l}_{m}} x_j' x_k' \cdots x_l' \tag{54}$$

with

$$\gamma_{i\underbrace{j\cdots kh\cdots l}_{m}} = \dot{\chi}_{i\underbrace{j\cdots kh\cdots l}_{m}} + \sum_{p=0}^{m-1} m(\chi_{i\underbrace{j\cdots k}_{p}\alpha} \chi_{\alpha\underbrace{h\cdots l}_{m-p}})_{(j\cdots kh\cdots l)}. \tag{55}$$

For instance, for the microacceleration tensor of lower order one gets:

$$\begin{aligned} &\gamma_{ij} = \dot{\chi}_{ij} + \chi_{i\alpha}\chi_{\alpha j}, \\ &\gamma_{ijk} = \dot{\chi}_{ijk} + \chi_{i\alpha}\chi_{\alpha jk} + 2(\chi_{ij\alpha}\chi_{\alpha k})_{(jk)}, \\ &\gamma_{ijkl} = \dot{\chi}_{ijkl} + \chi_{i\alpha}\chi_{\alpha jkl} + 3(\chi_{ij\alpha}\chi_{\alpha kl} + \chi_{ijk\alpha}\chi_{\alpha l})_{(jkl)}, \\ &\cdots . \end{aligned} \tag{56}$$

The generalization of (28) is easily written

$$\hat{p}_A = \hat{U}_i \gamma_i + \sum_{n=1}^{\infty} \hat{\chi}_{i\underbrace{p\cdots q}_{n}} \Gamma_{i\underbrace{p\cdots q}_{n}}, \tag{57}$$

with

$$\Gamma_{ip\cdots q} = \gamma_{ij} I_{jp\cdots q} + \gamma_{ijk} I_{jkp\cdots q} + \cdots . \tag{58}$$

Finally the formula which replaces (34) is

$$\hat{\mathscr{P}}_A = \mathscr{Q}(\hat{\mathscr{V}}, \mathscr{A}) = \int_S \rho \left(\hat{U}_i \gamma_i + \sum_{n=1}^{\infty} \hat{\chi}_{i\underbrace{p\cdots q}_{n}} \Gamma_{i\underbrace{p\cdots q}_{n}} \right) dv, \tag{59}$$

if we denote by $\mathscr{A}$ the set of acceleration fields $\gamma_i, \gamma_{ij}, \gamma_{ijk} \cdots$. It is essential (for a later application of conservation of energy) to check that *for real motions, the power of mass acceleration is the time derivative of the kinetic energy*:

$$\mathscr{Q}(\mathscr{V}, \mathscr{A}) = \frac{d}{dt}(\tfrac{1}{2}\mathscr{Q}(\mathscr{V}, \mathscr{V})). \tag{60}$$

For this purpose it is interesting to give an alternative expression for $\Gamma_{ip\cdots q}$. Going back to (52), (53) and introducing γ_i' directly in (25), one obtains (57) with

$$\begin{aligned}\Gamma_{ip\cdots q} &= \dot{\chi}_{ij} I_{jp\cdots q} + \dot{\chi}_{ijk} I_{jkp\cdots q} + \cdots \\ &\quad + \chi_{ij} A_{jp\cdots q} + \chi_{ijk} A_{jkp\cdots q} + \cdots,\end{aligned}$$

and

$$A_{\underbrace{jk\cdots l}_{m}p\cdots q} = m[\chi_{j\alpha} I_{\alpha k\cdots lp\cdots q} + \chi_{j\alpha\beta} I_{\alpha\beta k\cdots lp\cdots q} + \cdots]_{(jk\cdots l)}.$$

The left-hand side of (60) contains

(a) terms like

$$\dot{\chi}_{ij\cdots l}\chi_{ip\cdots q} I_{j\cdots lp\cdots q},$$

whose sum gives the terms of the right-hand side of (60) when the coefficients I in $\mathscr{Q}$ are taken as constants;

(b) terms like

$$\chi_{i\underbrace{p\cdots q}_{n}}\chi_{i\underbrace{j\cdots l}_{m}}[mA_{\underbrace{j\cdots l}_{m}p\cdots q} + nA_{\underbrace{p\cdots q}_{n}j\cdots l}].$$

But it is easily checked that, according to (47) and the properties of symmetry with respect to the indices,

$$\dot{I}_{\underbrace{ij\cdots l}_{m}\underbrace{p\cdots q}_{n}} = [mA_{\underbrace{j\cdots l}_{m}p\cdots q} + nA_{\underbrace{p\cdots q}_{n}j\cdots l}],$$

and these terms give the other terms in (60), i.e., those which are obtained by material derivatives of the coefficients I. *A similar calculation shows that Lemma* 4 *is still valid.*

Let the virtual motion of each point of the system be a uniform translation, one may first generalize (36). Differentiation of a virtual velocity field must be computed with the following formulas (see (56) above):

$$\begin{aligned}&(\dot{\hat{\chi}}_{ij}) + \hat{\chi}_{i\alpha}\chi_{\alpha j} = 0, \\ &(\dot{\hat{\chi}}_{ijk}) + \hat{\chi}_{i\alpha}\chi_{\alpha jk} + 2(\hat{\chi}_{ij\alpha}\chi_{\alpha k})_{(jk)} = 0, \\ &\cdots.\end{aligned}$$

After all these preliminaries of kinetics, it is easy to get the final result by application of the principle of virtual work (see the statement of § 4.2). The conclusions are given in the following theorem.

THEOREM 4. *The equations of statics of a general micromorphic medium are given by* (45). *The equations of dynamics are obtained by replacing f_i by $f_i - \rho\gamma_i$ and the $\Psi_{ij\cdots k}$ by $\Psi_{ij\cdots k} - \rho\Gamma_{ij\cdots k}$, where Γ may be expressed in terms of the acceleration fields and the velocity fields* (*and its derivatives*) *by* (58) *and* (56)—*or* (55).

This general theory is quite formal. For the applications, expansion (41) may be reduced to the first $n + 1$ terms—and the theory obtained in this way will be relative to *micromorphic medium of order n*. For instance, for $n = 3$, $\mathscr{V}$ is described by U_i, χ_{ij}, χ_{ijk}, χ_{ijkl}; the state of stresses by σ_{ij}, s_{ij}, s_{ijk}, s_{ijkl}, v_{ijk}, v_{ijkl}, v_{ijklm}; long range forces by f_i, Ψ_{ij}, Ψ_{ijk}, Ψ_{ijkl}; and generalized surface traction by T_i, M_{ij}, M_{ijk}, M_{ijkl}. Only the first four lines of (45) have to be written for equilibrium. As far as inertia forces are concerned, we keep γ_i, Γ_{ip}, Γ_{ipq}, Γ_{ipqr} only. It would not be difficult to write the jump relations and the energy equation. This matter is left to the reader.

6. Micropolar medium and other particular cases.

6.1. Micropolar medium. A very interesting particular case is obtained when assuming that the particle P (M) of the continuum moves as a rigid body. *The virtual motions which will be considered will also be assumed to adhere to this condition.* In this case, the micromorphic *medium is necessarily of order* 1, *and the tensor* χ_{ij} *is antisymmetric.* As a consequence, $\varkappa_{ijk}$ is also antisymmetric with respect to the indices i and j (see (9)).

Going back to § 4 and putting $\tau_{ij} = \sigma_{ij} + s_{ij}$, as in (14), (10) may be written

$$p = K_i U_i + \tau_{ij} U_{i,j} - s_{ij}\chi_{ij} + v_{ijk}\varkappa_{ijk},$$

and without loss of generality, s_{ij} and v_{ijk} may be assumed to be antisymmetric with respect to the indices i and j. As p is invariant, by superposition of a rigid body velocity field, one must have for any constant $\bar{U}_i$ and $\bar{\Omega}_{ij} = -\bar{\Omega}_{ji}$:

$$K_i\bar{U}_i + (\tau_{ij} - s_{ij})\bar{\Omega}_{ij} = 0;$$

then $K_i = 0$ and σ_{ij} is symmetric. As a result, *the tensor s_{ij} is nothing else than the antisymmetric part of the stress tensor* τ_{ij}. Similarly, in (12) and (13), Ψ_{ij} and M_{ij} may be assumed, without loss of generality, antisymmetric with respect to i and j: they define the long range volumic couple distribution and surface couple stress respectively. Again in $\hat{\mathscr{P}}_A$—see (34)—one has only to consider the antisymmetric part of $\Gamma_{ij} = \gamma_{ip} I_{pj}$, say $\Gamma_{[ij]}$. The equations of motion and boundary conditions for a micropolar medium may then be written

$$\begin{aligned} &f_i + \tau_{ij,j} = \rho\gamma_i, \qquad T_i = \tau_{ij}n_j, \\ &\tau_{[ij]} + v_{ijk,k} + \Psi_{ij} = \rho\Gamma_{[ij]}, \qquad M_{ij} = v_{ijk}n_k. \end{aligned} \tag{61}$$

It is obviously easy to get a more classical description when one introduces the adjoint tensors with respect to the indices i and j. Put

$$\begin{aligned} \chi_{ij} = -\varepsilon_{ijk}\chi_k, \quad M_{ij} = -\tfrac{1}{2}\varepsilon_{ijk}M_k, \quad \Psi_{ijk} = -\tfrac{1}{2}\varepsilon_{ijk}\Psi_k, \\ v_{ijk} = -\tfrac{1}{2}\varepsilon_{ijl}v_{lk}. \end{aligned} \tag{62}$$

Then,

$$K_{ij} = \chi_{ip}I_{pj} = -\varepsilon_{ip\alpha}\varkappa_\alpha I_{pj}.$$

As only the antisymmetric parts of K_{ij} and Γ_{ij} play a role, one may introduce

$$\sigma_k = -\varepsilon_{ijk}K_{ij} = J_{kj}\chi_j, \tag{63}$$

and then

$$\dot{\sigma}_k = -\varepsilon_{ijk}\Gamma_{ij} = J_{kj}\dot{\chi}_j \tag{64}$$

if one introduces the inertia tensor

$$J_{ij} = I_{pp}\delta_{ij} - I_{ij}. \tag{65}$$

The interpretation is quite easy: χ_k is a vector rotation rate, M_k and Ψ_k are couple densities, v_{ij} the couple stress density, σ_k the spin. The last line of (61) may then be written

$$\varepsilon_{kpq}\tau_{qp} + v_{kl,l} + \Psi_k = \rho\dot{\sigma}_k, \qquad M_k = v_{kl}n_l, \tag{66}$$

and the volumic kinetic energy is

$$\tfrac{1}{2}\rho(U_iU_i + J_{ij}\chi_i\chi_j) = \tfrac{1}{2}(\rho U_iU_i + \rho\sigma_i\chi_i). \tag{67}$$

Then, one recovers the main result given in the paper of Kafadar and Eringen (1971), in which many quite interesting interpretations and applications can be found.

6.2. Second gradient theory. The second gradient theory may be obtained very easily from the theory of micromorphic media of order 1 when one assumes that the particle is subject to the same deformation as the general continuum, i.e.,

$$\chi_{ij} = U_{i,j}. \tag{68}$$

Let us investigate briefly the consequences of such an assumption. It will be sufficient to use virtual motions which satisfy the condition (68)—say $\hat{\chi}_{ij} = \hat{U}_{i,j}$.

The equations of §§ 3 and 4 have been obtained by writing, for any subdomain $\mathscr{D}$, a relation such that

$$\int_{\mathscr{D}} (A_i\hat{U}_i + B_{ij}\hat{\chi}_{ij})\,dv + \int_{\partial\mathscr{D}} (E_i\hat{U}_i + F_{ij}\hat{\chi}_{ij})\,d\sigma = 0 \tag{69}$$

is valid whatever be the fields $\hat{U}_i$, $\hat{\chi}_{ij}$. In the present case these two fields are no longer independent; in order to apply the principle of virtual work, one must write the left-hand side of (69) in such a way that only independent quantities related to the virtual motions appear. First, one obtains easily for the volume integral,

$$\int_{\mathscr{D}} (A_i - B_{ij,j})\hat{U}_i\,dv + \int_{\partial\mathscr{D}} B_{ij}n_j\hat{U}_i\,d\sigma. \tag{70}$$

As the volume integral must be zero whatever the value of $\hat{U}_i$, one gets first the equation $A_i - B_{ij,j} = 0$. If one assumes equilibrium, this volume integral comes from $\hat{\mathscr{P}}_{(d)} + \hat{\mathscr{P}}_{(c)}$, and one has

$$A_i = f_i + \sigma_{ij} + s_{ij}, \qquad B_{ij} = \Psi_{ij} + s_{ij} + v_{ijk,k};$$

and writing

$$\tilde{\tau}_{ij} = \sigma_{ij} - v_{ijk,k} - \Psi_{ij}, \tag{71}$$

one gets

$$f_i + \tilde{\tau}_{ij,j} = 0. \tag{72}$$

The term s_{ij} disappears, as is quite clear when one looks at (11) taking account of

(68). Intrinsic stresses are defined by the symmetric tensor σ_{ij} and the tensor v_{ijk}, symmetric with respect to j and k.

Now consider the term in the surface integral of (69) arising from $\hat{\mathscr{P}}_{(c)}$ which is given by (13). Let us introduce, following Mindlin, the notation DK_i and D_jK_i for writing the normal derivative on $\partial\mathscr{D}$ of the vector field K_i and its tangential derivatives on $\partial\mathscr{D}$:

$$K_{i,j} = D_jK_i + n_jDK_i, \qquad DK_i = n_kU_{i,k}.$$

On $\partial\mathscr{D}$, $\hat{U}_i$ and $D\hat{U}_i$ may be chosen independently, but $D_j\hat{U}_i$ may be computed from the values of $\hat{U}_i$ on $\partial\mathscr{D}$. Then

$$M_{ij}\hat{U}_{i,j} = (M_{ij}\hat{U}_i)_{,j} - M_{ij,j}\hat{U}_i = D_j(M_{ij}\hat{U}_i) + M_{ij}n_jD\hat{U}_i - (D_jM_{ij})\hat{U}_i.$$

The divergence theorem for an open subdomain Σ of $\partial\mathscr{D}$ gives

$$\int_\Sigma D_lq_j\,d\sigma = \int_\Sigma n_jq_j(D_ln_l)\,d\sigma + \int_{\partial\Sigma} v_jq_j\,ds,$$

where t_i is the unit tangent vector to $\partial\Sigma$ and $v_j = \varepsilon_{jml}t_mn_l$, the unit normal vector to $\partial\Sigma$, tangent to $\partial\mathscr{D}$. Then

$$\hat{\mathscr{P}}_{(c)} = \int_{\partial\mathscr{D}} \{(T_i - D_jM_{ij} + (D_kn_k)M_{ij}n_j)\hat{U}_i + M_{ij}n_jD\hat{U}_i\}\,d\sigma$$

$$+ \int_\Gamma [\![M_{ij}v_j]\!]\hat{U}_i\,ds.$$

Here, Γ is a ridge, i.e., a line on $\partial\mathscr{D}$ (if any) where the tangent plane of $\partial\mathscr{D}$ is discontinuous. The only traction elements which may be defined on $\partial\mathscr{D}$ in terms of the stress tensors are then:

(i) a surface traction $\tilde{T}_i = T_i - D_jM_{ij} + (D_kn_k)M_{ij}n_j$,
(ii) a normal double traction $\tilde{N}_i = M_{ij}n_j$,
(iii) eventually a line traction $\tilde{R}_i = [\![M_{ij}v_j]\!]$,

and

$$\hat{\mathscr{P}}_{(c)} = \int_{\partial\mathscr{D}} (\tilde{T}_i\hat{U}_i + \tilde{N}_iD\hat{U}_i)\,d\sigma + \int_\Gamma \tilde{R}_i\hat{U}_i\,ds.$$

Finally, making a similar transformation for the other terms of the surface integral of (69) arising from $\hat{\mathscr{P}}_{(d)} + \hat{\mathscr{P}}_{(i)}$, and taking into account the surface integral of (70), one gets the boundary condition on any surface $\partial\mathscr{D}$:

$$\tilde{T}_i = \tilde{\tau}_{ij}n_j + (D_l\,n_l)v_{ijk}n_jn_k - D_j(v_{ijk}n_k), \tag{73}$$

$$\tilde{N}_i = v_{ijk}n_jn_k, \tag{74}$$

$$\tilde{R}_i = [\![v_{ijk}v_jn_k]\!], \tag{75}$$

which, with the field equations (71) and (72), form a complete set of equations of equilibrium for the second gradient theory (see Mindlin and Eshel (1968)) fully equivalent to the system given by Germain (1973).

The equation of motion may be derived by application of Theorem 2. This last statement shows the significance of the extra inertia terms one has to introduce in the dynamical equation.

6.3. Indeterminate couple stress theory. This model can be viewed as a micropolar medium subjected to the constraint

$$\chi_{ij} = \Omega_{ij} = \tfrac{1}{2}(U_{i,j} - U_{j,i}). \tag{76}$$

The particle has a rotation motion which is exactly the rotation of the general continuum. In order to obtain the corresponding equations, one can start again with (69) and assume that either $\hat{\chi}_{ij} = \hat{\Omega}_{ij}$ with the same B_{ij} and F_{ij} or $\hat{\chi}_{ij} = \hat{U}_{i,j}$, as in the preceding subsection, but by keeping only the antisymmetric parts of B_{ij} and F_{ij}. This last point of view leads immediately to the results we are looking for. Using the notation (62), (71) and (72) are replaced by

$$\begin{gathered} \bar{\tau}_{ij} = \sigma_{ij} + \tfrac{1}{2}\varepsilon_{ijl}v_{lk,k} + \tfrac{1}{2}\varepsilon_{ijk}\Psi_k, \\ f_i + \bar{\tau}_{ij,j} = 0. \end{gathered} \tag{77}$$

On the other hand, v_{ijk} is antisymmetric with respect to j and k. Thus (73) reduces to

$$\overline{T}_i = \bar{\tau}_{ij}n_j + \tfrac{1}{2}\varepsilon_{ijl}D_j(v_{lk}n_k). \tag{78}$$

The right-hand side of (74) is zero. Then $2\overline{N}_i = -\varepsilon_{ijk}M_kn_j = 0$. The couple M_i is then normal to $\partial\mathscr{D}$ and its intensity is fully indeterminate; it plays no role when one writes $\hat{\mathscr{P}}_{(c)}$, which reduces to the power of the surface tractions $\overline{T}_i$—at least if there is no ridge on $\partial\mathscr{D}$. Then (77) and (78) are the main equations of this theory which has many applications—see, for instance, Mindlin and Tiersten (1962). The relation of this theory with the micropolar theory has been previously emphasized in Eringen (1966a), (1966b). Of course the theories of § 6.1 and § 6.3 originate in the famous work of the Cosserat brothers (1909).

6.4. Microstructure without microstresses. Looking back again to a micromorphic medium, one may assume that the relative motion of the particles is possible without any expense of energy. In fact, if one assumes $v_{ijk} = 0$ in (10) and $\Psi_{ij} = 0$ in (12), (16) shows that also $s_{ij} = 0$ and thus we recover the classical theory where $\tau_{ij} = \sigma_{ij}$ defines completely the state of stress. Nevertheless, the parameters which define the orientation of the particle, if they do not appear in the equations of equilibrium, may be present in the definition of the specific internal energy, and consequently in the constitutive equations. They then play the role of hidden parameters. Such a model has recently been considered by Mandel (1971) in its general theory of viscoplasticity.

7. Comparison with other work. We conclude this paper with a few remarks dealing with the comparison of the present work with other theories presented in the literature.

The theory of micromorphic media of order one has been extensively considered by Eringen and his coworkers. The results of §§ 3 and 4 are then not new, but the method which is proposed seems to be original and is, in the author's opinion, simpler than the methods which have been proposed so far. This is why it has been possible to develop the general theory of § 5 which, to the author's knowledge, is completely new.

Mention must be made at this stage of another generalization proposed by Eringen (1968) and (1970) under the title, *Micromorphic continuum of higher grades.* Eringen has proposed to obtain the results of §§ 4 and 5 by a method of surface and volume averages on the basic concepts valid at the microscopic level. The generalization is obtained formally by taking higher moments in this process of averaging. The present author must confess that, in the present situation, the physical significance of such a formulation is not clear to him. The basic kinematics of the particle is still given by (8); how then is it possible to introduce a description of stresses which apparently is more refined than the one given in § 3 which is based on the fundamental duality between the kinematical description and the description of stress? What is the significance of the various micro-energies which are introduced? At first glance, at least, they seem to have no direct relation with the usual statement of the first principle of thermodynamics. If such a refinement is worth being considered, what would be its counterpart in the classical continuum theory? These are some of the questions which remain, until now, without answer, and which seem necessary to elucidate before such a theory may be accepted and used for the description of physical situations.

We now want to comment on a very interesting note by Green (1965) which intends to establish the connection between the micromorphic theory of order one and the theories of multipolar continuum introduced by Green and Rivlin (1964a, b) and (1965). The starting point of these theories is quite similar to the one on which the present paper is based (despite some difference in the words used for describing the same concepts): generalized forces are introduced by their power for a given kinematical description. Nevertheless these authors do not apply the principle of virtual power, but instead, and more simply, the first principle of thermodynamics or the equation of energy, supplemented by a condition of invariance under superposed rigid body motions. When applied to the classical continuum mechanics, this method provides at the same time the three statements of conservation of mass, momentum and energy. When applied to a more complex situation, this method does not give the same results as the method of virtual power: obviously the number of equations obtained by the former is less than the number obtained by the latter. More precisely, in case of micromorphic media, the last condition (17) is included in another one in which heat flow concepts are present as well as double surface traction M_{ij} and the microstress tensor v_{ijk}. This difference appears still more clearly when looking, for instance, at the paper of Green and Rivlin (1964a) devoted to what is called in the present terminology "theory of the second and higher gradients". If long range forces and generalized surface tractions are essentially those introduced here, the generalized stresses are different and consequently the basic dynamics equations are different and moreover less in number. Of course the number of equations which are missing are supplemented by a greater number of constitutive equations—for instance, when the material is assumed to be elastic. There is no reason to suspect this theory to be inconsistent, but the present author does not know if statements of existence, uniqueness and variational formulations can be established as has been done for the second gradient theory (Germain (1973a)). A further study of these questions and a more extended analysis of the physical significance of the theory of Green and Rivlin in comparison with the nth gradient theory would be very valuable.

A final remark on the work of Green and Rivlin is related to their use of the tetrahedron argument which they systematically apply in order to relate generalized surface tractions with generalized stress tensors. It is not clear that such an argument may be safely applied when high order derivatives are involved, as has been shown in detail in the above quoted paper on second gradient theory.

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