Linear elastic trusses leading to continua with exotic mechanical interactions



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# Linear elastic trusses leading to continua with exotic mechanical interactions

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**Abstract.** We study the statics of some trusses, i.e. networks of nodes linked by linear springs. The trusses are designed in such a way that a few number of floppy modes are present and remain when considering the homogenized limit of the truss. We then obtain linear elastic materials with exotic mechanical interactions which cannot be described in the classical framework of Cauchy stress theory. For aim of simplicity, the structures described here are two-dimensional. The extension to the 3D case does not present any difficulty.

## 1. Introduction

The theory of Cauchy stress is so efficient in describing mechanical interactions in usual materials that its foundations have seldom been reconsidered (see [3, 20, 19, 26, 28, 27, 37, 38, 16, 17, 21, 22, 23, 24, 25, 29, 30, 31, 32, 33, 40, 41]. Rapid progress in nanotechnologies makes now possible the design of new materials having exotic mechanical properties. One basic assumption of Cauchy stress theory is that the mechanical contact interaction between two parts of the material is only due to a surface density of contact forces concentrated on the “cut”, i.e. on the dividing surface [44, 34, 35]. In this paper we describe some microscopic structures leading at the macroscopic level to materials for which this basic assumption is no longer valid.

Microscopic (atomic) structures leading to high order gradient materials have already been described [1, 6, 42, 43]. The advantage of the structures presented here is that the non classic properties are preponderant and are not limited to producing thin boundary layers.

The microscopic structures we consider are simple is the sense that they consist only of linear elastic springs joining nodes (what is usually called “trusses”). The idea is to design structures with a few number of floppy modes, i.e. global deformations which do not change the length of any spring of the structure and so do not need any energy.

In the different sections we consider structures with an increasing degree of complexity. We compute the deformation energy in terms of the displacement of a network of nodes. This energy is generally self-explanatory : writing the homogenized energy which corresponds to a very dense network of nodes is straightforward even if its rigorous mathematical derivation is not so easy [5, 7, 9, 10, 11, 12, 13] (for a rigorous derivation of the homogenization of the structure presented in section 3.1 see [2]). It is not the aim of this paper to give such a rigorous mathematical analysis, we prefer to emphasize the mechanical features of the proposed structures.

## 2. The flexion beam and the flexion truss

### 2.1. The flexion beam

Let us begin by considering the very simple reticulated structure described in figure 1. When

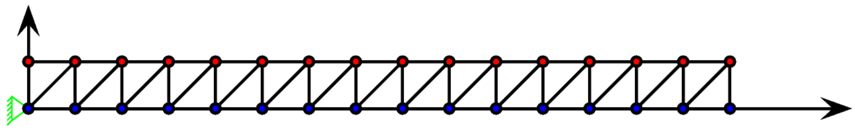

**Figure 1.** A beam truss.

the stiffness of elastic bars is suitably scaled, the homogenized behavior of the structure is one-dimensional and corresponds to a non-extensional flexion beam. This is not very difficult to check but the computation is much more easy when considering a slightly different structure (figure 2) in which we assume that the horizontal bars have infinite stiffness while all other bars have the same finite stiffness $k$.

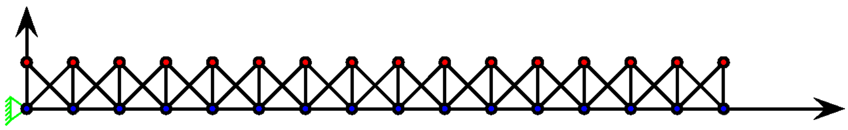

**Figure 2.** A alternative beam truss. The horizontal bars have infinite stiffness. All other bars have the same finite stiffness. The diagonal bars do not interact : this can be achieved either by using a 3D structure or by adding an extra node at the junction.

Indeed this alternative structure is made of a periodic superposition of basic elements described in figure 3. Thus its elastic energy $E$ is easy to write in terms of the displacements

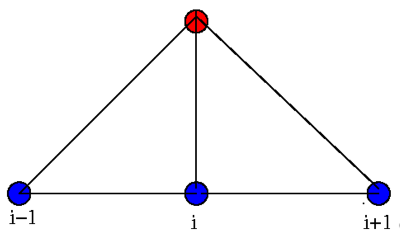


**Figure 3.** An element of the beam of fig.2.

of the nodes. Assuming that forces are applied on the lowest (blue) nodes only, we write the energy in terms of the displacements $u[i] = (u_1[i], u_2[i])$ $(i \in \{1, \ldots N\})$ of the blue nodes. (The $N$ nodes are numbered from the left to the right). The longitudinal displacement of the blue nodes has to vanish and the elastic energy contained in each element depends only on the transverse displacement $u_2$. It reduces to $(k/16)(u_2[i-1] - 2u_2[i] + u_2[i+1])^2$. The elastic energy of the beam then reads

$$E(u) = \sum_{i=2}^{N-1} (k/16)(u_2[i-1] - 2u_2[i] + u_2[i+1])^2$$

Let us assume that the total length of the structure is $\ell$ and let us denote $\epsilon = \ell/(N-1)$ the distance between two consecutive blue nodes. If the stiffness of the bars scales as $\epsilon^{-3}$ : $k = 16K\epsilon^{-3}$, then

$$E(u) = \epsilon \sum_{i=2}^{N-1} K \left( \frac{u_2[i-1] - 2u_2[i] + u_2[i+1]}{\epsilon^2} \right)^2.$$

When the beam is made of a large number of small cells, i.e. when $\epsilon$ tends to zero, then one can recognize in the expression of $E$ the finite difference approximation for the continuous energy

$$\tilde{E}(u) = \int_0^\ell K \left( \frac{\partial^2 u_2}{\partial x_1^2} \right)^2.$$

This is the energy of the well known linear flexion beam. Nevertheless let us make some remarks before using this structure to build a new 2D material : First, we have assumed that all bars are linear elastic bars. They have spring-like behavior or correspond to long range linear interactions : the possibility of buckling is not considered here. Secondly, external forces are exerted only on the lowest (blue) nodes, the upper (red) nodes are considered as “internal”. Finally note that, even when the first node is fixed (which corresponds to a Dirichlet condition $u(0) = 0$ in the continuous model) the truss is not isostatic. A floppy mode (rotation around node 1) remains. For instance, in order to counterbalance a vertical force applied in the middle of the beam, it is necessary to apply a suitable force on the second node (which corresponds in the continuous model to a torque applied at $x = 0$).

### *2.2. The flexion 2D material*

Now let us consider many ($M$) parallel beam structures similar to the one described in the previous section (each one with first node fixed) and let us link them by adding vertical bars which link each blue node with the blue node just above; the added vertical bars have stiffness $k'$; see figure 4. Note that, for sake of clearnesss, figure 4 shows a slightly different structure than the one we just described.

This structure clearly has a floppy mode : the deformation shown in figure 4 needs no energy as no bar is extended. It is the only floppy mode. Indeed it is easy to check that fixing an extra node makes the structure isostatic. At a first insight, one could think that we have here designed a degenerated classical material with a vanishing shear stiffness. It is easy to reject this interpretation by considering the dimension of the space of floppy modes in both cases.

The elastic energy of the whole structure is simply the sum of the elastic energies of the beams we just derived plus the energy concentrated in the additional vertical junctions. It writes

$$E(u) = \sum_{j=1}^{M} \sum_{i=2}^{N-1} (k/16)(u_2[i-1,j] - 2u_2[i,j] + u_2[i+1,j])^2 + \sum_{j=1}^{M-1} \sum_{i=1}^{N} k'(u_2[i,j] - u_2[i,j-1])^2$$

where $u[i,j]$ denotes the displacement of the $i$-th (blue) node of the $j$-th beam. With the previous scaling for $k$ and a suitable scaling for $k'$ : $k' = K'\epsilon$ we obtain by passing to the limit $\epsilon \to 0$ the following continuous energy :

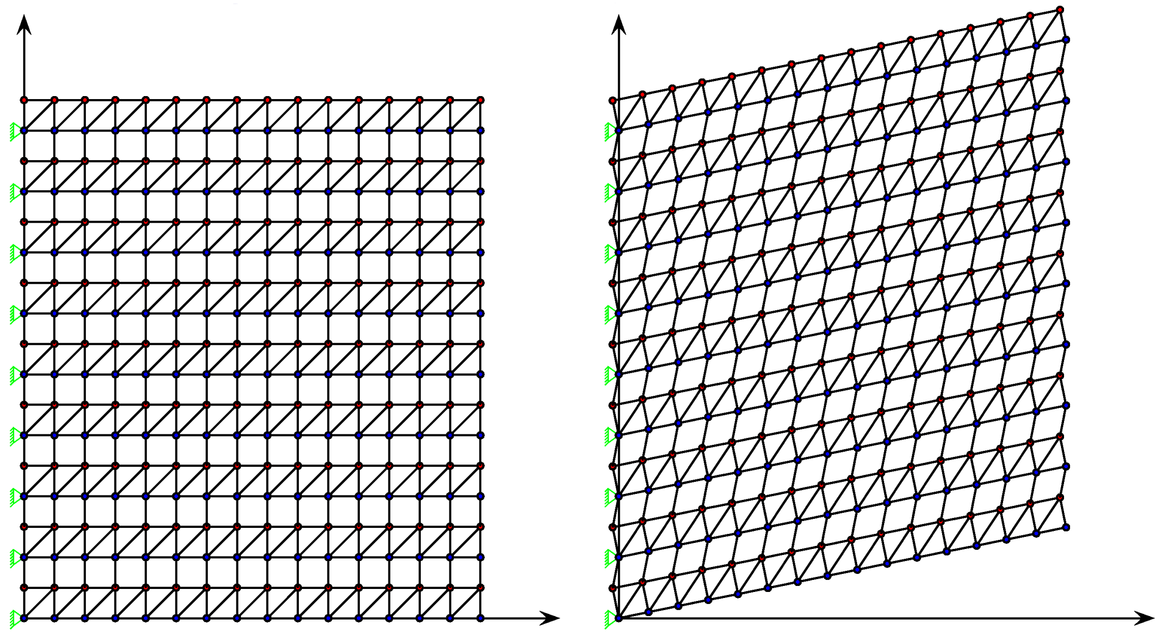

**Figure 4.** A 2D flexion structure. For the clearness of the figure, any red node has been joined with the blue one just above, but the computation is straightforward only when assuming that the blue nodes actually join the blue nodes just above. For the same reason we draw the first version of the flexion beam while the alternative version is used in the computation. The truss admits a floppy mode.

$$\tilde{E}(u) = \int_\Omega K \left(\frac{\partial^2 u_2}{\partial x_1^2}\right)^2 + K' \left(\frac{\partial u_2}{\partial x_2}\right)^2$$

subjected to the constraint $u_1 = 0$ in the whole domain $\Omega := (0, \ell) \times (0, \ell')$ and to the Dirichlet boundary condition $u = 0$ on the side $x_1 = 0$.

In order to understand the behavior of the continuum described by this elastic energy, let us consider the response of the system to some test load. We assume that one exerts a vertical force $f$ concentrated on a vertical surface $S$ in the middle of the domain (that is an academic example since applying such a concentrated force inside the material is not really easy). An equilibrium solution minimizes the total energy $\tilde{E} - \int_S f u_2 dx_2$. In a classical Cauchy material, an equilibrium state exists with (approximately) a constant shear deformation in the left hand part of the domain and no deformation in the right hand part (cf. figure 5). The situation is completely different for our new material. First, due to the aforementioned floppy mode, the Dirichlet condition is not sufficient for ensuring the existence of equilibrium. If the second node of all beams is also fixed (or alternatively if a suitable vertical force is applied on them) then a solution exists. This corresponds in the continuous model to adding an extra clamping constraint $\frac{\partial u_2}{\partial x_1} = 0$ on the side $S_l := \{x_1 = 0\}$ (or alternatively to applying a suitable surface density of torques on it, that is adding in the energy a term of the form $\int_{S_l} g \frac{\partial u_2}{\partial x_1} dx_2$ ). The equilibrium state is then very different from the classical case : the shear deformation varies linearly in the left hand part of the domain and remains constant but not vanishing in the right hand part (cf. figure 5).

Up to now the reader may be deceived. Indeed the interactions in the described material are

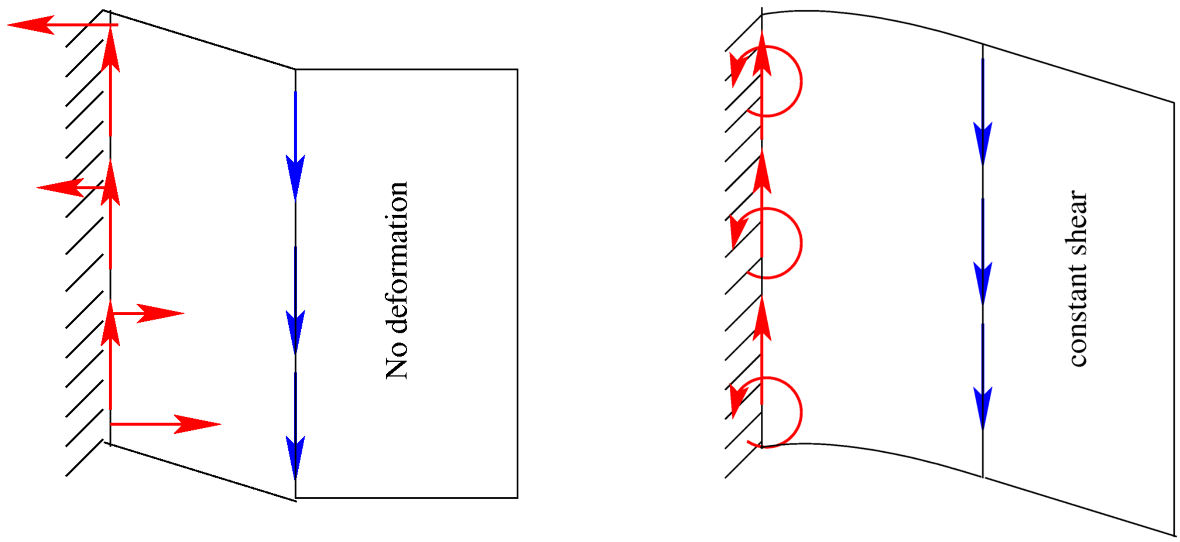


**Figure 5.** Comparison of the equilibrium states of classical (left hand drawing) and flexion (right hand drawing) materials subjected to a Dirichlet condition on the left hand side $S_l$ and a vertical force on a vertical middle surface $S$. In the classical case, the balance of forces is ensured by a suitable surface density of reaction forces on $S_l$ while, in the flexion case, a suitable density of torque has to be applied on $S_l$.

limited to surface densities of forces and torques which are not very exotic actions. The point is that at the microscopic level the displacement of the considered truss is a “local rotation”, the truss behaves like a Timochenko continuum [39] or a Cosserat continuum [14, 15] in the sense cleared up by Bleustein [8] (see also [18]). In the next section we build a structure endowed with local deformation instead of local rotation. Interactions will then be of a completely different nature.

## 3. The pantograph beam and the pantograph truss

### *3.1. The pantograph beam*

Now we build a structure based on the basic element described in figure 6 which possesses the same floppy mode as scissors. Indeed, for the clearness of the forthcoming figures we will simply draw a pair of scissors for representing this structure but remember that this is only a symbolic notation for a structure simply made of nodes joined by elastic bars.

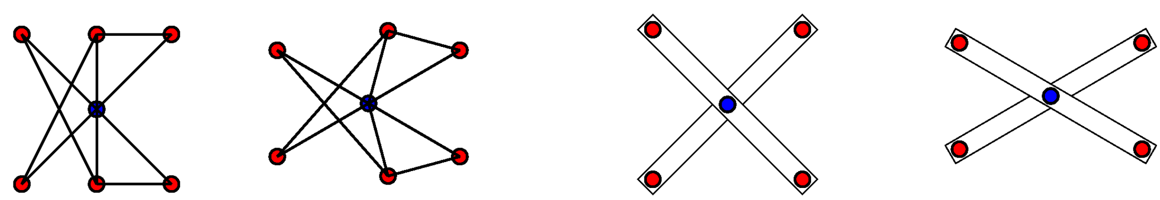

**Figure 6.** Basic element of the pantographic structures. We assume that the bars linking blue and red nodes have very high (infinite) stiffness corresponding to pure flexion (no extension) for the scissors. The structure is symbolized by two crossing rods. Its floppy mode plays an essential role.

By linking many such cells we get the pantographic structure described in figure 7. The

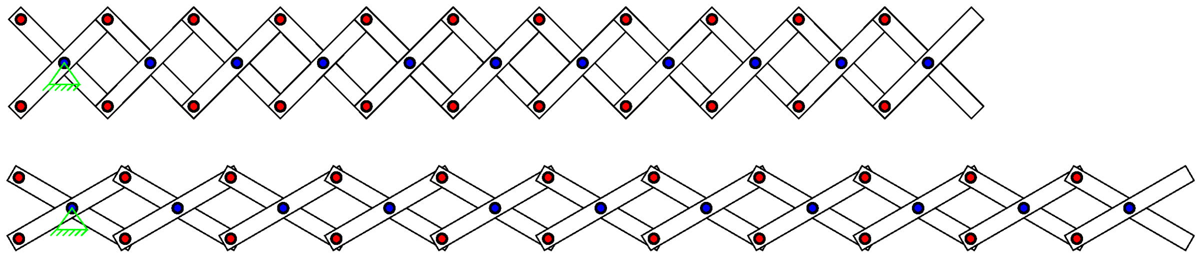

**Figure 7.** The pantographic structure and its extensional floppy mode.

remarkable feature of this structure is its extensional floppy mode (see fig. 7). Note that fixing the deformation of the first cell tends to fix the deformation of all cells. In particular, fixing the two first (blue) nodes makes the structure isostatic.

Computing the elastic energy in terms of the displacement of the blue nodes is straightforward. We get

$$E(u) = \sum_{i=2}^{N-1} k(u_2[i-1] - 2u_2[i] + u_2[i+1])^2 + k'(u_1[i-1] - 2u_1[i] + u_1[i+1])^2$$

We now recognize in $\epsilon^{-2}(u_1[i-1] - 2u_1[i] + u_1[i+1])$ the finite difference approximation of the second derivative of $u_1$ with respect to $x_1$. With a suitable scaling for $k$ and $k'$, the continuous limit ($N \to \infty$) model reads

$$\tilde{E}(u) = \int_0^\ell K\left(\frac{\partial^2 u_2}{\partial x_1^2}\right)^2 + K'\left(\frac{\partial^2 u_1}{\partial x_1^2}\right)^2$$

The transverse displacement $u_2$ and the axial one $u_1$ decouple. With respect to the transverse displacement, the behavior of the beam is exactly the behavior of the flexion beam described in the previous section. Owing to the symmetry of the terms involving $u_1$ and $u_2$ in the expression of the energy, every result valid for the transverse displacement can be transposed to the axial displacement, only the mechanical interpretation differs.

For instance, when the two first nodes are fixed, the displacement at the equilibrium under the action of a single axial force $F$ at a middle point point $x_1 = \ell/2$ minimizes $\tilde{E}(u) - Fu_1(\ell/2)$ under the constraints $u_1(0) = \frac{\partial u_1}{\partial x_1}(0) = 0$ : the deformation increases linearly from $x_1 = 0$ to $x_1 = \ell/2$ and remains constant but non zero for $x_1 > \ell/2$. Fixing the two first nodes corresponds in the continuous limit to imposing the constraint $\frac{\partial u_1}{\partial x_1}(0) = 0$. This can be replaced by the dual condition : two opposite axial forces applied at the two first nodes (see figure 8) which corresponds in the continuous limit to applying a new type of action. Two opposite axial forces are often said to have no action on a body but it is clearly not the case here as two such forces tend to modify the deformation of the first cell and as this deformation tends to propagate in the whole beam. Such "double force" is a new type of interaction [21, 22, 36] which needs a new symbol ( see figure 8).

### *3.2. The pantographic truss*

Consider now an array of many ($M$) horizontal pantographic beams (see figure 9), and link them using vertical pantographic beams. We fix the first node of each horizontal beam. Then we get a structure with four floppy modes (see figure 10).

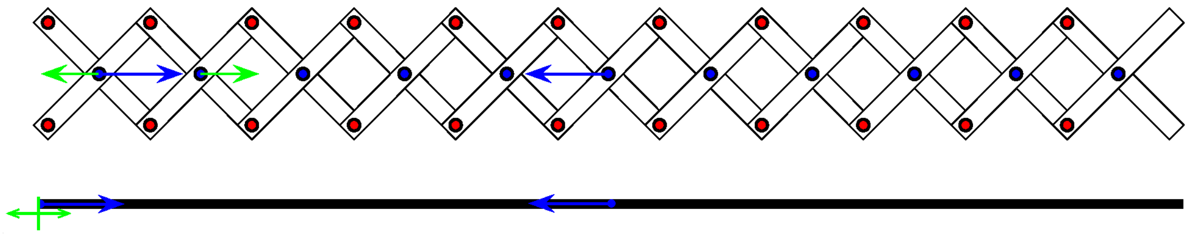

**Figure 8.** An axial force is applied at the middle point of a pantographic beam. On the left hand side of the pantograph, in addition to the reaction force, two opposite forces on the right hand side of the pantograph must be applied in order to enable the equilibrium of the structure. In the continuous limit (below) such additional action is a point-wise "double force".

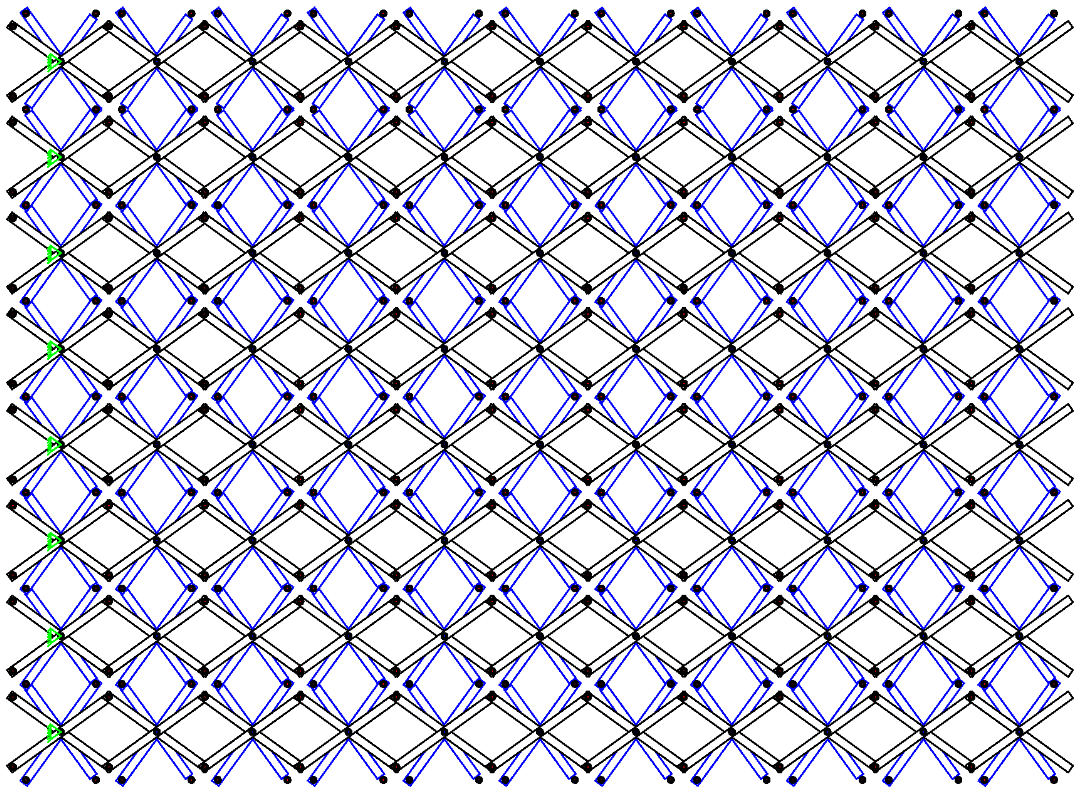

**Figure 9.** The pantographic 2D structure.

The elastic energy of the structure is the sum of the energies of all pantotographic beams:

$$
\begin{aligned}
E(u) = &\sum_{j=1}^{M}\sum_{i=2}^{N-1} k(u_1[i-1,j]-2u_1[i,j]+u_1[i+1,j])^2+k'(u_2[i-1,j]-2u_2[i,j]+u_2[i+1,j])^2\\
&+\sum_{j=2}^{M-1}\sum_{i=1}^{N} k(u_1[i,j-1]-2u_1[i,j]+u_1[i,j+1])^2+k'(u_2[i,j-1]-2u_2[i,j]+u_2[i,j+1])^2
\end{aligned}
$$

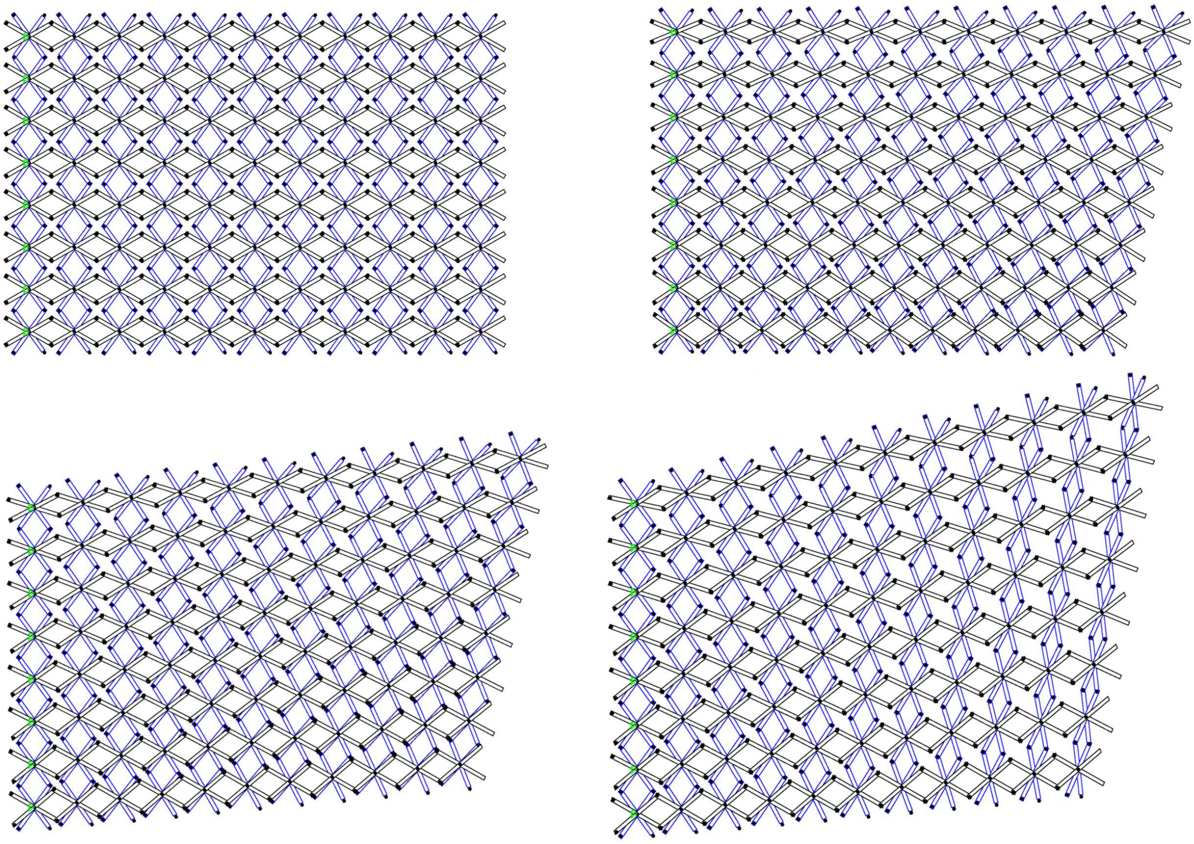

**Figure 10.** When imposing a Dirichlet condition on one side, the pantographic 2D structure has four floppy modes.

This clearly leads to the continuum model

$$\tilde{E}(u) = \int_\Omega K\left(\frac{\partial^2 u_2}{\partial x_1^2}\right)^2 + K'\left(\frac{\partial^2 u_2}{\partial x_2^2}\right)^2 + K'\left(\frac{\partial^2 u_1}{\partial x_1^2}\right)^2 + K\left(\frac{\partial^2 u_1}{\partial x_2^2}\right)^2$$

The floppy modes are of the form $u(x_1, x_2) = (ax_1x_2 + bx_1 + cx_2 + d, ex_1x_2 + fx_1 + gx_2 + h)$ which, taking into account the Dirichlet condition $u = 0$ on the boundary $x_1 = 0$, reduces to a four dimensional space of floppy modes:

$$u(x_1, x_2) = ((ax_2 + b)x_1, (ex_2 + f)x_1),$$

(see figure 10). Note that one can still act on the material at the fixed surface $x_1 = 0$ by applying a density $\mathcal{G}$ of torques (or fixing the "rotation $\frac{\partial u_2}{\partial x_1}$) or by applying a density $\mathcal{F}$ of "double-forces" (or fixing the "dilatation $\frac{\partial u_1}{\partial x_1}$). For instance, let us consider the extension to the 2D case of the situation described in figure 8. We apply a surface density of horizontal forces $f$ on the middle surface $S = \{x_1 = \ell/2\}$ adding in the energy the term $\int_S -fu_1dx_2$. The existence of an equilibrium is ensured only if we apply on the surface $x_1 = 0$ a surface density of horizontal forces $-f$ plus a surface density of double forces adding in the energy the terms $-\int_{S_l} fu_1dx_2$ and $-\int_{S_l} \mathcal{F}\frac{\partial u_1}{\partial x_1}dx_2$ with $\mathcal{F} = f\ell/2$).

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## 4. Third gradient structure

Up to now the continuous materials we have designed are second gradient material (see [21])that is materials with an elastic energy depending only on the first and second gradient of the displacement. It is possible, to the expense of some extra complexity, to design trusses leading to higher gradient material and consequently supporting more exotic mechanical interactions.

### *4.1. Third gradient flexion beam*

We first consider the basic element described in figure 11. Note that we build on the structure we already introduced (fig. 6) and remember that such a structure is only made of nodes and linear springs. The lowest horizontal bars are assumed to have infinite stiffness while all other ones have the same finite stiffness. One can easily check that the elastic energy

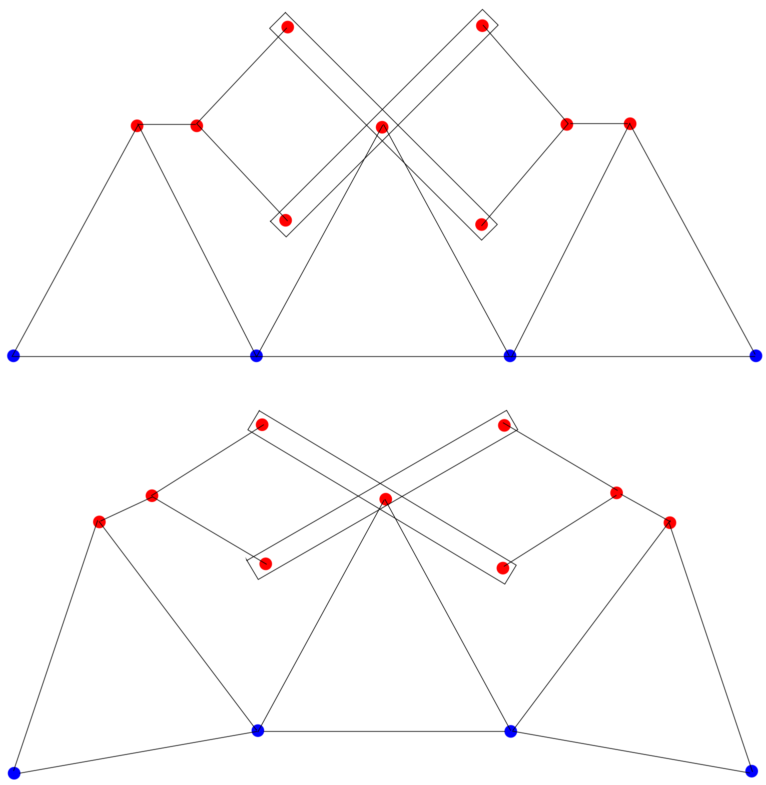
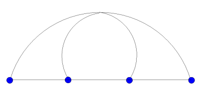


**Figure 11.** The basic element of the 3rd gradient beam and its major floppy mode.

of this element in terms of the displacements of the 4 lowest (red) nodes is proportional to $(u_2[i] - 3u_2[i+1] + 3u_2[i+2] - u_2[i+3])^2$.

In order to simplify the following drawings, we use the symbol  for representing this basic element. Then we construct a beam by link any four successive nodes of the beam by such basic elements (figure 12). It is non extensible ($u_1 = 0$) and its elastic energy reads

$$E(u) = \sum_{i=1}^{N-3} k(u_2[i] - 3u_2[i+1] + 3u_2[i+2] - u_2[i+3])^2.$$

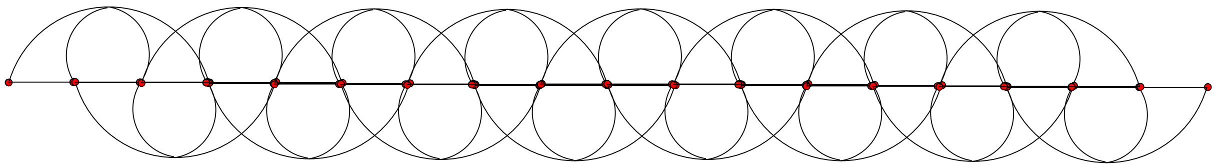


**Figure 12.** A 3rd gradient beam made by superposition of basic elements of figure 11.

When the two first nodes are fixed, a floppy mode remains which corresponds to deformations with constant curvature along the beam. This curvature is fixed for instance by fixing the displacement of the third node. One can also act on the beam by applying three vertical forces $(f, -2f, f)$ on three successive nodes. In the continuous limit we recognize the finite difference approximation of the third derivative of the transverse displacement and the energy reads

$$\tilde{E}(u) = \int_0^\ell K \left( \frac{\partial^3 u_2}{\partial x_1^3} \right)^3 .$$

*4.2. Third gradient 2D structure*

Consider now an array of many $(M)$ horizontal 3rd gradient beams (see figure 13), and link them using vertical elastic springs. We fix the two first nodes of each beam. Then we get a structure with one floppy mode. The elastic energy of the whole structure the sum of the elastic energies

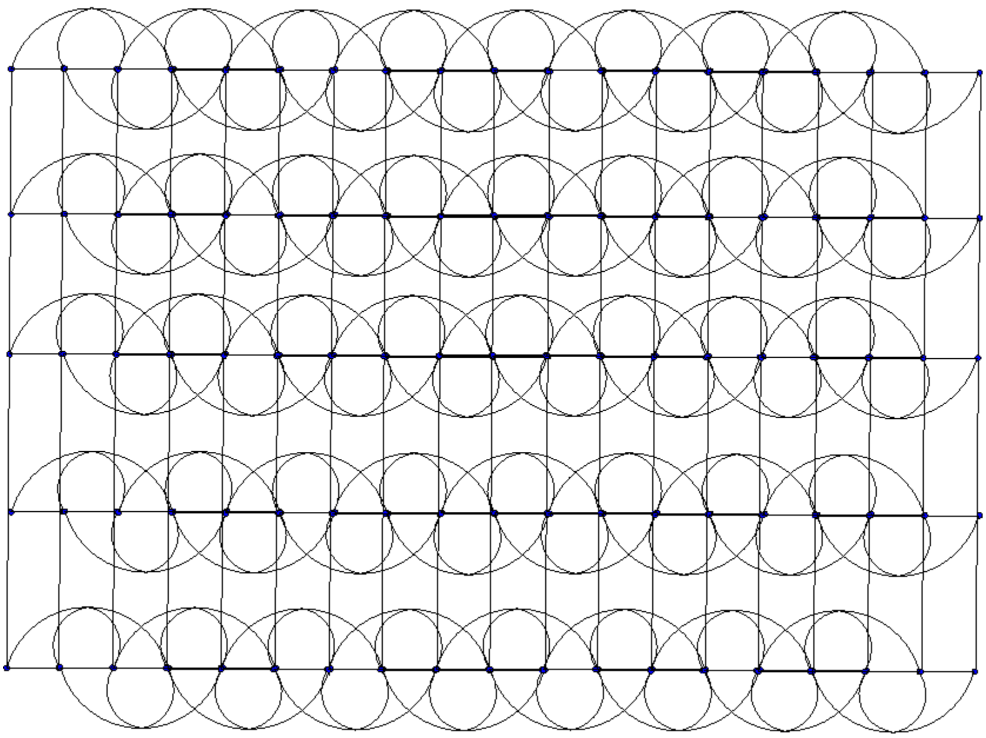


**Figure 13.** A 3rd gradient 2D structure.

of the beams we just derived plus the energy concentrated in the additional vertical junctions.

It takes the form

$$E(u) = \sum_{j=1}^{M}\sum_{i=1}^{N-3} k(u_2[i,j]-3u_2[i+1,j]+3u_2[i+2,j]-u_2[i+3,j])^2 + \sum_{j=1}^{M-1}\sum_{i=1}^{N} k'(u_2[i,j]-u_2[i,j-1])^2$$

and with a suitable scaling for $k$ and $k'$ we obtain the following continuous energy :

$$\tilde{E}(u) = \int_\Omega K\left(\frac{\partial^3 u_2}{\partial x_1^3}\right)^2 + K'\left(\frac{\partial u_2}{\partial x_2}\right)^2$$

with the constraints $u_1 = 0$ on the domain $\Omega$ and $u_2 = 0$, $\frac{\partial u_2}{\partial x_1} = 0$ on the boundary $x_1 = 0$.

Let us again consider the problem defined in section 2 (figure 5). The equilibrium solution is still different. First such a solution exists only if the third node of all beams is also fixed (or alternatively if a suitable vertical force is applied on them). This corresponds in the continuous model to adding an extra clamping constraint $\frac{\partial^2 u_2}{\partial x_1^2} = 0$ on the side $S_l := \{x_1 = 0\}$ (or alternatively to applying a suitable surface density of "triple-forces" on it, that is adding in the energy a term of the form $\int_{S_l} h\frac{\partial^2 u_2}{\partial x_1^2}dx_2$ ). This type of triple-forces can also be interpreted as "double-torques" and we use the symbol for them. In the equilibrium state the curvature increases in the left hand part of the domain and remains constant but not vanishing in the right hand part (cf. figure 14).

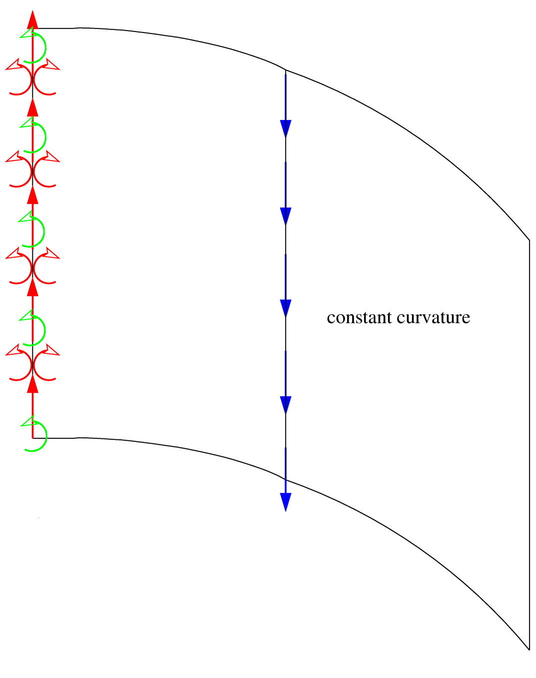



**Figure 14.** The equilibrium state of a flexion third gradient material clamped on the left hand side $S_l$ and submitted to a vertical force on the vertical middle surface.

## 5. Conclusion

We could carry on this enumeration of more and more intricate structures. For instance, building on the structural element introduced and symbolized in the previous section, we could consider

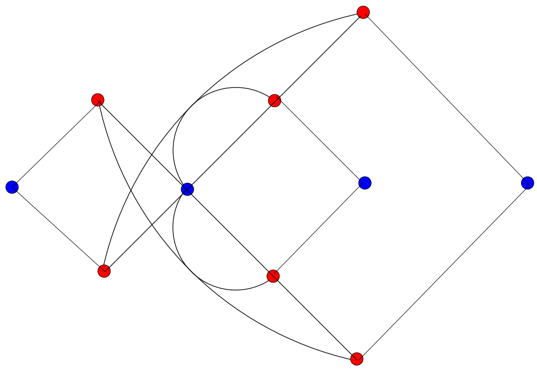


**Figure 15.** A structural element for a material sensitive to the second derivative of the dilatation.

the element described in figure 15. We let the reader check that it has a floppy mode which mimics the third derivative of $u_1$ with respect of $x_1$ (that is the second derivative of the dilatation) and that, introducing a new symbol to denote it, the element described in figure 16 has a floppy mode which mimics the fourth derivative of $u_2$ with respect of $x_1$. This enumeration

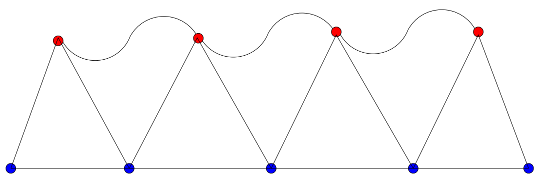


**Figure 16.** A structural element for a material sensitive to the fourth derivative of the transverse displacement.

would not be of great interest : the reader should yet be convinced by the examples we have presented that homogenization of trusses can lead to any high gradient materials.

In these materials, contact interactions between parts of the domain are far belong those described by the Cauchy stress theory. In our examples we have illustrated the possibility of surfaces densities of torques, “double-forces” or “triple-forces”. Our trusses give a clear microscopic interpretation of all these different actions : they are different distributional limits of highly oscillating forces. The denominations that P. Germain [21] gave to them are illustrated.

Let us recall that it has been rigorously proved in [4] that all types of high gradient materials can be achieved by homogenizing trusses.

The highest gradient terms are clearly active when the lower gradient terms do not play any role, that is when the structures possess floppy modes. The reader who prefer to work with non degenerated materials and thus to consider only trusses with no floppy mode (apart those corresponding to rigid motions) can add linear elastic links between each node and its neighbors. This will simply add a classical (first gradient) elastic term in the elastic energy. These supplementary links have to be much weaker than the links we have used for building our structures (the scaling with respect to $\epsilon$ is different). That explains why, for natural materials, these new types of interactions scarcely matter : to obtain a material in which a new type of action is preponderant, one must carefully build with very stiff links a structure having a floppy mode and then, possibly add some weaker links.

## References

[1] Abeyaratne R, Triantafyllidis N, 1984, *Trans. ASME J. Appl. Mech.* **51**, no. 3, 481–486.
[2] Alibert J-J, dell'Isola F and Seppecher P, 2003, *Math. Mech. Solids* **8** , no. 1, 51–73.
[3] Banfi C, Marzocchi A and Musesti A, 2006, *Ricerche di Matematica* **55** 299–310.
[4] Camar-Eddine M and Seppecher P, 2003, *Arch. Ration. Mech. Anal.*, **170** 3, 211 - 245.
[5] Cioranescu D and Rebnord D A, 1995, *Asymptotic Anal.*, **11**, 2, 107–130.
[6] Bardenhagen S and Triantafyllidis N, 1994, *J. Mech. Phys. Solids* **42** , no. 1, 111–139.
[7] Bensoussan A, Lions J L and Papanicolaou G, 1978, *Asymptotics Analysis for periodic Structures*, North-Holland Amsterdam.
[8] Bleustein J L, 1967, *International Journal of Solids and Structures*, **3** (6), pp. 1053-1057.
[9] Braides A and Gelli M S, 1999, *Preprint SISSA*, 41/99/M.
[10] Cioranescu D, Saint Jean Paulin J, 1989, *C.R.Acad.Sci. Paris I*, **308** no.2, 41-46.
[11] Cioranescu D, Saint Jean Paulin J, 1990, *Lecture notes in Control and Inform.Sci.*, **147**, Springer Verlag Berlin.
[12] Cioranescu D, Saint Jean Paulin J, 1992, *Lecture notes in Control and Inform.Sci.*, **178** Springer Verlag Berlin.
[13] Cioranescu D, Saint Jean Paulin J, 1999, *Applied Mathematical Sciences.*, **136** Springer Verlag New York.
[14] Cosserat E and Cosserat F, 1908, *Note sur la thorie de l'action euclidienne* Paris, Gauthier-Villars.
[15] Cosserat E and Cosserat F, 1909, *Théorie des Corps Déformables*, Paris, Hermann.
[16] Dell'Isola F and Seppecher P, 1995, *Comptes Rendus de l'Academie de Sciences - Serie IIb*, **321**, , pp. 303-308.
[17] Dell'Isola F and Seppecher P, 1997, *Meccanica*, **32**, pp. 33-52.
[18] Di Carlo A Rizzi N and Tatone A, 1990, *Meccanica*, **25**, 168-174.
[19] Degiovanni M, Marzocchi A and Musesti A, 2006, *Annali di Matematica***185**, 81–103.
[20] Degiovanni M, Marzocchi A and Musesti A, 1999, *Arch. Ration. Mech. Anal.* **147** 197–223.
[21] Germain P, 1973, *J. Mécanique*, **12**, 235–274.
[22] Germain P, 1973, *SIAM J. Appl. Math.*, **25** 3.
[23] Green A E and Rivlin R S, 1964, *Arch. Rational Mech. Anal.*, **16**, 325–353.
[24] Green A E and Rivlin R S, 1964, *Arch. Rational Mech. Anal.*, **17**, 113–147.
[25] Green A E and Rivlin R S, 1964, *Z. Angew. Math. Phys.*, **15**, 290–292.
[26] Lucchesi M, Silhav M and Zani N, 2008, *J Elasticity* **90**, 209–223.
[27] Marzocchi A and Musesti A, 2001, *Cont. Mech. Thermodyn.* **13**, 149–169.
[28] Marzocchi A and Musesti A, 2003, *Meccanica* **38**, 369–389.
[29] Maugin G and Metrikine A V, 2010, *Mechanics of Generalized Continua, One Hundred Years After the Cosserats*, Springer.
[30] Mindlin R D, 1964, *Arch. Rat. Mech. Analysis*, **16**, 51-78.
[31] Mindlin R D, 1965, *Int. J. Solids and Struct.* **1**, 4, 417-438.
[32] Mindlin R D and Eshel N N, 1968,*International Journal of Solids and Structures*, **4**, 1, 109-124.
[33] Mindlin R D and Tiersten H F, 1962, *Arch.Rat.Mech.Anal.*, **11**, 415-448.
[34] Noll W, 1959, *Proceeding of the Berkeley Symposium on the Axiomatic Method*, Amsterdam, 226–281.
[35] Noll W, 1973 *Arch. Rational Mech. Anal.*,**52** 62–92.
[36] Seppecher P, 1989, *C. R. Acad. Sci. Paris*, **309**, Srie II 497–502.
[37] Silhavy M, 1985, *Arch.Ration. Mech. Anal.*, **90** 195–211.
[38] Silhavy M, 1991, *Arch. Ration. Mech. Anal.*, **116** 223–255.
[39] Timoshenko S, 1976, *Strenght of Materials*, Krieger Publishing.
[40] Toupin R A, 1962 *Arch. Rat. Mech. Anal.*, **11**, 385-414.
[41] Toupin R A, 1964 *Arch. Rational Mech. Anal.*, **17** 85–112.
[42] Triantafyllidis N and Bardenhagen S, 1993, *J. Elasticity*, **33**.
[43] Triantafyllidis N and Bardenhagen S, 1996, *J. Mech. Phys. Solids*, **44** , no. 11, 1891–1928.
[44] Truesdell C A, 1977, *A First Course in Rational Continuum Mechanics, Vol. I General Concepts*, Academic Press.